Using Block Norms in Bicriteria Optimization

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Abstract

We propose to use block norms to generate nondominated solutions of multiple criteria programs and introduce the new concept of the oblique norm that is specially tailored to handle general problems. We show the applicability of oblique norms to deal with discrete or convex bicriteria programs and also discuss implications of using block norms in multiple criteria decision making.

Keywords: Bicriteria optimization, bicriteria programming, block norms, oblique norms, properly nondominated points.

1 Introduction

Compromise programming is based on the concept of identifying nondominated solutions of multiple criteria programs that are the closest to some utopia (ideal) point. Different norms have been used to measure the distance between the solutions and the utopia point. In particular, the family of L_p norms has been extensively studied by many researchers, including [Yu, 1973], [Zeleny, 1973], [Gearhart, 1979], [Wierzbicki, 1980], [Steuer and Choo, 1983], [Steuer, 1986], and many others. The l_∞ norm and the augmented l_∞ norm turned out to be very useful in generating nondominated solutions of general continuous or discrete multiple criteria programs and led to the well known weighted (augmented) Tchebycheff scalarization and its variations. [Kaliszewski, 1987] introduced a modified l_∞ norm and showed its applicability in generating nondominated solutions. Compromise programming was extended by [Szidarovszky et al., 1986] to composite programming using more

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than one value of p in the l_p distance. [Ballestero and Romero, 1998] analyzed connections between compromise programming and utility theory. [Carrizosa et al., 1996] proposed a new class of norms that contains the family of L_p norms to generate the set of points that have minimal distance to the utopia point with respect to at least one norm within this class of norms. Their approach leads to solving linear programs while generating nondominated solutions.

Not only have norms been beneficial in constructing scalarization approaches to multiple criteria programs but also become suitable tools supporting decision making. The choice of the utopia point and weights usually expresses decision maker's preferences in the objective space while selecting the most preferred nondominated solution. Applications of norm-based methods can be found in structural design [Miura and Chargin, 1996], water resource management [Bárdossy et al., 1985], manpower planning [Silverman et al., 1988], transportation and location [Ogryczak et al., 1988] and many other areas.

Motivated by the success of norm-based approaches in MCDM, we propose to apply block norms to generate nondominated solutions as well as to support the decision making process. The family of block norms, also called polyhedral norms, includes all the norms whose unit ball is a polyhedral set, so that the l_1 norm and the l_∞ norm are members of this family. In this paper, we introduce the concept of the oblique norm that can be viewed as a generalization of the augmented l_∞ norm. This new norm is designed to preserve capabilities of the l_∞ norm and the augmented l_∞ norm while allowing the decision maker more freedom in the choice of a distance measure.

In the next section we define the oblique norm and derive some properties useful for finding nondominated solutions. Section 3 contains the main results of the paper. We first examine relationships between nondominated solutions of a general multiple criteria program and optimal solutions of its scalarization by means of a block norm and an oblique norm. In the second part of this section we focus on bicriteria programs. In particular, we examine relationships between (properly) nondominated solutions of (finite) discrete problems and (polyhedral) convex problems and optimal solutions of related scalarizations by means of an oblique norm. At the end of this section we discuss practical implications of using block norms in MCDM and in Section 4 we highlight future research directions.

To facilitate further discussions, the following notation is used throughout the paper. Let $u, w \in \mathbb{R}^n$ be two vectors.

- We denote components of vectors by subscripts and enumerate vectors by superscripts.
- $u < w$ denotes $u_i < w_i$ for all $i = 1, \ldots, n$. $u \leq w$ denotes $u_i \leq w_i$ for all $i = 1, \ldots, n$, but $u \neq w$. $u \leqq w$ allows equality. The symbols $>, \geq, \geqq$ are used accordingly.

- Let $\mathbb{R}^n_{\geqq} := \{x \in \mathbb{R}^n : x \geqq 0\}$. If $S \subseteq \mathbb{R}^n$, then $S_{\geqq} := S \cap \mathbb{R}^n_{\geqq}$.
- $\langle u, w \rangle$ denotes the scalar product in $\mathbb{R}^n$: $\langle u, w \rangle = \sum_{i=1}^n u_i w_i$.
- $\text{conv}(S)$ denotes the convex hull of a set $S \subseteq \mathbb{R}^n$.
- $\text{int}(S)$ denotes the interior of $S \subseteq \mathbb{R}^n$.

We consider the following general multiple criteria program

$$\begin{array}{ll} \min & \{z_1 = f_1(x)\} \\ & \vdots \\ \min & \{z_n = f_n(x)\} \\ \text{s.t.} & x \in S, \end{array} \tag{1}$$

where $S \subseteq \mathbb{R}^m$ is the *feasible set* and $f_i(x), i = 1, \ldots, n$, are real-valued functions. We define the *set of all feasible criterion vectors* Z, the *set of all nondominated criterion vectors* N and the *set of all efficient points* E of (1) as follows

$$\begin{aligned} Z &= \{z \in \mathbb{R}^n : z = f(x), x \in S\} = f(S) \\ N &= \{z \in Z : \nexists \tilde{z} \in Z \text{ s.t. } \tilde{z} \leq z\} \\ E &= \{x \in S : f(x) \in N\}, \end{aligned}$$

where $f(x) = \big(f_1(x) \cdots f_n(x)\big)^T$. The set Z is assumed to be closed. The point $z^* \in \mathbb{R}^n$ with

$$z_i^* = \min\{f_i(x) : x \in S\} - \varepsilon_i \qquad i = 1, \ldots, n$$

is called the *ideal (utopia) criterion vector*, where the entries of $\varepsilon \in \mathbb{R}^n$ are small positive numbers. Without loss of generality we assume $z^* = 0$.

We define the set of properly nondominated solutions according to [Geoffrion, 1968]. A point $\bar{z} \in N$ is called *properly nondominated*, if there exists $M > 0$ such that for each $i = 1, \ldots, n$ and each $z \in Z$ satisfying $z_i < \bar{z}_i$ there exists a $j \neq i$ with $z_j > \bar{z}_j$ and

$$\frac{z_i - \bar{z}_i}{\bar{z}_j - z_j} \leq M.$$

Otherwise $\bar{z} \in N$ is called *improperly nondominated.* The set of all properly nondominated points is called N_p.

2 Oblique norms

In order to develop the new concept of oblique norms we first review some basic definitions about block norms. For a detailed introduction to norms and their properties we refer the reader to [Rockafellar, 1970], [Hiriart-Urruty and Lemaréchal, 1993a] and [Hiriart-Urruty and Lemaréchal, 1993b]. An overview of basic properties of block norms is also given in [Schandl, 1998].

Definition 2.1 A norm γ with a polyhedral unit ball in $\mathbb{R}^n$ is called a *block norm*. The vectors defined by the extreme points of the unit ball are called *fundamental vectors* and are denoted by v^i. The fundamental vectors defined by the extreme points of a facet of B span a *fundamental cone*.

Definition 2.2 Let $u \in \mathbb{R}^n$. The *reflection set* of u is defined as

$$R(u) := \{w \in \mathbb{R}^n : |w_i| = |u_i| \quad \forall i = 1, \dots, n\}.$$

Definition 2.3 [Bauer et al., 1961] A norm γ is said to be *absolute* if for any given $u \in \mathbb{R}^n$, all elements of $R(u)$ have the same distance from the origin with respect to γ, i. e.

$$\gamma(w) = \gamma(u) \quad \forall w \in R(u).$$

Note that the unit ball of an absolute norm has the same structure in every orthant, which is convenient as well as sufficient for multiple criteria programs as all nondominated solutions are located in the cone $z^* + \mathbb{R}^n_{\geqq}$ and one does not need to search the entire space $\mathbb{R}^n$.

Definition 2.4 A block norm γ with a unit ball B is called *oblique* if it has the following properties:

(i) γ is absolute.

(ii) $(z - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq} \cap \partial B = \{z\} \quad \forall z \in (\partial B)_{\geqq}$.

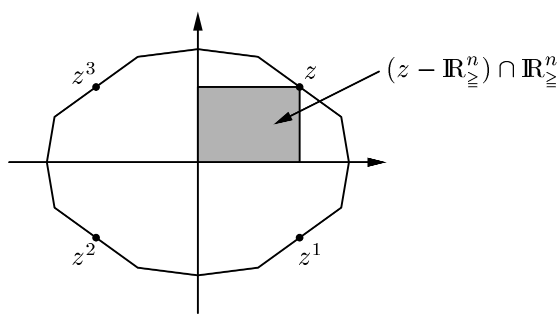

Fig. 1: Example of the unit ball of an oblique norm with $R(z) = \{z, z^1, z^2, z^3\}$

The following corollaries immediately result from Definitions 2.3 and 2.4.

Corollary 2.5 The number of fundamental vectors of an oblique norm γ in $B_{\geqq}$ is finite.

Corollary 2.6 If γ with the unit ball B is an oblique (absolute) norm, then $\tilde{\gamma}$ with the unit ball $\alpha B, \alpha > 0$ is also an oblique (absolute) norm.

The following lemmas are useful in developing our main results in the next section. Note that the condition (i) of Lemma 2.9 is identical with the condition (ii) of Definition 2.4.

Lemma 2.7 An oblique norm γ with the unit ball B has the following property:

$$(z - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq} \cap \partial(\gamma(z)B) = \{z\} \quad \forall z \in \mathbb{R}^n_{\geqq}.$$

Proof. Since $z \in \partial(\gamma(z)B)$, the statement follows directly from Definition 2.4 and Corollary 2.6. □

Lemma 2.8 An absolute norm γ with the unit ball B has the following property:

$$(z - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq} \subseteq \gamma(z)B_{\geqq} \quad \forall z \in \mathbb{R}^n_{\geqq}.$$

Proof. Consider first $z \in (\partial B)_{\geqq}$. It follows that $\gamma(z) = 1$. Since γ is absolute, all points in $R(z)$ are in B. Because of the convexity of B, we have $\text{conv}(R(z)) \subseteq B$. But $(z - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq}$ is a subset of $\text{conv}(R(z))$ and therefore also of $B_{\geqq}$.
The general case $z \in \mathbb{R}^n_{\geqq}$ follows again from Corollary 2.6. □

Lemma 2.9 Let γ be an absolute block norm with the unit ball B. Let $\mathcal{N}$ denote the set of outer normal vectors of all the facets of B. Let e^j be the j^{th} unit vector, $j = 1, \ldots, n$. Then the following two statements are equivalent:

(i) $(z - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq} \cap \partial B = \{z\} \quad \forall z \in (\partial B)_{\geqq}$.

(ii) $\langle \mathfrak{n}, e^j \rangle \neq 0 \quad \forall j = 1, \ldots, n$ and $\forall \mathfrak{n} \in \mathcal{N}$.

Proof.

(i) $\Rightarrow$ (ii) Let F be a facet of B with the normal vector $\mathfrak{n} \in \mathcal{N}$. Assume $\langle \mathfrak{n}, e^j \rangle = 0$ for some j. Then there exists a point $z \in F$ with $z_j \neq 0$ (otherwise F would not be a facet). Since γ is absolute, we can assume without loss of generality that $z \in \mathbb{R}^n_{\geqq}$. Define a point $\tilde{z}$ as follows:

$$\begin{aligned} \tilde{z}_k &= z_k \quad \forall k \neq j \\ \tilde{z}_j &= \tfrac{1}{2} z_j. \end{aligned}$$

Then $\tilde{z}$ is in $F \subseteq \partial B$, because γ is absolute. But we also have that

$$\tilde{z} \in (z - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq} \cap \partial B,$$

which is a contradiction to (i).

(ii) $\Rightarrow$ (i) Let $z \in (\partial B)_{\geqq}$ and assume there exists $\tilde{z} \neq z$ with

$$\tilde{z} \in (z - \mathbb{R}^n_{\geq}) \cap \mathbb{R}^n_{\geqq} \cap \partial B. \tag{2}$$

Because of Lemma 2.8 we have $\mathfrak{n} \geq 0$ for all normals of facets in $\mathbb{R}^n_{\geq}$. Together with $\langle \mathfrak{n}, e^j \rangle \neq 0$ for all j we even know that $\mathfrak{n} > 0$ for these same normals. Since we assumed that both z and $\tilde{z}$ are in ∂B, they are either on the same or on two different facets.

Assume first that z and $\tilde{z}$ are on the same facet F with the normal $\mathfrak{n}$. Consequently $\langle z - \tilde{z}, \mathfrak{n} \rangle = 0$, but since $z - \tilde{z} \geq 0$ and $\mathfrak{n} > 0$ it follows that $z = \tilde{z}$, a contradiction to our assumption.

Assume now that z and $\tilde{z}$ are on different facets, say F and $\tilde{F}$ with normals $\mathfrak{n}$ and $\tilde{\mathfrak{n}}$, respectively. Since $z \in B$ and $\tilde{z} \in \tilde{F}$, the definition of the outer normal yields $\langle z - \tilde{z}, \tilde{\mathfrak{n}} \rangle \leq 0$. But since $z - \tilde{z} \geq 0$ and $\tilde{\mathfrak{n}} > 0$ it follows again that $z = \tilde{z}$, a contradiction.

Thus $\tilde{z} \notin \partial B$ and assumption (2) was wrong. $\square$

3 Generating the Nondominated Set

3.1 General Results

We first show that for every nondominated point there exists a block norm so that this point is a unique minimizer of the related block-norm-scalarization. In the proof, to show the existence of the desired block norm we use the l_∞ norm, and thus not an oblique norm. The result gives another interpretation of the results on the weighted Tchebycheff approach in [Steuer, 1986] and illustrates the idea of introducing block norms to multiple criteria programming.

Theorem 3.1 Let $\bar{z} \in N$. Then there exists a block norm γ so that $\bar{z}$ uniquely minimizes

$$\min_{z \in Z} \gamma(z) = \min_{x \in S} \gamma(f(x)).$$

Proof. Recall that we assumed without loss of generality $z^* = 0$. Define the unit ball B of a block norm γ as $B = \operatorname{conv}(R(\bar{z}))$. Assume there is a $\tilde{z} \in Z$, $\tilde{z} \neq \bar{z}$ with $\gamma(\tilde{z}) \leq \gamma(\bar{z})$. From the construction of γ we have that $\tilde{z} \leqq \bar{z}$. Since $\tilde{z} \neq \bar{z}$, we have $\tilde{z}_i < \bar{z}_i$ for some i, which is a contradiction to $\bar{z} \in N$. Thus $\gamma(z) > \gamma(\bar{z})$ for all $z \in Z$. $\square$

We now focus on oblique norms and show that any optimal solution of the oblique-norm-scalarization of (1) is a nondominated solution of (1). The converse of this result is not true in general since oblique norms cannot be used to generate improperly nondominated points.

Theorem 3.2 Let γ be an oblique norm and let $\bar{z}$ be a solution of

$$\min_{z \in Z} \gamma(z) = \min_{x \in S} \gamma(f(x)).$$

Then $\bar{z} \in N$.

Proof. Assume $\bar{z} \notin N$. Then there exists $\tilde{z} \in Z$ with $\tilde{z} \leq \bar{z}$; therefore $\tilde{z} \in \left((\bar{z} - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq}\right) \setminus \{\bar{z}\}$. However, according to Lemma 2.7, we have $\{\bar{z}\} = \left((\bar{z} - \mathbb{R}^n_{\geqq}) \cap \mathbb{R}^n_{\geqq}\right) \cap \partial(\gamma(\bar{z})B)$. Thus $\tilde{z} \notin \partial(\gamma(\bar{z})B)$ and from Lemma 2.8, it follows that $\tilde{z} \in \operatorname{int}(\gamma(\bar{z})B)$. Therefore $\gamma(\tilde{z}) < \gamma(\bar{z})$, which is a contradiction to the minimality of $\bar{z}$. □

3.2 The Bicriteria Case

In this section we concentrate on bicriteria problems and show that there exists an oblique norm γ for every $z \in N_p \subseteq \mathbb{R}^2$ so that z uniquely minimizes

$$\min_{z \in Z} \gamma(z) = \min_{x \in S} \gamma(f(x)).$$

We study the cases where Z is a general discrete set, a finite discrete set, a convex polyhedral set and a general convex set. In each case we prove the existence of an oblique norm with the above mentioned property by constructing its unit ball.

Theorem 3.3 (Discrete case in $\mathbb{R}^2$) Let $Z \subseteq \mathbb{R}^2$ be discrete, $N_p \neq \varnothing$ and let $\bar{z} \in N_p$. Then there exists an oblique norm γ so that $\bar{z}$ uniquely minimizes

$$\min_{z \in Z} \gamma(z) = \min_{x \in S} \gamma(f(x)). \tag{3}$$

Proof. Consider the definition of properly nondominated points. For every $z \in N_p$ with $z_1 < \bar{z}_1$, we have $z_1 - \bar{z}_1 > -\bar{z}_1 > -\infty$, i.e. $z_1 - \bar{z}_1$ is finite. Thus $\bar{z}_2 - z_2 < 0$ cannot be arbitrarily close to zero. Therefore we can find a line through $\bar{z}$ with a slope smaller than 0 (and greater than -1) so that there does not exist a point $z \in N_p$ with $z_1 < \bar{z}_1$ below or on that line. Take the intersection point of this line and the z_2-axis as an extreme point of the unit ball B of γ and find an extreme point on the z_1-axis in an analogous way. The set of extreme points of B is then defined as the union of the reflection sets of the three mentioned points.

Due to the chosen slope of the boundary segments of B, the unit ball is convex and satisfies both conditions of Definition 2.4, so the resulting norm is an oblique norm. Since we constructed the boundary of B so that $\bar{z}$ is the only point in $N_p \cap B$, $\bar{z}$ minimizes (3) uniquely. □

Although we have given a general proof for the discrete case, it is interesting to demonstrate a construction of an oblique norm for the *finite* discrete case. The construction is described in Algorithm 3.4 while Lemma 3.5 and Theorem 3.6 show that the constructed norm is in fact an oblique norm so that $\bar{z}$ uniquely minimizes (3).

Algorithm 3.4 Let $Z \subseteq \mathbb{R}^2$ be discrete and finite, and let $\bar{z} \in N_p$.

Step 1: Finding the extreme points v of B with $v_1 \in [0, \bar{z}_1]$ and $v_2 \geq \bar{z}_2$.

If there does not exist a point $z \in N_p$ with $z_1 < \bar{z}_1$ below or on the line through $\bar{z}$ and $(0, z_2 + \alpha\bar{z}_1)$ where $0 < \alpha < 1$, then define $v^1 = \bar{z}$, $v^2 = (0, z_2 + \alpha\bar{z}_1)$ and goto Step 2.

Otherwise set $v^1 = \bar{z}$ and $i = 1$. Consider the following problem:

$$\begin{array}{ll} \min & z_2 \\ \text{s.t.} & 0 < z_1 < v_1^i \\ & z \in Z. \end{array} \tag{4}$$

Note that (4) is always feasible, because we consider it only if we have already found a point $z \in N_p$ with $z_1 < v_1^i$. Let v^{i+1} be the solution of (4).

If there does exist a point $z \in N_p$ with $z_1 < v_1^{i+1}$ below or on the line through v^i and v^{i+1}, then set $i = i + 1$ and consider again (4) to find subsequent extreme points. Otherwise redefine v^{i+1} as the intersection point of the z_2-axis and the line through v^i and v^{i+1}, i. e.

$$v^{i+1} \leftarrow \left(0, v_2^{i+1} - \frac{v_2^{i+1} - v_2^i}{v_1^{i+1} - v_1^i} v_1^{i+1}\right).$$

Step 2: Finding the extreme points v of B with $v_1 \geq \bar{z}_1$ and $v_2 \in [0, \bar{z}_2]$.

Get these extreme points in a similar way as in Step 1 by considering the following problem:

$$\begin{array}{ll} \min & z_1 \\ \text{s.t.} & 0 < z_2 < v_2^i \\ & z \in Z. \end{array} \tag{5}$$

Step 3: Finding the complete set of extreme points of B.

The entire set of extreme points of the unit ball B of γ is the union of the reflection sets of all the extreme points found in Steps 1 and 2.

Note that the procedure is finite, since Z and therefore N_p are both finite.

Lemma 3.5 The block norm constructed in Algorithm 3.4 is an oblique norm.

Proof. We first give two remarks:

(a) Each line segment between two consecutive extreme points v^{i+1} and v^i constructed in Step 1 has a negative slope, otherwise a point $z \in N_p$ with $z_1 < v_1^i$ and $z_2 \leq v_2^i$ would exist, which contradicts the construction of v^i using a nondominated point. An analogous result is valid for the points found in Step 2.

(b) The slope of the line segments between v^{i+1} and v^i constructed in Step 1 is always between 0 and -1 and increases with i. Since a slope change at v^{i+1} occurs only if there is a point $z \in N_p$ with $z_1 < v_1^{i+1}$ below the line through v^i and v^{i+1}, the slope can never decrease with i. An analogous result is valid for the points found in Step 2.

Because of remark (b), B is convex. Due to Step 3 of the algorithm, γ is an absolute norm. Due to remark (a) and Lemma 2.9, part (ii) of Definition 2.4 is satisfied, and by construction, part (i) of Definition 2.4 is satisfied as well. □

Theorem 3.6 (Finite discrete case in $\mathbb{R}^2$) Let $Z \subseteq \mathbb{R}^2$ be discrete and finite, let $\bar{z} \in N_p$. The point $\bar{z} \in N_p$ minimizes

$$\min_{z \in Z} \gamma(z) = \min_{x \in S} \gamma(f(x))$$

uniquely, where γ is the oblique norm constructed in Algorithm 3.4.

Proof. Follows directly from the construction of γ and Lemma 3.5. □

Theorem 3.7 (Convex polyhedral case in $\mathbb{R}^2$) Let $Z \subseteq \mathbb{R}^2$ be convex and polyhedral and let $\bar{z} \in N$. Then there exists an oblique norm γ so that $\bar{z}$ uniquely minimizes

$$\min_{z \in Z} \gamma(z) = \min_{x \in S} \gamma(f(x)). \tag{6}$$

Proof. Due to [Geoffrion, 1968], there exists a supporting line of Z at $\bar{z}$ with the normal vector $w > 0$. Define the two vectors $w^1 = (\alpha w_1, w_2)$ and $w^2 = (w_1, \alpha w_2)$ where $\alpha > 1$. Denote the line defined by the normal w^1 through $\bar{z}$ as l_1 and the line defined by the normal w^2 through $\bar{z}$ as l_2.

Take the intersection point of l_1 and the z_1-axis, the intersection point of l_2 and the z_2-axis, and the point $\bar{z}$ as extreme points of B in $\mathbb{R}^2_{\geqq}$ and get the entire set of extreme points of B by taking the union of the reflection sets of the three mentioned points.

Conditions (i) of Definition 2.4 is satisfied by construction. Since $w^1 > 0$ and $w^2 > 0$ and because of Lemma 2.9, B is convex and condition (ii) of Definition 2.4 is satisfied, so γ is oblique.

The point $\bar{z}$ minimizes (6) uniquely, because $\alpha > 1$ and therefore no other point of N can be in B. □

Theorem 3.8 (Convex case in $\mathbb{R}^2$) Let $Z \subseteq \mathbb{R}^2$ be convex and let $\bar{z} \in N_p$. Then there exists an oblique norm γ so that $\bar{z}$ uniquely minimizes

$$\min_{z \in Z} \gamma(z) = \min_{x \in S} \gamma(f(x)). \tag{7}$$

Proof. Since Z is convex and $\bar{z}$ is properly nondominated, there exists a supporting line of Z at $\bar{z}$ with a normal vector $w > 0$. We then proceed as in the proof of Theorem 3.7. □

3.3 Practical Implications

Having established theoretical foundations for applying block norms in bicriteria optimization we should turn our attention to the issue of enhancing the decision making process. Block norms can be viewed as a mathematical tool but also as a decision tool introducing a piecewise linear utility function in the objective space which minimized over the outcome set yields a most preferred nondominated solution. Piecewise linearity avoids computational difficulties when the utility function is nonlinear but on the other hand applies different utility to different regions of the objective space. As the number of the fundamental directions of a block norm and their length can be easily changed, the resulting utility function can be easily modified before the decision process starts or in the course of the process. This flexibility allows decision makers to change their preferences while searching for a most preferred solution.

Furthermore, block norms are dense in the set of all norms in $\mathbb{R}^n$, see [Ward and Wendell, 1985], so that any norm in $\mathbb{R}^n$ can be approximated arbitrarily close by a block norm, a feature again helpful in representing or approximating complex decision maker's preferences.

Last but not least, block norms can be helpful in exploring the objective space in several directions simultaneously, which can be beneficial in MCDM with multiple decision makers or in designing parallel algorithms for MCDM.

4 Conclusions

In this paper we introduced block norms into multiple criteria programming. We also defined oblique norms, a new class of block norms specially designed to generate properly nondominated solutions. These norms are absolute and have a unit ball whose boundary is determined by hyperplanes with normal vectors never parallel nor perpendicular to the coordinate axes of the objective space. This property makes the norms suitable to represent finite nonzero trade-offs between nondominated solutions.

We showed a general relationship between nondominated solutions and solutions of the scalarization by means of an oblique norm. Specific results are presented for bicriteria problems. We also briefly discussed the application of block norms in MCDM.

We will generalize the results of this paper for the multiple criteria case and will also study continuous nonconvex problems. In the future, we plan to develop block-norm-based approaches to MCDM which make use of these norms' flexibility and versatility.

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