# A Reduction Result for Location Problems with Polyhedral Barriers

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**Abstract**

In this paper we consider the problem of locating one new facility in the plane with respect to a given set of existing facilities where a set of polyhedral barriers restricts traveling. This non-convex optimization problem can be reduced to a finite set of convex subproblems if the objective function is a convex function of the travel distances between the new and the existing facilities (like e.g. the median and center objective functions). An exact Algorithm and a heuristic solution procedure based on this reduction result are developed.

*Keywords: location, non-convex optimization, barriers to travel*

# 1 Introduction

In times of increasing transportation costs and just-in-time delivery schedules, good locational decisions are needed in many different fields. The location of a warehouse with respect to a given set of customers or the location of an emergency facility in an expanding neighborhood are only two examples for a wide range of applications.
The development of realistic location models is a crucial step in every locational decision process. Especially in the case of planar location models we deal with a geometric representation of the problem, and the geographical reality has to be incorporated into this representation. Restrictions of different types occur in almost every real-world location problem since there are in general regions to exclude from placement of new facilities. These regions can also often not be used for transportation which can be modeled by the introduction of barrier regions in the plane $\mathbb{R}^2$. To give only some examples of possible barrier regions, consider military regions, mountain ranges, lakes, big rivers or highways, or, on a smaller scale, conveyor belts in an industrial plant.

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The increasing interest in location models incorporating restrictions and barrier regions is reflected in the recent literature. Katz and Cooper 1981 [11] developed a heuristic for the median problem in case that one circular barrier is given and distances are measured with the Euclidean distance function.
Likewise for the median problem Aneja and Parlar 1994 [1] and more recently Butt and Cavalier 1996 [3] developed heuristics for the case that the barriers are closed polyhedra and the distance is given by the $l_p$-metric. In the special case of the Manhattan metric $l_1$ discretization results were proven by Larson and Sadiq 1983 [15] and by Batta, Ghose and Palekar 1989 [2] for arbitrarily shaped barriers. A similar discretization was derived for a more general class of distance functions in Hamacher and Klamroth 1997 [6], namely the class of block norms. Related results and a short summary on the subject can also be found in the book edited by Drezner 1995 [4] and in particular in the survey on global optimization in location by Hansen, Jaumard and Tuy 1995 [9].
Klamroth 1996 [12] considered the median problem for the case that the barrier is a line with a finite number of passages. A reduction of the non-convex original problem to a polynomial number of unrestricted median problems was given for any metric induced by a norm. This approach was extended to the multiple criteria case in Klamroth and Wiecek 1998 [14].

In this paper we develop a reduction result that implies a general solution strategy for location problems with polyhedral barriers. We consider objective functions that are convex functions of distances between a set of existing facilities and one new facility. This definition of the objective function includes for example the well known median (Weber) and center objective functions as well as ordered Weber objective functions.

Let a finite set of convex, closed, polyhedral and pairwise disjoint barriers $\{B_1, \ldots, B_N\}$ be given in $\mathbb{R}^2$, representing those regions in the plane where neither trespassing nor location of new facilities is allowed. We denote the union of these barrier regions by $\mathcal{B} := \bigcup_{i=1}^{N} B_i$ and the finite sets of extreme points and facets of $\mathcal{B}$ by $\mathcal{P}(\mathcal{B})$ and $\mathcal{F}(\mathcal{B})$, respectively. The feasible region $F$ for new locations is given by

$$F := \mathbb{R}^2 \setminus \operatorname{int}(\mathcal{B}).$$

A finite set of existing facilities $\mathcal{E}x := \{Ex_m \in F : m \in \mathcal{M}\}$, $\mathcal{M} = \{1, \ldots, M\}$ is given in a connected subset of the feasible region $F$.
Furthermore we assume that a distance measure $d$ induced by a norm $\| \bullet \|_d$ is given by $d(X,Y) = \| Y - X \|_d$ for all $X, Y \in \mathbb{R}^2$. Taking the restriction due to the barrier regions into account, we can find the corresponding barrier distance function $d_{\mathcal{B}}(X,Y)$ for two points $X, Y \in F$ as the length of a shortest path between $X$ and $Y$ not intersecting the interior of a barrier. More formally, let $P$ be a *permitted $X$-$Y$-path* in $F$, i.e. a curve connecting $X$ and $Y$ not intersecting the interior of a barrier. Furthermore, let $p$ be a piecewise continuous differentiable parameterization of $P$, $p : [a,b] \to \mathbb{R}^2$ with $a, b \in \mathbb{R}$,

$a < b$, $p(a) = X$, $p(b) = Y$ and $p([a,b]) \cap \mathrm{int}(\mathcal{B}) = \emptyset$. Then $d_{\mathcal{B}}(X,Y)$ can be defined as

$$d_{\mathcal{B}}(X,Y) := \min \left\{ \int_a^b \| p'(t) \|_d \, \mathrm{d}t \ : \ P \text{ permitted } X\text{-}Y\text{-path} \right\}. \tag{1}$$

A permitted $X$-$Y$-path with length $d_{\mathcal{B}}(X,Y)$ will be called a *d-shortest permitted $X$-$Y$-path.* Additionally, we call two points $X$ and $Y$ in $F$ *d-visible* if they satisfy $d_{\mathcal{B}}(X,Y) = d(X,Y)$, i.e. the distance between $X$ and $Y$ is not lengthened by the barrier regions.
Note that for $d_{\mathcal{B}}$ the triangle inequality is satisfied (provided it holds for the original distance function $d$, which is guaranteed by the fact that $d$ is induced by a norm), but that $d_{\mathcal{B}}$ is in general not positively homogeneous.

Using the distance measure $d_{\mathcal{B}}$ as defined in (1), we consider the following general location problem:

$$\begin{aligned} \min \quad & f_{\mathcal{B}}(X) = f(d_{\mathcal{B}}(X, Ex_1), \ldots, d_{\mathcal{B}}(X, Ex_M)) \\ \text{s.t.} \quad & X \in F, \end{aligned} \tag{2}$$

where $f$ is any convex and nondecreasing function of the barrier distance $d_{\mathcal{B}}$ between the new facility $X$ and the existing facilities in $\mathcal{E}x$. Well known examples are the median objective function $f_{\mathcal{B}}(X) = \sum_{m \in \mathcal{M}} w_m d_{\mathcal{B}}(X, Ex_m)$ and the center objective function $g_{\mathcal{B}}(X) = \max_{m \in \mathcal{M}} w_m d_{\mathcal{B}}(X, Ex_m)$ where the positive weights $w_m$ represent the demand of the facility $Ex_m$, $m \in \mathcal{M}$.
Note that the barrier distance $d_{\mathcal{B}}$ is in general not convex and that therefore $f_{\mathcal{B}}$ is also in general not convex.

To simplify further notation we will use the classification $(Pos1/Pos2/Pos3/Pos4/Pos5)$ of location problems as introduced in Hamacher 1995 [5] or Hamacher and Nickel 1996 [7] (see Hamacher and Nickel 1999 [8] for an overview). Following their notation, problem (2) is classified as $(1/P/\mathcal{B}/d_{\mathcal{B}}/f$ convex), where $Pos1$ gives the number of new facilities sought (1 for a single-facility problem), $Pos2$ denotes the type of location problem ($P$ for planar location problems), $Pos3$ contains special assumptions ($\mathcal{B}$ for barrier regions), $Pos4$ contains the information about the distance function ($d_{\mathcal{B}}$ in case of barrier distances) and $Pos5$ indicates the objective function, which in this case is a convex function of the distances between the new and the existing facilities ($f$ convex).

In the following section some basic properties and concepts related to shortest paths in the presence of polyhedral barriers will be derived. In Section 3 a reduction result is developed that interrelates location problems with polyhedral barriers with a set of unrestricted location problems. The algorithmic consequences of this result are discussed in Section 4, and the paper is concluded with Section 5.

# 2 Shortest paths in the presence of barriers

In this section we focus on shortest permitted paths in the presence of barriers where the set of barriers $\mathcal{B}$ consists of pairwise disjoint convex polyhedral sets.

Let $d$ be a given distance function induced by a norm $\| \bullet \|_d$. The set of points $Y \in F$ that are not $d$-visible from a point $X \in F$ is called the *shadow of* $X$ *with respect to* $d$, i.e.

$$shadow_d(X) := \{Y \in F \ : \ d_{\mathcal{B}}(X,Y) > d(X,Y)\}.$$

In Figure 1 two examples are given for the Euclidean metric $l_2$ and the Manhattan metric $l_1$, respectively.

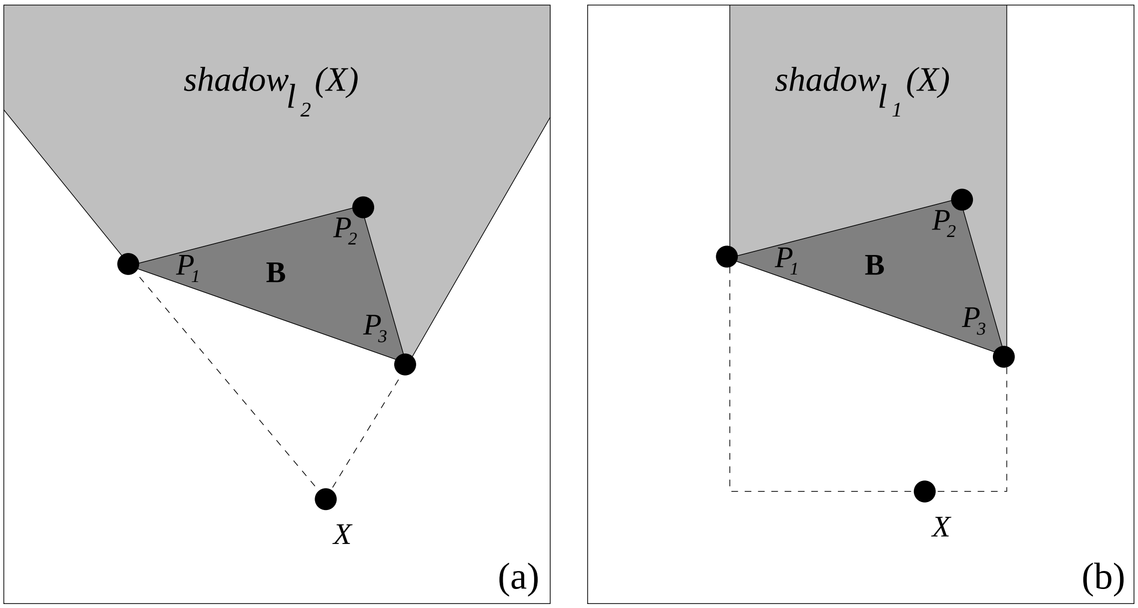


Figure 1: Part (a) depicts $shadow_{l_2}(X)$ whereas part (b) shows $shadow_{l_1}(X)$.

Note that for some choices of $d$ a point that is $d$-visible may not be $l_2$-visible, i.e. not visible in the usual sense of straight line visible. On the other hand every pair of $l_2$-visible points is also $d$-visible if $d$ is a distance function induced by a norm. This result is a generalization of an earlier result in [6] where it was proven for block norms.

**Lemma 1** *Let $d$ be a distance function induced by a norm. Then*

$$shadow_d(X) \subseteq shadow_{l_2}(X), \qquad X \in F.$$

*Furthermore if $X, Y \in F$ are $l_2$-visible, $X \neq Y$, then the straight line segment connecting $X$ and $Y$ is a $d$-shortest permitted path with length $d(X,Y)$.*

**Proof:** Wlog let $X = 0$ be the origin and let $Y \in F$ be a point that is $l_2$-visible from $X$. Then the straight-line segment connecting $X$ and $Y$ is a permitted path $P$ given by the parameterization $p : [0,1] \to I\!\!R^2$, $p(t) = t \cdot Y$, $t \in [0,1]$. Using (1), the length of $P$ can be calculated as

$$\begin{aligned} d_{\mathcal{B}}(0,Y) &\leq \int_0^1 \| p'(t) \|_d \, dt = \int_0^1 \| \frac{d}{dt}(tY) \|_d \, dt = \int_0^1 \| Y \|_d \, dt = \| Y \|_d \\ &= d(0,Y). \end{aligned}$$

This inequality implies that the subregion of $F$ that is $l_2$-visible from a point $X \in F$ is also $d$-visible from $X$, which in turn implies that $shadow_d(X) \subseteq shadow_{l_2}(X)$ for all $X \in F$.

□

In the case that distances are measured by an $l_p$-metric, $1 \leq p \leq \infty$ Viegas and Hansen 1985 [17] showed that for any two points $X, Y \in F$, $X \neq Y$ there always exists an $l_p$-shortest permitted path connecting $X$ and $Y$ that is a piecewise linear path with breaking points only in extreme points of a barrier. This property was generalized for block norms in [6]. The following result shows that it also holds for any other distance function $d$ that is induced by a norm.

**Lemma 2** *Let $d$ be a distance function induced by a norm and let $X, Y \in F$. Then there exists a $d$-shortest permitted path $SP$ connecting $X$ and $Y$ with the following property.*

*Property 1:* *$SP$ is a piecewise linear path with breaking points only in extreme points of barriers.*

**Proof:** Let $X, Y \in F$ and let $SP$ be any $d$-shortest permitted path connecting $X$ and $Y$ in $F$ that does not satisfy Property 1. Note that, since the set of barriers and correspondingly the set of extreme points $\mathcal{P}(\mathcal{B})$ of barriers is finite, $SP$ can be partitioned by a finite set of points so that two consecutive points on $SP$ are $l_2$-visible. Lemma 1 therefore implies that the straight line segment connecting two consecutive points on $SP$ is a $d$-shortest permitted path connecting these two points. We can therefore construct a piecewise linear path $SP'$ with a finite set of breaking points that has the same length as $SP$. A $d$-shortest permitted path $SP''$ with Property 1 can be constructed from $SP'$ similar to the construction given in [17] for $l_p$-distances:
Let $[T_{i-1}, T_i]$ and $[T_i, T_{i+1}]$ be two consecutive straight line segments of $SP'$. First assume that $T_{i-1}$ and $T_{i+1}$ are $l_2$-visible. Then the two segments $[T_{i-1}, T_i]$ and $[T_i, T_{i+1}]$ can be replaced by one straight line segment $[T_{i-1}, T_{i+1}]$ without increasing the length of $SP'$. Otherwise, using again Lemma 1, the breaking point $T_i$ can be moved along $[T_{i-1}, T_i]$ or along $[T_i, T_{i+1}]$ towards $T_{i-1}$ or $T_{i+1}$, respectively, without increasing the length of $SP'$, until one of these line segments becomes tangent to a barrier.
While iterating both operations every extreme point of a barrier which lies on $SP'$ is interpreted as a breaking point $T_i$ even if the straight line segment $[T_{i-1}, T_{i+1}]$ is part of $SP'$. Thus the iteration of both operations yields a path $SP''$ with the desired property after a finite number of steps since every breaking point of $SP'$ which is not yet an extreme point of a barrier can be moved towards $X$, $Y$, or an extreme point of a barrier.

□

An immediate consequence of Lemma 2 is that the barrier distance $d_{\mathcal{B}}(X,Y)$, $X,Y \in F$ can be calculated with respect to a so-called *intermediate point* $I_{X,Y} \neq Y$, i.e. a breaking point on a $d$-shortest permitted $X$-$Y$-path with Property 1 so that $I_{X,Y}$ is $d$-visible from $Y$. (Note that in case that $X$ and $Y$ are $d$-visible the intermediate point $I_{X,Y}$ equals $X$.)

**Corollary 1** *Let $d$ be a distance function induced by a norm and let $X,Y \in F$. Furthermore let $SP$ be a $d$-shortest permitted $X$-$Y$-path with Property 1 and let the point $I_{X,Y} \neq Y$ be a breaking point on $SP$ that is $d$-visible from $Y$. Then*

$$d_{\mathcal{B}}(X,Y) = d_{\mathcal{B}}(X, I_{X,Y}) + d(I_{X,Y}, Y).$$

Note that the intermediate points $I_{X,Y}$ are not necessarily unique. Furthermore, as a result of Lemma 2, an intermediate point $I_{X,Y}$ can always be chosen such that it is not only $d$-visible from $Y$, but also $l_2$-visible from $Y$.

A visibility graph as proposed in Butt and Cavalier 1996 [3] can be used to determine distances between the existing facilities and all those points that are candidates for intermediate points on a $d$-shortest permitted path between an existing facility and a point $X \in F$. The node set of this visibility graph $G$ is given by $V(G) := \mathcal{E}x \cup \mathcal{P}(\mathcal{B})$. Two nodes $v_i, v_j \in V(G)$ are connected by an edge of length $d(v_i, v_j)$ if the corresponding points in the plane are $d$-visible and have distance $d(v_i, v_j)$. In Figure 2 an example is given for the case that distances are measured by the Manhattan metric $l_1$.

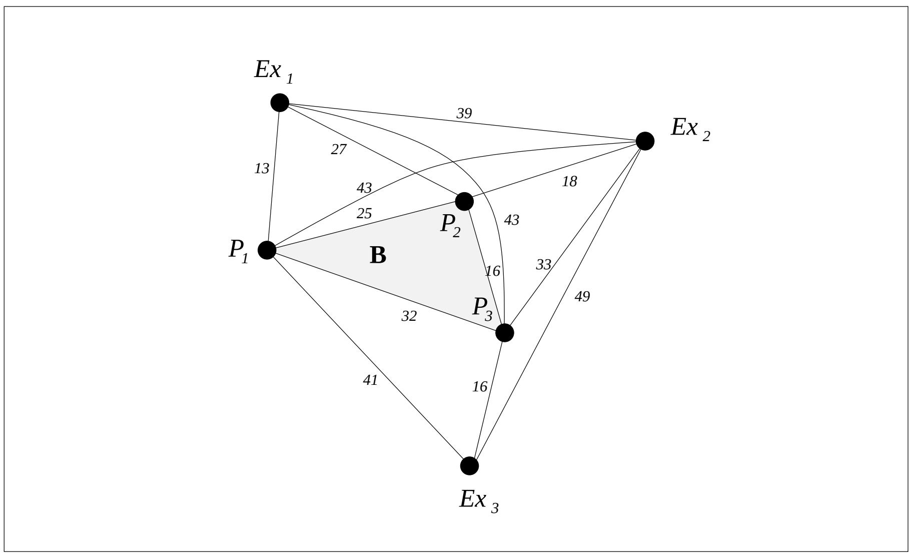


Figure 2: The visibility graph for an example problem where distances are measured with respect to $l_1$.

The barrier distance $d_{\mathcal{B}}(Ex_m, X)$ between an existing facility $Ex_m \in \mathcal{E}x$ and a point $X \in F$ can now be calculated as

$$d_{\mathcal{B}}(Ex_m, X) = d_G(Ex_m, I_{Ex_m,X}) + d(I_{Ex_m,X}, X), \tag{3}$$

where $d_G(Ex_m, I_{Ex_m,X})$ denotes the length of a shortest path between $Ex_m$ and the intermediate point $I_{Ex_m,X}$ in the visibility graph $G$.

Another consequence of Lemmas 1 and 2 is that the boundary of $shadow_d(X)$,

$$\begin{aligned}\partial(shadow_d(X)) \quad := \quad & \{Y \in F \,:\, N_\varepsilon(Y) \cap shadow_d(Y) \neq \emptyset \\ & \text{and } N_\varepsilon(Y) \not\subseteq shadow_d(Y) \ \forall \varepsilon > 0\},\end{aligned}$$

where $N_\varepsilon(Y) := \{Z \in \mathbb{R}^2 : d(Z,Y) < \varepsilon\}$, is piecewise linear for any distance function $d$ that is induced by a norm. Therefore $shadow_d(X)$ has a simple analytic representation for all $X \in F$.

Obviously those parts of $\partial(shadow_d(X))$ that are part of the boundary of a barrier region are piecewise linear. For all other parts of $\partial(shadow_d(X))$, consider a point $Y$ on $\partial(shadow_d(X))$ and let $I_{X,Y}$ be an intermediate point on a $d$-shortest permitted $X$-$Y$-path with Property 1. Note that in this case $Y$ is $d$-visible from $X$. If all the points $Z$ on the line-segment starting at $I_{X,Y}$, passing through $Y$ and ending as soon as it intersects the interior of a barrier are $d$-visible from $X$ and thus not in $shadow_d(X)$, then $\partial(shadow_d(X))$ is piecewise linear. To simplify further discussion we assume wlog that $X = 0$ is the origin.

**Lemma 3** *Let $d$ be a distance function induced by a norm. Furthermore let $Y \in F$ be a point that is $d$-visible from the origin and let $I := I_{0,Y}$ be an intermediate point on a $d$-shortest permitted $0$-$Y$-path with Property 1. Let $Z = I + \lambda(Y \Leftrightarrow I)$, $\lambda \geq 0$ be a point in $F$ such that $Z$ is $l_2$-visible from $I$. Then $Z$ is $d$-visible from the origin.*

**Proof:** First assume that $Y$ is $l_2$-visible from the origin. Then $I = 0$ and thus $Z$ is also $l_2$-visible and $d$-visible from the origin.
Now consider the case that $Y$ is not $l_2$-visible from the origin and thus $I \neq 0$. Then $d(0,I) + d(I,Y) = d(0,Y)$ since $I$ is a point on a $d$-shortest permitted $0$-$Y$-path.
Assume that there exist $\lambda, \mu \in [0,1]$ such that $d(0,Z) < \lambda d(0,I) + \mu d(I,Y)$ where $Z = \lambda I + \mu(Y \Leftrightarrow I)$. Using the triangle inequality we obtain

$$\begin{aligned} d(0,Y) &= d(0, \lambda I + (1 \Leftrightarrow \lambda)I + \mu(Y \Leftrightarrow I) + (1 \Leftrightarrow \mu)(Y \Leftrightarrow I)) \\ &\leq d(0,Z) + (1 \Leftrightarrow \lambda)d(0,I) + (1 \Leftrightarrow \mu)d(I,Y) \\ &< \lambda d(0,I) + \mu d(I,Y) + (1 \Leftrightarrow \lambda)d(0,I) + (1 \Leftrightarrow \mu)d(I,Y) \\ &= d(0,I) + d(I,Y), \end{aligned}$$

contradicting the assumption that $I$ is a point on a $d$-shortest $0$-$Y$-path.
Thus $d(0,Z) = \lambda d(0,I) + \mu d(I,Y)$ for all $\lambda, \mu \in [0,1]$, which, using $\lambda = 1$, proves the result for all $\mu \in [0,1]$.
The remaining case is that $\lambda = 1$ but $\mu > 1$, i.e. that $Z = I + \mu(Y \Leftrightarrow I)$, $\mu > 1$. Assume that there exists $\mu > 1$ such that $d(0,Z) < d(0,I) + \mu d(I,Y)$. It follows that

$$\begin{aligned} & \mu d(0, \frac{1}{\mu} I + (Y \Leftrightarrow I)) &<& \quad d(0,I) + \mu d(I,Y) \\ \Leftrightarrow \quad & d(0, \frac{1}{\mu} I + (Y \Leftrightarrow I)) &<& \quad d(0, \frac{1}{\mu} I) + d(I,Y) \end{aligned}$$

which completes the proof using the inequalities derived for the previously discussed case.

□

Lemma 3 implies that the boundary of $shadow_d(X)$ is piecewise linear for all points $X \in F$. Note that $shadow_d(X)$ is nevertheless not necessarily convex as can be seen in Figure 1.

**Corollary 2** *Let $d$ be a distance function induced by a norm and let $X \in F$ be a feasible point. Then $\partial(shadow_d(X))$ is piecewise linear.*

# 3 Reducing the non-convex barrier problem to a set of convex location problems

For the median problem with polyhedral barriers and the Euclidean distance function Butt and Cavalier [3] proposed a partitioning of the feasible region into a finite set of subregions $R_k \subseteq F$ such that the shortest barrier distance from every point $X \in R_k$ to all of the existing facilities in $Ex_m \in \mathcal{E}x$ can be calculated with respect to the same intermediate points $I_{Ex_m,X}$, $m \in \mathcal{M}$. Using this partitioning an optimal solution to the original problem can be found by solving a finite set of convex subproblems on each of the subregions $R_k$. Since this approach is not efficient in practice due to the nonlinearity of the boundaries of the regions $R_k$ (their determination is difficult especially as the number of barrier regions and existing facilities increases) a heuristic method is suggested in [3] that avoids the explicit calculation of the subregions $R_k$. Starting with some initial solution $X \in F$, the procedure iteratively solves unrestricted median problems with respect to the intermediate points corresponding to the current solution. This approach is computationally very efficient and an optimal solution of the problem is found in the majority of cases, however, an optimal solution cannot be guaranteed by this procedure.

A different partitioning of the feasible region is suggested in this paper which also avoids the determination of nonlinear boundaries but which still allows the development of an exact solution procedure to solve the non-convex barrier problem. This partitioning uses a smaller number of subregions and, moreover, the boundaries of all subregions are piecewise linear. A major drawback though is that the objective function is not necessarily convex on each of the subregions. We will prove a reduction result that nevertheless implies an exact algorithm based on this grid tessellation of the feasible region.

Consider the grid $\mathcal{G}_d$ in the plane that is defined by the boundaries of the shadows of all existing facilities and of all extreme points of the barrier regions, plus all facets of the barrier regions, i.e.

$$\mathcal{G}_d := \left( \bigcup_{X \in \mathcal{E}x \cup \mathcal{P}(\mathcal{B})} \partial(shadow_d(X)) \right) \cup \mathcal{F}(\mathcal{B}). \tag{4}$$

Since the barriers are convex polyhedra and since the boundary of $shadow_d(X)$ is piecewise linear for all $X \in F$ (Corollary 2), the grid $\mathcal{G}_d$ consists of a finite set of line segments in $F$.

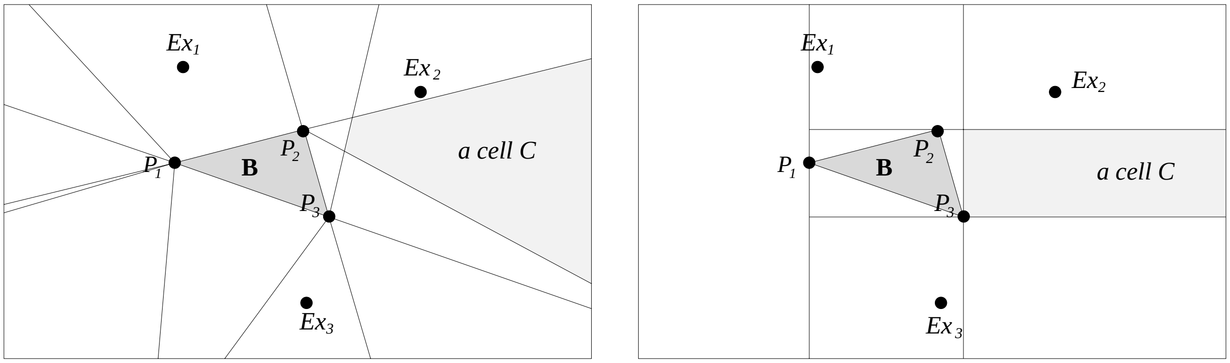


Figure 3: The grids $\mathcal{G}_{l_2}$ and $\mathcal{G}_{l_1}$, respectively, for the example problem.

The set of *cells of* $\mathcal{G}_d$, i.e. the set of smallest (not necessarily convex or closed) polyhedra not intersected by a line segment in $\mathcal{G}_d$, is denoted by $\mathcal{C}(\mathcal{G}_d)$.

Similar to the representation of the barrier distance with respect to intermediate points in Corollary 1, the objective function $f_{\mathcal{B}}(X)$ can be rewritten for every point $X$ in a cell $C \in \mathcal{C}(\mathcal{G}_d)$.

**Corollary 3** *Let $C \in \mathcal{C}(\mathcal{G}_d)$ be a cell and let $X \in C$ be a feasible solution of $(1/P/\mathcal{B}/d_{\mathcal{B}}/f$ convex). Then*

$$f_{\mathcal{B}}(X) = f_X(X), \tag{5}$$

*where*

$$f_X(Y) \;:=\; f(d(Y, I_1) + c_1, \ldots, d(Y, I_M) + c_M), \qquad Y \in I\!R^2, \tag{6}$$

*and*

$$\begin{aligned} c_m &:= d_{\mathcal{B}}(I_m, Ex_m) \\ &= d_G(I_m, Ex_m), \qquad m = 1, \ldots, M \end{aligned} \tag{7}$$

*and where $I_m := I_{Ex_m,X} \neq X$ $(m \in \mathcal{M})$ is an intermediate point on a d-shortest permitted $X$-$Ex_m$-path with Property 1 that is d-visible from $X$.*

Note that $f_X(Y)$ is convex in $I\!R^2$ since it can be interpreted as the composition of the convex, nondecreasing function $f$ and the convex functions $d(Y, I_m) + c_m$, $m \in \mathcal{M}$, where $c_m$ is a constant not depending on the choice of $Y$.

The reformulation of the objective function $f_{\mathcal{B}}$ given in Corollary 3 will be used in the following to interrelate the non-convex problem $(1/P/\mathcal{B}/d_{\mathcal{B}}/f$ convex) to a finite set of corresponding convex problems of type $(1/P/\bullet/d/f_X)$.

**Theorem 1** *Let $C \in \mathcal{C}(\mathcal{G}_d)$ be a cell and let $X^*_{\mathcal{B}} \in \mathrm{int}(C)$ be an optimal solution of $(1/P/\mathcal{B}/d_{\mathcal{B}}/f$ convex). Then $X^*_{\mathcal{B}}$ is an optimal solution of the corresponding convex problem*

$$\begin{aligned} \min \quad & f_{X^*_{\mathcal{B}}}(Y) \\ \text{s.t.} \quad & Y \in I\!R^2, \end{aligned} \tag{8}$$

*where $f_{X^*_{\mathcal{B}}}(Y)$ is defined according to (6) and (7).*

**Proof:** Let $X^*_{\mathcal{B}} \in \text{int}(C)$ and let $f_{X^*_{\mathcal{B}}}(Y)$ be defined according to (6) and (7). Furthermore let $I_m \neq X^*_{\mathcal{B}}$, $m \in \mathcal{M}$ be the corresponding intermediate points on $d$-shortest permitted $X^*_{\mathcal{B}}$-$Ex_m$-paths with Property 1 that are $d$-visible from $X^*_{\mathcal{B}}$. Since $\text{int}(C)$ is not intersected by the boundary of the shadow of any candidate for an intermediate point (i.e. an existing facility or an extreme point of a barrier), the intermediate points $I_m$, $m \in \mathcal{M}$ are $d$-visible for all points $Y \in C$. Thus the inequality

$$f_{\mathcal{B}}(Y) = f_Y(Y) \leq f_{X^*_{\mathcal{B}}}(Y) \tag{9}$$

holds for all $Y \in C$. Assume that there exists a point $Y^* \in C$ such that

$$f_{X^*_{\mathcal{B}}}(Y^*) < f_{X^*_{\mathcal{B}}}(X^*_{\mathcal{B}}).$$

Using (9), we can calculate that

$$f_{\mathcal{B}}(Y^*) = f_{Y^*}(Y^*) \leq f_{X^*_{\mathcal{B}}}(Y^*) < f_{X^*_{\mathcal{B}}}(X^*_{\mathcal{B}}) = f_{\mathcal{B}}(X^*_{\mathcal{B}}),$$

contradicting the optimality of $X^*_{\mathcal{B}}$.
Using the fact that $f_{X^*_{\mathcal{B}}}(Y)$ is a convex function of $Y$ in $\mathbb{R}^2$ and that $X^*_{\mathcal{B}} \in \text{int}(C)$, we can conclude that $X^*_{\mathcal{B}}$ minimizes $f_{X^*_{\mathcal{B}}}(Y)$ in $\mathbb{R}^2$.

□

Theorem 1 implies that any problem of type $(1/P/\mathcal{B}/d_{\mathcal{B}}/f$ convex) can be reduced to a finite set of convex subproblems within each cell in $\mathcal{C}(\mathcal{G}_d)$ even though the original objective function $f_{\mathcal{B}}(X)$ is in general non-convex within the cells.
Note that Theorem 1 can be generalized to the case that the objective function $f_{\mathcal{B}}(X)$ is a non-convex function of the barrier distances. Nevertheless in this case the resulting subproblems are also non-convex and the problem difficulty is not reduced as in the convex case.

In some applications it may be beneficial to consider the grid $\mathcal{G}_{l_2}$ instead of the grid $\mathcal{G}_d$ for a given distance function $d$, especially in the case that the construction of $\mathcal{G}_{l_2}$ is simpler than that of $\mathcal{G}_d$. This is possible for any distance function $d$ induced by a norm since Lemmas 1 and 2 imply the following reformulation of Theorem 1:

**Corollary 4** *Let $d$ be a distance function induced by a norm. Furthermore, let $C \in \mathcal{C}(\mathcal{G}_{l_2})$ be a cell in the grid $\mathcal{G}_{l_2}$ and let $X^*_{\mathcal{B}} \in \text{int}(C)$ be an optimal solution of $(1/P/\mathcal{B}/d_{\mathcal{B}}/f$ convex). Then $X^*_{\mathcal{B}}$ is an optimal solution of the corresponding problem*

$$\begin{aligned} \min \quad & f_{X^*_{\mathcal{B}}}(Y) \\ \text{s.t.} \quad & Y \in \mathbb{R}^2, \end{aligned} \tag{10}$$

*where $f_{X^*_{\mathcal{B}}}(Y)$ is defined according to (6) and (7) and the intermediate points $I_m$ $(m \in \mathcal{M})$ are chosen such that they are $l_2$-visible from $X^*_{\mathcal{B}}$.*

An important consequence of Theorem 1 is that many of the general properties of unrestricted location problems can be transferred to the restricted case ($1/P/\mathcal{B}/d_{\mathcal{B}}/f$ convex) if $X^*_{\mathcal{B}} \notin \mathcal{G}_d$.

As an example consider location problems for which the set of optimal solutions lies within the convex hull of the existing facilities in the unrestricted case. Defining the *iterative convex hull* $R_{\mathcal{B}}$ of the existing facilities and the barrier regions as the smallest convex subset of $F$ such that $\partial R_{\mathcal{B}} \cap \mathrm{int}(\mathcal{B}) = \emptyset$ (see [6] for the construction of $R_{\mathcal{B}}$), the following analogous result can be proven in the restricted case:

**Theorem 2** *Let $X^*_{\mathcal{B}} \notin \mathcal{G}_d$ be an optimal solution of ($1/P/\mathcal{B}/d_{\mathcal{B}}/f$ convex). Then*

$$X^*_{\mathcal{B}} \in R_{\mathcal{B}}$$

*if, for the corresponding unrestricted problem ($1/P/\bullet/d/f_X$) with objective function $f_X$ as defined in (6) and (7), the set of optimal solutions is contained in the convex hull of the existing facilities.*

**Proof:** Let $X^*_{\mathcal{B}}$ be an optimal solution of ($1/P/\mathcal{B}/d_{\mathcal{B}}/f$ convex) such that $X^*_{\mathcal{B}} \in \mathrm{int}(C)$ for some cell $C \in \mathcal{C}(\mathcal{G}_d)$.
Suppose that $X^*_{\mathcal{B}} \notin R_{\mathcal{B}}$. Wlog we assume that there exists no barrier in $\mathbb{R}^2 \setminus R_{\mathcal{B}}$ since this assumption does not increase the objective value of any point $X \in F$. Applying Theorem 1 we can follow that $X^*_{\mathcal{B}}$ is an optimal solution of problem (8) with respect to some intermediate points $I_m \in \mathcal{E}x \cup \mathcal{P}(\mathcal{B})$, $m \in \mathcal{M}$. This problem is an unrestricted location problem of type ($1/P/\bullet/d/f_X$) and thus $X^*_{\mathcal{B}} \in \mathrm{conv}\{I_m : m \in \mathcal{M}\} \cap F$. Since $R_{\mathcal{B}}$ is the convex hull of all existing facilities and all barrier sets intersected with the feasible region $F$, we can conclude that

$$\mathrm{conv}\{I_m : m \in \mathcal{M}\} \cap F \subseteq \mathrm{conv}(\mathcal{E}x \cup \mathcal{P}(\mathcal{B})) \cap F \subseteq R_{\mathcal{B}}.$$

$\square$

Other consequences of Theorem 1 are e.g. the discretization results developed for the median problem with Manhattan- or block norm distances (see [2, 6, 15]).

# 4 Algorithmic consequences

Reducing a problem of type ($1/P/\mathcal{B}/d_{\mathcal{B}}/f$ convex) to a set of convex subproblems with respect to Theorem 1 (or Corollary 4), two different cases may occur. An optimal solution $X^*_{\mathcal{B}}$ of a problem of type ($1/P/\mathcal{B}/d_{\mathcal{B}}/f$ convex) may either be located on the grid $\mathcal{G}_d$ or in the interior of a cell $C \in \mathcal{C}(\mathcal{G}_d)$. In the first case $X^*_{\mathcal{B}}$ can be easily found by applying a line search procedure on the line segments of $\mathcal{G}_d$. In the latter case $X^*_{\mathcal{B}}$ is the optimal solution of a corresponding unrestricted problem (8).

Thus a two step algorithm can be suggested to solve problems of type ($1/P/\mathcal{B}/d_{\mathcal{B}}/f$ convex). In a first step, a line search procedure is applied on each line segment of the grid $\mathcal{G}_d$. In a second step, a local minimum is sought in the interior of a cell in $F \setminus \mathcal{G}_d$ by solving convex subproblems (8) for all feasible reformulations $f_{\mathcal{B}}(Y) = f_X(Y)$ of the objective function. For each solution $Y^*$ of one of these subproblems feasibility has to be tested, i.e. it has to be verified whether $f_{\mathcal{B}}(Y^*) = f_X(Y^*)$.

**Algorithm 1**

**Input:** *Location problem* $(1/P/\mathcal{B}/d_{\mathcal{B}}/f$ *convex).*

Step 1: *Construct the grid* $\mathcal{G}_d$.

Step 2: *Find the minimum of* $(1/P/\mathcal{B}/d_{\mathcal{B}}/f$ *convex) on* $\mathcal{G}_d$.

Step 3: *For all feasible reformulations of the objective function, i.e. for all feasible assignments of intermediate points to existing facilities, do:*

- *(a) Find an optimal solution* $Y^*$ *of the corresponding unrestricted problem* $\min f_X(Y)$, $Y \in \mathbb{R}^2$.
- *(b) If* $f_{\mathcal{B}}(Y^*) = f_X(Y^*)$, *the solution* $Y^*$ *is a candidate for an optimal solution.*

Step 4: *Determine the set of global minima from the candidate set found in Steps 2 and 3.*

**Output:** *Set of optimal solutions of* $(1/P/\mathcal{B}/d_{\mathcal{B}}/f$ *convex).*

Note that the grid $\mathcal{G}_d$ can also be replaced by the grid $\mathcal{G}_{l_2}$ (see Corollary 4).

The time complexity of Steps 1 and 2 of Algorithm 1 depends on the size of the grid $\mathcal{G}_d$ (or $\mathcal{G}_{l_2}$, respectively) and thus on the number of existing facilities, the number of extreme points of the barrier regions and the choice of the distance function $d$. In the case that distances are measured by the Euclidean distance function $l_2$, the number of line segments in $\mathcal{G}_{l_2}$ is bounded by $(\,|\mathcal{E}x| + |\mathcal{P}(\mathcal{B})|\,) \cdot |\mathcal{P}(\mathcal{B})|$.
The overall time complexity of Algorithm 1 is in general dominated by Step 3. If no additional information is available to reduce the possible assignments of existing facilities to intermediate points, the number of subproblems is exponential in the number of existing facilities and in the number of extreme points of the barrier regions. For specially shaped barrier regions better results are nevertheless available. As an example consider the case that the barrier is given by one line with a finite number of passages. In this case it was shown in [12] that a polynomial number of subproblems is sufficient to determine an optimal solution.

A slight modification of this algorithm was implemented in [16] for the Euclidean distance function and one circular or polyhedral barrier. In case of a circular barrier, tangents to the circle are used to define the boundary of the shadow of the barrier; see [13] for a more detailed discussion. In this implementation all the convex subproblems are solved by an adaption of the method of Hooke and Jeeves [10].

To compare Algorithm 1 with the results of other authors, we used an example problem introduced in [11]. In this problem, five existing facilities with weights 1 are given at the coordinates $Ex_1 = (-8.0, -6.0)$, $Ex_2 = (-7.0, 13.0)$, $Ex_3 = (-1.0, -5.0)$, $Ex_4 = (6.6, -0.5)$, $Ex_5 = (4.4, 10.0)$, and one circular barrier with radius 2 centered at $(0.0, 0.0)$ is located within the considered region. We approximated the optimal solution at the point $X_1 = (-1.18602, 2.06044)$ with an objective value of $z_1 = 48.2548$. This result only slightly improves the solution $X_2 = (-1.2016, 2.0776)$ with $z_2 = 48.2560$ as found in [3], whereas in [11] only an approximate solution $X_3 = (-0.08130, 2.4833)$ with $z_3 = 48.3524$ was determined.

Since Algorithm 1 is computationally expensive if no additional information is available on the structure of the problem, a heuristic strategy can alternatively be applied that, in a large number of cases, still finds the optimal solution of $(1/P/\mathcal{B}/d_{\mathcal{B}}/f$ convex) in a remarkably smaller number of iterations. Instead of evaluating all the theoretically possible assignments of existing facilities to intermediate points, a sample set $S$ of (not necessarily equidistant) grid points can be constructed in $R_{\mathcal{B}}$. All the points in this sample set are used as starting points for an unrestricted location problem (8). As in Algorithm 1, the corresponding optimal solution $Y^*$ is used as a candidate for the optimal solution of $(1/P/\mathcal{B}/d_{\mathcal{B}}/f$ convex) if $Y^*$ is feasible, i.e. if $f_{\mathcal{B}}(Y^*) = f_X(Y^*)$.

**Algorithm 2**

**Input:** *Location problem* $(1/P/\mathcal{B}/d_{\mathcal{B}}/f$ *convex).*

Step 1: *Construct the grid* $\mathcal{G}_d$.

Step 2: *Find the minimum of* $(1/P/\mathcal{B}/d_{\mathcal{B}}/f$ *convex) on* $\mathcal{G}_d$.

Step 3: *Define a sample set* $S$ *of grid points in* $R_{\mathcal{B}}$.

Step 4: *For each grid point* $X \in S$ *do:*

- *(a) Find an optimal solution* $Y^*$ *of the corresponding unrestricted problem* $\min f_X(Y)$, $Y \in \mathbb{R}^2$.
- *(b) If* $f_{\mathcal{B}}(Y^*) = f_X(Y^*)$, *the solution* $Y^*$ *is a candidate for an optimal solution.*

Step 5: *Determine the best solution found in Steps 2 and 4.*

**Output:** *Approximation of the optimal solution of* $(1/P/\mathcal{B}/d_{\mathcal{B}}/f$ *convex).*

An optimal solution of a problem of type $(1/P/\mathcal{B}/d_{\mathcal{B}}/f$ convex) can be approximated with increasing accuracy by refining the sample set $S$.

**Theorem 3** *For any problem of type* $(1/P/\mathcal{B}/d_{\mathcal{B}}/f$ *convex), Algorithm 2 yields an optimal solution if the sample set of grid points* $S$ *is chosen sufficiently fine.*

**Proof:** If an optimal solution of $(1/P/\mathcal{B}/d_{\mathcal{B}}/f$ convex) is located on the grid $\mathcal{G}_d$, then this solution is found in Step 2 of the algorithm.

Otherwise, consider a partitioning of the grid $\mathcal{G}_d$ into a finite set of subregions $R_k \subseteq F$, $k = 1, \ldots, K$ as discussed at the beginning of Section 3. These subregions can be chosen such that the shortest barrier distance from every point $X \in R_k$ to all the existing facilities in $Ex_m \in \mathcal{E}x$ can be calculated with respect to the same intermediate points and so that every subregion has a nonempty interior. (Note that subregions with empty interior can be discarded since they must be contained in the boundary of subregions with nonempty interior.) Furthermore, let $\varepsilon$ be maximal with the property that a ball of radius $\varepsilon$ can be included in the interior of every subregion $R_k$, $k = 1, \ldots, K$. If a sample set $S$ of equidistant grid points with step length $\varepsilon$ is chosen, every subregion contains at least one grid point in its interior, thus ensuring that every relevant assignment of existing facilities to intermediate points is considered in the procedure.

$\square$

The sample set $S$ can be chosen in many alternative ways. An intuitive option is to select sample points from the grid $\mathcal{G}_d$. Using this option, the following figure shows the computation times of an implementation of Algorithm 2 in [16] for a set of randomly generated test problems.

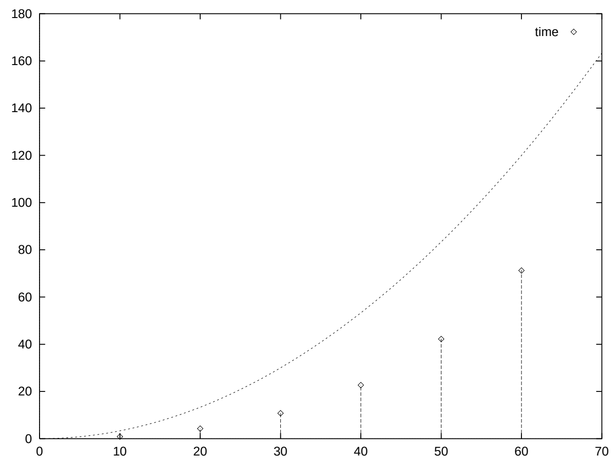


Figure 4: Average cpu time in seconds for problems randomly generated in [16] with $M = 10, 20, \ldots, 60$ existing facilities and one barrier region. For comparison, the function $f(M) = \frac{1}{30}M^2$ is included in the graph.

The computations were performed on a machine of type i586 Linux 2.0.30 with 32MB RAM and 100MHz. The solutions were compared to an approximate global optimum obtained by evaluating the objective function at a finite set of equidistant points. In none of the cases this second solution was better than the solution obtained by Algorithm 2.
However, a high solution quality implies a large number of iterations and thus a decreasing efficiency of the algorithm. A large sample set improves the quality of the solution but on the other hand it is proportional to the number of iterations of the procedure.
In [16] the performance of Algorithm 2 was compared for different sizes of the sample set $S$. If for example only 10% of the intersection points in $\mathcal{G}_d$ are selected (according to

their objective value, i.e. the best 10% of the intersection points are chosen), Algorithm 1 converged to an optimal solution in all of 54 example problems with one circular barrier, whereas in the case of a thin rectangular barrier an optimal solution was found only in 25 out of 38 example problems.

Summarizing the discussion above we conclude that, if computation time is our major concern, the iterative procedure developed in [3] is preferable since it determines a solution that has a high probability to being optimal in a very small number of iterations. If on the other hand the quality of the solution is our main interest, Algorithm 2 (or Algorithm 1) can be applied to verify these results with an accuracy and computation time specified by the user with the choice of the sample set $S$.

It is an interesting open question whether it is possible to construct a small sample set $S$ which still guarantees the detection of an optimal solution of the barrier problem.

# 5 Conclusions

In this paper a reduction result for a general class of planar location problems with polyhedral barriers is developed that allows the exact solution of this type of non-convex optimization problems by solving a finite number of related convex location problems. This result is as well of theoretical as of practical interest. It allows the transfer of theoretical results for unrestricted planar location problems to the restricted case and it also yields exact and heuristic Algorithms to solve planar location problems with barriers.

Future research topics include the investigation of special cases like simple barrier shapes (e.g. rectangles or circles) and specific distance functions (e.g. the Manhattan metric $l_1$ or the more general class of block norms). Furthermore reduction based methods could be combined with modern solution techniques to develop efficient implementations of the suggested algorithms. Further generalizations as e.g. to problems in $\mathbb{R}^n$, to multi-facility problems or to problems with more than one objective function seem to be possible and should be discussed in more detail in the future.

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