A Bi-Objective Median Location Problem with a Line Barrier

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Abstract

The multiple objective median problem (MOMP) involves locating a new facility with respect to a given set of existing facilities so that a vector of performance criteria is optimized. A variation of this problem is obtained if the existing facilities are situated on two sides of a linear barrier. Such barriers like rivers, highways, borders, or mountain ranges are frequently encountered in practice. In this paper, theory of an MOMP with line barriers is developed. As this problem is non-convex but specially-structured, a reduction to a series of convex optimization problems is proposed. The general results lead to a polynomial algorithm for finding the set of efficient solutions. The algorithm is proposed for bi-criteria problems with different measures of distance.

1 Introduction

Planar location problems have been intensively studied over the last two decades due to their increasing importance in modern life. Growing population and increased economic demand gave rise to studies on choosing an

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optimal site for such facilities as shopping centers, schools, fire stations, etc. Development of personal computers required higher integration of electronic circuits which led to a similar problem of optimal locating of electronic elements. In a major part of these applications, especially in regional and social planning, several decision makers with different priorities are involved in the locational decision which causes a growing need for efficient solution strategies for location models including multiple objective functions. The median problem is one of the most extensively studied problems in the location literature due to a variety of applications. For an overview on location models with a single performance criterion as well as with multiple criteria see e.g. [6, 8, 23, 31].

However, as modern life encounters an ever growing concentration in many areas and aspects, more recent location models often deal with obstacles or barriers. Consider various applications with areas where positioning of a new facility is not allowed (see e.g. [5, 11, 12]) or with regions where trespassing is prohibited. Such barriers may be for example determined by buildings, lakes, or mountain ranges. The idealized case that the barriers are linear and have only a finite set of passages is a special case frequently encountered in practice. Line barriers with passages may be rivers, border lines, highways, mountain ranges or, on a smaller scale, conveyer belts in an industrial plant.

In this paper, the multiple objective median problem (MOMP) is extended by the concept of line barriers, which significantly increases the complexity of the problem but makes the model a more realistic representation for many applications. An example of a concrete application of this model can be found in the city of Halle in Germany where the location of a day care facility for children was sought in a neighborhood divided by a two lane highway with only two bridges for pedestrians (see [20]). The city council decision makers modeled the problem with respect to two conflicting criteria the first of which was based on the accessibility of the day care facility to the children in the neighborhood itself whereas the second criterion modeled the accessibility of the facility to children using public transportation.

The literature on restricted location problems is very limited and focused on some particular types of distance metrics and barrier shapes, all considered for the single criterion case. See e.g. [26] for an introduction to location problems with barriers. One circle as a barrier and the Euclidean distance were studied in [18] while closed polyhedra as barriers and the l_p-metric were examined in [1, 4]. Line barriers with passages have been treated in the case of the Manhattan metric l_1 [21, 2] for which arbitrarily shaped barriers can be handled, and for arbitrary metrics induced by norms in [19].

The authors believe that this paper is the first to study multiple objective restricted location problems.

The problem we consider is based on the MOMP, also referred to as the *multiple objective Weber problem* or the *multiple objective mini-sum problem.* It can be formulated as

$$\text{vmin}_{X \in F} \; [f_1(X), \ldots, f_Q(X)] \qquad (Q \geq 2) \tag{1}$$

where the variable X denotes the location of a new facility in the feasible region $F \subseteq I\!R^2$. The Q individual criteria measure the performance of a locational decision in F with respect to a finite set of existing facilities $\mathcal{E}x = \{Ex_1, Ex_2, \ldots, Ex_M\}$ represented by points in $I\!R^2$. Each objective is given as a median function, i.e. the weighted sum of distances from the new facility to the existing facilities in $\mathcal{E}x$. Thus

$$f_q(X) = \sum_{m=1}^{M} w_{q,m} d_q(X, Ex_m), \qquad q = 1, \ldots, Q, \tag{2}$$

with positive weights $w_{q,m}$, $q = 1, \ldots, Q$, $m = 1, \ldots, M$. As each decision maker may consider different ways of transportation, distances may be measured differently in each objective. Thus for each criterion $q \in \{1, \ldots, Q\}$, d_q is an arbitrary distance function induced by a norm.

Solving (1) is understood as generating its efficient (Pareto) solutions. A feasible point $X_E \in F$ is said to be an efficient solution of (1) if there is no other point $X \in F$ such that $f(X) \leq f(X_E)$, i.e.:

$$\begin{array}{llll} & \forall q \in \{1, \ldots, Q\} & & f_q(X) \leq f_q(X_E) \\ \text{and} & \exists q \in \{1, \ldots, Q\} & s.t. & f_q(X) < f_q(X_E). \end{array} \tag{3}$$

Let $\mathcal{X}_E$ denote the set of efficient solutions of (1) and let $\mathcal{Y}_E$ denote the image of $\mathcal{X}_E$ in the objective space, that is $\mathcal{Y}_E = f(\mathcal{X}_E)$, where $f = [f_1, \ldots, f_Q]$. $\mathcal{Y}_E$ is referred to as the set of nondominated solutions of (1).

When each objective function of (2) is minimized individually over F, the set of optimal solutions, denoted by $\mathcal{X}_q$, is found:

$$\mathcal{X}_q = \{\arg\min_{X \in F} \; f_q(X)\}, \qquad q = 1, \ldots, Q.$$

We also define the utopia point $U = [U_1, \ldots, U_Q]$, where $U_q = \min_{X \in F} f_q(X)$, i.e. $U_q = f_q(\mathcal{X}_q)$, $q = 1, \ldots, Q$.

With respect to the classification scheme for location problems proposed in [8, 13] this problem has the classification $1/P/\bullet/d/Q{-}\sum$. This is the classification of a single-facility location problem (1 in the first position) in the plane (P in the second position) with no special assumptions and constraints ($\bullet$ in the third position), d as a vector of distance functions $d_1, \ldots, d_Q$ (d in the fourth position) and Q criteria which can all be given as median functions ($Q - \sum$ in the fifth position). We will use this classification scheme in the following to achieve a simple description of the different problems mentioned.

The MOMP with barriers is a special case of (1) where the travel distances d_q (compare (2)) are lengthened due to one or several barriers in the plane. For a given finite set of closed polyhedral barrier sets

$$\mathcal{B} = \{B_1, B_2, \ldots, B_b\} \subset \mathbb{R}^2$$

we define the feasible region $F := \mathbb{R}^2 \setminus int(\bigcup_{i=1}^{b} B_i)$ as that region where new facilities can be located. Furthermore we denote by $d_{\mathcal{B}}(X, Y)$ the length of a shortest path (with respect to d) from X to Y not crossing a barrier.

Thus the MOMP can now be restated as the *multiple objective median problem with barriers* $1/P/\mathcal{B}/d_{\mathcal{B}}/Q - \sum$:

$$\begin{array}{rl} \text{vmin} & [f_1(X), \ldots, f_Q(X)] \\ \text{s.t.} & X \in F \end{array} \tag{4}$$

with the individual objective functions given by

$$f_q(X) = \sum_{m=1}^{M} w_{q,m}\, d_{q,\mathcal{B}}(X, Ex_m), \qquad q = 1, \ldots, Q, \tag{5}$$

where

$$d_{q,\mathcal{B}}(X, Y) := \inf_{\substack{r \in \mathbb{N} \\ T_1, \ldots, T_r \in F}} \sum_{i=1}^{r-1} d_q(T_i, T_{i+1}), \qquad X, Y \in F, \tag{6}$$

with $T_1 = X$, $T_r = Y$ and r intermediate points $T_i \in F$ $(i = 1, \ldots, r)$ such that there exists a feasible path (not crossing $\mathcal{B}$) from T_i to T_{i+1} with length $d_q(T_i, T_{i+1})$.

The set of efficient solutions of (4) is denoted by $\mathcal{X}_{E,\mathcal{B}}$ and the set of nondominated solutions of (4) is denoted by $\mathcal{Y}_{E,\mathcal{B}}$.

Note that $1/P/\mathcal{B}/d_{\mathcal{B}}/Q - \sum$ has a solution only if all existing facilities are located in the same *connected* component of F.

Observe that the objective functions of (4) may not be convex since in general the distance measures $d_{q,\mathcal{B}}$ are not positively homogeneous ($q \in \{1,\ldots,Q\}$). Consequently, the multiple objective problem may not have features possessed by convex multiple objective programs. In general, the efficient set $\mathcal{X}_{E,\mathcal{B}}$ may not be connected, and the set $\mathcal{Y}_{E,\mathcal{B}} + I\!R^2_{\geq}$ may be neither convex (that is, one may encounter nondominated solutions in a duality gap) nor the set $\mathcal{Y}_{E,\mathcal{B}}$ may be connected. Here connectedness of the set is understood as defined in [3].

As (4) is a non-convex multiple objective program, it may feature globally as well as locally efficient solutions that can be found by means of some suitable scalarizations specially developed to handle non-convexity. All the globally efficient solutions can be found by means of the lexicographic weighted Tchebycheff approach (see [27]) while the locally efficient solutions can be generated using the augmented Lagrangian approach (see [28]). In order to avoid treating (4) in this general methodological framework and to obtain specific and more effective approaches, we focus on the special case of line barriers with passages but still consider a large class of metrics including the class of l_p metrics, which transforms (4) to problem $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/Q - \sum$. In Section 2, we show that $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/Q$–$\sum$ has a special structure that allows to develop conceptual results and specific approaches to finding the efficient solutions. In Section 3, an algorithm is proposed for the bi-objective case, i.e. for the problem $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/2 - \sum$ and the algorithm components are discussed for different measures of distance. Section 4 includes an illustrative example and the paper is concluded in Section 5.

2 General Results

The following mathematical model will be used for the MOMP with line barriers $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/Q-\sum$:

Let $L := \{(x,y) \in I\!R^2 \mid y = ax + b\}$ be a linear barrier and let $\{P_n \in L \mid n \in \mathcal{N} := \{1,\ldots,N\}\}$ be a set of points on L, i.e. the set of passages through L. Then

$$\mathcal{B}_L := L \setminus \{P_1, \ldots, P_N\}$$

is called a *line barrier with passages* or shortly *line barrier*. (The case that the barrier is a vertical line, which is not included in this description, can be easily transformed to this definition.)

The feasible region F for new locations is defined as the union of the two closed half-planes F^1 and F^2 on both sides of $\mathcal{B}_L$. Here $F^1 \cup F^2 = I\!R^2$ because the line $y = ax + b$ belongs to both half-planes F^1 and F^2. As all results can be easily transferred to the case that the line barrier has a finite width, for simplification, this model will be used in the following although a new location placed directly on the barrier is not allowed in reality.

Furthermore, a finite number of existing facilities $Ex^i_m \in F^i$, $m \in \mathcal{M}^i := \{1, \ldots, M^i\}$ are given in each half-plane F^i, $i = 1, 2$, represented by points in $I\!R^2$. A vector of positive weights $w^i_{q,m} := w_q(Ex^i_m) \in I\!R_+$, $q = 1, \ldots, Q$, is associated with each existing facility Ex^i_m representing the demand of Ex^i_m in the individual criterion. As in the more general problem formulation (4), different distance functions induced by norms are permitted for the individual criteria.

Given a distance function d_q (for criterion q) and the barrier model as above, the distance function $d_{q,\mathcal{B}_L}$ results from (6), where the infimum can be replaced by the minimum.

$$d_{q,\mathcal{B}_L}(X, Y) := \min_{\substack{r \in I\!N \\ T_1, \ldots, T_r \in F}} \sum_{i=1}^{r-1} d_q(T_i, T_{i+1}), \qquad X, Y \in F, \tag{7}$$

with intermediate points T_i, $i = 1, \ldots, r$ defined as in case of (6). This leads immediately to the following description of $d_{q,\mathcal{B}_L}$ (compare [19]):

Lemma 1 *Let d_q be a metric induced by a norm and $i, j \in \{1, 2\}$, $i \neq j$. Then for every $q \in \{1, \ldots, Q\}$*

$$d_{q,\mathcal{B}_L}(X, Y) = \begin{cases} d_q(X, Y) & \text{if} \quad X, Y \in F^i \\ d_q(X, P_{n(q,X,Y)}) + d_q(P_{n(q,X,Y)}, Y) & \text{if} \quad X \in F^i,\ Y \in F^j, \end{cases}$$

where $n(q, X, Y)$ denotes the index of a passage located on a shortest path from X to Y with respect to criterion q.

Note that the triangle inequality holds for $d_{q,\mathcal{B}_L}$ even though $d_{q,\mathcal{B}_L}$ is not positively homogeneous. Consequently, in general $d_{q,\mathcal{B}_L}$ is not a distance function induced by a norm.

As shown in [19] for the corresponding single objective problem, Lemma 1 can be used to rewrite the vector objective function evaluated at a point $X \in F^i$ with respect to each criterion $q \in \{1, \ldots, Q\}$:

Lemma 2 *Let $d = [d_1, \ldots, d_Q]$ be a vector of metrics induced by norms, $X \in F^i$ and $i, j \in \{1, 2\}$, $i \neq j$. Then for each existing facility Ex^j_m there exist passages $P_{n(q,X,Ex^j_m)}$ such that*

$$\begin{pmatrix} f_1(X) \\ \vdots \\ f_Q(X) \end{pmatrix} = \begin{pmatrix} f^i_{1,X}(X) \\ \vdots \\ f^i_{Q,X}(X) \end{pmatrix} + \begin{pmatrix} g^j_{1,X} \\ \vdots \\ g^j_{Q,X} \end{pmatrix}, \tag{8}$$

where

$$f^i_{q,X}(Y) = \sum_{m=1}^{M^i} w^i_{q,m} d_q(Y, Ex^i_m) + \sum_{m=1}^{M^j} w^j_{q,m} d_q(Y, P_{n(q,X,Ex^j_m)}), Y \in F^i, \tag{9}$$

$$g^j_{q,X} = \sum_{m=1}^{M^j} w^j_{q,m} d_q(P_{n(q,X,Ex^j_m)}, Ex^j_m) \tag{10}$$

for $q = 1, \ldots, Q$.

Lemma 2 reveals that the MOMP with line barriers $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/Q - \sum$ is closely related to the unrestricted MOMP. Observe also that the right hand side of (8) takes on different values depending on what passage points have been used to evaluate the distance from a point X to the existing facilities located in the opposite half-plane while passing through those passage points. Due to the definition of $n(q, X, Ex^j_m)$, we have that $f^i_{q,Y}(Y) + g^j_{q,Y} \leq f^i_{q,X}(Y) + g^j_{q,X}$ for all $X, Y \in F^i$ and $q = 1, \ldots, Q$.

Consequently, the MOMP with line barriers can be decomposed into a finite series of unrestricted MOMPs with respect to the facilities in one half-plane and the passage points connecting the two half-planes. Note that the second term in the right-hand-side of (8) denoted by $[g^j_{1,X}, \ldots, g^j_{Q,X}]$ is only implicitly dependent on the location of a new facility and does not directly influence the minimization of the objective $[f_1, \ldots, f_Q]$.

Let the unrestricted MOMPs be denoted by(P_k^i) $(i \in \{1,2\},\ k \in I\!N)$ and have the following form:

$$\begin{array}{rl} \text{vmin} & [f_1(X), \ldots, f_Q(X)] \\ \text{s.t.} & X \in F_k^i, \end{array} \tag{11}$$

where F_k^i is a part of the half-plane F^i $(i,j \in \{1,2\},\ i \neq j)$ such that the passage points $P_{n(q,X,Ex_m^j)}$ located on a shortest path from a point $X \in F_k^i$ to a facility $Ex_m^j \in F^j$ are the same for all points $X \in F_k^i$ $(q = 1, \ldots, Q,$ $m = 1, \ldots, M^j)$. Observe that the objective functions of problem (11) are identical with those of problem (4), however the feasible set F_k^i may not be convex.

Defining $n(q,k,Ex_m^j)$ to be $n(q,X,Ex_m^j)$ where X is an arbitrary point in F_k^i, we observe that there exist passages $P_{n(q,k,Ex_m^j)}$ depending only on F_k^i such that for all $X \in F_k^i$

$$\begin{aligned} d_{q,\mathcal{B}_L}(X, Ex_m^j) &= d_q(X, P_{n(q,k,Ex_m^j)}) + d_q(P_{n(q,k,Ex_m^j)}, Ex_m^j), \\ & \qquad\qquad q = 1, \ldots, Q,\ m = 1, \ldots, M^j. \end{aligned}$$

Consequently, the term $[g_{1,k}^j, \ldots, g_{Q,k}^j] := [g_{1,X}^j, \ldots, g_{Q,X}^j]$ (with an arbitrary point $X \in F_k^i$) is constant for all $X \in F_k^i$. Furthermore we have that $\bigcup_k F_k^i = F^i$. Note that the number of regions F_k^i is finite because there exists only a finite number of possible combinations of passage points $P_{n(q,\bullet,Ex_m^j)}$ as we have that $n(q,\bullet,Ex_m^j) \in \{1, \ldots, N\}$ $(q = 1, \ldots, Q,\ m = 1, \ldots, M^j)$.

Using this decomposition of the feasible region F, the vector objective function of (11) is given by

$$[f_1(X), \ldots, f_Q(X)] = [f_{1,k}^i(X) + g_{1,k}^j, \ldots, f_{Q,k}^i(X) + g_{Q,k}^j],\ \ X \in F_k^i, \tag{12}$$

where for $q = 1, \ldots, Q$

$$f_{q,k}^i(X) = \sum_{m=1}^{M^i} w_{q,m}^i d_q(X, Ex_m^i) + \sum_{m=1}^{M^j} w_{q,m}^j d_q(X, P_{n(q,k,Ex_m^j)}),\ X \in F_k^i, \tag{13}$$

$$g_{q,k}^j = \sum_{m=1}^{M^j} w_{q,m}^j d_q(P_{n(q,k,Ex_m^j)}, Ex_m^j). \tag{14}$$

Solving problem (11) is still a complex task since finding the feasible sets F_k^i is computationally expensive. Therefore we relax the constraint $X \in F_k^i$

to $X \in F^i$ which makes every subproblem (P_k^i) a convex constrained multiple objective problem for which connectedness of its efficient set is a well known result from the literature [29].

In fact, the constraint $X \in F^i$ can be completely omitted for a large class of distance functions since it is automatically satisfied if the set of efficient solutions of a corresponding unconstrained problem always lies within the convex hull of the existing facilities, or, less restrictively, within their smallest enclosing axis-parallel rectangle if L is an axis-parallel line. The former is for example satisfied for all l_p distance functions with $p \in (0, \infty)$ whereas the latter holds for the l_1 distance function. Note that the case of the l_∞ distance function can be transformed to the case of l_1 distances by using a linear transformation of the problem. In these cases, every subproblem (P_k^i) becomes a convex unconstrained multiple objective problem.

Let $\mathcal{X}_{E,k}^i$ and $\mathcal{Y}_{E,k}^i$ denote the set of efficient solutions and nondominated solutions of the relaxed problem (P_k^i), respectively.

Individual minimization of each objective function $f_{q,k}^i(X)$ over the feasible set F^i produces the set of optimal solutions:

$$\mathcal{X}_{q,k}^i := \{\arg \min_{X \in F^i} \; f_{q,k}^i(X)\}, \qquad q = 1, \ldots, Q,$$

and the optimal solution value:

$$y_{q,k}^i = \min_{X \in F^i} f_{q,k}^i(X) + g_{q,k}^j, \qquad q = 1, \ldots, Q.$$

Having the efficient set of each convex subproblem available, we can specify their relationship with the efficient set of the non-convex problem (4) with line barriers. Similarly, the nondominated set of this problem can be described by means of the nondominated set of the convex problems.

Theorem 1

(i)

$$\mathcal{X}_{E,\mathcal{B}_L} \subseteq \bigcup_{\substack{i=1,2;\\ k}} \mathcal{X}_{E,k}^i$$

(ii)

$$\mathcal{Y}_{E,\mathcal{B}_L} = \text{vmin} \bigcup_{\substack{i=1,2;\\ k}} \mathcal{Y}_{E,k}^i.$$

Proof:

(i) Let $X^* \in F^i$ $(i,j \in \{1,2\},\ i \neq j)$ be an efficient solution of $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/Q - \sum$. From Lemma 2 and (12) we have that there exists a $k \in I\!N$ such that

$$[f_1(X^*), \ldots, f_Q(X^*)] = [f^i_{1,k}(X^*) + g^j_{1,k}, \ldots, f^i_{Q,k}(X^*) + g^j_{Q,k}].$$

Assume that $X^* \notin \mathcal{X}^i_{E,k}$. Then there is a point $X^\circ \in \mathcal{X}^i_{E,k}$, $X^\circ \neq X^*$, such that

$$[f^i_{1,k}(X^\circ), \ldots, f^i_{Q,k}(X^\circ)] \leq [f^i_{1,k}(X^*), \ldots, f^i_{Q,k}(X^*)].$$

Adding $[g^j_{1,k}, \ldots, g^j_{Q,k}]$ to both sides of this inequality we therefore obtain

$$[f_1(X^\circ), \ldots, f_Q(X^\circ)] \leq [f_1(X^*), \ldots, f_Q(X^*)],$$

contradicting that $X^* \in \mathcal{X}_{E,\mathcal{B}_L}$.

(ii) Part (ii) results from part (i) and the definition of efficient solutions.

□

Theorem 1 provides the new information about the efficient sets and non-dominated sets of problem (4) with line barriers and of the subproblems (P^i_k), which will be used in the next section in the development of an algorithm for finding these sets in the bi-objective case.

3 Methodology for the case of two criteria

In this section we study the bi-objective median problem with a line barrier which we formulate as

$$\text{vmin}_{X \in F}\, [f_1(X), f_2(X)], \tag{15}$$

where f_1 and f_2 are defined by (5). Furthermore the distance functions of both criteria are identical throughout this section, i.e. $d_1 = d_2$.

Using Theorem 1, a straightforward algorithm to find the efficient set $\mathcal{X}_{E,\mathcal{B}_L}$ can be proposed. The algorithm first checks for all existing facilities in either half-plane F^i and for all possible passages to the opposite half-plane

F^j, and then determines the set of efficient solutions of the corresponding relaxed problems (P_k^i). From the union of all the efficient sets $\mathcal{X}_{E,k}^i$ of the subproblems (P_k^i), the efficient solutions of the original problem, referred to as *globally efficient solutions*, have to be determined. This can be done by constructing the *lower envelope* of all the nondominated solutions of the subproblems in the objective space.

However, a polynomial algorithm for $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/2{-}\sum$ can be proposed if the idea of reducing this non-convex problem to a finite set of the relaxed problems (P_k^i) is used more efficiently. Due to the definition of the relaxed problems (P_k^i), their number depends upon the number of the passages and existing facilities and in total there are $O(N^M)$ subproblems, where $M := M^1+M^2$. We will show that considering a smaller number of the subproblems is sufficient to find the globally efficient set $\mathcal{X}_{E,\mathcal{B}_L}$. This smaller number will be additionally reduced by applying a reduction procedure eliminating subproblems whose nondominated sets are dominated by nondominated sets of other subproblems. We now discuss the details of this approach.

Without loss of generality we assume that the passages are in consecutive order, i.e. there is no passage between P_n and P_{n+1} for $1 \leq n \leq N-1$. Let $D_n^j(m)$ denote the *difference of distances* between an existing facility Ex_m^j and every two adjacent passages P_n and P_{n+1} defined as follows: for $j \in \{1,2\}$ and $n = 1,\ldots,N-1$:

$$D_n^j(m) := d(Ex_m^j, P_n) - d(Ex_m^j, P_{n+1}), \qquad m \in \mathcal{M}^j.$$

Since d is a metric induced by a norm, a shortest path SP from an existing facility $Ex_m^j \in F^j$ to a point $X \in F^i$ has to pass through one of the passages $P_1,\ldots,P_N$ depending on the following condition:

$$\begin{aligned}
P_1 \in SP &\Leftarrow D_1^j(m) < d(P_2,X) - d(P_1,X)\\
P_n \in SP &\Leftarrow (d(P_n,X) - d(P_{n-1},X) < D_{n-1}^j(m))\\
&\qquad \wedge (D_n^j(m) < d(P_{n+1},X) - d(P_n,X)), \quad n = 2,\ldots,N-1,\\
P_N \in SP &\Leftarrow d(P_N,X) - d(P_{N-1},X) < D_{N-1}^j(m).
\end{aligned}$$

This condition must be satisfied for all $X \in F^i$ since d is a metric induced by a norm and thus $d(Ex_m^j, P)$ and $d(X,P)$ are convex functions of a passage $P \in L$, moving on the line L, for all $Ex_m^j \in F^j$. Therefore the problem

$$\begin{aligned}
\min \quad & d(Ex_m^j, P) + d(P,X)\\
\text{s.t.} \quad & P \in \{P_1,\ldots,P_N\}
\end{aligned}$$

either has a unique minimum, or two or more adjacent passages achieve the same minimum value.

This analysis leads to the following observations. If a shortest path from a point $X \in F^i$ to an existing facility $Ex_m^j \in F^j$ passes through the passage point P_n, then for the shortest path from any existing facility $Ex_{\bar{m}}^j$ to X with $D_{n-1}^j(\bar{m}) \geq D_{n-1}^j(m)$ a passage $P_{\bar{n}}$ with $\bar{n} < n$ cannot be optimal. Analogously, the shortest path from any existing facility $Ex_{\hat{m}}^j$ to X with $D_n^j(\hat{m}) \leq D_n^j(m)$ through a passage $P_{\hat{n}}$ with $\hat{n} > n$ cannot be optimal. We conclude that not all of the $O(N^M)$ possible combinations of existing facilities and passage points have to be considered because a majority of these combinations will not lead to efficient solutions. In fact, the number of subproblems (P_k^i) can be reduced to $O(\binom{M+N-1}{N-1})$. This is polynomial in the number of existing facilities M if the number N of passage points is constant, which is a realistic assumption. The resulting selection procedure was developed for a single objective median problem in [19] and can also be applied to the discussed bi-objective case.

After the selection of an appropriate set of subproblems (P_k^i) is completed, the set of globally efficient solutions has to be determined from the sets $\mathcal{X}_{E,k}$ of efficient solutions of the selected subproblems.

Let $List(P_k^i)$ be a list of all currently selected subproblems. If M is the overall number of existing facilities, and N is the total number of passages, then $List(P_k^i)$ contains not more than $L := \binom{M+N-1}{N-1}$ selected subproblems. Since only a small number of these subproblems contribute to the globally nondominated solutions, a reduction procedure is developed which reduces the number of subproblems a second time before the globally nondominated solutions are finally determined as the lower envelope of the remaining sets $\mathcal{Y}_{E,k}^i$. We now turn our attention to the reduction procedure.

Consider a problem (P_k^i) and its efficient and nondominated sets $\mathcal{X}_{E,k}^i$, $\mathcal{Y}_{E,k}^i$. Since (P_k^i) is a convex problem, $\mathcal{Y}_{E,k}^i$ is a curve spanned between the points A_k^i and B_k^i where

$$A_k^i = (a_{1,k}^i, a_{2,k}^i) \qquad \text{and} \qquad B_k^i = (b_{1,k}^i, b_{2,k}^i)$$

and

$$
\begin{aligned}
a^i_{1,k} &= \min_{X\in F^i} f_1(X) \\
a^i_{2,k} &= f_2\left(\arg\left(\operatorname{lex}\min_{X\in F^i}\left[f_1(X), f_2(X)\right]\right)\right) \\
b^i_{2,k} &= \min_{X\in F^i} f_2(X) \\
b^i_{1,k} &= f_1\left(\arg\left(\operatorname{lex}\min_{X\in F^i}\left[f_2(X), f_1(X)\right]\right)\right).
\end{aligned}
$$

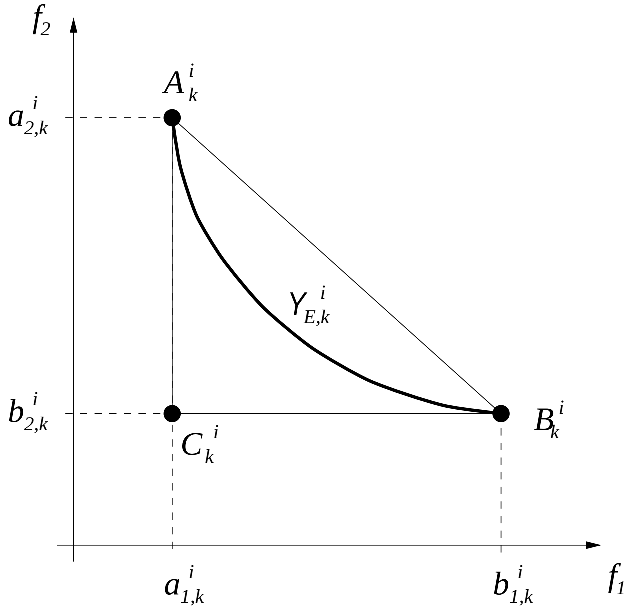

Figure 1: The nondominated set $\mathcal{Y}^i_{E,k}$ of a convex problem (P^i_k).

As illustrated in Figure 1, the nondominated curve is contained in the triangle T^i_k with vertices A^i_k, B^i_k, C^i_k, where $C^i_k = (a^i_{1,k}, b^i_{2,k})$. Observe that the examination of the mutual location of the triangles T^i_k will help eliminate those problems (P^i_k) whose nondominated sets are dominated by nondominated sets of other subproblems.

Figure 2 shows four of many possible locations of the nondominated curves for two arbitrary problems (P^i_k) and $(P^{\bar{i}}_{\bar{k}})$, $i, \bar{i} \in \{1, 2\}$. In particular, Figure 2a shows that one of the two problems can be eliminated while Figure 2b presents an irreducible case. Figure 2c and d show that only subsets of the two nondominated sets may be in the globally nondominated set.

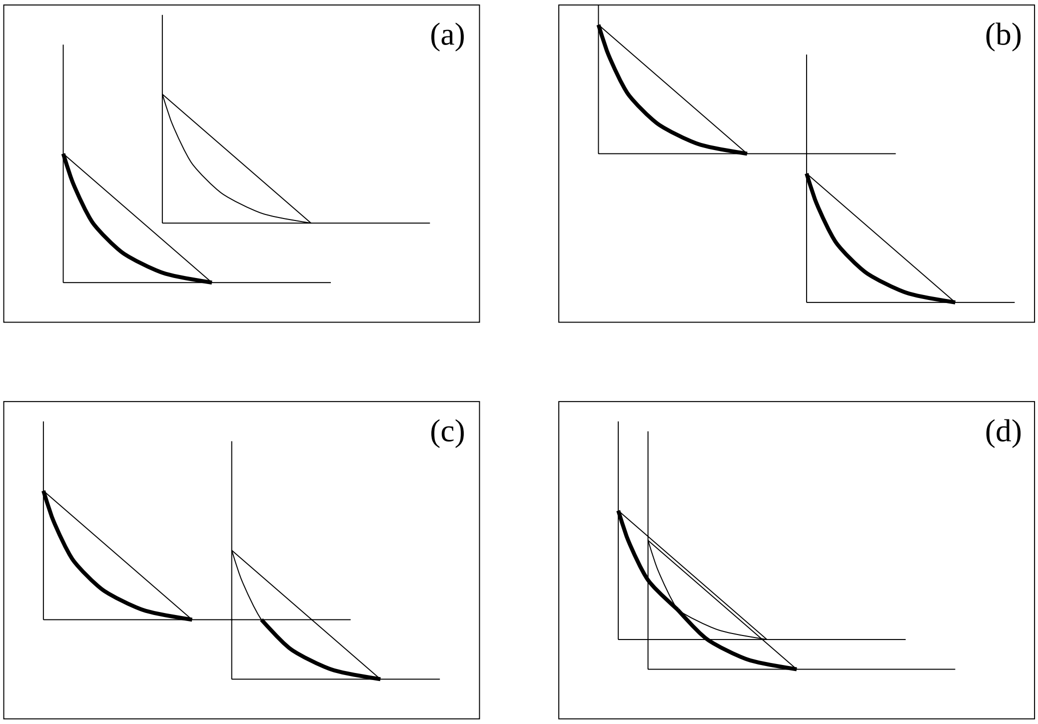

Figure 2: Some examples for possible locations of the triangles A^i_k, B^i_k, C^i_k and $A^{\bar{i}}_{\bar{k}}, B^{\bar{i}}_{\bar{k}}, C^{\bar{i}}_{\bar{k}}$ for two different subproblems (P^i_k) and $(P^{\bar{i}}_{\bar{k}})$ in the objective space $(i, \bar{i} \in \{1, 2\})$. The bold curves represent the sets of globally nondominated solutions, respectively.

These observations will be incorporated into the reduction procedure as follows:

In the first part of the procedure, the Hershberger algorithm [16], that finds the lower envelope of a collection of line segments in linear time, is used to determine the lower envelope of the segments $\overline{A^i_k B^i_k}$ of all subproblems in $List(P^i_k)$. Since our goal is to find a superset of the nondominated sets of the subproblems, we add an auxiliary horizontal line at point B^i_k and an auxiliary vertical line at point A^i_k (this is equivalent to finding $\partial(\overline{A^i_k B^i_k} + \mathbb{R}^2_{\geq})$) of each individual segment $\overline{A^i_k B^i_k}$ to eliminate points coming from other subproblems but dominated by the points of subproblem (P^i_k).

After the lower envelope is found, all the subproblems contributing to it are selected and stored in a second list $\underline{List}(P^i_k)$.

In the second step of the procedure all those subproblems (P_k^i) are added to the list $\underline{List}(P_k^i)$ for which at least one point (i.e. the point C_k^i) is not dominated by the lower envelope.

Summarizing, the following procedure is obtained:

Reduction Procedure:

Let $I\!R^2_{\geq} := \{(x,y) \in I\!R^2 \ : \ x \geq 0, \ y \geq 0\}$.

Input: $List(P_k^i)$, Segments $\overline{A_k^i B_k^i}$.

Step 1 Construct $\partial(\overline{A_k^i B_k^i} + I\!R^2_{\geq})$ for all subproblems in $List(P_k^i)$.

Step 2 Apply the Hershberger algorithm to find the lower envelope of these line segments and half-lines.

Step 3 Identify those subproblems in $List(P_k^i)$ whose corresponding segments $\overline{A_k^i B_k^i}$ contribute to the lower envelope. Let $\underline{List}(P_k^i)$ be the list of these subproblems and remove them from $List(P_k^i)$.

Step 4 For every remaining subproblem $(P_k^i) \in List(P_k^i)$ check whether C_k^i is dominated by the lower envelope. If it is not dominated, add (P_k^i) to $\underline{List}(P_k^i)$.

Output: Reduced list of subproblems $\underline{List}(P_k^i)$.

The time complexity of the reduction procedure is $O(r) = O(L \log L)$ which is the complexity of the Hershberger algorithm. For many location problems the savings resulting from the reduction procedure will be substantial, however they cannot be theoretically guaranteed.

The reduction procedure eliminates only those subproblems (P_k^i) whose nondominated sets are entirely dominated by the nondominated set of another subproblem (P_k^i) (see Figure 2a). Cases with partial reductions (see Figure 2c, d) are subject to further investigation.

Theorem 2

$$\mathcal{Y}_{E,\mathcal{B}_L} = \text{vmin} \bigcup_{\underline{List}(P_k^i)} \mathcal{Y}^i_{E,k}.$$

Proof: Assume that there exists a subproblem $(P_{\bar{k}}^{\bar{i}}) \in List(P_k^i)$ such that $Y_{E,\bar{k}}^{\bar{i}}$ is globally nondominated, but $Y_{E,\bar{k}}^{\bar{i}} \notin \bigcup_{\underline{List}(P_k^i)} \mathcal{Y}_{E,k}^i$.

Since $Y_{E,\bar{k}}^{\bar{i}} \notin \bigcup_{\underline{List}(P_k^i)} \mathcal{Y}_{E,k}^i$, the corresponding point $C_{\bar{k}}^{\bar{i}}$ of the triangle $T_{\bar{k}}^{\bar{i}}$ of this subproblem is dominated by some point D in the lower envelope found by the Hershberger algorithm. Therefore there exists a point $Y_{E,\hat{k}}^{\hat{i}} \in \bigcup_{\underline{List}(P_k^i)} \mathcal{Y}_{E,k}^i$, $Y_{E,\hat{k}}^{\hat{i}} \neq Y_{E,\bar{k}}^{\bar{i}}$, dominating D and thus dominating $Y_{E,\bar{k}}^{\bar{i}}$, which contradicts the assumption. This proves that $\mathcal{Y}_{E,\mathcal{B}_L} \subseteq \bigcup_{\underline{List}(P_k^i)} \mathcal{Y}_{E,k}^i$ which implies the desired result.

$\square$

Recall that our ultimate goal is to determine the set of globally efficient and globally nondominated solutions $\mathcal{X}_{E,\mathcal{B}_L}$ and $\mathcal{Y}_{E,\mathcal{B}_L}$ from the solutions $\mathcal{X}_{E,k}^i$ and $\mathcal{Y}_{E,k}^i$ of the individual subproblems. For this purpose the sets $\mathcal{X}_{E,k}^i$ and $\mathcal{Y}_{E,k}^i$ have to be found by available algorithms for the corresponding subproblems $(P_k^i) \in \underline{List}(P_k^i)$ (for example, see [14]). Clearly, as the bi-objective median problem with a line barrier is a generalization of the corresponding unrestricted subproblems, we cannot expect to find better solution techniques for that problem than those known for the unrestricted problem.

We first discuss solution approaches for the case when distances are measured by the l_1-distance function or by more general block norms. In this case, each problem (P_k^i) involves two piece-wise linear objective functions and its nondominated set $\mathcal{Y}_{E,k}^i$ is a piece-wise linear curve. The problem (P_k^i) can be converted into a bi-objective linear problem and the parametric cost simplex method (see Geoffrion [7]) can be then applied to exactly determine the nondominated set. Equivalently, the procedure of Nickel and Wiecek ([24]) specially designed for bi-objective piece-wise linear programs can be used.

Given the nondominated sets of all the problems $(P_k^i) \in \underline{List}(P_k^i)$, we can determine the globally nondominated points, as proposed in [10], by means of the Hershberger algorithm [16]. As this algorithm finds a lower envelope of a collection of line segments, we again add an auxiliary horizontal line at point B_k^i and a vertical line at point A_k^i of every triangle T_k^i to eliminate points coming from other subproblems but dominated by the points of the subproblem (P_k^i). After the lower envelope has been found, these auxiliary lines are eliminated. The resulting lower envelope of the sets $\mathcal{Y}_{E,k}^i$ of all the subproblems $(P_k^i) \in \underline{List}(P_k^i)$ equals the set of globally nondominated solutions $\mathcal{Y}_{E,\mathcal{B}_L}$.

For other distance functions (such as l_p-distance functions, $p \in (1, \infty)$) only approximation algorithms are known even in the unrestricted single criterion case and consequently the nondominated sets $\mathcal{Y}^i_{E,k}$ can be only approximated with a prescribed accuracy ε. The block sandwich method proposed by Yang and Goh [32] can produce piece-wise linear upper and lower approximations of the nondominated sets. The method requires to (approximately) solve scalarizations of the subproblems of the type

$$\begin{aligned} \min \quad & \lambda(f^i_{1,k}(X) + g^j_{1,k}) + (1-\lambda)(f^i_{2,k}(X) + g^j_{2,k}) \\ \text{s.t.} \quad & X \in F^i \end{aligned} \tag{16}$$

with weights $\lambda \in [0, 1]$. A quadratic convergence property of this algorithm is established in [32], that is, the total number of optimization problems required to attain a prescribed approximation error is less than a constant multiple of the square root of the inverse of the given error.

The respective scalarized subproblems (16) can be solved by applying the Weiszfeld algorithm [30] in the case of l_p-distances, $p \in (0, \infty)$. For an overview of solution procedures for various kinds of single objective planar median problems we refer to [6]. In particular, the points A^k_i and B^k_i can be found by solving the corresponding single objective median problems.

Alternatively, in order to approximate the nondominated sets of the subproblems for nonlinear distance functions one can use methods developed for general bi-objective problems which produce discrete approximating sets of points (see Jahn and Merkel [17], Payne [25], and Helbig [15]). When connected, those points can become the input to the Hershberger algorithm [16]. As the nondominated sets are convex curves, one can also use the hyper-ellipse approach of Li et al. [22] specially designed for convex bi-objective problems. This approach produces a hyper-ellipse whose equation analytically represents the nondominated set.

Whatever the method to approximate the nondominated sets of the subproblems is, these sets become again the input to the Hershberger algorithm [16] as this algorithm also finds a lower envelope of a collection of segments of more general curves in the plane.

The discussion above leads to a polynomial algorithm for solving the bi-objective median problem with a line barrier:

Algorithm for solving $1/P/\mathcal{B}_L/d_{\mathcal{B}_L}/2-\sum$:

Step 1 Apply the selection procedure and create a list $List(P_k^i)$ of selected subproblems (P_k^i).

Step 2 For every subproblem $(P_k^i) \in List(P_k^i)$: find the triangle T_k^i.

Step 3 Apply the reduction procedure and create a reduced list of subproblems $\underline{List}(P_k^i)$.

Step 4 Construct the lower envelope of the sets $\mathcal{Y}_{E,k}^i$ corresponding to the subproblems $(P_k^i) \in \underline{List}(P_k^i)$ or of their (piece-wise linear or convex) approximations determined with a prescribed error ε.

Output: The lower envelope is an exact representation of the set $\mathcal{Y}_{E,\mathcal{B}_L}$ or an approximation of the set $\mathcal{Y}_{E,\mathcal{B}_L}$ with error ε, depending on the available solution procedures for the corresponding unrestricted median subproblems.

If M is the number of existing facilities, N is the number of passages, $L = \binom{M+N-1}{N-1}$ is an upper bound on the number of subproblems in $List(P_k^i)$ after the application of the selection procedure, and the overall complexity of the proposed algorithm is $O(s+r+h)$ where

$O(s) = O\left(NM\log M + L\right)$	is the complexity of the selection procedure,
$O(r) = O(L\log L)$	is the complexity of the reduction procedure and
$O(h)$	is the complexity of the solution of the subproblems and of the subsequent application of the Hershberger algorithm.

4 Example

In the following example we consider a location problem with the classification $1/P/\mathcal{B}_L/(l_1)_{\mathcal{B}_L}/2-\sum$ as given in Figure 3, where distances are measured according to the Manhattan metric l_1.

For the analogous unrestricted median problem of type $1/P/\bullet/l_1/2-\sum$ exact algorithms are given in [14]. These algorithms are implemented in LOLA, the Library of Location Algorithms [9], which will be used to find the exact efficient and nondominated sets of $1/P/\mathcal{B}_L/(l_1)_{\mathcal{B}_L}/2-\sum$.

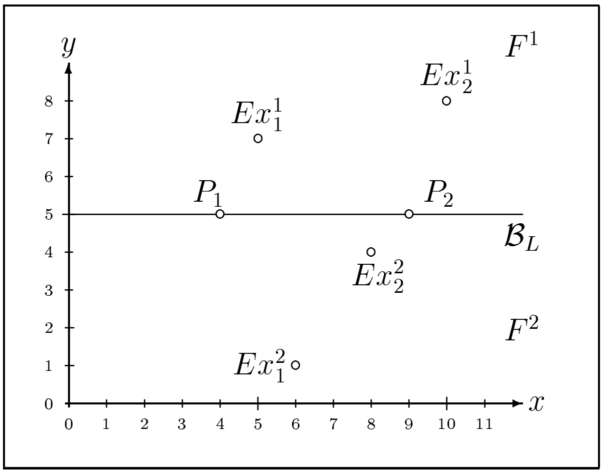

Figure 3: The example problem with the classification $1/P/\mathcal{B}_L/(l_1)_{\mathcal{B}_L}/2-\sum$.

Let the line barrier

$$\mathcal{B}_L := \{(x,y) \in \mathbb{R}^2 \mid y = 5\} \setminus \{P_1 = (4,5), P_2 = (9,5)\}$$

divide the plane into the two half-planes F_1 and F_2. Furthermore four existing facilities are given on both sides of $\mathcal{B}_L$ with coordinates and weights as listed in Table 1. Thus $\mathcal{M}^1 = \mathcal{M}^2 = \{1,2\}$ and $M^1 = M^2 = 2$.

Existing facility Ex_m^i		$w_{1,m}^i$	$w_{2,m}^i$	$D_1^i(m)$
Ex_1^1	$(5,7)$	8	2	-3
Ex_2^1	$(10,8)$	5	6	5
Ex_1^2	$(6,1)$	10	1	-1
Ex_2^2	$(8,4)$	7	4	3

Table 1: Existing facilities with their weights and the values of $D_1^i(m) = d(Ex_m^i, P_1) - d(Ex_m^i, P_2)$.

In step 1 of the algorithm presented above, the selection procedure is applied and $List(P_k^i)$ includes six subproblems (P_k^i) listed in Table 2 that are further investigated. At the end of step 3, the reduced list $\underline{List}(P_k^i)$ includes (P_0^1), (P_0^2), (P_1^2).

(P_k^i)	Weights of P_1 $\tilde{w}_1$	$\tilde{w}_2$	P_2 $\tilde{w}_1$	$\tilde{w}_2$	Weights of existing facilities
(P_0^1)	0	0	17	5	$\tilde{w}_q(Ex_m^1) := w_q(Ex_m^1),\ q \in \{1,2\},\ m \in \mathcal{M}^1,$ $\tilde{w}_q(Ex_m^2) := 0,\ q \in \{1,2\},\ m \in \mathcal{M}^2$
(P_1^1)	10	1	7	4	$\tilde{w}_q(Ex_m^1) := w_q(Ex_m^1),\ q \in \{1,2\},\ m \in \mathcal{M}^1,$ $\tilde{w}_q(Ex_m^2) := 0,\ q \in \{1,2\},\ m \in \mathcal{M}^2$
(P_2^1)	17	5	0	0	$\tilde{w}_q(Ex_m^1) := w_q(Ex_m^1),\ q \in \{1,2\},\ m \in \mathcal{M}^1,$ $\tilde{w}_q(Ex_m^2) := 0,\ q \in \{1,2\},\ m \in \mathcal{M}^2$
(P_0^2)	0	0	13	8	$\tilde{w}_q(Ex_m^1) := 0,\ q \in \{1,2\},\ m \in \mathcal{M}^1,$ $\tilde{w}_q(Ex_m^2) := w_q(Ex_m^2),\ q \in \{1,2\},\ m \in \mathcal{M}^2$
(P_1^2)	8	2	5	6	$\tilde{w}_q(Ex_m^1) := 0,\ q \in \{1,2\},\ m \in \mathcal{M}^1,$ $\tilde{w}_q(Ex_m^2) := w_q(Ex_m^2),\ q \in \{1,2\},\ m \in \mathcal{M}^2$
(P_2^2)	13	8	0	0	$\tilde{w}_q(Ex_m^1) := 0,\ q \in \{1,2\},\ m \in \mathcal{M}^1,$ $\tilde{w}_q(Ex_m^2) := w_q(Ex_m^2),\ q \in \{1,2\},\ m \in \mathcal{M}^2$

Table 2: Weights of the existing facilities $\mathcal{E}x = \{Ex_1^1, Ex_2^1, Ex_1^2, Ex_2^2, P_1, P_2\}$ of the six selected subproblems (P_k^i) of type $1/P/\bullet/l_1/2-\sum$.

Sub-problem	Efficient solutions of the subproblems $\mathcal{X}_{E,k}^i$
(P_0^1)	$\{(x,y) \in I\!R^2 \mid (x=9) \wedge (5 \le y \le 7)\}$
(P_1^1)	$\{(x,y) \in I\!R^2 \mid (5 \le x \le 9) \wedge (y=5)\}$ $\cup \{(x,y) \in I\!R^2 \mid (x=9) \wedge (5 \le y \le 7)\}$
(P_2^1)	$\{(x,y) \in I\!R^2 \mid (4 \le x \le 5) \wedge (5 \le y \le 7)\}$
(P_0^2)	$\{(x,y) \in I\!R^2 \mid (8 \le x \le 9) \wedge (4 \le y \le 5)\}$
(P_1^2)	$\{(x,y) \in I\!R^2 \mid (6 \le x \le 8) \wedge (y=4)\}$ $\cup \{(x,y) \in I\!R^2 \mid (x=8) \wedge (4 \le y \le 5)\}$
(P_2^2)	$\{(x,y) \in I\!R^2 \mid (4 \le x \le 6) \wedge (4 \le y \le 5)\}$

Table 3: Efficient solutions of the six subproblems in $List(P_k^i)$.

For illustrative reasons, we include the sets of efficient solutions (see Table 3) and nondominated solutions (see Figure 4) of all the subproblems in $List(P_k^i)$ which were determined using LOLA [9].

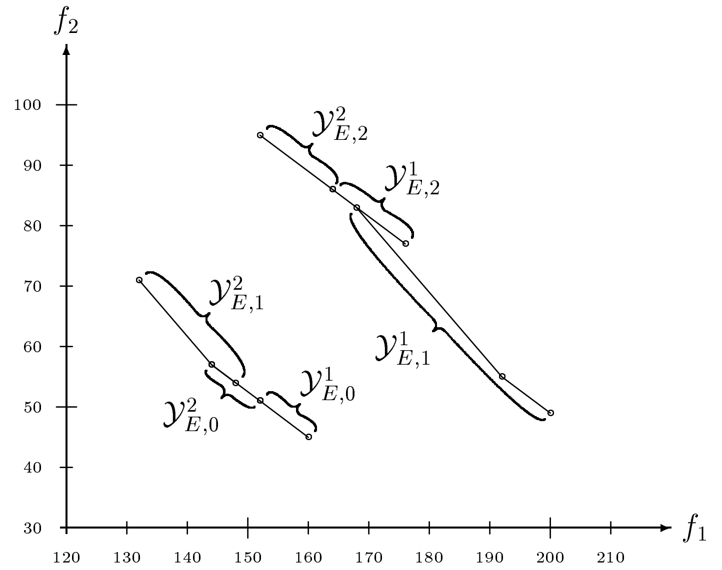

Figure 4: Nondominated solutions of the six subproblems in $List(P_k^i)$.

Step 4 yields the set of globally nondominated solutions $\mathcal{Y}_E$ and the set of globally efficient solutions $\mathcal{X}_E$ of this example problem:

$$\begin{aligned}
\mathcal{Y}_E &= \mathcal{Y}_{E,0}^1 \cup \mathcal{Y}_{E,0}^2 \cup \mathcal{Y}_{E,1}^2 \\
\mathcal{X}_E &= \mathcal{X}_{E,0}^1 \cup \mathcal{X}_{E,0}^2 \cup \mathcal{X}_{E,1}^2 \\
&= \{(x,y) \in I\!R^2 \,|\, ((x=9) \wedge (5 \leq y \leq 7)) \\
&\quad \vee ((8 \leq x \leq 9) \wedge (4 \leq y \leq 5)) \\
&\quad \vee ((6 \leq x \leq 8) \wedge (y=4))\}.
\end{aligned}$$

The set of globally efficient solutions $\mathcal{X}_E$ is graphed in Figure 5.

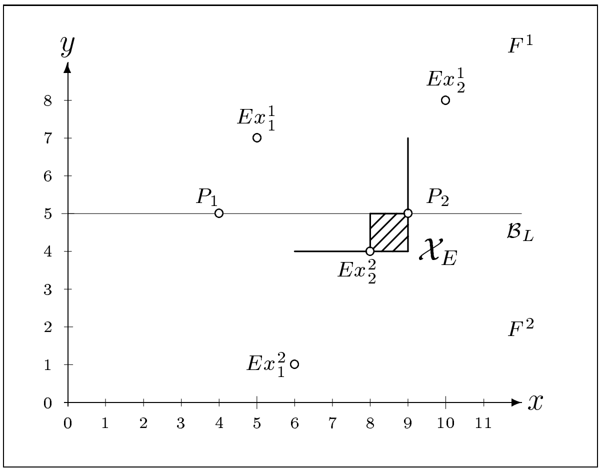

Figure 5: Efficient solution $\mathcal{X}_E$ of the example problem of type $1/P/\mathcal{B}_L/(l_1)_{\mathcal{B}_L}/2-\sum$.

5 Conclusions

This paper studies the multiple objective median problem with a line barrier. The primary goal of this pioneering research is the analytical determination of the efficient set of the problem. The structure of the efficient set is first examined in order to motivate the design of special algorithms. The theoretical analysis shows that the original non-convex problem can be decomposed to a series of multiple objective convex subproblems.

An algorithm for solving the bi-criteria median problem with a line barrier and different distance measures is developed. The nondominated set or an approximation of the nondominated set of the original problem is determined as the lower envelope of the nondominated sets of the subproblems, depending on the given distance function. The complexity of the algorithm depends on the complexity of the methods used to solve the subproblems but if the chosen method has polynomial complexity (such as the block sandwich method of Yang and Goh ([32]), then the algorithm is also polynomial. An illustrative example is included.

The proposed methodology produces solution approaches to the bi-criteria restricted median problem as good as they can be for the corresponding single criterion unrestricted problem. The algorithm gives exact solutions (i.e. finds all efficient/nondominated points) for problems with linear measures of distance whose nondominated set is piece-wise linear but may be non-convex. The authors are not aware of another algorithm in the literature producing exact solutions for non-convex bi-objective problems.

More research is needed to efficiently design the reduction procedure eliminating some of the subproblems. Currently, the procedure checks only for the nondominated sets that are entirely dominated by nondominated sets of other subproblems. Cases with partial domination should also be considered.

Clearly, other location problems with barriers should be studied in the multiple objective framework. Complexity of those problems, however, may heavily affect the ability to approximate their efficient sets. In this case, one may be interested in obtaining partial information about the efficient solutions and in designing tools for choosing a most preferred solution as the optimal one.

Furthermore, not only location problems can lead to non-convex multiple objective problems decomposable to a series of convex problems. This class of non-convex multiple objective problems should be explored independently of their applications.

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