A Wavefront Approach to Center Location Problems with Barriers

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Abstract

Center location problems have many applications, for example, in the public sector, and various different algorithms have been developed for their solution. This paper suggests a novel solution strategy to the problem that is based on the propagation of circular wavefronts. The approach is discussed theoretically and implemented both as a physical experiment and as a computer simulation.

Keywords: *location, barriers to travel, wavefronts*

1 Introduction

Wavefront approaches have been frequently and very successfully used in computational geometry for the computation of Voronoi diagrams (see, for example, Chew and Drysdale, 1985) and for the solution of shortest path problems (see, for example, Mitchell (1992), Hershberger and Suri (1999), and Mitchell (1998) for an overview). Since planar location problems with barriers heavily depend on efficient computations of shortest paths, an application also in this context appears natural. In location theory, a similar idea based on the variation of circular disks centered at existing facility locations was suggested by Brady and Rosenthal (1980) and Brady and Rosenthal (1983) who developed an interactive graphical solution procedure for center problems with constrained feasible regions (see Figure 1).

Moreover, wavefront approaches suggest very general notions of distance since generalized wavefronts can be used to model any distance function defined by a convex unit ball. If, in particular, Euclidean distances are used, the unit balls become circular and resemble the circular wavefronts of water waves.

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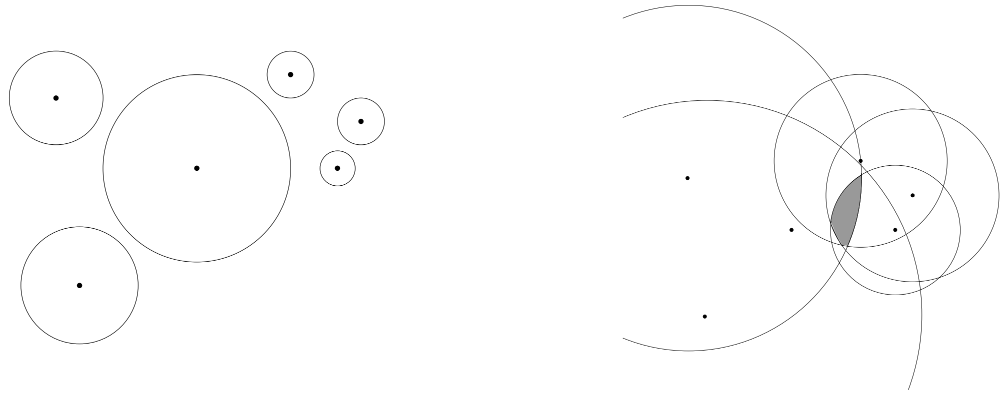

Figure 1: Varying circular disks centered at existing facility locations as suggested in Brady and Rosenthal (1980). The shaded region in the right figure highlights candidate locations for the center problem from which all existing facilities can be reached within at most five units of time.

In a general planar (continuous) location model, we consider a finite set of *existing facility locations* in the plane $I\!R^2$, denoted by $E = \{\mathbf{e}_j = (x_j, y_j)^T : j = 1, \ldots, m\}$, and want to locate one *new facility* $\mathbf{x} = (x, y)^T \in I\!R^2$ such that some location objective function is minimized. These objective functions are generally convex, nondecreasing functions of the distances between the new and the existing facilities, of which the well-known center objective is only one example.

The selection of an appropriate distance function is crucial in the design of planar location models representing specific applications. This involves not only the choice of a metric or of an appropriate unit ball defining a distance measure, but also the consideration of possible restrictions imposed by *areas forbidden for new location*, *congested regions* or *barrier regions*. Barriers may be used to model, for example, lakes, mountain ranges, military regions or existing buildings and production lines in an industrial plant or warehouse. Congested and forbidden regions are more suitable to model, for example, densely populated neighborhoods.

Despite their practical relevance, location problems with barriers have only recently received some attention. Most of the research so far has focused on the Weber problem and suggested different variations and modifications of this problem (see, for example, Drezner *et al.*, 2002, for a recent survey). Other objective functions, including the center objective, have benn rarely considered.

Many approaches to the center problem with barriers are based on rectilinear or block norm distances which allow for problem decompositions and discretizations. Dearing *et al.* (2002) considered the center problem with polyhedral barriers and rectilinear distances and derived a dominating set result for the problem. Extending similar ideas to a more general class of location problems, Segars Jr. (2000) and

Dearing and Segars Jr. (2002a,b) developed a decomposition of the feasible set into convex domains on which the objective function of a location problem with barriers and rectilinear distances is convex and can be optimized using methods from convex optimization. Similarly, Klamroth (2001) showed for the case of polyhedral barrier sets and arbitrary norm distances that an optimal solution of the non-convex barrier problem can be found by solving a finite number of related convex subproblems.

An alternative line of research has its roots in the field of computational geometry. For the case that n pairwise disjoint axis-aligned rectangles are given as barriers, Choi *et al.* (1998) presented an $O(n^2 m \log^2 m)$-time algorithm, based on parametric search, for the center problem with rectilinear distances. Ben-Moshe *et al.* (2001) recently improved this result for the unweighted case by giving an $O(nm \log(n+m))$-time algorithm.

A completely different approach to handle the non-convexity of the objective function of the center problem with barriers is the application of general global optimization methods, see, for example, Hansen *et al.* (1995).

This paper considers the center problem with polyhedral barriers and suggests to use propagating wavefronts for its solution. The problem is formally introduced in the following section and its relation to wavefront approaches as known from computational geometry is discussed. Based on experiments realized in the context of the graduate projects of L. Frieß and M. Sprau at the University of Kaiserslautern, Germany (1999), a wave tank experiment is described in Section 3 that models the unweighted center problem with barriers and Euclidean distances. The experiment nicely illustrates the problem and can be used, for example, in undergraduate mathematics or engineering education. It leads to a computer simulation in Section 4 that extends the idea of propagating wavefronts from circular water waves to general shapes and in particular to polyhedral distances (block norm distances).

2 The Center Problem and Wavefronts

Let $E = \{\mathbf{e}_j = (x_j, y_j)^T : j = 1, \dots, m\}$ denote a finite set of existing facilities in the plane $\mathbb{R}^2$. A positive weight w_j, $j = 1, \dots, m$, is associated with each existing facility $\mathbf{e}_j$, representing, for example, the importance or demand of this facility. Travel cost or speed is modeled based on some given distance function d, but is restricted by a finite set of barriers. We assume that the barrier sets are given by a finite number of compact polyhedral sets $B_i \subseteq \mathbb{R}^2$, $i = 1, \dots, n$, with pairwise disjoint interiors, that is, $\text{int}(B_i) \cap \text{int}(B_j) = \emptyset$ for $i \neq j$. We denote by $\mathcal{B} = \bigcup_{i=1}^n B_i$ the union of all barrier sets. The *feasible region* or *free space* where traveling as well as facility location is permitted is given by $\mathcal{F} = \mathbb{R}^2 \setminus \text{int}(\mathcal{B})$. To avoid infeasibility we assume furthermore that the feasible set is connected and that $E \subseteq \mathcal{F}$.

The *center problem* without barriers can be formulated as

$$\min_{\mathbf{x}\in\mathbb{R}^2} \{f(\mathbf{x}) = \max\{w_1 d(\mathbf{x},\mathbf{e}_1),\ldots,w_m d(\mathbf{x},\mathbf{e}_m)\}\}$$

where we assume that d is a distance function induced by a norm $\|\bullet\|_d\colon \mathbb{R}^2 \to \mathbb{R}_+$, that is $d(\mathbf{x},\mathbf{y}) = \|\mathbf{y}-\mathbf{x}\|_d$, $\mathbf{x},\mathbf{y}\in\mathbb{R}^2$. Analogously, the *center problem with barriers* is written as

$$\min_{\mathbf{x}\in\mathcal{F}} \{f_{\mathcal{F}}(\mathbf{x}) = \max\{w_1 d_{\mathcal{F}}(\mathbf{x},\mathbf{e}_1),\ldots,w_m d_{\mathcal{F}}(\mathbf{x},\mathbf{e}_m)\}\}$$

where $d_{\mathcal{F}}$ is the *barrier distance* corresponding to the given distance function d. The barrier distance $d_{\mathcal{F}}$ for two points $\mathbf{x},\mathbf{y}\in\mathcal{F}$ is given by the length of a shortest path connecting $\mathbf{x}$ and $\mathbf{y}$ not intersecting with the interior of $\mathcal{B}$. More formally, let P be a *permitted* $\mathbf{x}$-$\mathbf{y}$-*path* in $\mathcal{F}$, i.e. a curve connecting $\mathbf{x}$ and $\mathbf{y}$ not intersecting with the interior of $\mathcal{B}$. Furthermore, let p be a piecewise continuously differentiable parameterization of P, $p:[a,b]\to\mathbb{R}^2$ with $a,b\in\mathbb{R}$, $a<b$, $p(a)=\mathbf{x}$, $p(b)=\mathbf{y}$ and $p([a,b])\cap\operatorname{int}(\mathcal{B})=\emptyset$. Then $d_{\mathcal{F}}(\mathbf{x},\mathbf{y})$ can be defined as

$$d_{\mathcal{F}}(\mathbf{x},\mathbf{y}) := \inf\left\{\int_a^b \|p'(t)\|_d \,\mathrm{d}t \;:\; P \text{ permitted } \mathbf{x}\text{-}\mathbf{y}\text{-path}\right\}. \tag{1}$$

Note that for $d_{\mathcal{F}}$ the triangle inequality is satisfied, but $d_{\mathcal{F}}$ is in general not positively homogeneous, i.e., it may not satisfy $d_{\mathcal{F}}(\lambda\mathbf{x},\lambda\mathbf{y}) = |\lambda| d_{\mathcal{F}}(\mathbf{x},\mathbf{y})$ for all $\lambda\in\mathbb{R}$. A permitted $\mathbf{x}$-$\mathbf{y}$-path with length $d_{\mathcal{F}}(\mathbf{x},\mathbf{y})$ will be called a *shortest permitted* $\mathbf{x}$-$\mathbf{y}$-*path*. Due to the non-convexity of the feasible set as well as the barrier distance, the center problem with barriers is non-convex and hence not solvable by known classical location algorithms.

An optimal solution $\mathbf{x}^*$ of the center problem without barriers satisfies

$$w_j d(\mathbf{x}^*,\mathbf{e}_j) \;\le\; f(\mathbf{x}^*) \qquad \forall j=1,\ldots,m. \tag{2}$$

Moreover, the optimal objective value $f(\mathbf{x}^*)$ is minimal with this property. Conversely,

$$\mathbf{x}^* \;\in\; \mathbf{e}_j + f(\mathbf{x}^*)\cdot\frac{1}{w_j}\cdot C_d \qquad \forall j=1,\ldots,m, \tag{3}$$

where $C_d := \{\mathbf{x}\in\mathbb{R}^2 : \|\mathbf{x}\|_d \le 1\}$ denotes the unit ball of the given distance function and $\mathbf{e}_j + z\cdot\frac{1}{w_j}\cdot C_d = \{\mathbf{x}\in\mathbb{R}^2 : w_j d(\mathbf{x},\mathbf{e}_j) \le z\} =: L_{\le}(z,j)$ denotes the *level set of* $\mathbf{e}_j$ *at level* z, $z\in\mathbb{R}_+$ and $j=1,\ldots,m$ (see Francis, 1967). The corresponding *level curve of* $\mathbf{e}_j$ *at level* z, $L_=(z,j) := \{\mathbf{x} \in I\!R^2 : w_j d(\mathbf{x}, \mathbf{e}_j) = z\}$ is equal to the boundary of $L_\leq(z,j)$, $j = 1, \ldots, m$. As for property (2), the optimal objective value $z^* = f(\mathbf{x}^*)$ is minimal with property (3).

Hence an optimal solution of the center problem without barriers can be found by expanding balls C_d at rates $\frac{1}{w_j}$ and centered at the existing facilities $\mathbf{e}_j$ for all $j = 1, \ldots, m$, or, in other words, by expanding the corresponding level sets or wavefronts. As soon as all expanded level sets have a nonempty intersection, i.e., $\bigcap_{j=1}^m L_\leq(z,j) \neq \emptyset$ for some $z \in I\!R_+$, an optimal solution is found in this intersection. The optimal objective value $z^* = f(\mathbf{x}^*)$ then satisfies

$$z^* = \min \left\{ z \in I\!R_+ : \bigcap_{j=1}^m L_\leq(z,j) \neq \emptyset \right\},$$

and the set of optimal solutions X^* is given by

$$X^* = \bigcap_{j=1}^m L_\leq(z^*, j).$$

The concept of level curves and level sets transfers to problems including barriers if the unconstrained distance d is replaced by the barrier distance $d_\mathcal{F}$. In this case, (2) transforms to

$$w_j d_\mathcal{F}(\mathbf{x}^*_\mathcal{F}, \mathbf{e}_j) \leq f_\mathcal{F}(\mathbf{x}^*_\mathcal{F}) \qquad \forall j = 1, \ldots, m, \tag{4}$$

where $\mathbf{x}^*_\mathcal{F}$ is an optimal solution of the center problem with barriers. Let $L_{\leq,\mathcal{F}}(z,j) := \{\mathbf{x} \in \mathcal{F} : w_j d_\mathcal{F}(\mathbf{x}, \mathbf{e}_j) \leq z\}$ be the level set of $\mathbf{e}_j$ at level z accounting for the barrier regions, and let $L_{=,\mathcal{F}}(z,j) := \{\mathbf{x} \in \mathcal{F} : w_j d_\mathcal{F}(\mathbf{x}, \mathbf{e}_j) = z\}$ be the corresponding level curve, $j = 1, \ldots, m$. Similar to the unconstrained case, $L_{\leq,\mathcal{F}}(z,j)$ is the set of all those points in the feasible set $\mathcal{F}$ that can be reached from $\mathbf{e}_j$ on a permitted path of length at most $\frac{1}{w_j} z$. Hence condition (4) is equivalent to

$$\mathbf{x}^*_\mathcal{F} \in L_{\leq,\mathcal{F}}(f_\mathcal{F}(\mathbf{x}^*_\mathcal{F}), j) \qquad \forall j = 1, \ldots, m, \tag{5}$$

and $f_\mathcal{F}(\mathbf{x}^*_\mathcal{F})$ is minimal with this property.

An optimal solution of the center problem with barriers can now again be found by expanding level sets (or wavefronts) $L_{\leq,\mathcal{F}}(z,j)$, $j = 1, \ldots, m$, until $\bigcap_{j=1}^m L_{\leq,\mathcal{F}}(z,j) \neq \emptyset$ for some $z \in I\!R_+$. The optimal objective value $z^*_\mathcal{F} = f_\mathcal{F}(\mathbf{x}^*_\mathcal{F})$ satisfies

$$z^*_\mathcal{F} = \min \left\{ z \in I\!R_+ : \bigcap_{j=1}^m L_{\leq,\mathcal{F}}(z,j) \neq \emptyset \right\}.$$

3 The “Center Experiment”

3.1 Wavefront Models

In the case that distances are measured by the Euclidean metric $d = l_2$, expanding level sets or, equivalently, wavefronts, can be realized using water waves. If a water wave is induced in a wave tank at the location of an existing facility $\mathbf{e}_j$, $j \in \{1, \ldots, m\}$, circular wavefronts propagate at a constant velocity unless they hit an obstacle in the water, or unless the depth of the water changes (see, for example, Feynman *et al.*, 1998, for the physical details about water waves). Due to the unperturbed superposition of water waves, waves that are simultaneously induced at several existing facilities propagate independently of one another. Thus the points of first intersection of all propagating circular wavefronts correspond to optimal solutions of the unweighted center problem with this set of existing facilities. Note that this approach implicitly assumes that the weights of all existing facilities are equal and thus their respective level sets expand at a uniform rate. We will assume throughout the remainder of this section that $w_j = 1$ for all $j = 1, \ldots, m$.

If barriers are given within the feasible set, appropriately shaped objects forming islands in the water can be used to realize the barrier distance $l_{2,\mathcal{F}}$. The wavefront of a water wave induced at the location of an existing facility $\mathbf{e}_j$ then corresponds to the level set $L_{\leq,\mathcal{F}}(z, j)$. This correspondence follows from Huygen's principle (see, for example, Feynman *et al.*, 1998) which can be summarized as follows:

Huygen's principle: Every point on a propagating wavefront (e.g., the crest of a waterwave) can be considered a *point source* which induces a new *elementary wave* with the same velocity and the same wavelength as the original wave. The outer envelope of the wavefronts of all elementary waves yields the wavefront at a later point of time.

Note that, in the case of nonconvex obstacles, the reflections of those parts of a wavefront that hit, for example, a “pocket” of an obstacle, will not contribute to this outer envelope. Figure 2 shows an application of Huygen's principle to the propagation of circular wavefronts approaching and passing a gap in an obstacle.

In addition to barrier sets, congested regions where traveling is allowed at a lower speed (or at a higher cost, respectively) can be included into the experiment by making use of the fact that water waves have a higher velocity in deep water than in shallow water. A congested region can thus be realized by changing the depth of the water using appropriately shaped objects that are put on the bottom of the tank (see Gamito and Musgrave, 2002, for models and simulations of wave refraction over shallow water). The correlations between the unweighted center problem and the center experiment are summarized in Table 1.

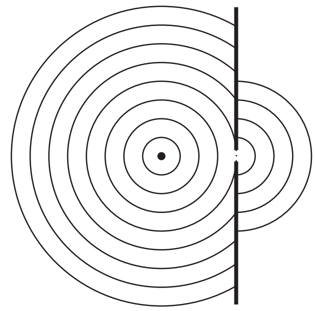

Figure 2: Huygen's principle applied to circular wavefronts approaching a gap in an obstacle

Unweighted Center Problem	Center Experiment
modeling horizon in $\mathbb{R}^2$	wave tank filled with water
barrier sets	islands of the shape of the barriers
congested regions	shallow regions of the shape of the congested regions
existing facilities	point sources where circular waves are simultaneously induced at time $t_0 = 0$
level curves $L_{=,\mathcal{F}}(z,j)$	wavefronts corresponding to point source j at time $t(z) = cz$ for some problem dependent constant c
set of optimal solutions $X^*_{\mathcal{F}}$	area of first intersection of all wavefronts

Table 1: The unweighted center problem and the center experiment

3.2 Experimental Setup

The center experiment was realized using two wave tanks. The first of these wave tanks, wave tank 1, was a standard model manufactured by Leybold-Heraeus GmbH & Co (Köln, Germany) that is frequently used in high school and undergraduate teaching. The wave tank consists of a flat plastic tank with gentle sides, four adjusting screws at the corners and a mirror of the size 31 x 21 cm^2 at the bottom of the tank.

The reflections of the light of a light source installed almost centrally over wave tank 1 are projected to the ceiling of the room where the projection can be recorded using a standard video recorder. If the wave tank is filled with water, propagating water waves are visualized by (circular) shadows due to the different refraction of light in the wave crests and troughs.

The second wave tank, wave tank 2, is a construction particularly designed for classroom use. It was optimized with respect to installation time and flexibility. The size of the glass bottom of the rectangular aluminum tank exactly fits the size of a standard overhead projector. If filled with water, the light of the overhead projector goes through the water and projects the picture to the wall where it can be easily followed by the students. Figure 3 shows this second wave tank together with its dimensions.

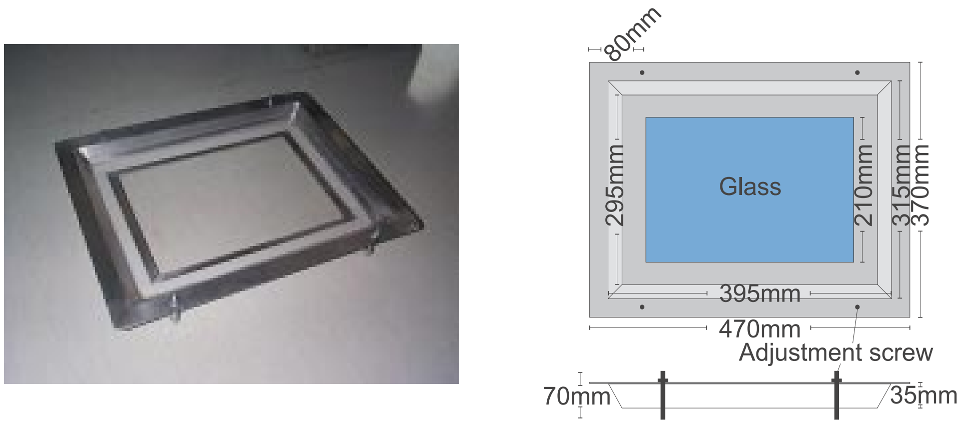

Figure 3: Wave tank 2 for use on an overhead projector

In order to induce waves simultaneously at the location of all existing facilities, steel tips are attached to a polystyrene glass plate at the location of the existing facilities and the plate is installed above the wave tank. In its initial position the steel tips reach into the water. The waves are induced by lifting the plate using a lifting mechanism based on four electro magnetic lifters at the corners of the mechanism, see Figure 4.

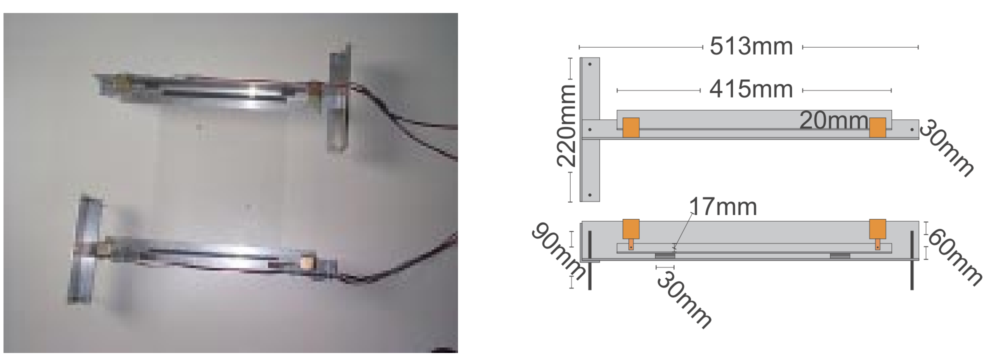

Figure 4: Lifting mechanism used with wave tanks 1 and 2. The photograph shows two arms with two electro magnetic lifters each holding the polystyrene glass plate. Two existing facilities are realized by two attached steel tips. The technical drawing illustrates one arm of the lifting mechanism with the electro magnetic lifters represented by the darker shaded squares.

3.3 Experimental Results

Figure 5(a) shows an example problem with three existing facilities $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. The three shaded rectangular regions represent congested regions where traveling is allowed at a slightly lower speed. The black rectangle represents a barrier set where traveling is completely prohibited. The corresponding setup in wave tank 1, recorded from the ceiling with a video camera, is shown in Figure 5(b). Since the light source cannot be installed exactly centrally above the wave tank, two images of each object in the tank are projected to the ceiling: One direct image on the mirror and a second image resulting from the light reflected by the mirror. The resulting projection error that particularly occurs for the steel tips representing the existing facilities has to be taken into account in the evaluation of the experiment.

In addition to the existing facilities, the projections of the three congested regions are clearly visible. They are realized using rectangular aluminum blocks that are slightly covered by the water in the tank. However, the projection of the barrier set, one of the aluminum blocks that is not covered by water, is only hardly visible. The blurred picture results from the surface tension of water that forms a concave surface upon contact with the barrier. Due to the different refraction of light in concave and convex parts of the surface that was so useful for the visualization of wavefronts, the boundary of the barrier set appears unclear. This problem can be reduced by using additives for the reduction of surface tension like, for example, dish washer detergent. Chemical substances like Glydol N 193 (produced by Zschimmer & Schwarz, Lahnstein, Germany) or different barrier materials can alternatively be used, but a complete neutralization of the effect is hard to achieve in practice.

Figure 5: Example problem (left) with three congested regions (grey rectangles) and one barrier set (black rectangle) and the corresponding experimental setup filmed from wave tank 1 (right)

Some phases of the course of the experiment are shown in Figure 6. In all pictures, the existing facilities, barrier sets and congested regions were outlined to better visualize the location of the existing facilities and the congested regions and barrier set.

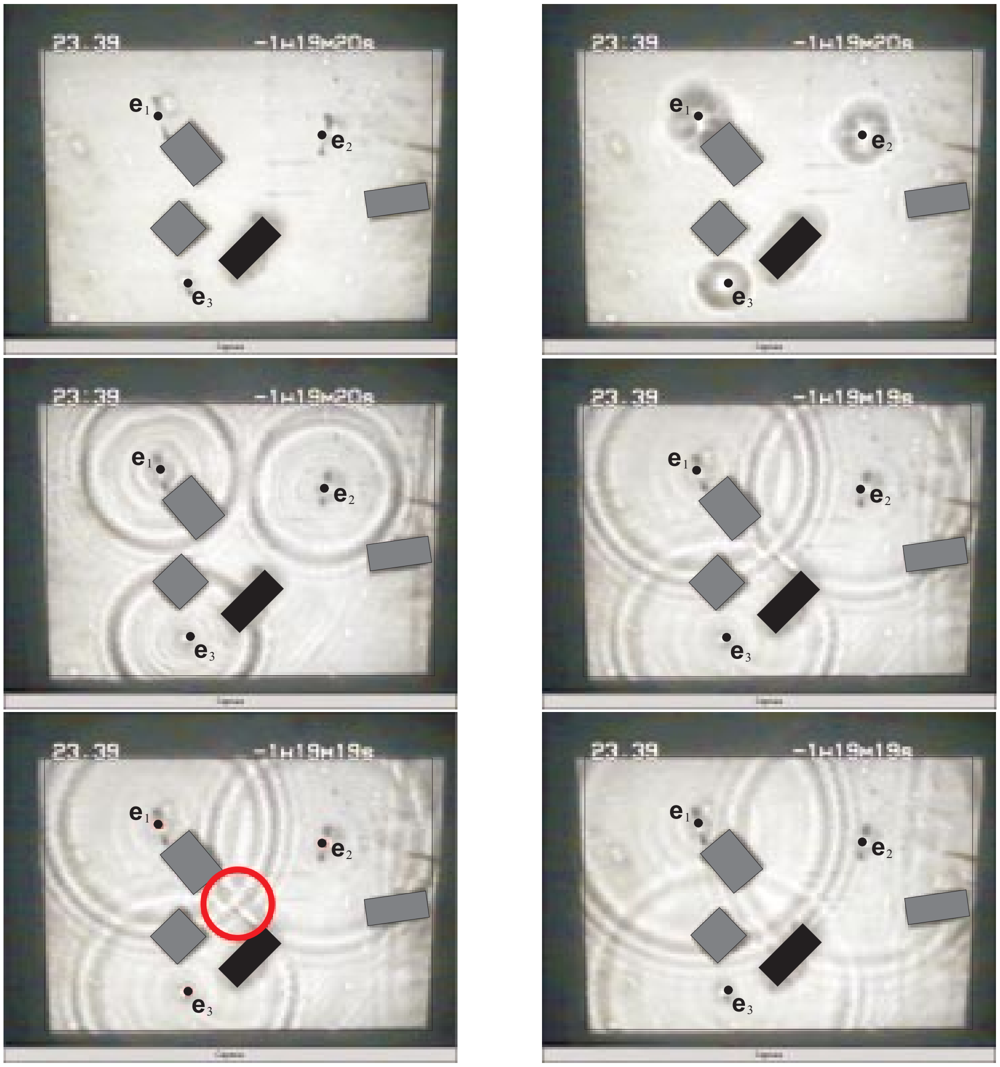

Figure 6: Six consecutive phases of the experiment (from left to right and top to bottom): (a) initial situation, the steel tips still reach into the water; (b) three circular wavefronts have been induced simultaneously; (c) and (d) propagation of wavefronts; (e) the intersection of all three wavefronts is nonempty for the first time, the intersection is highlighted by a circle; and (f) further propagation of the wavefronts

The congested regions have almost no influence on the solution since the height of the blocks representing the congested region is small as compared to the total depth of the water. On the other hand, the barrier region cannot be passed and affects the propagation of the wavefronts induced at facility $\mathbf{e}_3$, see Figure 6(f). In Figure 6(e) the region of first intersection of all three wavefronts is found and highlighted by a circle. The size of the circle reflects the estimated solution error that is inherent to the experiment due to various reasons, of which the problems with the surface tension and the non-central location of the light source are only two examples.

4 Simulation

The center experiment suggests a computer simulation that imitates the propagation of wavefronts for the solution of the center problem. This is possible not only for the Euclidean metric $d = l_2$ and equally weighted existing facilities, but also for other convex polyhedral unit balls and for general weightings of the existing facilities. A prototype implementation used regular and symmetric polytopes that define block norm distances. Due to computational limitations, the Euclidean unit ball was also approximated by a polytope with up to 60 extreme points. Moreover, one convex polyhedral barrier set or forbidden region can be included into the model.

The simulation was implemented in C++ and using some geometrical objects from the LEDA 3.8 program library developed at the Max-Planck-Institute for Computer Science, Saarbrücken, Germany.

The basic structure of the algorithm can be summarized as follows: After an initialization phase where elementary waves are initiated at every existing facility, the algorithm iterates between an *expansion phase* and an *intersection phase*. In the expansion phase all active wavefronts are expanded by a given constant, and in the intersection phase the intersection set storing the intersection of wavefronts induced by different existing facilities is updated accordingly. The algorithm terminates as soon as the intersection set contains a nonempty intersection of wavefronts of all existing facilities.

If polyhedral barrier sets restrict traveling, then the expansion of wavefronts includes the identification of events at which an expanding wavefront hits a barrier boundary. As soon as an extreme point of a barrier polyhedron is reached, a new elementary wave (cf. Figure 2) is initiated at this point and its expansion is combined with the expansion of all other wavefronts of the corresponding existing facility.

While the prototype implementation can handle only one convex polyhedral barrier set or forbidden region, first computational results were very promising, so that an extension to more general problem settings, including also congested regions, seems within reach. Figure 7 shows four phases of the simulation for the Euclidean metric $d = l_2$. Figure 8 shows the simulation applied to a problem with rectilinear distances $d = l_1$.

Since the Euclidean unit ball is approximated by a regular $2n$-gon in the current implementation the set of optimal solutions found by the algorithm is still subject to an approximation error. If a regular $2n$-gon is used for the approximation, $n \geq 2$,

Figure 7: Phases of the simulation for a Euclidean center problem with two existing facilities and one barrier set

and if the maximal Euclidean distance between an existing facility and the optimal solution set is given by r, the error of the coordinates of the intersection polytope defining the optimal set can be bounded by $2 \cdot r \cdot (1 - \cos \frac{\pi}{2n})$. In addition, the expansion of wavefronts at constant, discrete steps induces a small approximation error which can be minimized by reducing the step length to a small constant. To speed up the procedure and at the same time improve the solution quality, future implementations should however be based on a continuous expansion of wavefronts similar to that implemented in recent shortest path algorithms (see Mitchell *et al.*, 1992; Hershberger and Suri, 1999). Here, wavefronts are expanded until certain *events* occur, that is, until two wavefronts intersect or until a wavefront intersects with a barrier boundary.

5 Conclusions

Expanding level sets or, equivalently, propagating circular wavefronts can be used for the solution of center location problems with barriers and / or congested and forbidden regions. A physical experiment was described and a related computer

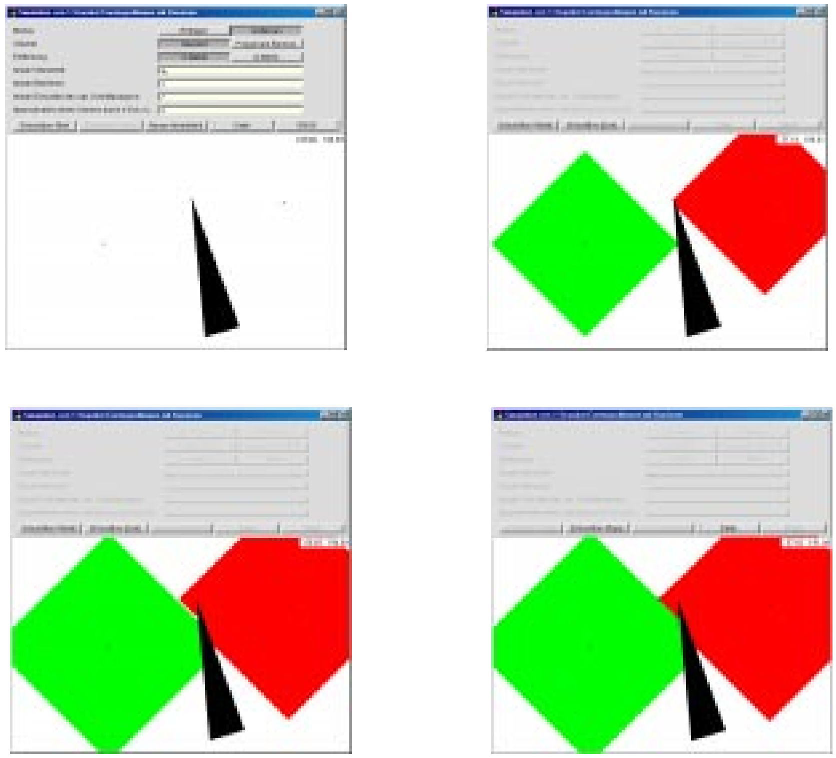

Figure 8: Phases of the simulation for a center problem with l_1 distances, two existing facilities and one barrier set

simulation was presented. The results suggest that wavefront approaches are a strong tool not only for the determination of shortest paths or Voronoi diagrams, but also in the context of location theory. Future research should focus on further refinements of the procedure and efficient implementations for general problem settings. Moreover, three-dimensional generalizations could be imagined where waves are initiated inside a deep-water tank. This idea raises many interesting questions as, for example, on how to stipulate waves below the surface of the water and on how to monitor the propagation of the resulting wave fronts.

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