Planar Weber Location Problems with Barriers and Block Norms

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Abstract

The Weber problem for a given finite set of existing facilities $\mathcal{E}x = \{Ex_1, Ex_2, \ldots, Ex_M\} \subset I\!R^2$ with positive weights w_m $(m = 1, \ldots, M)$ is to find a new facility $X^* \in I\!R^2$ such that $\sum_{m=1}^{M} w_m d(X, Ex_m)$ is minimized for some distance function d. In this paper we consider distances defined by block norms.

A variation of this problem is obtained if barriers are introduced which are convex polyhedral subsets of the plane where neither location of new facilities nor traveling is allowed. Such barriers like lakes, military regions, national parks or mountains are frequently encountered in practice.

From a mathematical point of view barrier problems are difficult, since the presence of barriers destroys the convexity of the objective function. Nevertheless, this paper establishes a discretization result: One of the grid points in the grid defined by the existing facilities and the fundamental directions of the polyhedral distances can be proved to be an optimal location. Thus the barrier problem can be solved with a polynomial algorithm.

1 Introduction

Location Theory, like many other branches of Operations Research, is driven by two forces: On one hand decisions in management, economy, production planning etc. contain many facets which are related to "locating facilities". On the other hand location theory is by its own right an interesting and challenging part of mathematics with an ever increasing set of problems which may or may not have a real-world background.

In this paper we develop some results which seem to be both of theoretical and practical importance: We use block norms (polyhedral norms, symmetric polyhedral gauges) to evaluate distances and we introduce barriers which restrict the available area for locating facilities and cannot be crossed while going from one facility to some other ("no trespassing" property).

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Gauge distances have been introduced by Minkowski 1967 [16]. Within location theory Durier and Michelot 1985 [6] showed a discretization result for location problems with polyhedral gauges which will be reviewed later on. Nickel 1995 [18] showed that also location problems with restrictions (i.e. regions which can be crossed but cannot be used for placement of new facilities) can be discretized. The importance of polyhedral gauges and, in particular, block norms, in evaluating distances in real-world contexts was pointed out by Ward and Wendell 1985 [20] and Brimberg and Love 1996 [3].

Restrictions are part of virtually all real-world location problems, since there are in general regions to exclude from placement of new facilities. In most cases these regions can also not be used for transportation such that barrier problems are realistic models for location problems occurring in practice. They have been considered by Katz and Cooper 1981 [13] if the barrier is a single circular region and distances are measured with the Euclidean distance function, and by Klamroth 1996 [14] for the case that the barrier is a line with passages and the distance function is derived from a norm. Aneja and Parlar 1994 [1] and recently Butt and Cavalier 1996 [4] developed heuristics for the case that the barriers are closed polyhedra and the distance is given by the l_p-metric. In the special case of the Manhattan metric l_1 discretization results where proved by Larson and Sadiq 1983 [15] and by Batta, Ghose and Palekar 1989 [2] for arbitrarily shaped barriers.
In this paper upper bounds for location problems with barriers are provided. Furthermore, planar location problems with poluhedral barriers and block norms are studied and a discretization result similar to those given in [2, 15] is developed.

In the following section we will show how to compute lower and upper bounds for barrier problems. The bounds are obtained from the solution of restricted problems which use the barrier as restricting set but allow trespassing (Section 3). In Section 4 it is shown that the barrier problem can be reduced to a discrete location problem (i.e. a location problem with a discrete set of possible locations). In the next section we start with a formal introduction of the problem.
Throughout the paper we use the classification $Pos1/Pos2/Pos3/Pos4/Pos5$ of location problems as introduced in Hamacher 1995 [8] or Hamacher and Nickel 1996 [11] (see Hamacher and Nickel 1999 [12] for an overview). In this classification scheme, $Pos1$ indicates the number of new facilities (e.g. 1 in the case of a single-facility problem), $Pos2$ gives the type of the location problem (e.g. P in the case of planar location problems), $Pos3$ contains special assumptions (e.g. forbidden regions $\mathcal{R}$ or barriers $\mathcal{B}$ in the planar case or a $\bullet$ if no special assumptions are to be made), $Pos4$ gives the distance function in the planar case (e.g. l_1 or l_2) and $Pos5$ indicates the objective function (e.g. $\sum$ for Median problems and max for Center problems). As an example, the unrestricted Weber problem with Euclidean distances will be classified as $1/P/\bullet/l_2/\sum$.

2 Location problems with block norms and barriers

Let $\mathcal{B} = \{B_1, \ldots, B_N\}$ be a set of convex, closed and pairwise disjoint barriers in the plane, i.e. regions where neither trespassing nor location of new facilities is allowed. Note that boundedness of the barrier regions is not required in this definition. Unbounded barriers may occur for example in the modeling of oceans and rivers, or, on a msaller scale, in the case that a large barrier region intersects a small modeling horizon.
The feasible region F for new locations is given by

$$F := \mathbb{R}^2 \setminus \operatorname{int}(\mathcal{B}).$$

Furthermore a finite number of existing facilities $Ex_m \in F$, $m \in \mathcal{M} = \{1, \ldots, M\}$ is given in a connected subset of the feasible region F. With each existing facility a positive weight $w_m := w(Ex_m)$ is associated representing the demand of facility Ex_m.
The major difference to unrestricted planar location problems becomes clear in the definition of the distance measure: Let the given distance function d be derived from a norm $\|\bullet\|$. Then the distance $d_{\mathcal{B}}(X, Y)$ between two points $X, Y \in F$ is defined as the length of a shortest path (with respect to the given distance function d) from X to Y not crossing a barrier. Formally, let p be a piecewise continuous differentiable parametrization $p : [a, b] \to \mathbb{R}^2$, $a, b \in \mathbb{R}$, $a < b$, of a *permitted path* connecting X and Y, i.e. a curve not intersecting the interior of a barrier, $p([a, b]) \cap \operatorname{int}(\mathcal{B}) = \emptyset$, with $p(a) = X$ and $p(b) = Y$. Then $d_{\mathcal{B}}$ is given by

$$d_{\mathcal{B}}(X, Y) := \min \left\{ \int_a^b \|p'(t)\| \, \mathrm{d}t \ : \ p \text{ permitted path connecting } \ X \text{ and } \ Y \right\}.$$

Any path connecting X and Y with length $d_{\mathcal{B}}(X, Y)$ not intersecting the interior of $\mathcal{B}$ is called a *d-shortest permitted path connecting X and Y.*
Note that for $d_{\mathcal{B}}$ the triangle inequality is satisfied (provided it holds for the original distance function d), but that $d_{\mathcal{B}}$ is in general not positively homogeneous.

Using this problem formulation the Weber problem can be restated: While the unrestricted Weber problem $1/P/\bullet/d/\sum$ is to find a new facility $X \in \mathbb{R}^2$ minimizing $f(X) = \sum_{i=1}^{M} w_m d(X, Ex_m)$, the Weber problem with barriers $1/P/\mathcal{B}/d_{\mathcal{B}}/\sum$ is to find a new facility $X^*_{\mathcal{B}} \in F$ such that

$$f_{\mathcal{B}}(X) := \sum_{i=1}^{M} w_m d_{\mathcal{B}}(X, Ex_m)$$

is minimized.

From the definition of $d_{\mathcal{B}}$ follows that $f_{\mathcal{B}}$ is in general not convex. Due to this basic difference to unrestricted planar location problems most of the methods developed in planar location theory cannot be used to handle problems of the type $1/P/\mathcal{B}/d_{\mathcal{B}}/\sum$ in general. (It should be noted, that in a correct classification of this problem the properties of $\mathcal{B}$

stated at the beginning of this section could be specified. We will not do this to simplify the denotation.)

As already mentioned our main purpose will be to develop concepts for the case that distances are measured by block norms.
A block norm (polyhedral norm, symmetric polyhedral gauge) is given by a symmetric convex polyhedron $\mathcal{P}$ in the plane $I\!R^2$ containing the origin $0 = (0,0)$ in its interior. It is well known [16] that $\mathcal{P}$ defines a norm $\| \bullet \|$ given by

$$\| X \| := \min_{\lambda \in I\!R_+} \{\lambda : X \in \lambda \mathcal{P}\}.$$

With $d_1, \ldots, d_\delta$ we denote the extreme points of $\mathcal{P}$ and call them *fundamental directions* (see Figure 1). If X is in the cone $\mathcal{C}(d_i, d_{i+1})$ spanned by d_i and d_{i+1}, then $X = \| X \| \, Z$, where Z is the intersection point of the boundary $\partial\mathcal{P}$ of $\mathcal{P}$ with the line segment connecting 0 and X. Hence with $\mu \in [0,1]$ we get

$$Z = \mu d_i + (1-\mu) d_{i+1}$$

and thus

$$\begin{aligned} \frac{1}{\| X \|} \cdot X &= \mu d_i + (1-\mu) d_{i+1} \\ \Rightarrow \qquad X &= \mu \| X \| \, d_i + (1-\mu) \, \| X \| \, d_{i+1}. \end{aligned}$$

On the other hand $X \in \mathcal{C}(d_i, d_{i+1})$ implies

$$X = \alpha_i d_i + \alpha_{i+1} d_{i+1}$$

for two scalars $\alpha_i, \alpha_{i+1} \in I\!R_+$. Since the representation of X in terms of d_i and d_{i+1} is unique, we have

$$\begin{aligned} \mu \, \| X \| &= \alpha_i \qquad \text{and} \\ (1-\mu) \, \| X \| &= \alpha_{i+1}, \end{aligned}$$

which implies $\| X \| = \alpha_i + \alpha_{i+1}$. Thus only the two fundamental directions d_i and d_{i+1} need to be used to determine $\| X \|$ for any point $X \in \mathcal{C}(d_i, d_{i+1})$.
Obviously, we can interpret $\| X \|$ as the distance $\gamma(0, X)$ between 0 and X and extend this definition to define the *gauge distance*

$$\gamma(X, Y) := \gamma(0, Y - X) = \| Y - X \|$$

between any two points $X, Y \in I\!R^2$. Due to the preceding discussion the gauge distance can be represented by a (d_i, d_{i+1})-staircase path using only the two fundamental directions d_i and d_{i+1} with Euclidean length $\alpha_i \, \| d_i \|_2$ and $\alpha_{i+1} \, \| d_{i+1} \|_2$ in direction d_i and d_{i+1}, respectively (see Figure 2).

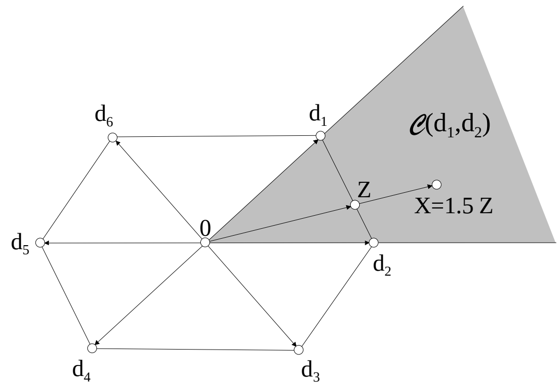

Figure 1: A block norm with six fundamental directions

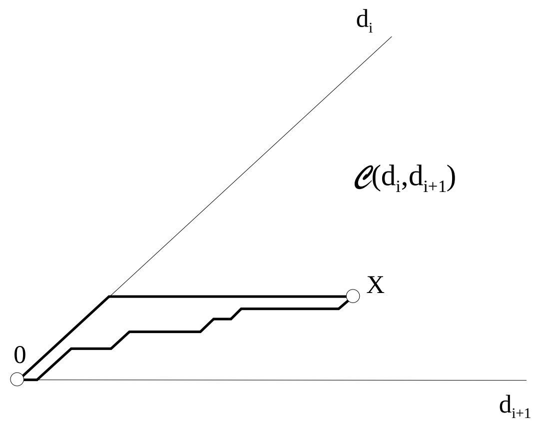

Figure 2: Two possible (d_i, d_{i+1})-staircase paths representing $\gamma(0, X)$

Next, we consider the situation, where a barrier B is given which cannot be trespassed, i.e. the set of permitted paths between two points X and Y in F consists only of those paths not intersecting the interior of B. A *shortest permitted path* (connecting X and Y) is one whose length is equal to the *barrier-gauge distance* $\gamma_B(X, Y) \geq \gamma(X, Y)$ for some points $X, Y \in F$. We restrict ourselves to barriers which are convex closed subsets of $\mathbb{R}^2$. In this situation we consider $X \in \mathcal{C}(d_i, d_{i+1})$ and distinguish three cases in which $B \cap \mathcal{C}(d_i, d_{i+1}) \neq \emptyset$ (see Figure 3):

Case a: The lines $L_i := \{\lambda d_i \ : \ \lambda \geq 0\}$ and $L_{i+1} := \{\lambda d_{i+1} \ : \ \lambda \geq 0\}$ both contain points of B.

Case b: Only one of the lines, say L_{i+1}, contains a point of B.

Case c: Neither L_i nor L_{i+1} contain points of B.

In all cases B separates $\mathcal{C}(d_i, d_{i+1})$ into two parts: One part in which there exists a permitted path from 0 to X, i.e. a path not intersecting the interior of B, with length $\gamma(0, X)$ and

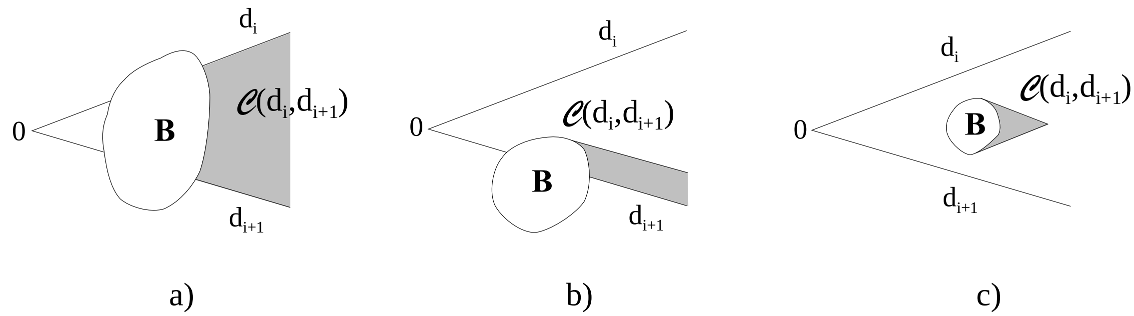

Figure 3: Three cases in which a barrier B changes the distance between 0 and X. The shaded area is the set of points for which $\gamma_B(0,X) > \gamma(0,X)$.

one part where this is not true. We call the latter part the *non γ-visible* part or *γ-shadow* of $\mathcal{C}(d_i, d_{i+1})$ while the former is the *γ-visible part* of $\mathcal{C}(d_i, d_{i+1})$. It should be noted that this visibility concept needs to refer to the underlying distance γ: Some γ-visible points are obviously non-visible in the usual sense (i.e. non l_2-visible).

In all three cases the non γ-visible part of $\mathcal{C}(d_i, d_{i+1})$ has to be determined differently. The non γ-visible part of $\mathcal{C}(d_i, d_{i+1})$ in case (a) equals the non l_2-visible part of $\mathcal{C}(d_i, d_{i+1})$. In case (b) let wlog $B \cap d_{i+1} \neq \emptyset$. Then the non γ-visible part of $\mathcal{C}(d_i, d_{i+1})$ is a subset of the non l_2-visible part of $\mathcal{C}(d_i, d_{i+1})$. It is the region bounded by ∂B, d_{i+1} and the tangent on B in $\mathcal{C}(d_i, d_{i+1})$ parallel to d_{i+1}. Analogously, in case (c) the non γ-visible points are non l_2-visible and the corresponding subset of $\mathcal{C}(d_i, d_{i+1})$ is bounded by ∂B and two tangents on ∂B parallel to d_i and d_{i+1}, respectively.
In all cases the set of non γ-visible points is also non l_2-visible, and it can be easily shown that this is also true in general:

Corollary 1 *Every point that is l_2-visible from the origin is also γ-visible from the origin. Furthermore in this case the straight line segment connecting the origin and X is a shortest permitted path from the origin to X with respect to γ.*

Proof: Let X be a point that is l_2-visible from the origin with $\gamma(0,X) = \alpha_i + \alpha_{i+1}$. Then the straight-line segment connecting the origin and the point X is a permitted path from the origin to X given by $p : [0,1] \to I\!R^2$, $p(0) = 0$, $p(1) = X$ and $p(t) = tX$, $t \in [0,1]$. The length of this path is given by

$$\int_0^1 \| p'(T) \| \, \mathrm{d}t = \lim_{n\to\infty} \sum_{k=1}^{n} \gamma(\frac{k-1}{n}X, \frac{k}{n}X) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\gamma(0,X) = \gamma(0,X).$$

□

In the special case that all barriers are convex polyhedra, the relation between γ-visibility and l_2-visibility can be used to obtain a simpler description of $\gamma_{\mathcal{B}}$. The following lemma is a generalization of a result of Viegas and Hansen 1985 [19] for the rectilinear distance function:

Lemma 1 *Let $X, Y \in \mathbb{R}^2 \setminus \operatorname{int}(\mathcal{B})$ where $\mathcal{B}$ is a finite set of polyhedral barriers. Then there exists a γ-shortest permitted path SP from X to Y with the following property:*

$$SP \text{ is a piecewise linear path with breakpoints only in extreme points of barriers.} \tag{1}$$

Proof: Let, therefore, SP be a piecewise linear path from X to Y which is a γ-shortest permitted path connecting X and Y, for which (1) is not true. Note that such a path always exists since any γ-shortest permitted path between X and Y can be partitioned by a finite set of points such that two consecutive points are l_2-visible and since Corollary 1 therefore implies that the straight line segment connecting two consecutive points is a γ-shortest path. Then a γ-shortest permitted path SP' with property (1) can be constructed in the following way:
Let $[T_{i-1}, T_i]$ and $[T_i, T_{i+1}]$ be two consecutive straight line segments of SP. If T_{i-1} and T_{i+1} are l_2-visible, $[T_{i-1}, T_i]$ and $[T_i, T_{i+1}]$ can be replaced by $[T_{i-1}, T_{i+1}]$ without increasing the length of SP. If T_{i-1} and T_{i+1} are not l_2-visible, the breakpoint T_i can be moved along $[T_{i-1}, T_i]$ or along $[T_i, T_{i+1}]$ towards T_{i-1} or T_{i+1}, respectively, without increasing the length of SP, until one of these line segments becomes tangent of a barrier. Due to the triangle inequality for γ this change does not increase the length of SP.
While iterating both operations every extreme point of a barrier located on SP is interpreted as a breakpoint T_i even if $[T_{i-1}, T_{i+1}]$ is a straight line segment. Thus the iteration of both operations yields a path SP' with the desired property since every breakpoint of SP which is no extreme point of a barrier can be moved towards X, Y or an extreme point of a barrier, respectively.

□

3 Bounds for barrier problems

In order to obtain bounds for the barrier problem $1/P/\mathcal{B}/\gamma_{\mathcal{B}}/\sum$ it is relaxed to a restricted location problem: While it is still forbidden to place a new facility in $\operatorname{int}(\mathcal{B})$ trespassing is allowed. This problem, classified as $1/P/\mathcal{R} = \mathcal{B}/\gamma/\sum$ can be solved by an algorithm developed in Hamacher and Nickel 1994,1995 [9, 10] for the special case of $\gamma = l_1$ and $\gamma = l_\infty$ and in Nickel 1995 [17] for general polyhedral gauges. An optimal location $X^*_{\mathcal{R}}$ of the restricted problem is obtained by solving first the unrestricted problem $1/P/\bullet/\gamma/\sum$. If an optimal location X^* of the unrestricted problem is feasible, i.e., $X^* \not\subseteq \operatorname{int}(\mathcal{B})$, then $X^*_{\mathcal{R}} = X^*$ (see Figure 4, $\mathcal{B} = \{B_a\}$). Otherwise it can be shown that $X^*_{\mathcal{R}}$ is the best of the at most δM many intersection points of fundamental directions with the boundary $\partial\mathcal{B}$ of $\mathcal{B}$ (see Figure 4, $\mathcal{B} = \{B_b\}$).

Lemma 2 *Let $z^*_{\mathcal{B}}$ be the optimal objective value of the barrier problem $1/P/\mathcal{B}/\gamma_{\mathcal{B}}/\sum$ and let $X^*_{\mathcal{R}}$ be an optimal solution of the restricted problem $1/P/\mathcal{R} = \mathcal{B}/\gamma/\sum$. Then*

$$f(X^*_{\mathcal{R}}) = \sum_{i=1}^{M} w_i \gamma(Ex_i, X^*_{\mathcal{R}}) \leq z^*_{\mathcal{B}} \leq \sum_{i=1}^{M} w_i \gamma_{\mathcal{B}}(Ex_i, X^*_{\mathcal{R}}) = f_{\mathcal{B}}(X^*_{\mathcal{R}}).$$

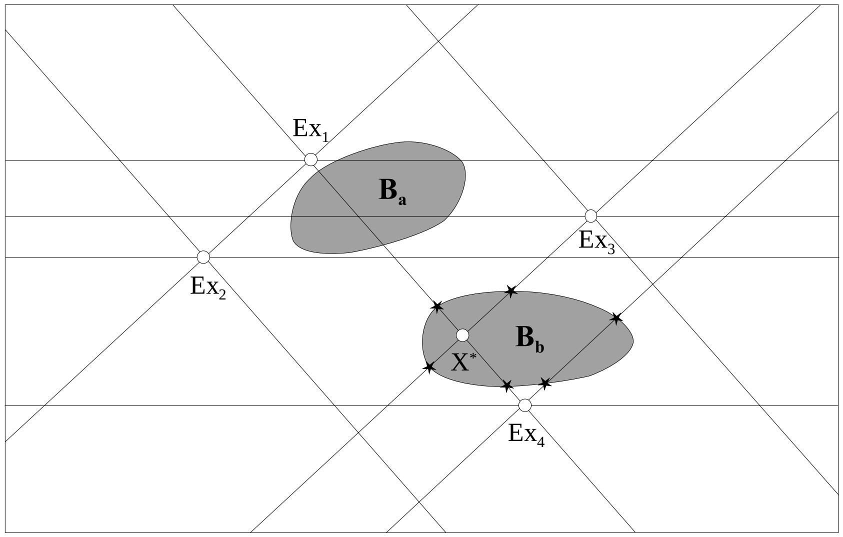

Figure 4: In $1/P/\mathcal{R} = \{B_a\}/\gamma/\sum$ we have $X^*_{\mathcal{R}} = X^*$. In $1/P/\mathcal{R} = \{B_b\}/\gamma/\sum$ one of the intersection points marked by stars is the optimal solution $X^*_{\mathcal{R}}$.

Proof: The second inequality is trivial. For the first one let $X^*_{\mathcal{B}}$ be an optimal solution of the barrier problem. Since $X^*_{\mathcal{R}}$ is an optimal solution of the restricted problem and since $f(X) \leq f_{\mathcal{B}}(X)$ for all $X \in F$ we have

$$\begin{aligned} f(X^*_{\mathcal{R}}) &\leq f(X^*_{\mathcal{B}}) \\ &\leq f_{\mathcal{B}}(X^*_{\mathcal{B}}) \\ &= z^*_{\mathcal{B}}. \end{aligned}$$

□

An immediate consequence of the preceding lemma is the next result.

Corollary 2 *Let $X^*_{\mathcal{R}}$ be an optimal solution of the restricted problem $1/P/\mathcal{R} = \mathcal{B}/\gamma/\sum$. If $\gamma(Ex_i, X^*_{\mathcal{R}}) = \gamma_{\mathcal{B}}(Ex_i, X^*_{\mathcal{R}})$ for all $i = 1, \ldots, M$, then $X^*_{\mathcal{R}} = X^*_{\mathcal{B}}$ is an optimal solution of $1/P/\mathcal{B}/\gamma_{\mathcal{B}}/\sum$.*

In the example given in Figure 4 optimality cannot be shown for the problem $1/P/\mathcal{R} = \{B_a\}/\gamma/\sum$ since $X^*_{\mathcal{R}} = X^*$ and $\gamma(Ex_1, X^*_{\mathcal{R}}) < \gamma_{\mathcal{B}_a}(Ex_1, X^*_{\mathcal{R}})$.

A different approach to derive bounds for the barrier problem makes use of the visibility graph of the problem to interrelate the barrier problem with a network location problem. For this purpose let the set of barriers be a set of polyhedra with extreme points $\mathcal{P}(\mathcal{B}) := \{p_i : \ i = 1, \ldots, P\}$. Then the embedded visibility graph is defined by $G = (V, E)$ with node set $V(G) = \mathcal{E}x \cup \mathcal{P}(\mathcal{B})$ and weights $w(v) = 0$ if $v = p \in \mathcal{P}(\mathcal{B})$ and $w(v) = w(Ex_m)$

if $v = Ex_m \in \mathcal{E}x$. Any two nodes $v_i, v_j \in V(G)$ which are γ-visible in the embedding of G in F are connected by an edge of length $\gamma(v_i, v_j)$. With $d(u,v)$ the length of a shortest network path between u and v is denoted. Then the node network location problem $1/G/\bullet/d(V,V)/\sum$ on G is defined by $\min_{v \in V(G)} f_G(v)$ with

$$f_G(v) = \sum_{u \in V(G)} w(v) d(u,v).$$

Lemma 3 *Let polyhedra with extreme points $\mathcal{P}(\mathcal{B}) := \{p_i : \ i = 1, \ldots, P\}$ be given as barriers and let $\mathcal{E}x$ be a set of existing facilities in the feasible region. Furthermore let G be the visibility graph of the existing facilities and the extreme points of the barriers as defined above. If X_G^* is an optimal solution of the node network location problem $1/G/\bullet/d(V,V)/\sum$ on G, then the corresponding point X_G^* of the embedding of G in the plane is feasible for $1/P/\mathcal{B}/\gamma_{\mathcal{B}}/\sum$ and*

$$f_{\mathcal{B}}(X_{\mathcal{B}}^*) \leq f_G(X_G^*).$$

Proof: The feasibility of X_G^* is trivial because $X_G^* \in V(G) = \mathcal{E}x \cup \mathcal{P}(\mathcal{B})$. The upper bound on the optimal objective value of the barrier problem follows from

$$\begin{aligned}
f_{\mathcal{B}}(X_{\mathcal{B}}^*) &= \min_{X \in F} \sum_{m=1}^{M} w_m \gamma_{\mathcal{B}}(Ex_m, X) \\
&\leq \min_{X \in \mathcal{E}x \cup \mathcal{P}(\mathcal{B})} \sum_{m=1}^{M} w_m \gamma_{\mathcal{B}}(Ex_m, X) \\
&= \min_{X \in V(G)} \sum_{v \in V(G)} w(v) d(v, X) \\
&= f_G(X_G^*).
\end{aligned}$$

□

An example for the application of Lemma 3 is given in Figure 5.

Even though the bounds derived above may lead to good approximations or even to optimal solutions in many applications, a solution with proven quality can in general not be expected. In fact, examples can be constructed for which these bounds become arbitrarily bad with respect to the optimal solution of the problem:
For the upper bound based on the relaxation to restricted location problems $1/P/\mathcal{R} = \mathcal{B}/\gamma/\sum$, consider an arbitrarily long but narrow barrier region and three existing facilities with equal weights, two of which are located on the same side of the barrier. Obviously, an optimal solution of $1/P/\mathcal{B}/\gamma_{\mathcal{B}}/\sum$ would then be located on the same side of the barrier as the two existing facilities, whereas the barrier may be placed so that the solution of $1/P/\mathcal{R} = \mathcal{B}/\gamma/\sum$ is located on the opposite side of the barrier. Since increasing the length of the barrier increases the error of the approximation, the corresponding bound may become arbitrarily weak.

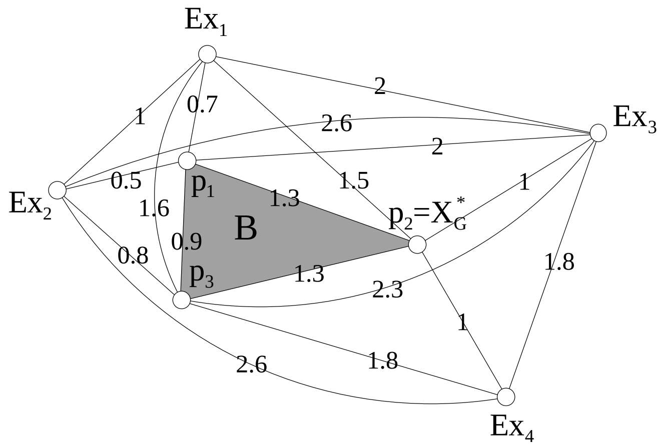

Figure 5: The visibility graph G for a barrier problem with the block norm introduced in Figure 1. If the weights of all existing facilities are equal to one, the optimal solution of the node network location problem on G is $X_G^* = p_2$ with objective value $f_G(X_G^*) = 5.3$.

For the upper bound based on the visibility graph of the problem, an example with similar properties can be constructed even without introducing a barrier region. Let three existing facilities with equal weights be located on the corner points of a triangle with equal side lengths. Then the optimal solution of $1/P/\bullet/\gamma/\sum$ is located at the center of the triangle, whereas any solution of $1/G/\bullet/d(V,V)/\sum$ is located in a corner point of the triangle. Increasing the size of the triangle again leads to an arbitrarily bad bound.

However, location problems with barriers are in general hard, nonconvex optimization problems for which bounds are crucial in order to develop efficient solution procedures. Since the bounds given in this section can be easily calculated and since they perform well in practice, their application can significantly facilitate the solution of problems of the type $1/P/\mathcal{B}/\gamma_{\mathcal{B}}/\sum$.

4 A finite dominating set for barrier problems with block norms

Discretization of planar location problems with polyhedral gauges, a special case of which are block norms, to discrete location problems was already successful for different kinds of problems. Durier and Michelot 1985 [6] showed that in the case of the unrestricted Weber problem with polyhedral gauges $1/P/\bullet/\gamma/\sum$ the fundamental directions rooted at the existing facilities EX_m, $m \in \mathcal{M}$, (*construction lines*) define a grid tessalation of the plane such that the set of optimal locations is a cell, a line connecting two adjacent grid points of a cell or a single grid point. If none of these optimal locations is feasible for the restricted

Weber problem with convex forbidden regions and polyhedral gauges $1/P/\mathcal{R}/\gamma/\sum$ then Nickel 1995 [17] showed that it is sufficient to consider only the intersection points of construction lines and the boundary $\partial\mathcal{R}$ of the forbidden set $\mathcal{R}$. Both results are heavily based on the fact that the objective function is convex and linear in each cell.

Although both of these properties are in general not satisfied in the barrier problem $1/P/\mathcal{B}/\gamma_{\mathcal{B}}/\sum$ we will show in this section that, nevertheless, a tessalation of the plane yielding an optimal grid point for $1/P/\mathcal{B}/\gamma_{\mathcal{B}}/\sum$ can be found. This can be done in polynomial time.

With $\mathcal{P}(\mathcal{B})$ and $\mathcal{F}(\mathcal{B})$ we denote the set of extreme points and facettes of the convex barrier polyhedra, respectively. Moreover let $\mathcal{E}x$ be the set of existing facilities. For any $X \in \mathcal{E}x \cup \mathcal{P}(\mathcal{B})$ and for any fundamental direction d_i $(i = 1, \ldots, \delta)$ let

$$(X + d_i)_{\mathcal{B}} := \{X + \lambda d_i \ : \ \lambda \in I\!R_+; \ (X + \mu d_i) \cap \operatorname{int}(\mathcal{B}) = \emptyset \ \forall 0 \leq \mu \leq \lambda\}$$

be the set of points in the plane which are l_2-visible from X in the fundamental direction d_i. Then

$$\mathcal{G} := \left(\bigcup_{X \in \mathcal{E}x \cup \mathcal{P}(\mathcal{B})} \ \bigcup_{i=1}^{\delta} (X + d_i)_{\mathcal{B}} \right) \cup \mathcal{F}(\mathcal{B})$$

defines a grid in $I\!R^2$. The intersection points of lines in $\mathcal{G}$ define the set $\mathcal{P}(\mathcal{G})$ of grid points and $\mathcal{C}(\mathcal{G})$ is the set of resulting cells in F, i.e. the set of smallest convex polyhedra with extreme points in $\mathcal{P}(\mathcal{G})$ (see Figure 6).

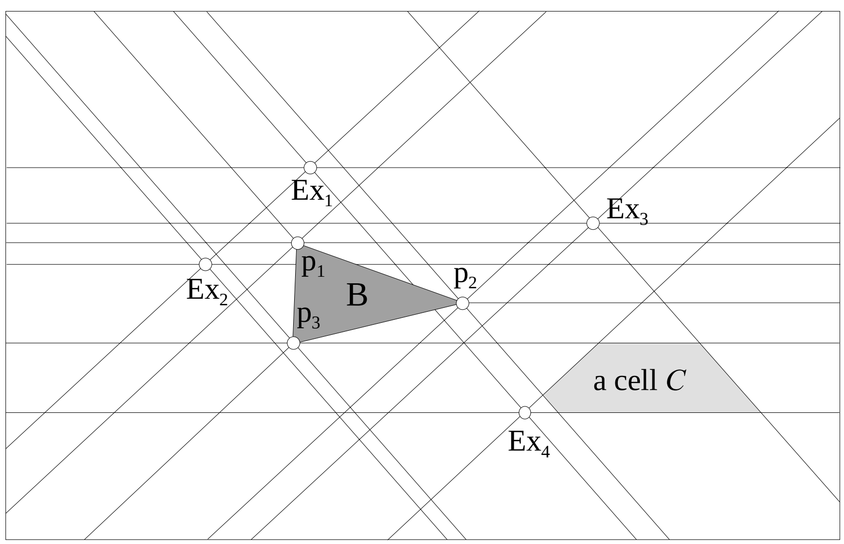

Figure 6: The grid $\mathcal{G}$ for the barrier problem introduced in Figure 5.

Note that $\mathcal{G}$ is constructed such that each existing facility in $\mathcal{E}x$ and each extreme point in $\mathcal{P}$ of a barrier, which is γ-visible from any point in the interior of a cell C, is γ-visible from all points of C.

Theorem 1 *One of the grid points of $\mathcal{G}$ is optimal for $1/P/\mathcal{B}/\gamma_{\mathcal{B}}/\sum$.*

Proof: Let $C \in \mathcal{C}(\mathcal{G})$ be a cell and let $X \in C$ such that X is not a grid point. For any $Ex_m \in \mathcal{E}x$ we know by Lemma 1 that there exists a γ-shortest path SP from X to Ex_m with property (1), i.e. SP is a piecewise linear path $SP = (X = T_0, T_1, \ldots, T_{k-1}, T_k = Ex_m)$ with breakpoints T_i $(i = 1, \ldots, k-1)$ only in extreme points of a barrier.
Let $I_m := T_1$ where $I_m = Ex_m$ if $k = 1$ (and, consequently, if SP is a straight line) and $I_m \in \mathcal{P}(\mathcal{B})$ otherwise. By definition of SP the grid point I_m is l_2-visible as well as γ-visible from X (see Figure 7). Since $\gamma_{\mathcal{B}}(X, Ex_m) = \gamma(X, I_m) + \gamma_{\mathcal{B}}(I_m, Ex_m)$ the objective function for X can be written as

$$f_{\mathcal{B}}(X) = \underbrace{\sum_{m\in\mathcal{M}} w_m \gamma(X, I_m)}_{=:f_X(X)} + \underbrace{\sum_{m\in\mathcal{M}} w_m \gamma_{\mathcal{B}}(I_m, Ex_m)}_{=:K \text{ (constant for fixed } X)}. \tag{2}$$

For any other points $Y \in C$ we have $\gamma_{\mathcal{B}}(Y, Ex_m) \leq \gamma(Y, I_m) + \gamma_{\mathcal{B}}(I_m, Ex_m)$ since I_m is γ-visible from any point of the cell C and thus

$$f_{\mathcal{B}}(Y) \leq f_X(Y) + K \qquad \forall Y \in C,$$

where equality holds for $X = Y$. Here $f_X(Y)$ is the objective function of an unrestricted Weber problem $1/P/\bullet/\gamma/\sum$ with existing facilities $\{I_m : m \in \mathcal{M}\}$.
Ward and Wendell 1985 [20] proved for this problem $1/P/\bullet/\gamma/\sum$ that the level curves $L_{=}(z, f_X, C) := \{Y \in C \ : \ f_X(Y) = z\}$ are linear in the cell C. (Note that the cell C of the grid $\mathcal{G}$ is contained in a cell C_X of the analogous grid $\mathcal{G}_X$ of this unrestricted Weber problem $1/P/\bullet/\gamma/\sum$.) From the convexity of C it follows that there must exist a grid point $I^* \in \mathcal{P}(\mathcal{G})$ of C such that $f_X(I^*) \leq f_X(X)$. Hence

$$\begin{aligned} f_{\mathcal{B}}(I^*) &\leq f_X(I^*) + K \\ &\leq f_X(X) + K \\ &= f_{\mathcal{B}}(X) \end{aligned}$$

proving the result of Theorem 1.

□

It should be noted that this result is known (Larson and Sadiq 1983 [15]) for rectilinear distances ($\gamma = l_1$). Their proof heavily relies on the fact that the objective function is convex within each cell, a fact which is not needed in the preceding proof. Moreover, Larson and Sadiq 1983 [15] proved in the rectilinear case that for any point X in a cell C there exists an l_1-shortest path from X to Ex_m passing through a corner point of C which

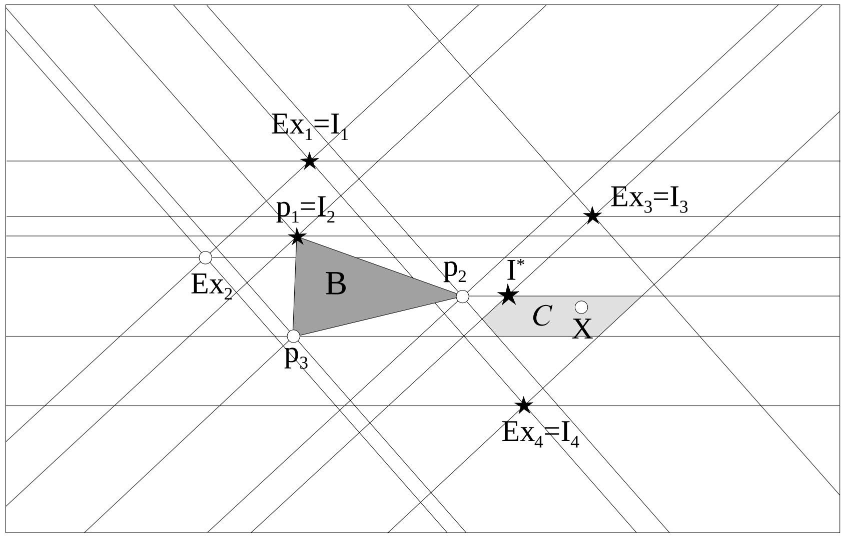

Figure 7: For a point $X \in C$ the corresponding intersection points I_m, $m = 1, \ldots, 4$ and I^* are marked by stars.

is not true in general for block norms. In Figure 7 there exists for example no γ-shortest path from X to Ex_3 passing through a corner point of C.
The methods used in the proof of Theorem 1 will be generalized in the following to derive a stronger result for the set of optimal solutions of $1/P/\mathcal{B}/\gamma_{\mathcal{B}}/\sum$.

Corollary 3 *The set $\mathcal{X}^*_{\mathcal{B}}$ of optimal solutions of $1/P/\mathcal{B}/\gamma_{\mathcal{B}}/\sum$ can be partitioned into subsets that are either*

- *grid points of $\mathcal{G}$,*
- *facets of cells of $\mathcal{G}$ or*
- *complete cells of $\mathcal{G}$.*

Proof: Using the same decomposition of the objective function as in the proof of Theorem 1 and using the linearity of the objective function of the corresponding unrestricted Weber problem, this result can be proven analogous to Theorem 1.

□

Theorem 1 leads to the formulation of a simple and efficient algorithm that computes at least one optimal solution of the barrier problem $1/P/\mathcal{B}/\gamma_{\mathcal{B}}/\sum$. The algorithm is based on the discretization of the problem to the set of grid points $\mathcal{P}(\mathcal{G})$.

Construction Line Algorithm for $1/P/\mathcal{B}/\gamma_{\mathcal{B}}/\sum$:

1. Compute the grid $\mathcal{G}$.

2. Determine the set of all grid points $\mathcal{P}(\mathcal{G})$.

3. **Output:** $X^*_{\mathcal{B}} \in \operatorname{argmin}\{f_{\mathcal{B}}(I) : I \in \mathcal{P}(\mathcal{G})\}$.

The size of the candidate set $\mathcal{P}(\mathcal{G})$ constructed during this algorithm is bounded by the number of intersection points of construction lines in $\mathcal{G}$. Since the total number of construction lines is bounded by $O((M+P)\delta)$ and since each construction line may intersect every other construction line at most once, the size of the candidate set is bounded by $|\mathcal{P}(\mathcal{G})| \leq O((M+P)^2\delta^2)$. This implies that the construction line algorithm solves problems of the type $1/P/\mathcal{B}/\gamma_{\mathcal{B}}/\sum$ in polynomial time.

We will show in the following that the size of the candidate set can be reduced by omitting a large set of points which cannot be optimal. This will be done by restricting the optimal solution to a subset $F_{\mathcal{B}}$ of the feasible region F.

Theorem 2 *Let $F_{\mathcal{B}}$ be the smallest closed convex subset of F such that $\mathcal{E}x \subset F_{\mathcal{B}}$ and $\partial F_{\mathcal{B}} \cap \operatorname{int}(\mathcal{B}) = \emptyset$. Then there exists at least one optimal solution of the barrier problem $1/P/\mathcal{B}/\gamma_{\mathcal{B}}/\sum$ in a grid point in $F_{\mathcal{B}}$.*

Proof: Let $\mathcal{X}^*_{\mathcal{B}}$ be the set of optimal locations of $1/P/\mathcal{B}/\gamma_{\mathcal{B}}/\sum$. Suppose that $(\mathcal{X}^*_{\mathcal{B}} \cap F_{\mathcal{B}}) = \emptyset$ and choose some $X^* \in \mathcal{X}^*_{\mathcal{B}}$ with $f_{\mathcal{B}}(X^*) = z^*$. Wlog we assume that there exists no barrier in $\mathbb{R}^2 \setminus F_{\mathcal{B}}$ (this assumption cannot increase the objective value of any point $X \in \mathbb{R}^2 \setminus F_{\mathcal{B}}$). For each existing facility $Ex_m \in \mathcal{E}x$ there exists a γ-shortest path to X^* that intersects the boundary $\partial(F_{\mathcal{B}})$ of $F_{\mathcal{B}}$ in a first point I_m such that I_m is l_2-visible from X^* (Lemma 1). All these intermediate points I_m, $m \in \mathcal{M}$, are therefore located on those faces $F^i(F_{\mathcal{B}})$ $(i = 1, \ldots, k)$ of $\partial(F_{\mathcal{B}})$ that are l_2-visible from X^* (see Figure 8).
As $F_{\mathcal{B}}$ is the convex hull of a set of points and a set of convex polyhedra, $F_{\mathcal{B}}$ itself is also a convex polyhedron. Furthermore $X^* \notin F_{\mathcal{B}}$ and the supporting hyperplanes h^i defining the faces f^i divide $\mathbb{R}^2$ into two halfplanes H^i_1 and H^i_2 such that $X^* \in H^i_1$ and $F_{\mathcal{B}} \subset H^i_2$ $(i = 1, \ldots, k)$. Hence for each existing facility Ex_m $(m \in \mathcal{M})$ the straight line connecting X^* and I_m intersects h^i in a point I^i_m (see Figure 8).
The objective function value $f_{\mathcal{B}}(X^*)$ can therefore be determined as

$$f_{\mathcal{B}}(X^*) = \underbrace{\sum_{m=1}^{M} w_m \gamma(X^*, I^i_m)}_{=: f^i(X^*)} + \underbrace{\sum_{m=1}^{M} w_m \gamma_{\mathcal{B}}(I^i_m, Ex_m)}_{=: \kappa^i \text{ (constant for each } i)}; \qquad i \in \{1, \ldots, k\}.$$

For $i \in \{1, \ldots, k\}$ κ^i is constant and f^i is the objective function of an unrestricted Weber problem $1/P/\bullet/\gamma/\sum$ with existing facilities I^i_m, $m \in \mathcal{M}$, which has at least one optimal solution in $\operatorname{conv}\{I^i_m : m \in \mathcal{M}\}$ (see Durier and Michelot 1985 [6]).
Now consider the node network location problem $1/T^i/\bullet/d(V,V)/\sum$ on the tree T^i defined by the node set $V(T^i) = \{I^i_m : m \in \mathcal{M}\}$ and weights $w(v) = w(Ex_m)$ if $v = I^i_m$, $m \in \mathcal{M}$.

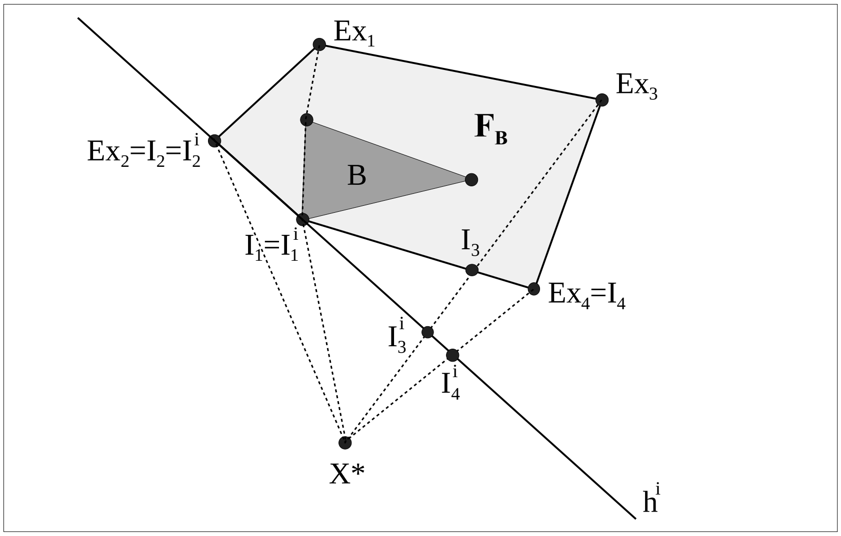

Figure 8: The intermediate points I_m and I_m^i for point $X^* \notin F_\mathcal{B}$ in the example problem.

Two nodes $I_m^i, I_n^i \in V(T^i)$ are connected by an edge of length $\gamma(I_m^i, I_n^i)$ if the corresponding points I_m^i and I_n^i of the planar embedding of T^i on h^i are consecutive points on h^i.
The optimal solution X^i of this node network location problem is also optimal for the unrestricted Weber problem with objective function f^i and satisfies

$$f_\mathcal{B}(X^i) \leq f_G(X^i) + \kappa^i = f^i(X^i) + \kappa^i \leq f_\mathcal{B}(X^*).$$

Furthermore Goldman 1971 [7] proved that a node $X^i \in V(T^i)$ is an optimal solution of the node network location problem on a tree network T^i if and only if it has both of the following properties:

$$\begin{aligned}\sum_{v \in V^i} w(v) + w(X^i) &\geq \frac{1}{2} \sum_{v \in V(T^i)} w(v)\\ \sum_{v \in \bar{V}^i} w(v) + w(X^i) &\geq \frac{1}{2} \sum_{v \in V(T^i)} w(v),\end{aligned}$$

where V^i and $\bar{V}^i$ are the two disjoint connected components of $V(T^i)$ resulting from the removal of node X^i. These two properties only depend on the weights of the nodes and on their order on h^i which is identical for all $i \in \{1, \ldots, k\}$. Thus there exists an index $m \in \mathcal{M}$ such that $X^i = I_m^i$ is an optimal solution of $1/T^i/\bullet/d(V,V)/\sum$ for all $i \in \{1, \ldots, k\}$ and

$$f_\mathcal{B}(I_m^i) \leq f_\mathcal{B}(X^*); \qquad i \in \{1, \ldots, k\}.$$

As the point I_m^i has to be located on the boundary of $\partial(F_\mathcal{B})$ for at least one index $i \in \{1, \ldots, k\}$, this fact is contradicting the assumption $(\mathcal{X}_\mathcal{B}^* \cap F_\mathcal{B}) = \emptyset$. Thus using Corollary 3 it can be concluded that there exists at least one optimal grid point in $F_\mathcal{B}$. $\square$

The set $F_{\mathcal{B}}$ of Theorem 2 can be found by the following algorithm:

Algorithm to construct $F_{\mathcal{B}}$:

1. Let $F := \text{conv}(\mathcal{E}x)$.
2. While there exists a barrier $B_i \in \mathcal{B}$ such that $\partial F \cap \text{int}(B_i) \neq \emptyset$ set $F := \text{conv}(F, B_i)$.
3. **Output:** $F_{\mathcal{B}} := F$.

Figure 9 indicates the reduced number of points that have to be investigated during the construction line algorithm if Theorem 2 is applied. However, the theoretical bound on the size of the candidate set $\mathcal{P}(\mathcal{G})$ is not affected by Theorem 2 since e.g. in the case of unbounded barriers a reduction may not be possible.

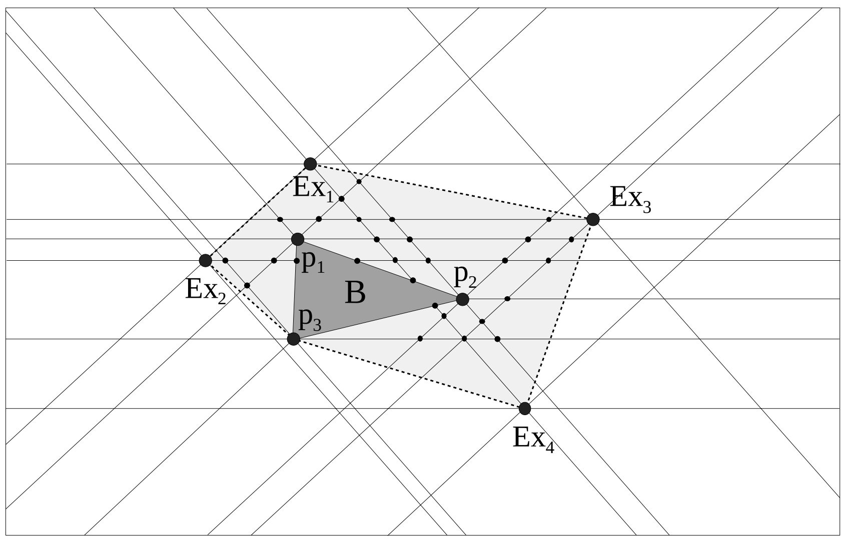

Figure 9: Applying Theorem 2, the candidate set of the example problem can be reduced from 83 candidate points to only 35 candidate points in the set $F_{\mathcal{B}}$.

5 Conclusion and future research

In this paper we proved a discretization result for location problems with barriers and block norms. This result implies a polynomial algorithm to solve this problem.

If the summation of the weighted distances in this paper is replaced by the maximization we obtain a class of problems which has so far been unsolved. In [5] we deal with this barrier center problem of the type $1/P/\mathcal{B}/(l_1)_{\mathcal{B}}/\max$. Other research topics include the analysis of level curves for barrier problems which will be used to tackle multi-criteria location problems with barriers and block norms.

Even though we mainly focused on block norms in this paper the results can be generalized to the more general class of polyhedral gauges. In Figure 10 a), an example of a polyhedral gauge is given that is not a block norm.

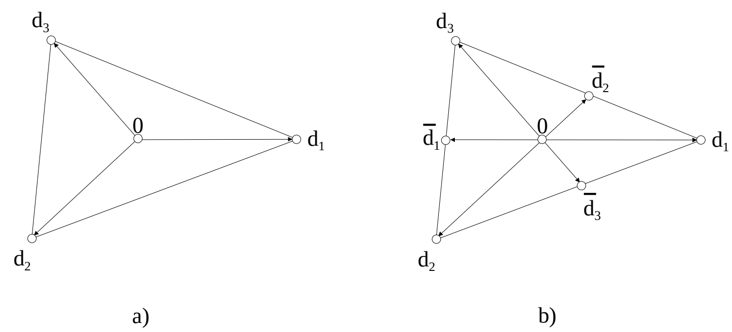

Figure 10: a) A polyhedral gauge with three fundamental directions, and b) its symmetric extension.

However, since polyhedral gauges may be nonsymmetric, some additional considerations have to be made. In Figure 11 the grid as defined in Section 4 is shown, using the three fundamental directions d_1, d_2, d_3 from Figure 10 a).

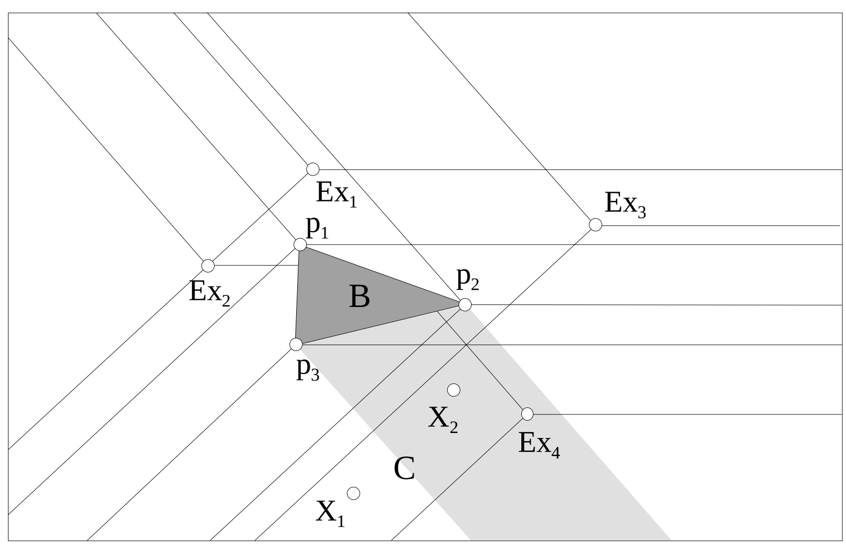

Figure 11: The grid $\mathcal{G}$ for the example problem with the polyhedral gauge given in Figure 10 a).

As illustrated by the shaded region in Figure 11, this construction implies that an existing facility may be γ-visible only from parts of a cell and not from the complete cell. Consider for example the existing facility Ex_1, the cell C and the two points X_1, X_2 in C. Then the shortest distance from X_2 to Ex_1 is lengthened by the barrier B, i.e. Ex_1 is not γ-visible from X_2, whereas the same existing facility, Ex_1, is γ-visible from the point X_1. (Note that, in the case of nonsymmetric distance functions, the distance from a point X to a point Y may be different from the distance from Y to X.)

In order to overcome this difficulty which becomes relevant in the proof of Theorem 1, the *symmetric extension* of a given nonsymmetric polyhedral gauge as shown in Figure 10 b) can be used. The basic idea is to introduce the redundant fundamental directions $\overline{d}_1, \overline{d}_2$ and $\overline{d}_3$ pointing in the opposite directions of the given fundamental directions d_1, d_2 and d_3. These additional fundamental directions do not influence the distance measure with respect to the given polyhedral gauge γ. But their existence implies a finer grid $\overline{\mathcal{G}}$ in which every existing facility that is γ-visible from some point in the interior of a cell is also γ-visible from every other point in the same cell.

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