A unified model for Weber problems with continuous and network distances

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Abstract

Continuous location problems and network location problems are generally viewed as completely different classes of problems. We will show in this paper that despite the classical distinction between continuous and discrete optimization, there are many similarities that can be exploited both for the development of new location models and for the derivation of theoretical properties and solution methods. This interrelation gives rise to a new line of research combining ideas from the fields of continuous and network location.

Keywords: *Weber problem, mixed distances, barriers, embedded networks, mixed integer bilinear programming*

1 Introduction

Characteristic for location problems is that the objective function depends on the *distances* between new and existing facilities. It models, for example, the travel cost between existing and new facilities, or the general accessibility of the new facilities. Two classical approaches can be distinguished:

Continuous Models assume that existing as well as new facilities can be represented by points in the Euclidean plane $\mathbb{R}^2$ (or — as needed, for example, if antennas are to be located — in the three dimensional space $\mathbb{R}^3$). The new location can be placed anywhere in some specified *feasible region* which often coincides with the complete plane. Travel costs are commonly modeled by some distance metric like the Euclidean metric or the Manhattan metric.

Network Models are based on a given transportation network. The existing facilities are represented by nodes of this network, and new facilities can be placed either only on the nodes or on nodes and edges of the network. A cost value can be associated with every edge of the network modeling, for example, the travel time between the respective nodes. Accordingly, shortest paths in the network serve as a *network distance* function for network location problems.

The need for realistic representations of distance measures in location problems is reflected in the recent literature. Continuous models have been extended by various types of restrictions and constraints in order to better incorporate the geographic reality into the geometric representation. Location problems with *forbidden regions* have been extensively studied and can be considered relatively well-solved (for an overview, see Hamacher and Nickel, 1995). On the other hand, problems involving physical barriers or congestions still give rise to many open questions that are caused by the non-convexity of the objective function (see Sarkar et al. (2004) for problems with congestions and Klamroth (2002) for a detailed survey on location problems with barriers).

If on the other hand a road or transportation network serves as the basis for a (network or discrete) location model, a balance between the size of the network (and the resulting computational complexity) and the accuracy of the model has to be found (see, for example Drezner and Hamacher, 2002). Moreover, the topology of the underlying network has a profound impact on the optimal facility locations.

Very little has been done to include continuities and/or additional modeling parameters in network location models. Batta and Palekar (1988) extended a network location model by adding so-called mega nodes which can be entered and left only at a finite set of access points. Inside a (not necessarily convex) mega node distances are measured based on rectilinear paths while network distances are used outside the mega nodes. It is shown that this problem can be reduced to a 1-median or p-median problem, respectively, on a suitably extended network. Erkut (1992) added a finite candidate set for new locations outside a given transportation network. Travel distances are measured partly on the network, but — in order to model continuous propagations, for example, of polluted air from an industrial plant — continuous metrics are used in addition to the network metric in the objective function. Blanquero et al. (2000) further extended this model by defining a convex feasible region for new locations replacing the finite candidate set. The focus in both papers is mainly on model formulations and on existence results. Similarly, Berman et al. (2000) consider a transportation network together with a set of points of potential hazards which may be located either on or off the network. Eight different routing and location problems are modeled and solved. In the two comprised location problems, the new location is restricted to the network while the distance to the points of hazard is measured using squared Euclidean (continuous) distances. Drezner and Wesolowsky (1996) present an obnoxious facility location problem where one new facility is to be located in the (continuous) convex hull of a planar network such that the weighted Euclidean distance to any node or arc of the network is maximized. A graphical solution approach based on growing forbidden regions around arcs and nodes of the network is presented, and implemented in the framework of a bisection search algorithm.

A continuous location problem based on the superposition of a (polyhedral) gauge distance function and a finite set of so-called rapid transit lines modeling, for example, a high-speed transportation network, is suggested by Carrizosa and Rodriguez-Chia (1997) who derive a mixed integer programming formulation for the problem; see Section 1.2 for a more detailed review of this approach.

In this paper, we survey the main properties of continuous location problems with added discontinuities (barriers) on one hand, and of network location problems (rapid transit lines) with added continuities on the other hand. Based on the representation of both problem classes as mixed integer programming problems, a unified problem formulation is suggested that can be viewed as a unifying umbrella under which continuous location models *and* network location models can be described.

In the following section, we will first review some useful results for Weber problems with barriers. Based on mixed integer programming formulations for problems with polyhedral barriers and making use of the visibility graph of the problem, we will then span the bridge to generalized network location models as suggested by Carrizosa and Rodriguez-Chia (1997) where continuous choices for new facility locations also outside the underlying transportation network are allowed, leading to the formulation of a unified model for both problems. Section 2 is devoted to the analysis of theoretical properties of the unified model. Special attention is given to the case that continuous distances are measured by block norms since in this case further simplifications of the mathematical formulation are possible that facilitate the development of exact solution approaches.

1.1 The Weber problem with polyhedral barriers

Given a finite set of existing facilities $\mathcal{A} = \{a_1, \ldots, a_n\} \subset \mathbb{R}^2$ with positive weights $w_1, \ldots, w_n \in \mathbb{R}$, the classical, continuous *Weber problem* is to find one new facility $x \in \mathbb{R}^2$ such that the weighted sum of distances between x and the existing facilities at $a_1, \ldots, a_n$ is minimized:

$$\begin{array}{ll} \min & f(x) = \sum_{m=1}^{n} w_m d(x, a_m) \\ \text{s.t.} & x \in \mathbb{R}^2. \end{array} \tag{1}$$

In order to obtain a realistic estimate for the distances between the new location and the existing facilities, we assume that a finite set of polyhedral barriers is given in $\mathbb{R}^2$ and that traveling is prohibited in the interior of these barriers. Barriers may model, for example, rivers, lakes, mountain ranges, or, on a smaller scale, conveyor belts or large machines in an industrial plant.

Let $\{B_1, \ldots, B_b\}$ denote a finite set of pairwise disjoint polyhedral barrier sets with a finite number of extreme points in $\mathbb{R}^2$, and let $\mathcal{B} = \bigcup_{i=1}^{b} B_i$ be the union of these barriers. The *feasible region* is given by $\mathcal{F} := \mathbb{R}^n \setminus \text{int}(\mathcal{B})$. To avoid infeasible cases we assume that $\mathcal{F}$ is connected and that all existing facilities $a_1, \ldots, a_n$ are in $\mathcal{F}$. Moreover, let d be a metric which is induced by a norm $\|\cdot\|_d : \mathbb{R}^2 \to \mathbb{R}$. Then the *barrier distance* $d_{\mathcal{B}}(x, y)$ between two points $x, y \in \mathcal{F}$ is defined as the length of a shortest path between x and y which does not intersect the interior of $\mathcal{B}$. Formally, let P be a *permitted x-y-path* in $\mathcal{F}$, i.e., a curve connecting x and y and not intersecting the interior of a barrier. Let p be a piecewise continuous differentiable parameterization of P, with $p : [a, b] \to \mathbb{R}^2$, $a, b \in \mathbb{R}$,

$a < b$, $p(a) = x$, $p(b) = y$ and $p([a, b]) \cap \mathcal{B} = \emptyset$. Then $d_{\mathcal{B}}$ is given by

$$d_{\mathcal{B}}(x, y) := \min \left\{ \int_a^b \|p'(t)\|_d \mathrm{d}t : P \text{ permitted } x\text{-}y\text{-path} \right\}$$

A permitted x-y-path with length $d_{\mathcal{B}}(x, y)$ is called a *d-shortest permitted x-y-path*. It can be shown (see, for example, Klamroth, 2001) that $d_{\mathcal{B}}$ is a metric on the feasible region $\mathcal{F}$, and that there always exists a d-shortest permitted x-y-path with the following property:

Barrier Touching Property (BTP):
There always exists a d-shortest permitted path connecting x and y that is a piecewise linear path with breaking points only in extreme points of barriers.

Two points x and y in $\mathcal{F}$ are called *d-visible* if they satisfy $d_{\mathcal{B}}(x, y) = d(x, y)$. The set of points $y \in \mathcal{F}$ that are not d-visible from a point $x \in \mathcal{F}$ is called the *shadow of x with respect to d*, i.e.,

$$\text{shadow}_d(x) := \{y \in \mathcal{F} : d_{\mathcal{B}}(x, y) > d(x, y)\}.$$

Using barrier distances in the problem formulation of the Weber problem (1), we can now formulate the *Weber problem with polyhedral barriers*:

$$\begin{aligned} \min \quad & f_{\mathcal{B}}(x) = \sum_{m=1}^{n} w_m d_{\mathcal{B}}(x, a_m) \\ \text{s.t.} \quad & x \in \mathcal{F}. \end{aligned} \tag{2}$$

Note that while the objective function of the unconstrained Weber problem (1) is convex, the Weber problem with barriers (2) is a non-convex problem.

Based on the barrier touching property, Klamroth (2001) developed a reduction result for Weber problems with polyhedral barriers that decomposes the feasible region into subregions based on visibility properties:

Definition 1.1 *Let $x \in \mathcal{F}$ be a candidate site for the new facility location and let $a_m \in \mathcal{A}$ be one of the existing facilities. An* intermediate point *i_{x,a_m} is a point different from x that is an existing facility or an extreme point of a barrier that lies on a d-shortest permitted x-a_m-path with the barrier touching property and that is d-visible from x.*

If x and a_m are d-visible, the intermediate point i_{x,a_m} can be chosen as a_m. If $a_m = x$, then $i_{x,a_m} := a_m$.

Intermediate points are not necessarily unique and depend on the prescribed metric. Consider the example of one triangular barrier illustrated in Figure 1 for two different metrics, namely the Manhattan metric, i.e., $d(x, y) = l_1(x, y) = |y_1 - x_1| + |y_2 - x_2|$, and the Euclidean metric, i.e., $d(x, y) = l_2(x, y) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$. The dotted x-a_m-path is an l_1-shortest as well as an l_2-shortest permitted x-a_m-path with the barrier touching property. With respect to the Manhattan metric, i^1_{x,a_m} and i^2_{x,a_m} are candidates for intermediate

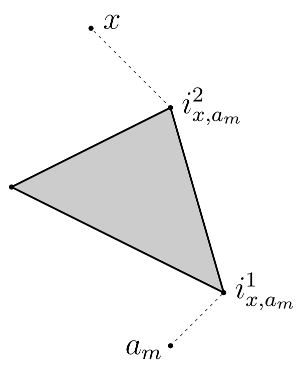

Figure 1: Intermediate points for a triangular barrier with respect to different metrics.

points since they are both l_1-visible from x, whereas i^2_{x,a_m} is the unique intermediate point on an l_2-shortest permitted x-a_m-path.

We say that an intermediate point i_{x,a_m} *is assigned to* a_m if a d-shortest permitted x-a_m-path with the barrier touching property passes through i_{x,a_m}.

Definition 1.1 implies that

$$d_{\mathcal{B}}(x, a_m) = d(x, i_{x,a_m}) + d_{\mathcal{B}}(i_{x,a_m}, a_m). \tag{3}$$

Since only existing facilities and extreme points of barriers are candidates for intermediate points, the constant distances $d_{\mathcal{B}}(i_{x,a_m}, a_m)$ can be computed in a preprocessing phase and stored in a distance matrix D. If now the feasible region $\mathcal{F}$ is decomposed into a finite number of subregions such that the same subset of candidates for intermediate points is d-visible from a complete subregion, subproblems are obtained that consist of the selection of a new facility location x in one subregion *and* the assignment of appropriate (visible) intermediate points to this facility. A subdivision of the feasible region $\mathcal{F}$ with the desired property is obtained by introducing a grid $\mathcal{G}_d \subset \mathbb{R}^2$ that is composed of the boundaries of the shadows of all existing facilities in $\mathcal{A}$ and of all points in the set $\mathcal{P}(\mathcal{B})$ of the extreme points of the barrier regions plus the facets of the barrier regions $\mathcal{F}(\mathcal{B})$:

Definition 1.2 *The grid*

$$\mathcal{G}_d := \left(\bigcup_{x \in \mathcal{A} \cup \mathcal{P}(\mathcal{B})} \partial\big(\text{shadow}_d(x)\big) \right) \cup \mathcal{F}(\mathcal{B})$$

is called the visibility grid *with respect to $\mathcal{A}$ and $\mathcal{B}$. The set of cells of $\mathcal{G}_d$, i.e., the set of all polyhedra with nonempty interior in $\mathcal{F}$ induced by $\mathcal{G}_d$, that are not intersected by a line segment in $\mathcal{G}_d$, is denoted by $\mathcal{C}(\mathcal{G}_d)$.*

As was shown in Klamroth (2001), the Weber problem with polyhedral barriers can be formulated as a mixed integer programming problem for each of the cells $C \in \mathcal{C}(\mathcal{G}_d)$. Within a cell $C \in \mathcal{C}(\mathcal{G}_d)$ the corresponding subproblem consists of finding an optimal location for the new facility and assigning the optimal intermediate points to the existing facilities $a_1, \ldots, a_n$:

Let $C \in \mathcal{C}(\mathcal{G}_d)$ be a given cell and let $\mathcal{I} := \{i_1, \ldots, i_k\} \subseteq \mathcal{A} \cup \mathcal{P}(\mathcal{B})$ be that subset of candidates for intermediate points that are d-visible from all points in C. The binary variables $y_{rm},\ r = 1, \ldots, k,\ m = 1, \ldots, n$, are defined as

$$y_{rm} = \begin{cases} 1, & i_r \text{ is used as intermediate point } i_{x,a_m}, \\ 0, & i_r \text{ is not used as intermediate point } i_{x,a_m}, \end{cases} \qquad r = 1, \ldots, k,\ m = 1, \ldots, n.$$

Then the Weber problem with barriers restricted to C can be written as

$$\begin{aligned} \min\quad & \sum_{m=1}^{n} w_m \left(\sum_{r=1}^{k} y_{rm} \big(d(x, i_r) + d_{\mathcal{B}}(i_r, a_m) \big) \right) \\ \text{s.t.}\quad & \sum_{r=1}^{k} y_{rm} = 1, \quad m = 1, \ldots, n, \\ & x \in C, \\ & y_{rm} \in \{0, 1\}, \quad r = 1, \ldots, k,\ m = 1, \ldots, n, \end{aligned} \tag{B}$$

where the barrier distances $d_{\mathcal{B}}(i_r, a_m)$ are constant for each pair $(i_r, a_m) \in \mathcal{I} \times \mathcal{A}$.

An optimal solution $x^*_{\mathcal{B}} \in C$ of problem (2) is an optimal solution of problem (B) (Klamroth, 2001). Hence, problems of type (2) can be represented by $|\mathcal{C}(\mathcal{G}_d)|$ mixed integer programming problems, each of them restricted to a cell $C \in \mathcal{C}(\mathcal{G}_d)$. The optimal solution of (2) is the minimum of the optimal solutions of these mixed integer problems.

1.2 The Weber problem with embedded networks

In order to formulate a unified model for Weber problems with polyhedral barriers on the one hand and for Weber problems with embedded networks on the other hand, we will review the model and extend the problem formulation introduced by Carrizosa and Rodriguez-Chia (1997). It will be shown that the *Weber problem with embedded networks* can be formulated as a problem of the same mathematical structure as the Weber problem with polyhedral barriers as described in Section 1.1.

The Weber problem with embedded networks introduced by Carrizosa and Rodriguez-Chia (1997) extends the classical Weber problem (1) by allowing traveling in the plane $\mathbb{R}^2$ as well as on an embedded transportation network. The network may be used to represent rapid transit lines as given, for example, by a subway system. If a network connection is used whenever this improves the overall travel time, the resulting distance measure is a mixture of a metric d induced by a norm in $\mathbb{R}^2$ and the network distance on the transportation network. The objective is to place one new facility (or several new facilities) in the plane such that the sum of transportation costs is minimized.

More formally, let $\mathcal{A} = \{a_1, \ldots, a_n\} \subseteq \mathbb{R}^2$ be the set of existing facilities, $x \in \mathbb{R}^2$ the new facility and let d be a metric induced by a norm in $\mathbb{R}^2$. The embedded network $G(\mathcal{N}, E)$ is given by a finite set of nodes $\mathcal{N} = \{n_1, \ldots, n_o\} \subseteq \mathbb{R}^2$ and undirected edges $(n_p, n_q) \in E,\ n_p, n_q \in \mathcal{N}$. The cost of an edge $(n_p, n_q) \in E$ is given by $k(n_p, n_q)$. Note that the embedding of G in the plane $\mathbb{R}^2$ and in particular the coordinates of the nodes in $\mathcal{N}$ play a central role for the computation of the overall distance function. Different from Carrizosa and Rodriguez-Chia (1997) we do not assume that G is a complete graph, or even a connected graph.

Let d_G denote the network distance in G, i.e., $d_G(n_p, n_q)$ is the cost of a minimum cost network path between two nodes n_p and n_q in G. We set $d_G(n_p, n_q) = \infty$ whenever n_p and n_q are nodes from two disconnected components of G. Not only in the case that G is disconnected we may encounter practical situations where traveling between two nodes $n_p, n_q \in \mathcal{N}$ is faster if the network connection is not used, i.e.,

$$d_G(n_p, n_q) \gtrless d(n_p, n_q), \quad n_p, n_q \in \mathcal{N}. \tag{4}$$

This assumption, that extends the model of Carrizosa and Rodriguez-Chia (1997), seems to be realistic since, for example, a public transportation network may not directly connect each station with every other station and sometimes it is faster to walk between two subway stations rather than taking the corresponding subway line.

For convenience we suppose that the metric d and the cost function on the edges in E are given in the same units. Then the *direct transportation cost* c between two points $x, y \in \mathbb{R}^2$ can be defined as

$$c(x,y) := \begin{cases} d(x,y) & \text{if } x \notin \mathcal{N} \ \vee \ y \notin \mathcal{N}, \\ d_G(x,y) & \text{if } x, y \in \mathcal{N} \text{ and } d_G(x,y) \le d(x,y), \\ d(x,y) & \text{if } x, y \in \mathcal{N} \text{ and } d_G(x,y) > d(x,y). \end{cases} \tag{5}$$

Consequently, the *minimum transportation cost* $d_{\mathcal{G}}$ between two points $x, y \in \mathbb{R}^2$ that allows multiple changes between network travel and continuous travel is given by

$$d_{\mathcal{G}}(x,y) := \inf_{t_1,\ldots,t_r \in \mathbb{R}^2:\, r \text{ finite}} c(x,t_1) + c(t_1,t_2) + \cdots + c(t_{r-1},t_r) + c(t_r,y). \tag{6}$$

It is easy to see that $d_{\mathcal{G}}$ defines a metric on $\mathbb{R}^2$.

Based on this definition of a distance function, the *Weber problem with embedded networks* can be formulated as

$$\begin{aligned} \min \quad & f_G(x) = \sum_{m=1}^{n} w_m d_{\mathcal{G}}(x, a_m) \\ \text{s.t.} \quad & x \in \mathbb{R}^2. \end{aligned} \tag{7}$$

As in the case of the Weber problem with polyhedral barriers (2), the objective function of problem (7) is non-convex since the distance function $d_{\mathcal{G}}$ is non-convex. The problem is further complicated by the fact that, due to (4), there may exist minimum cost paths with more than two transshipment points between network travel and continuous travel. To overcome this difficulty and to simplify the evaluation of the distance function $d_{\mathcal{G}}$ we introduce an extended network G' that yields the same overall distance function:

Definition 1.3 *Let d be a metric induced by a norm in $\mathbb{R}^2$, let $G = (\mathcal{N}, E)$ be an embedded transportation network, and let $d_{\mathcal{G}}$ be the minimum transportation cost in $\mathbb{R}^2$ according to (6). Then the* extended transportation network $G' = (\mathcal{I}, E')$ *consists of*

(a) the node set $\mathcal{I} := \mathcal{N} \cup \mathcal{A}$, i.e., all existing facilities are included in $\mathcal{I}$,

(b) edges (a_m, n_p) of cost $d(a_m, n_p)$ for all $a_m \in \mathcal{A} \backslash \mathcal{N}$, $n_p \in \mathcal{N}$, i.e., each existing facility that is not in $\mathcal{N}$ is connected to all nodes in $\mathcal{N}$,

(c) edges (n_p, n_q) of cost $k(n_p, n_q)$ for all $(n_p, n_q) \in E$ with $d(n_p, n_q) \geq k(n_p, n_q)$,

(d) edges (n_p, n_q) of cost $d(n_p, n_q)$ for all $n_p, n_q \in \mathcal{N}$ with $d(n_p, n_q) < d_G(n_p, n_q)$, i.e., if traveling between two nodes is faster if the transportation network G is not used, the corresponding edge length is set to $d(n_p, n_q)$. (If the edge (n_p, n_q) is contained in E its edge length is only updated, otherwise a new edge with this length is added to E'.)

Observe that G' is a connected graph. Moreover, the network distance $d_{G'}$ in G' that assigns the cost of a minimum cost network path in G' to each pair of nodes in G' satisfies

$$d_{G'}(x, y) \leq d(x, y) \quad \forall\, x \in \mathcal{N},\, y \in \mathcal{I}. \tag{8}$$

It is easy to see that the network distance $d_{G'}$ has all properties of a metric in G'.

The following lemma shows that G' can indeed be used to represent minimum transportation costs between nodes in $\mathcal{N}$ and $\mathcal{I}$, respectively:

Lemma 1.1 *Let $x \in \mathcal{N}$ and $y \in \mathcal{I}$. Then $d_{G'}(x, y) = d_{\mathcal{G}}(x, y)$.*

Proof. Let $x \in \mathcal{N}$ and $y \in \mathcal{I}$. The definition of the extended transportation network immediately implies that $d_{G'}(x, y) \geq d_{\mathcal{G}}(x, y)$.

To show that also $d_{G'}(x, y) \leq d_{\mathcal{G}}(x, y)$, let $t_0 := x$, $t_{r+1} := y$, and let $t_1, \ldots, t_r \in \mathbb{R}^2$ be a set of transshipment points on a minimum cost path with respect to $d_{\mathcal{G}}$ (c.f. (6)), i.e.,

$$d_{\mathcal{G}}(x, y) = c(x, t_1) + c(t_1, t_2) + \cdots + c(t_{r-1}, t_r) + c(t_r, y),$$

where $c(t_i, t_{i+1})$ denotes the direct transportation cost according to (5), $i = 0, \ldots, r$.
Case 1: $t_1, \ldots, t_r \in \mathcal{N}$. Then (5) and Definition 1.3 (b), (c) and (d) directly imply that $c(t_i, t_{i+1}) = d_{G'}(t_i, t_{i+1})$, $i = 0, \ldots, r$, and hence $d_{G'}(x, y) \leq d_{\mathcal{G}}(x, y)$.
Case 2: $\exists s \in \{1, \ldots, r\} \;:\; t_s \notin \mathcal{N}$. Then (5) together with the fact that d satisfies the triangle inequality in $\mathbb{R}^2$ yield

$$c(t_{s-1}, t_s) + c(t_s, t_{s+1}) = d(t_{s-1}, t_s) + d(t_s, t_{s+1}) \geq d(t_{s-1}, t_{s+1}) \geq c(t_{s-1}, t_{s+1}),$$

and hence

$$d_{\mathcal{G}}(x, y) \geq c(x, t_1) + \cdots + c(t_{s-2}, t_{s-1}) + c(t_{s-1}, t_{s+1}) + c(t_{s+1}, t_{s+2}) + \cdots + c(t_r, y).$$

After finitely many iterations of this procedure we obtain Case 1. □

Consequently, the minimum transportation cost from an arbitrary point $x \in \mathbb{R}^2$ to an existing facility $a_m \in \mathcal{A} \subseteq \mathcal{I}$ can be represented by a minimum transportation cost path that enters the network G' at an *access node* denoted by $i_{x,a_m} \in \mathcal{I}$ and continues on the network G' until it reaches the existing facility a_m. This access node is not necessarily unique, and it may coincide with the existing facility a_m if no edge of the network G is used on the minimum cost path from x to a_m. Carrizosa and Rodriguez-Chia (1997) showed a similar result for the original transportation network G.

Lemma 1.2 *The minimum transportation cost from $x \in \mathbb{R}^2$ to any existing facility $a_m \in \mathcal{A}$ can be computed as*

$$d_{\mathcal{G}}(x, a_m) = d(x, i_{x,a_m}) + d_{G'}(i_{x,a_m}, a_m), \tag{9}$$

where $i_{x,a_m} \in \mathcal{I}$ is the first node of a minimum cost path from x to a_m that is a node of G' and that does not leave G' between i_{x,a_m} and a_m.

Proof. Since $a_m \in \mathcal{I}$, every path from x to a_m enters G' at some point. Let $i_{x,a_m} \in \mathcal{I}$ be the first node of a minimum transportation cost path from x to a_m that is a node of G' and that does not leave G' between i_{x,a_m} and a_m.
Case 1: $i_{x,a_m} = a_m$. Then $d_{\mathcal{G}}(x, a_m) = d_{\mathcal{G}}(x, i_{x,a_m}) + 0 = d(x, i_{x,a_m}) + d_{G'}(i_{x,a_m}, a_m)$.
Case 2: $i_{x,a_m} \in \mathcal{N}$. Using Lemma 1.1 we obtain

$$d_{\mathcal{G}}(x, a_m) = d_{\mathcal{G}}(x, i_{x,a_m}) + d_{\mathcal{G}}(i_{x,a_m}, a_m) = d(x, i_{x,a_m}) + d_{G'}(i_{x,a_m}, a_m).$$

Case 3: $i_{x,a_m} \in \mathcal{I} \setminus \mathcal{N}$, $i_{x,a_m} \neq a_m$. In this case, the choice of the access node implies that the next node on the corresponding minimum cost path to a_m is another node of G', say node n_i. Since according to Definition 1.3 all existing facilities in $\mathcal{I} \setminus \mathcal{N}$ are only connected to nodes in $\mathcal{N}$, we have $n_i \in \mathcal{N}$ and, using again Lemma 1.1,

$$\begin{aligned} d_{\mathcal{G}}(x, a_m) &= d_{\mathcal{G}}(x, i_{x,a_m}) + d_{\mathcal{G}}(i_{x,a_m}, n_i) + d_{\mathcal{G}}(n_i, a_m) \\ &= d(x, i_{x,a_m}) + \underbrace{d_{G'}(i_{x,a_m}, n_i) + d_{G'}(n_i, a_m)}_{\geq d_{G'}(i_{x,a_m}, a_m)} \\ &\geq d(x, i_{x,a_m}) + d_{G'}(i_{x,a_m}, a_m). \end{aligned}$$

From the definition of the minimum transportation cost $d_{\mathcal{G}}$ given by (6) follows

$$d_{\mathcal{G}}(x, a_m) \leq d(x, i_{x,a_m}) + d_{G'}(i_{x,a_m}, a_m)$$

and combining both inequalities yields

$$d_{\mathcal{G}}(x, a_m) = d(x, i_{x,a_m}) + d_{G'}(i_{x,a_m}, a_m). \qquad \square$$

Lemma 1.2 implies that minimum transportation cost distances can be decomposed into a continuous part $d(x, i_{x,a_m})$ and a constant part $d_{G'}(i_{x,a_m}, a_m)$ similar to the case of barrier distances, c.f. (3). The constant distances $d_{G'}(i_{x,a_m}, a_m)$ can be computed in a preprocessing phase and stored in a distance matrix D. Consequently, the Weber problem with embedded networks can be formulated as a mixed integer programming problem with continuous location variables $x \in \mathbb{R}^2$ and binary decision variables $y_{rm} \in \{0,1\}$, $r = 1,\dots,k := |\mathcal{I}|$ and $m = 1,\dots,n$, assigning an optimal access node to each x-a_m-path:

$$y_{rm} = \begin{cases} 1, & i_r \text{ is used as access node } i_{x,a_m}, \\ 0, & i_r \text{ is not used as access node } i_{x,a_m}, \end{cases} \quad r = 1\dots,k, \quad m = 1,\dots,n.$$

This yields the following formulation for the Weber problem with embedded networks:

$$\begin{array}{ll} \min & \sum_{m=1}^{n} w_m \Big(\sum_{r=1}^{k} y_{rm} \big(d(x,i_r) + d_{G'}(i_r,a_m) \big) \Big) \\ \text{s.t.} & \sum_{r=1}^{k} y_{rm} = 1, \quad m = 1,\dots,n, \\ & x \in \mathbb{R}^2, \\ & y_{rm} \in \{0,1\}, \quad r = 1,\dots,k, \; m = 1,\dots,n. \end{array} \tag{N}$$

Note that in this problem formulation the number of binary variables was reduced by a factor of $|\mathcal{N}|$ as compared to the formulation given in Carrizosa and Rodriguez-Chia (1997). This reduction was possible due to the definition of the extended network G' (Definition 1.3) and the fact, that only one access node onto G' has to be considered for paths from a new location at x to an existing facility at $a_m \in \mathcal{A}$ (Lemma 1.2).

1.3 A unified model

The mathematical structure of problems (B) and (N) is very similar. Using this similarity both problems can be represented by the same, unified model:

$$\begin{array}{ll} \min & \sum_{m=1}^{n} w_m \Big(\sum_{r=1}^{k} y_{rm} \big(d(x,i_r) + \alpha_{rm} \big) \Big) \\ \text{s.t.} & \sum_{r=1}^{k} y_{rm} = 1, \quad m = 1,\dots,n, \\ & x \in X, \\ & y_{rm} \in \{0,1\}, \quad r = 1,\dots,k, \; m = 1,\dots,n, \end{array} \tag{U}$$

where $\alpha_{rm} \in \mathbb{R}$, $r = 1\dots,k$, $m = 1,\dots,n$, are given constants, and $X \subseteq \mathbb{R}^2$, $X \neq \emptyset$, is a closed set, the feasible region for new location. The set $\mathcal{I} = \{i_1,\dots,i_k\} \subset \mathbb{R}^2$ is a finite set of given facilities.

If model (U) represents one of the problems (B) or (N), respectively, then $\mathcal{I}$ contains the candidates for the intermediate points with respect to the cell C or for the access nodes onto the extended network G', respectively, and α_{rm} and X are defined in a problem dependent way as specified in Table 1.

Table 1: The definition of α_{rm} and X depends on the problem.

	(B)	(N)
α_{rm}	$d_{\mathcal{B}}(i_r, a_m)$	$d_{G'}(i_r, a_m)$
X	$C \in \mathcal{C}(\mathcal{G}_d)$	$\mathbb{R}^2$

Theorem 1.3 *The Weber problem with polyhedral barriers (B) and the Weber problem with embedded networks (N) can be represented by the unified model (U).*

Proof. Follows immediately from the definition of problems (B), (N), and (U), respectively. □

In addition to problems (B) and (N), the unified model (U) can also be used to represent other, structurally different, types of location problems. One example is the *concentrator location problem with storage costs* (C): Let n commodities and k warehouses i_r be given in the plane $\mathbb{R}^2$. Then the objective is to optimally locate a distribution center, and to optimally assign warehouses to which the commodities are distributed from this center. In addition to the transportation cost between the center and the respective warehouses measured by a prescribed metric d, the storage of commodity m in warehouse i_r causes a fixed cost of $\alpha_{rm} \geq 0$. The total cost is given by the sum of all transportation and storage costs. Note that concentrator location problems (C) differ from barrier problems (B) and embedded network problems (N) in the sense that the transit costs α_{rm} do not represent any distances.

The unified model is particularly interesting since it combines features from continuous and network location models. Since most practical applications are neither exclusively continuous nor discrete, it facilitates both the modeling of location problems as well as their solution since algorithms developed for the unified model are widely applicable. Examples for possible solution strategies for the unified model are given in Section 2.

In order to represent the different problem types (B), (N), or (C) by the unified model (U), the problem specific characteristics have to be incorporated into the specification of the sets $\mathcal{I}$ and X and of the parameters α_{rm} in (U). The main difference between problem (B) on one hand and problems (N) and (C) on the other hand can be seen in the fact that in the presence of barriers traveling is not permitted everywhere in $\mathbb{R}^2$. This is reflected in the respective definition of the feasible region X for new location, see item 3. below. In the following a summary of the respective differences and similarities between the three problem types is given.

1. (B) $d(x,y) \leq d_{\mathcal{B}}(x,y) \quad \forall\, x,y \in \mathbb{R}^2$,

   (N) $d(x,y) \gtrless d_{\mathcal{G}}(x,y) \quad \forall\, x,y \in \mathbb{R}^2$,

   i.e., barriers can only increase the length of a shortest x-y-path, whereas embedded networks can increase or reduce the transportation cost between x and y.

   (C) The distance measure d and the transit costs α_{rm} are not correlated.

2. (B) $i_r \in \mathcal{I} \subseteq \mathcal{P}(\mathcal{B}) \cup \mathcal{A}$,
   i.e., not necessarily all existing facilities and barrier extreme points are contained in the set of intermediate points $\mathcal{I}$ that are visible from a point $x \in \mathbb{R}^2$.

   (N) $i_r \in \mathcal{I} = \mathcal{N} \cup \mathcal{A}$,
   i.e., all existing facilities and all nodes of G are contained in the set of possible access nodes $\mathcal{I}$, independently of the location of $x \in \mathbb{R}^2$.

   (C) $i_r \in \mathcal{I} \subseteq \mathbb{R}^2$.

3. (B) $x \in C$,
   i.e., x is restricted to lie in a bounded polyhedron $C \subseteq \mathbb{R}^2$. Therefore, an optimal solution of (B) is in general only a local optimal solution of the Weber problem with polyhedral barriers (2). A global optimal solution of the Weber problem with polyhedral barriers can be generated by solving problem (B) for all cells $C \in (\mathcal{G}_d)$.

   (N) $x \in \mathbb{R}^2$,
   i.e., an optimal solution of (N) is a global optimal solution of the Weber problem with embedded networks (7).

   (C) $x \in \mathbb{R}^2$,
   i.e., an optimal solution of (C) is a global optimal solution of the concentrator problem with storage costs.

2 Properties of the unified model

The main difficulty of the Weber problem with barriers and of the Weber problem with embedded networks is the non-convexity of the objective function. To overcome this difficulty this section is devoted to the derivation of general properties of the unified model (U) that facilitate the development of solution methods for both problems.

2.1 Relation to classical Weber problems

Even though the unified model (U) is more general than the Weber problem with polyhedral barriers (B) and the Weber problem with embedded networks (N) in the sense that not every problem instance of (U) originates from a problem of type (B) or (N), the unified model (U) shares one of the central properties of these two original models: An optimal solution of *any* instance of problem (U) can be found by solving a *finite* series of Weber problems (1) with some constraints on the feasible region for new location:

Theorem 2.1 *Any problem of type (U) can be solved by solving a finite series of Weber problems (1) with a finite set of existing facilities $\mathcal{A} \subseteq \{i_1, \ldots, i_k\}$, $|\mathcal{A}| \leq k$, and with the additional constraint that x is restricted to the feasible region for new location X.*

Proof. For any feasible assignment $\bar{y}$ of binary values to the variables y with $\sum_{r=1}^{k} \bar{y}_{rm} = 1$, $m = 1, \ldots, n$, the optimal values of x can be found by solving a Weber problem (1) with existing facilities at i_r, $r = 1, \dots, k$, and weights $\bar{w}_r := \sum_{m=1}^{n} \bar{y}_{rm} w_m$, and with the additional constraint $x \in X$. Since only finitely many feasible assignments for y exist, the result follows. $\square$

Note that $\bar{w}_r$ may be zero for some values of r. Then the corresponding existing facilities i_r have no impact on the solution of the related Weber problem (1) and can be omitted.

Theorem 2.1 relates the unified model (U) to the classical Weber problem (1) with a feasible region or, as better known from the literature, to the Weber problem with forbidden regions, see Hamacher and Nickel (1995). This relation will be used in the following sections to transfer properties of the classical Weber problem (1) to the unified model (U).

2.2 Convex hull properties

Particularly, if general solution methods are applied to problem (U), a reduction of the set of optimal locations to a smaller subset of $\mathbb{R}^2$ can significantly improve the computational efficiency. Corresponding results for problems (B) and (N), respectively, are based on the convex hull of the set $\mathcal{I}$ (see Klamroth, 2001; Carrizosa and Rodriguez-Chia, 1997) and can be extended to the unified model (U).

Theorem 2.2 *Let* $\mathrm{conv}(\mathcal{I}) \subseteq X$ *and let* d *be a metric induced by a norm such that the unconstrained Weber problem (1) has the* convex hull property*, i.e., the set of optimal solutions of (1) is contained in the convex hull* $\mathrm{conv}(\mathcal{A})$ *of the existing facilities. Then every optimal solution of problem (U) is contained in the convex hull* $\mathrm{conv}(\mathcal{I})$ *of* $\mathcal{I}$.

Proof. According to Theorem 2.1, the solution of problem (U) can be reduced to the solution of a finite number of Weber problems (1) with the feasible set X and with existing facilities that form different subsets of the set $\mathcal{I}$. Hence, the result follows from the assumption. $\square$

The assumption of Theorem 2.2 is satisfied for a large class of metrics. A well-known example is the class of l_p metrics with $1 < p < \infty$, see Juel and Love (1983). On the other hand there exist some metrics as for example the l_1 and the l_∞ metric for which only a weaker convex hull property holds. However, a similar result to that given in Theorem 2.2 can be proven in this case:

Theorem 2.3 *Let* $\mathrm{conv}(\mathcal{I}) \subseteq X$ *and let* d *be a metric induced by a norm such that the unconstrained Weber problem (1) has the* weak convex hull property*, i.e., at least one optimal solution of (1) is contained in the convex hull* $\mathrm{conv}(\mathcal{A})$ *of the existing facilities. Then at least one optimal solution of problem (U) is contained in the convex hull* $\mathrm{conv}(\mathcal{I})$ *of* $\mathcal{I}$.

Proof. Analogous to Theorem 2.2. $\square$

Note that Juel and Love (1983) and Wendell and Hurter (1973) showed that at least the weak convex hull property is satisfied for *all* Weber problems (1) with a metric d induced by a norm.

2.3 Integrality of the solution

Besides the non-convexity of the objective function, a further difficulty of the unified model (U) are the integrality constraints on the variables y_{rm}, $r = 1, \dots, k$, $m = 1, \dots, n$. We will show in this section that the integrality constraints $y_{rm} \in \{0, 1\}$ can be relaxed to $0 \le y_{rm} \le 1$ for all $r = 1, \dots, k$, $m = 1, \dots, n$.

For this purpose consider the continuous relaxation of the unified model (U):

$$\begin{aligned} \min \quad & \sum_{m=1}^{n} \left(\sum_{r=1}^{k} y_{rm} w_m \big(d(x, i_r) + \alpha_{rm} \big) \right) \\ \text{s.t.} \quad & \sum_{r=1}^{k} y_{rm} = 1, \qquad m = 1, \dots, n, \\ & x \in X, \\ & 0 \le y_{rm} \le 1, \quad r = 1, \dots, k, \ m = 1, \dots, n. \end{aligned} \tag{10}$$

Theorem 2.4 *If the set of optimal solutions of (10) is nonempty, then there exists at least one optimal solution x^*, y^* of (10) which satisfies*

$$y^*_{rm} \in \{0, 1\}, \quad r = 1, \dots, k, \ m = 1, \dots, n.$$

Proof. Let x^*, y^* be an optimal solution of (10). Suppose that y^* is not integer, i.e.,

$$\exists\, t \in \{1, \dots, n\}, \ j, l \in \{1, \dots, k\} : 0 < y^*_{jt} < 1, \ 0 < y^*_{lt} < 1 \text{ and } y^*_{jt} + y^*_{lt} \le 1.$$

Hence the objective value of x^*, y^* can be computed as

$$y^*_{jt} w_t \big(d(x^*, i_j) + \alpha_{jt} \big) + y^*_{lt} w_t \big(d(x^*, i_l) + \alpha_{lt} \big) \\ + \underbrace{\sum_{\substack{m=1 \\ m \ne t}}^{n} \sum_{r=1}^{k} y^*_{rm} w_m \big(d(x^*, i_r) + \alpha_{rm} \big) + \sum_{\substack{r=1 \\ r \ne j,l}}^{k} y^*_{rt} w_t \big(d(x^*, i_r) + \alpha_{rt} \big)}_{=:C}$$

Case 1: One of the paths from x^* and a_t through the intermediate points i_j and i_l, respectively, is shorter/cheaper than the other. Without loss of generality suppose that

$$d(x^*, i_j) + \alpha_{jt} < d(x^*, i_l) + \alpha_{lt}.$$

Inserting this inequality into the objective function leads to

$$y^*_{jt} w_t \big(d(x^*, i_j) + \alpha_{jt} \big) + y^*_{lt} w_t \big(d(x^*, i_l) + \alpha_{lt} \big) + C \ > \ \underbrace{(y^*_{jt} + y^*_{lt})}_{=:\bar{y}_{jt}} w_t \big(d(x^*, i_j) + \alpha_{jt} \big) + C.$$

Since the solution $\bar{x}, \bar{y}$ with $\bar{x} := x^*$, $\bar{y}_{rm} := y^*_{rm}$ $\forall (r, m) \notin \{(j, t), (l, t)\}$, $\bar{y}_{jt} := y^*_{jt} + y^*_{lt}$ and $\bar{y}_{lt} := 0$ is feasible for (10), this contradicts the optimality of x^*, y^*.

Case 2: Both paths from x^* to a_t through the intermediate points i_j and i_t, respectively, have the same length:

$$d(x^*, i_j) + \alpha_{jt} = d(x^*, i_l) + \alpha_{lt}.$$

Define a new solution $\bar{x}, \bar{y}$ of (10) as $\bar{x} := x^*$, $\bar{y}_{rm} := y^*_{rm}$ $\forall (r,m) \notin \{(j,t),(l,t)\}$, $\bar{y}_{jt} := y^*_{jt} + y^*_{lt}$ and $\bar{y}_{lt} := 0$. The objective value of $\bar{x}, \bar{y}$ is the same as of x^*, y^*, and $\bar{y}$ has at least one more integer component. After finitely many iterations of this procedure either Case 1 or an integer optimal solution is obtained. □

2.4 The case of block norms

If the prescribed metric d is induced by a block norm, then the piecewise linearity of the resulting distance function allows for further simplifications of the objective function of the unified model (U). We consider a block norm $\|\cdot\|_S$ whose unit ball S is a polytope with extreme points $v_g \in \mathbb{R}^2$, $g = 1, \dots, s$. Following the definition of Ward and Wendell (1985), block norm distances are given by

$$d(x, i_r) = \min\left\{ \sum_{g=1}^{s} \beta_{gr} : i_r - x = \sum_{g=1}^{s} \beta_{gr} v_g,\ \beta_{gr} \geq 0 \right\}. \tag{11}$$

Using (11) yields the following formulation of (U):

$$\begin{array}{lll}
\min & \sum_{m=1}^{n} w_m \left(\sum_{r=1}^{k} y_{rm} \left(\sum_{g=1}^{s} \beta_{gr} + \alpha_{rm} \right) \right) & \\
\text{s.t.} & \sum_{r=1}^{k} y_{rm} = 1, & m = 1, \dots, n, \\
& x \in X, & \\
& y_{rm} \in \{0,1\}, & r = 1, \dots, k,\ m = 1, \dots, n, \\
& i_{rp} - x_p = \sum_{g=1}^{s} \beta_{gr} v_{gp}, & r = 1, \dots, k,\ p = 1, 2, \\
& \beta_{gr} \geq 0, & g = 1, \dots, s,\ r = 1, \dots, k,
\end{array} \tag{UB}$$

where $i_r = (i_{r1}, i_{r2})^T$, $i_r \in \mathcal{I}$, are the candidates for the intermediate points with respect to the cell C or for the access nodes onto the extended network G', respectively, and $v_g = (v_{g1}, v_{g2})^T$, $g = 1, \dots, s$, are the fundamental directions of the prescribed block norm.

If we additionally assume that the feasible set X for new location is a bounded polyhedron, (UB) is a mixed integer bilinear programming problem with linear constraints. Note that this assumption is not very restrictive even though it is not explicitly contained in the formulation of (U). If $X = \mathbb{R}^2$, Theorems 2.2 and 2.3 imply that the (redundant) constraint $x \in \text{conv}(\mathcal{I})$ can be added to the model without changing the optimal objective value. According to Theorem 2.4, the binary constraints on y can be omitted such that a bilinear programming problem is obtained.

There are several methods to generate exact solutions of problem (UB). One is geometrically motivated and based on discretization due to the construction line grid (see Pfeiffer and Klamroth, 2005, for details). Another approach is to linearize the bilinear objective function and then to solve the resulting mixed integer linear programming problem. We have applied two different linearization methods to problem (UB). A first approach using the Reformulation Linearization Technique (RLT) introduced by Sherali and Adams (1998) is discussed by Pfeiffer and Klamroth (2005). This transformation results in a mixed integer linear programming problem with $\mathcal{O}(kn)$ binary variables, $\mathcal{O}(k^2ns)$ continuous variables and $\mathcal{O}(k^2ns)$ linear constraints.

A second approach that is based on the linearization technique proposed by Chang and Chang (2000) is presented in the following. It turns out to be well-suited for problem (UB) since the assignment constraints on the binary variables $\sum_{r=1}^{k} y_{rm} = 1,\ m = 1, \ldots, n$, can be incorporated in order to generate a more compact linearized problem. The transformation yields the following problem

$$
\begin{array}{lllr}
\min & \sum\limits_{m=1}^{n} w_m z_m & & \\
\text{s.t.} & z_m \geq \left(\alpha_{rm} + \sum\limits_{g=1}^{s} \beta_{gr} \right) - M(1 - y_{rm}), & r = 1, \ldots, k,\ m = 1, \ldots, n, & \text{(i)} \\
& \sum\limits_{r=1}^{k} y_{rm} = 1, & m = 1, \ldots, n, & \\
& x \in X, & & \text{(UBL)} \\
& y_{rm} \in \{0, 1\}, & r = 1, \ldots, k,\ m = 1, \ldots, n, & \\
& i_{rp} - x_p = \sum\limits_{g=1}^{s} \beta_{gr} v_{gp}, & r = 1, \ldots, k,\ p = 1, 2, & \\
& \beta_{gr} \geq 0, & g = 1, \ldots, s,\ r = 1, \ldots, k, & \\
& z_m \geq 0, & m = 1, \ldots, n, & \text{(ii)}
\end{array}
$$

with a constant M chosen sufficiently large, e.g.,

$$
M = \max_{\substack{m=1,\ldots,n \\ r=1,\ldots,k \\ x \in X}} \left\{ \alpha_{rm} + d(x, i_r) \right\}.
$$

The formulation (UBL) contains a set of n additional continuous variables $z_m,\ m = 1, \ldots, n$, together with n corresponding nonnegative constraints, see (ii) in (UBL). The additional $k \cdot n$ constraints (i) guarantee the equivalence between (UB) and (UBL). For fixed m, exactly one binary variable y_{rm} out of the group of k variables is equal to one. For this pair (r, m) constraint (i) is active and forces z_m to be greater than or equal to the composed distance between the new facility x and the existing facility a_m related to the intermediate point i_r. Since the objective is to minimize over the weighted sum of the variables z_m, this yields an equivalent representation. (UBL) is a mixed integer linear programming problem with $\mathcal{O}(kn)$ binary variables, $\mathcal{O}(n + ks)$ continuous variables and $\mathcal{O}(kn)$ linear constraints. Both linearization techniques, the RLT and formulation (2.4), are applied to an illustrative example.

2.4.1 Example

The following example problem shows a location problem in the city of Nuremberg, Germany, with two different sets of existing facilities, where one new location according to the Weber objective has to be found. Walking distances in the inner part of the city can be well approximated by the rectilinear distance function l_1. Since in addition the subway system provides an alternative mode of transportation, the location problem can be well represented by a Weber problem with embedded networks (N).

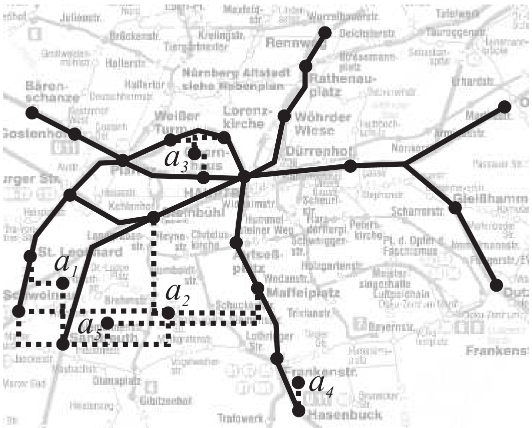

Figure 2: Example for a Weber problem with embedded networks in the city of Nuremberg, Germany.

The network G of the example problem shown in Figure 2 consists of the main lines of the public transportation system in the city of Nuremberg, Germany, with estimated edge costs of one third of the (Euclidean) track length. Distances outside the network are measured by the l_1-metric, i.e., $d(x, y) = l_1(x, y)\ \forall x, y \in \mathbb{R}^2$. For five existing facilities $a_1, \ldots, a_5$ with equal demand $w_m = 1,\ m = 1, \ldots, 5$, the extended network G' is illustrated in Figure 2, where only those edges that are relevant for the computation of the distance matrix $D_{G'}$ (solid and dotted lines) are shown.

An optimal solution for this example problem is generated by solving the linearized problem obtained by using RLT as well as formulation (UBL) derived in Section 2.4 with CPLEX 9.1, see Figure 3. The solution of the problem with six existing facilities $a_1, \ldots, a_6$ with equal demand $w_m = 1,\ m = 1, \ldots, 6$, is shown in Figure 4. The optimal paths are represented by dashed lines. The computational time for the problem with five existing facilities is in the case of the problem linearized by RLT 254 seconds and for formulation (UBL) 10.1 seconds. The problem with six existing facilities is solved in the RLT case in 912 seconds and using model (UBL) in 119 seconds. We can conclude that in this example, the linearization technique of Chang and Chang (2000) requires significantly fewer variables and constraints, and hence computational time.

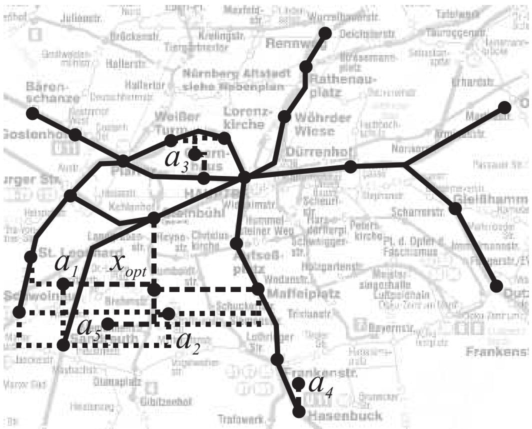

Figure 3: Optimal solution x_{opt} for the example problem from Figure 3.

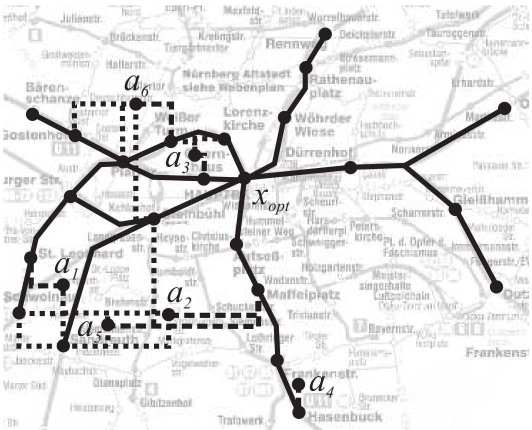

Figure 4: Optimal solution x_{opt} for the example with six existing facilities.

3 Conclusions and future research

In this paper we have discussed a unified model for Weber problems with distance measures that combine continuous and network distances in a very general way. Two special cases of this model are Weber problems with polyhedral barriers which fall under the class of continuous location problems, and Weber problems with embedded networks which are closely related to network location problems. Using the same problem formulation for continuous problems on one hand and network problems on the other hand opens up new possibilities for model development as well as solution techniques. We derive theoretical properties of the unified model and suggest algorithmic approaches for the case that continuous distances are measured by block norms.

The solution methods presented in this paper are exact solution methods and applicable only to small problem instances. This is caused by the fact that problem (UBL) is in general strongly non-convex and mixed integer. Therefore, future research should focus on modeling issues, including the derivation of further theoretical properties, stronger problem formulations and valid inequalities, as well as heuristic approaches as, for example, iterative location-allocation heuristics (see, for example, Fleischmann, 2004), decomposition methods (Plastria, 1992) or evolutionary algorithms (Bischoff and Klamroth, 2005).

Different transformations of the ideas presented in this paper to objective functions other than the Weber objective suggest themselves. One example are multifacility location problems as discussed in the case of Weber problems with embedded networks by Carrizosa and Rodriguez-Chia (1997). Other examples include the center objective as well as ordered Weber functions and multicriteria models.

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