Planar location problems with block distance and barriers

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Abstract

This paper considers one facility planar location problems using block distance and assuming barriers to travel. Barriers are defined as generalized convex sets relative to the block distance. The objective function is any convex, nondecreasing function of distance. Such problems have a non-convex feasible region and a non-convex objective function. The problem is solved by modifying the barriers to obtain an equivalent problem and by decomposing the feasible region into a polynomial number of convex subsets on which the objective function is convex. It is shown that solving a planar location problem with block distance and barriers requires at most a polynomial amount of additional time over solving the same problem without barriers.

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1 Introduction

Continuous, planar, one facility location models assume a finite set of *existing facility locations* in the plane $I\!R^2$, denoted by $E = \{\mathbf{e}_j = (x_j, y_j)^T : j = 1, \ldots, m\}$ with the objective of locating one *new facility* $\mathbf{x} = (x, y)^T \in I\!R^2$ in order to minimize some function of the distances between $\mathbf{x}$ and the $\mathbf{e}_j$ for all $j = 1, \ldots, m$.

A wide array of distance functions are available to represent travel distance between $\mathbf{x}$ and $\mathbf{e}_j$. Models using the l_2 distance (Euclidean), the l_1 distance (rectilinear), and l_p for $1 \leq p \leq \infty$ are well studied. The location models considered here assume block distances which are denoted by $d_p(\mathbf{x}, \mathbf{x}')$ for a given integer $p \geq 2$ and $\mathbf{x}, \mathbf{x}' \in I\!R^2$. Block distances are defined explicitly in Section 2.

The models considered here also assume barriers, denoted by $B_i, i = 1, \ldots, n$, which are subsets of the plane through which travel is forbidden. Barriers are defined explicitly later, but may represent lakes, mountains or any restricted areas that prohibit tresspassing. Let $\mathcal{B} = \bigcup_{i=1}^n B_i$. Then the *feasible region* for the location of a new facility $\mathbf{x}$ is denoted by $\mathcal{F} = I\!R^2 \setminus \text{int}(\mathcal{B})$. The set $\mathcal{F}$ is also the set through which travel is permitted. We assume that $\mathcal{F}$ is connected and that $E \subseteq \mathcal{F}$. The feasible region $\mathcal{F}$ is typically not convex.

For a set $S \subseteq I\!R^2$ and points $\mathbf{x}, \mathbf{x}' \in S$, the shortest feasible block distance between $\mathbf{x}$ and $\mathbf{x}'$ with travel restricted to S is denoted by $d_{p,S}(\mathbf{x}, \mathbf{x}')$ which is the length, measured by the distance $d_p(\mathbf{x}, \mathbf{x}')$, of a shortest path that lies entirely in S. The distance $d_{p,S}(\mathbf{x}, \mathbf{x}')$ is a metric on S, but is typically a nonconvex function of $\mathbf{x}$. In particular, $d_{p,\mathcal{F}}(\mathbf{x}, \mathbf{e}_j)$ measures the shortest feasible block distance between $\mathbf{x}$ and $\mathbf{e}_j$ restricted to the feasible set $\mathcal{F}$.

The objective function is determined by any convex, nondecreasing function f of the feasible distances $d_{p,\mathcal{F}}(\mathbf{x}, \mathbf{e}_j), j = 1, \ldots, m$. For convenience, denote $d_{p,\mathcal{F}}(\mathbf{x}, E) = (d_{p,\mathcal{F}}(\mathbf{x}, \mathbf{e}_1), \ldots, d_{p,\mathcal{F}}(\mathbf{x}, \mathbf{e}_m))$. Then the objective function is given by $f(d_{p,\mathcal{F}}(\mathbf{x}, E))$. Two well known special cases are the sum of distances, which is called the median problem, and the maximum of distances, which is called the center problem.

The *planar location problem with barriers* is written as

$$\min_{\mathbf{x} \in \mathcal{F}} f(d_{p,\mathcal{F}}(\mathbf{x}, E)).$$

Figure 1 shows an example problem with five existing facilities and five barriers.

Despite their practical relevance, location problems with barriers have received relatively little attention in the location literature. Most authors have concentrated on special barrier shapes and/or special distance functions. Barriers were first considered in location models by Katz and Cooper (1981) who developed a heuristic solution procedure for the median problem with Euclidean distance and one circular barrier. In the case that all barrier sets are polytopes, a visibility graph of the existing facilities and the extreme points of the barrier polytopes can be constructed. The visibility graph was used by Aneja and Parlar (1994), Butt (1994) and Butt and Cavalier (1996) to efficiently evaluate solution points in a heuristic algorithm for the median problem using Euclidean distance. For the case of polyhedral barrier sets, Klamroth (2001a) and Klamroth (2001b) showed that an optimal solution of the non-convex barrier problem can be found by solving a finite (and in the case of line barriers a polynomial) number of related unconstrained subprob-

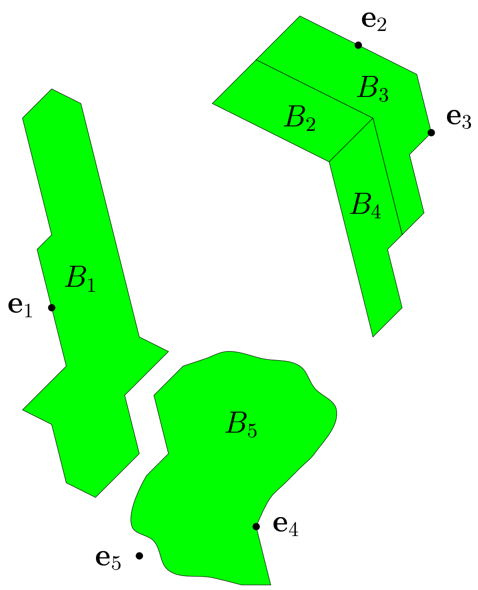

Figure 1: Example of a planar location problem with five existing facilities and five barriers

lems. This result was generalized to the multi-criteria case in Klamroth and Wiecek (2002).

For several classes of location problems with barriers, finite dominating sets have been constructed that are shown to contain optimal solutions. Larson and Sadiq (1983) identified an easily determined finite dominating set for the median problem with rectilinear distances. Their results were later generalized by Batta *et al.* (1989) who included forbidden regions into the model, and by Savaş *et al.* (2002) who located finite size facilities acting as barriers themselves. Similar finite dominating sets were developed by Hamacher and Klamroth (2000) for the median problem with general block norm distances and by Dearing *et al.* (2002) for the center problem with rectilinear distances.

The computational efficiency of these methods was improved by Segars Jr. (2000) and Dearing and Segars Jr. (2002a,b) who showed that a much smaller dominating set is sufficient to solve the problem. Klamroth (2002) provides an overview of results on location with barriers up to 2002.

For the special case that n pairwise disjoint axis-aligned rectangles are given as barriers, Kusakari and Nishizeki (1997) presented an output-sensitive $O((k+n)\log n)$-time algorithm for the median problem, assuming that the number of existing facilities m is small. This result was improved to $O(n\log n + k)$ by Choi *et al.* (1998) who also gave an $O(n^2 m \log^2 m)$-time algorithm, based on parametric search, for the center problem. Ben-Moshe *et al.* (2001) recently improved this last result for the unweighted center problem by giving an $O(nm\log(n+m))$-time algorithm.

General global optimization methods are applied by Hansen *et al.* (1995) to the nonconvex objective function of barrier location problems. Krau (1996) and Hansen *et al.* (2000) generalized the big square small square method, a geometrical branch and bound algorithm, to handle the median problem with polyhedral barrier sets as well as forbidden regions, and Fliege (1997) suggested modeling the physical barriers by suitable barrier functions (in the sense of nonlinear optimization).

This paper extends the results of Dearing and Segars Jr. (2002a,b) from l_1 distances to block distances. In the following section, block distances and the related concepts of shortest paths and visibility are formally introduced. Section 3 introduces barriers and discusses modifications of barriers. Equivalence results for location problems with different - but related - barrier sets are proved in Section 4. Section 5 presents a specific modification of the barriers that leads to a maximal reduction of the feasible set. In Section 6 the feasible set is partitioned into convex subsets on which the objective function is convex, which leads to a solution procedure. An example problem is worked in Section 7. Extensions are discussed in Section 8.

2 Block Distance and Paths

Block distance is defined in the plane with respect to a symmetric polytope as its unit ball. The polytope is assumed to have $2p$ distinct extreme points, for an integer $p \geq 2$, that are called *fundamental directions* and denoted by $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{2p}$ where $\mathbf{b}_{p+k} = -\mathbf{b}_k$ for $k = 1, \ldots, p$. Assume the fundamental directions are ordered counter clockwise and for notational convenience, let $\mathbf{b}_{2p+k} = \mathbf{b}_k$ for $k = 1, \ldots, p$.

For each fundamental direction $\mathbf{b}_k = (b_{1,k}, b_{2,k})^T$, $k = 1, \ldots, 2p$, let $\mathbf{b}_k^+$ denote the orthogonal (row) vector given by $\mathbf{b}_k^+ = (b_{2,k}, -b_{1,k})$.

For any two vectors $\mathbf{b}, \mathbf{b}' \in I\!R^2$, let $\Gamma(\mathbf{b}, \mathbf{b}')$ denote the cone in $I\!R^2$ generated by $\mathbf{b}$ and $\mathbf{b}'$ with its vertex at the origin. Observe that $\Gamma(\mathbf{b}_k, \mathbf{b}_{k+1}) = \{\mathbf{x} : \mathbf{b}_k^+\mathbf{x} \leq 0, \mathbf{b}_{k+1}^+\mathbf{x} \geq 0\}$. Also, for any two points $\mathbf{x}_o$ and $\mathbf{x}_d \in I\!R^2$, $\mathbf{x}_d - \mathbf{x}_o \in \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$ for some $k = 1, \ldots, 2p$, so that

$$\mathbf{x}_d - \mathbf{x}_o = \alpha_{od}\mathbf{b}_k + \beta_{od}\mathbf{b}_{k+1}$$

for some unique, nonnegative scalars α_{od} and β_{od}.

Definition: *The* block distance *between the points,* $\mathbf{x}_o$ *and* $\mathbf{x}_d$ *with respect to a given set of fundamental directions* $\mathbf{b}_1, \ldots, \mathbf{b}_{2p}$*, is denoted by* $d_p(\mathbf{x}_o, \mathbf{x}_d)$ *and is defined as*

$$d_p(\mathbf{x}_o, \mathbf{x}_d) = \alpha_{od} + \beta_{od}$$

where α_{od} *and* β_{od} *are nonnegative scalars so that*

$$\mathbf{x}_d - \mathbf{x}_o = \alpha_{od}\mathbf{b}_k + \beta_{od}\mathbf{b}_{k+1}$$

for some $k = 1, \ldots, 2p$.

Block distances are a special case of norm distances (see, for example, Minkowski, 1911) and were introduced to location models by Witzgall (1964), and Ward and Wendell (1985). They can also be viewed as a generalization of *distances in fixed orientations* as introduced in Widmayer *et al.* (1987) who assumed that all fundamental directions have unit length, that is $||\mathbf{b}_k|| = 1$, $k = 1, \ldots, 2p$ where $||\mathbf{b}_k||$ is the Euclidean norm of $\mathbf{b}_k$.

For any $\mathbf{x}$ and $\mathbf{x}_o$ such that $\mathbf{x} - \mathbf{x}_o \in \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$, an alternative expression for the block distance is given by $d_p(\mathbf{x}_o, \mathbf{x}) = \mathbf{b}_k^0(\mathbf{x} - \mathbf{x}_o)$ where $\mathbf{b}_k^0$ denotes the *polar direction* (a row vector) determined by $\mathbf{b}_k$ and $\mathbf{b}_{k+1}$ (see Ward and Wendell, 1985). In fact, $\mathbf{b}_k^{0T} = \mathbf{e}^T D_{k,k+1}^{-1}$ where $D_{k,k+1}$ is the matrix with columns $\mathbf{b}_k$ and $\mathbf{b}_{k+1}$ and $\mathbf{e} = (1,1)^T$. The unit ball centered at $\mathbf{x}_o$ for the block distance d_p is the polytope in $I\!R^2$ whose extreme points are $\mathbf{x}_o + \mathbf{b}_k$, for $k = 1, \ldots, 2p$. The unit ball may also be expressed as the set $\{\mathbf{x} : \mathbf{b}_k^0(\mathbf{x} - \mathbf{x}_o) \leq 1, k = 1, \ldots, 2p\}$, and the facet between each consecutive pair of directions $\mathbf{x}_o + \mathbf{b}_k$ and $\mathbf{x}_o + \mathbf{b}_{k+1}$ of the unit ball centered at $\mathbf{x}_o$ is the line $\mathbf{b}_k^0\mathbf{x} = \mathbf{b}_k^0\mathbf{x}_o + 1$ for $k = 1, \ldots, 2p$.

Block distances are used to model travel distance in which travel directions are restricted to the fundamental directions. The l_1 distance is an example of a block distance with $p = 2$ and fundamental directions given by $\mathbf{b}_1 = \boldsymbol{\varepsilon}_1$, $\mathbf{b}_2 = \boldsymbol{\varepsilon}_2$, $\mathbf{b}_3 = -\boldsymbol{\varepsilon}_1$, and $\mathbf{b}_4 = -\boldsymbol{\varepsilon}_2$, where $\boldsymbol{\varepsilon}_i$ is the i^{th} unit vector in $I\!R^2$. Figure 2 illustrates the fundamental directions of two block distances in $I\!R^2$: the l_1 distance ($p = 2$) and a block distance with $p = 4$, referred to here as the 'example block distance'. The example block distance is used in the example problem of Figure 1 and throughout the paper.

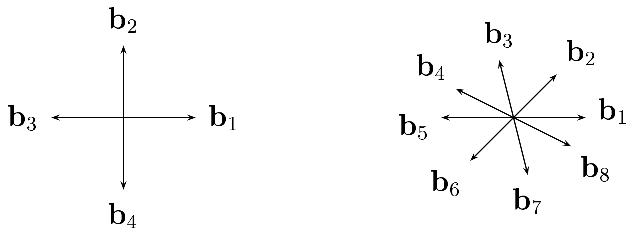

Figure 2: Examples of block distances in $I\!R^2$

For the example block distance, Figure 3 shows the cone $\Gamma(\mathbf{b}_3, \mathbf{b}_4)$, the orthogonal vectors $\mathbf{b}_3^+$ and $\mathbf{b}_4^+$, the polar vector $\mathbf{b}_3^0$ and a vector $\mathbf{x}_d - \mathbf{x}_o \in \Gamma(\mathbf{b}_3, \mathbf{b}_4)$ with $\alpha_{od} = .5$ and $\beta_{od} = 1.5$ so that $\mathbf{x}_d - \mathbf{x}_o = .5\mathbf{b}_3 + 1.5\mathbf{b}_4$ and $d_p(\mathbf{x}_o, \mathbf{x}_d) = 2$.

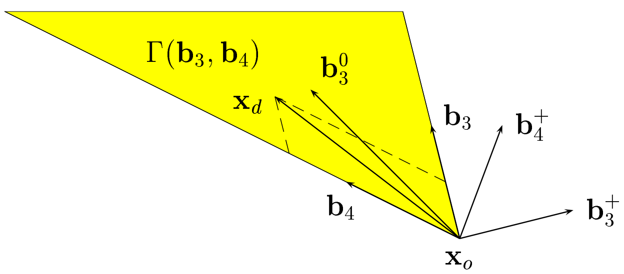

Figure 3: Vectors $\mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_3^+, \mathbf{b}_4^+, \mathbf{b}_3^0$, and $\mathbf{x}_d - \mathbf{x}_o \in \Gamma(\mathbf{b}_3, \mathbf{b}_4)$

A *path* from an origin point, $\mathbf{x}_o = (x_o, y_o)^T$ to a destination point $\mathbf{x}_d = (x_d, y_d)^T$, denoted $P(\mathbf{x}_o, \mathbf{x}_d)$, is a rectifiable, Jordan arc (see Apostol (1960)) in $I\!R^2$ whose points $(x, y)^T$ are defined by a continuous, vector-valued, one-to-one function $\delta = (u, v)^T$ on the interval $[0, 1]$ with:

$$\begin{aligned} \delta(0) &= (u(0), v(0))^T &&= (x_o, y_o)^T \\ \delta(t) &= (u(t), v(t))^T &&= (x, y)^T \qquad for \quad t \;\in\; (0,1) \\ \delta(1) &= (u(1), v(1))^T &&= (x_d, y_d)^T. \end{aligned}$$

For each point $t_i \in (0, 1)$, the point $\mathbf{x}_i = \delta(t_i)$ is an *intermediate point* of $P(\mathbf{x}_o, \mathbf{x}_d)$, and is written $P(\mathbf{x}_o, \mathbf{x}_i, \mathbf{x}_d)$. A partition $\mathcal{P}$ of $[0, 1]$ is a finite set of points $\{t_0, t_1, \ldots, t_{n(\mathcal{P})}\} \subset [0, 1]$ with $0 = t_0 \leq t_1 \leq \ldots \leq t_{n(\mathcal{P})} = 1$. Each partition $\mathcal{P}$ of $[0, 1]$ yields a set of intermediate points $\mathbf{x}_i = \delta(t_i)$, for $t_i \in \mathcal{P}$, of $P(\mathbf{x}_o, \mathbf{x}_d)$. For each partition $\mathcal{P}$, let $\Delta(\mathcal{P}) = max_{t_i \in \mathcal{P}}(t_i - t_{i-1})$.

Definition: *The* length of a path $P(\mathbf{x}_o, \mathbf{x}_d)$ *in terms of the distance d_p is defined as:*

$$d_p(P(\mathbf{x}_o, \mathbf{x}_d)) = \lim_{\Delta(\mathcal{P}) \to 0} \sum_{i=1}^{n(\mathcal{P})} d_p(\mathbf{x}_{i-1}, \mathbf{x}_i).$$

The assumption that $P(\mathbf{x}_o, \mathbf{x}_d)$ is a Jordan arc ensures that $d_p(P(\mathbf{x}_o, \mathbf{x}_d))$ exists and is finite. Note that for any intermediate point $\mathbf{x}_i$, $d_p(P(\mathbf{x}_0, \mathbf{x}_i, \mathbf{x}_n))$ $=d_p(P(\mathbf{x}_0, \mathbf{x}_i)) + d_p(P(\mathbf{x}_i, \mathbf{x}_n))$. Also, for any path $P(\mathbf{x}_o, \mathbf{x}_d)$, $d_p(P(\mathbf{x}_o, \mathbf{x}_d)) \geq d_p(\mathbf{x}_o, \mathbf{x}_d)$.

Definition: *A path $P(\mathbf{x}_o, \mathbf{x}_d)$ is a* d_p shortest path *if and only if:*

$$d_p(P(\mathbf{x}_o, \mathbf{x}_d)) = d_p(\mathbf{x}_o, \mathbf{x}_d).$$

Observe that if $P(\mathbf{x}_o, \mathbf{x}_d)$ is a d_p shortest path from $\mathbf{x}_o$ to $\mathbf{x}_d$ with intermediate point $\mathbf{x}_i$, then $P(\mathbf{x}_o, \mathbf{x}_i)$ is a d_p shortest path from $\mathbf{x}_o$ to $\mathbf{x}_i$, $P(\mathbf{x}_i, \mathbf{x}_d)$

is a d_p shortest path from $\mathbf{x}_i$ to $\mathbf{x}_d$, and

$$d_p(P(\mathbf{x}_o, \mathbf{x}_i, \mathbf{x}_d)) = d_p(\mathbf{x}_o, \mathbf{x}_i) + d_p(\mathbf{x}_i, \mathbf{x}_d).$$

If $\mathbf{x}_d = \mathbf{x}_o + \lambda_0 \mathbf{b}_k$ for some $\lambda_0 > 0$ and some $k = 1, \ldots, 2p$, then the path $P(\mathbf{x}_o, \mathbf{x}_d)$ that coincides with the ray $\mathbf{x}_o + \lambda \mathbf{b}_k$ for $\lambda \geq 0$ is the unique d_p shortest path from $\mathbf{x}_o$ to $\mathbf{x}_d$ and has length λ_0.

Lemma 1 *Suppose $P(\mathbf{x}_o, \mathbf{x}_d)$ is a d_p shortest path from $\mathbf{x}_o$ to $\mathbf{x}_d$ and suppose $\mathbf{x}_d - \mathbf{x}_o \in \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$ for some $k = 1, \ldots, 2p$. Then for all intermediate points $\mathbf{x}_i, \mathbf{x}_i - \mathbf{x}_o \in \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$.*

Proof: Suppose that for some intermediate point $\mathbf{x}_i$, $\mathbf{x}_i - \mathbf{x}_o \notin \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$. Then either $\mathbf{b}_k^+(\mathbf{x}_i - \mathbf{x}_o) > 0$, or $\mathbf{b}_{k+1}^+(\mathbf{x}_i - \mathbf{x}_o) < 0$. Suppose $\mathbf{b}_k^+(\mathbf{x}_i - \mathbf{x}_o) > 0$. Since $\mathbf{x}_d - \mathbf{x}_o \in \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1}) = \{\mathbf{x} : \mathbf{b}_k^+ \mathbf{x} \leq 0, \mathbf{b}_{k+1}^+ \mathbf{x} \geq 0\}$, then $\mathbf{b}_k^+(\mathbf{x}_d - \mathbf{x}_o) \leq 0$, and $\mathbf{b}_k^+ \mathbf{x}_i > \mathbf{b}_k^+ \mathbf{x}_o \geq \mathbf{b}_k^+ \mathbf{x}_d$. Then the ray $\mathbf{x}_o + \lambda \mathbf{b}_k$ must intersect the continuous arc $P(\mathbf{x}_i, \mathbf{x}_d)$ at some intermediate point, say $\mathbf{x}_r$. Let $P^*(\mathbf{x}_o, \mathbf{x}_r)$ be the path coincident with the ray $\mathbf{x}_o + \lambda \mathbf{b}_k$ so that $P^*(\mathbf{x}_o, \mathbf{x}_r)$ is the unique d_p shortest path between $\mathbf{x}_o$ and $\mathbf{x}_r$. This contradicts the assumption that $P(\mathbf{x}_o, \mathbf{x}_i, \mathbf{x}_r)$ is a d_p shortest path between $\mathbf{x}_o$ and $\mathbf{x}_r$. Thus $\mathbf{b}_k^+(\mathbf{x}_i - \mathbf{x}_o) \leq 0$. A similar contradiction is reached for the case $\mathbf{b}_{k+1}^+(\mathbf{x}_i - \mathbf{x}_o) < 0$ and the result is proved.

□

Note that the converse of Lemma 1 is not necessarily true. However, the following condition provides an equivalent characterization of a d_p shortest path.

Definition: *Given two points* $\mathbf{x}_o$ *and* $\mathbf{x}_d$*, suppose* $\mathbf{x}_d - \mathbf{x}_o \in \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$*. A path* $P(\mathbf{x}_o, \mathbf{x}_d)$ *is* monotone *if for all intermediate points* $\mathbf{x}_i, \mathbf{x}_j$ *determined by* $t_i, t_j \in (0,1)$ *with* $t_i < t_j$*, the vector* $\mathbf{x}_j - \mathbf{x}_i \in \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$*.*

Lemma 2 *A path* $P(\mathbf{x}_o, \mathbf{x}_d)$ *in* $\mathbb{R}^2$ *is a* d_p *shortest path from* $\mathbf{x}_o$ *to* $\mathbf{x}_d$ *if and only if it is monotone.*

Proof: Suppose $P(\mathbf{x}_o, \mathbf{x}_d)$ is monotone and suppose $\mathbf{x}_d - \mathbf{x}_o \in \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$ for some $k = 1, \ldots, 2p$ so that $\mathbf{x}_d - \mathbf{x}_o = \alpha\mathbf{b}_k + \beta\mathbf{b}_{k+1}$ for some nonnegative α and β and $d_p(\mathbf{x}_o, \mathbf{x}_d) = \alpha + \beta$. Let $\mathcal{P}$ be a partition of $[0,1]$ with $0 = t_0 \leq t_1 \leq \cdots \leq t_{n(\mathcal{P})} = 1$ so that $\mathbf{x}_o = \mathbf{x}_0 = \delta(t_0)$ and $\mathbf{x}_d = \mathbf{x}_{n(\mathcal{P})} = \delta(t_{n(\mathcal{P})})$ with intermediate points $\mathbf{x}_i = \delta(t_i)$ for $i = 1, \ldots, n(\mathcal{P}) - 1$. Let $\Delta(\mathcal{P}) = \max_{t_i \in \mathcal{P}}(t_i - t_{i-1})$. Since $P(\mathbf{x}_o, \mathbf{x}_d)$ is monotone, for each pair of consecutive intermediate points $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$, $\mathbf{x}_i - \mathbf{x}_{i-1} \in \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$ so that $\mathbf{x}_i - \mathbf{x}_{i-1} = \alpha_i\mathbf{b}_k + \beta_i\mathbf{b}_{k+1}$ with $\alpha_i \geq 0$ and $\beta_i \geq 0$ for $i = 1, \ldots, n(\mathcal{P})$.

Thus

$$\begin{aligned}
\mathbf{x}_d - \mathbf{x}_o &= \mathbf{x}_{n(\mathcal{P})} - \mathbf{x}_{n(\mathcal{P})-1} + \mathbf{x}_{n(\mathcal{P})-1} - \cdots + \mathbf{x}_1 - \mathbf{x}_o \\
&= \sum_{i=1}^{n(\mathcal{P})} \{\alpha_i\mathbf{b}_k + \beta_i\mathbf{b}_{k+1}\} \\
&= (\sum_{i=1}^{n(\mathcal{P})} \alpha_i)(\mathbf{b}_k) + (\sum_{i=1}^{n(\mathcal{P})} \beta_i)(\mathbf{b}_{k+1}).
\end{aligned}$$

This implies

$$\alpha = \sum_{i=1}^{n(\mathcal{P})} \alpha_i \qquad \text{and} \qquad \beta = \sum_{i=1}^{n(\mathcal{P})} \beta_i$$

for any partition $\mathcal{P}$ since α and β are unique in the representation $\mathbf{x}_d - \mathbf{x}_o = \alpha\mathbf{b}_k + \beta\mathbf{b}_{k+1}$.

Thus

$$\begin{aligned}
d_p(P(\mathbf{x}_o, \mathbf{x}_d)) &= \lim_{\Delta(\mathcal{P})\to 0} \sum_{i=1}^{n(\mathcal{P})} (d_p(\mathbf{x}_{i-1}, \mathbf{x}_i)) \\
&= \lim_{\Delta(\mathcal{P})\to 0} \sum_{i=1}^{n(\mathcal{P})} (\alpha_i + \beta_i) \\
&= \lim_{\Delta(\mathcal{P})\to 0} (\alpha + \beta) \\
&= \alpha + \beta \\
&= d_p(\mathbf{x}_o, \mathbf{x}_d)
\end{aligned}$$

so that $P(\mathbf{x}_o, \mathbf{x}_d)$ is a d_p shortest path.

For the converse, suppose $P(\mathbf{x}_o, \mathbf{x}_d)$ is a d_p shortest path from $\mathbf{x}_o$ to $\mathbf{x}_d$ and suppose $\mathbf{x}_d - \mathbf{x}_o \in \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$ for some $k = 1, \ldots, 2p$, with $\mathbf{x}_d - \mathbf{x}_o = \alpha_d \mathbf{b}_k + \beta_d \mathbf{b}_{k+1}$ and $\alpha_d \geq 0$ and $\beta_d \geq 0$.

Choose $0 \leq t_i < t_j \leq 1$ and consider the intermediate points $\mathbf{x}_i = \delta(t_i)$ and $\mathbf{x}_j = \delta(t_j)$. The object is to show that $\mathbf{x}_j - \mathbf{x}_i \in \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$ so that $P(\mathbf{x}_o, \mathbf{x}_d)$ is monotone. Consider the path $P(\mathbf{x}_i, \mathbf{x}_d)$ which is a d_p shortest path from $\mathbf{x}_i$ to $\mathbf{x}_d$. Furthermore, $\mathbf{x}_d - \mathbf{x}_i \in \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$. This follows because $P(\mathbf{x}_d, \mathbf{x}_o)$ is a d_p shortest path with $\mathbf{x}_o - \mathbf{x}_d \in \Gamma(-\mathbf{b}_k, -\mathbf{b}_{k+1})$, and by Lemma 1, for any intermediate point $\mathbf{x}_i$, $\mathbf{x}_i - \mathbf{x}_d \in \Gamma(-\mathbf{b}_k, -\mathbf{b}_{k+1})$, or $\mathbf{x}_d - \mathbf{x}_i \in \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$. Applying Lemma 1 to $P(\mathbf{x}_i, \mathbf{x}_d)$, implies that for all intermediate points $\mathbf{x}_j$, $\mathbf{x}_j - \mathbf{x}_i \in \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$.

□

Observe that any straight line path $P(\mathbf{x}_o, \mathbf{x}_d)$ from $\mathbf{x}_o$ to $\mathbf{x}_d$ can be represented by a linear function as follows:

$$\mathbf{x} = \delta(t) = \mathbf{x}_o + t(\mathbf{x}_d - \mathbf{x}_o) \qquad \text{for} \qquad 0 \leq t \leq 1.$$

Let $\mathbf{b}_k, \mathbf{b}_{k+1}$ be an adjacent pair of fundamental directions such that the vector $\mathbf{x}_d - \mathbf{x}_o = \alpha \mathbf{b}_k + \beta \mathbf{b}_{k+1}$. Then for all $t_i, t_j \in [0, 1]$ with $t_i < t_j$, the vector

$$\begin{aligned} \mathbf{x}_j - \mathbf{x}_i &= (t_j - t_i)(\mathbf{x}_d - \mathbf{x}_o) \\ &= (t_j - t_i)(\alpha \mathbf{b}_k + \beta \mathbf{b}_{k+1}) \\ &= (t_j - t_i)\alpha \mathbf{b}_k + (t_j - t_i)\beta \mathbf{b}_{k+1} \end{aligned}$$

so that $\mathbf{x}_j - \mathbf{x}_i \in \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$ for all intermediate points $\mathbf{x}_i$ and $\mathbf{x}_j$, and hence a straight line path is always a d_p shortest path.

Definition: *A* path of fundamental directions, *$P(\mathbf{x}_o, \mathbf{x}_d)$, from $\mathbf{x}_o$ to $\mathbf{x}_d$ (not necessarily a d_p shortest path) has the property that there exists a finite set of intermediate points $\mathbf{x}_i$, determined by $t_i \in [0, 1]$ with $t_{i-1} < t_i$, for $i = 1, \ldots, n$, such that for each consecutive pair of intermediate points $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$, there is some fundamental direction $\mathbf{b}_k$ such that $\mathbf{x}_i - \mathbf{x}_{i-1} = \lambda_i \mathbf{b}_k$ with $\lambda_i > 0$, and for all intermediate points $\mathbf{x}_{ir} = \delta(t_{ir})$ with $t_{i-1} \leq t_{ir} \leq t_i$, then $\mathbf{x}_{ir} - \mathbf{x}_{i-1} = \lambda_{ir} \mathbf{b}_k$ for some $0 \leq \lambda_{ir} \leq \lambda_i$.*

That is, a path of fundamental directions $P(\mathbf{x}_o, \mathbf{x}_d)$ proceeds from $\mathbf{x}_o$ to $\mathbf{x}_d$ in a sequence of connected straight line segments each of which is parallel to some fundamental direction.

Definition: *A* staircase path *$P(\mathbf{x}_o, \mathbf{x}_d)$ is a path of fundamental directions such that exactly two adjacent fundamental directions are used.*

A staircase path consists of straight line segments that alternate between two adjacent fundamental directions. Observe that a staircase path is a d_p shortest path. Figure 4 illustrates a path of fundamental directions and a staircase path with respect to the example block distance d_4 shown in Figure 2.

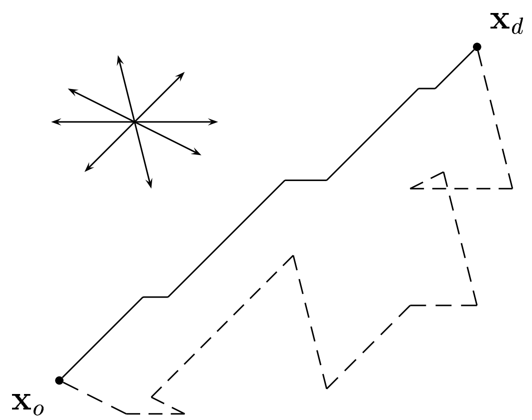

Figure 4: A path of fundamental directions (dashed) and a staircase path (solid) in the fundamental directions $\mathbf{b}_1$ and $\mathbf{b}_2$

Definition: *A* feasible path with respect to a set $S \subseteq \mathbb{R}^2$, *denoted* $P_S(\mathbf{x}_o, \mathbf{x}_d)$, *is a path from* $\mathbf{x}_o$ *to* $\mathbf{x}_d$ *contained in* S. *That is* $P_S(\mathbf{x}_o, \mathbf{x}_d) \subseteq S$.

Definition: *The* d_p distance with respect to S *between* $\mathbf{x}_o$ *and* $\mathbf{x}_d$, *denoted* $d_{p,S}(\mathbf{x}_o, \mathbf{x}_d)$, *is defined as the length of a* d_p *shortest feasible path with respect to* S. *A path* $P_S(\mathbf{x}_o, \mathbf{x}_d)$ *is a* d_p *shortest feasible path with respect to* S *if and only if* $d_{p,S}(\mathbf{x}_o, \mathbf{x}_d) = d_p(P_S(\mathbf{x}_o, \mathbf{x}_d))$.

Definition: *If* $d_{p,S}(\mathbf{x}_o, \mathbf{x}_d) = d_p(\mathbf{x}_o, \mathbf{x}_d)$ *then* $\mathbf{x}_o$ *and* $\mathbf{x}_d$ *are said to be* d_p visible *with respect to the set* S.

Definition: *A set* $S \subseteq \mathbb{R}^2$ *is said to be* d_p visible *if for all pairs of points* $\mathbf{x}_o, \mathbf{x}_d \in S$, $\mathbf{x}_o$ *and* $\mathbf{x}_d$ *are* d_p *visible, that is,* $d_{p,S}(\mathbf{x}_o, \mathbf{x}_d) = d_p(\mathbf{x}_o, \mathbf{x}_d)$.

The following property of d_p visible sets is used subsequently.

Lemma 3 *Let $S \subseteq I\!R^2$ be a d_p visible set and let $\mathbf{x}_o$ and $\mathbf{x}_d$ be distinct points in S such that $\mathbf{x}_d - \mathbf{x}_o = \lambda_0 \mathbf{b}_k$ for some $k = 1, \ldots, 2p$ and $\lambda_0 > 0$. Then the straight line segment connecting $\mathbf{x}_o$ and $\mathbf{x}_d$ is contained in S.*

Proof: The straight line segment connecting $\mathbf{x}_o$ and $\mathbf{x}_d$ is the unique d_p shortest path between $\mathbf{x}_o$ and $\mathbf{x}_d$. Since S is d_p visible, this path must be contained in S.

□

The next lemma gives the converse of Lemma 3 with the additional assumption that the set S is connected. Figure 5 illustrates a counter-example to the converse of Lemma 3 for the l_1 distance; that is, $S = S_1 \cup S_2$ with the property that any vertical or horizontal line segment connecting two points in S is contained in S, but S is not l_1 visible.

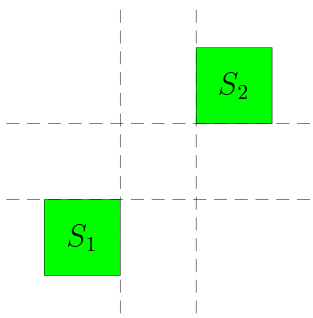

Figure 5: A counter-example to the converse of Lemma 3 using l_1.

Lemma 4 *Let $S \subseteq I\!R^2$ be a connected set, and suppose that for any pair of points $\mathbf{x}_o, \mathbf{x}_d \in S$ satisfying $\mathbf{x}_d - \mathbf{x}_o = \lambda_0 \mathbf{b}_k$ for some $k = 1, \ldots, 2p$ and $\lambda_0 > 0$, the line segment connecting $\mathbf{x}_o$ and $\mathbf{x}_d$ is contained in S. Then S is d_p visible.*

Proof: Let $\mathbf{x}_o, \mathbf{x}_d \in S$ and suppose that $\mathbf{x}_d - \mathbf{x}_o \in \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$ for some $k = 1, \ldots, 2p$. Since S is connected, there exists a feasible path $P_S(\mathbf{x}_o, \mathbf{x}_d)$. If $P_S(\mathbf{x}_o, \mathbf{x}_d)$ is monotone, it is a shortest feasible path so that $\mathbf{x}_o$ and $\mathbf{x}_d$ are visible. If $P_S(\mathbf{x}_o, \mathbf{x}_d)$ is not monotone, there exists $t_i < t_j \in [0, 1]$ corresponding to intermediate points $\mathbf{x}_i$ and $\mathbf{x}_j$ such that $\mathbf{x}_j - \mathbf{x}_i \notin \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$. If $\mathbf{b}_k^+ \mathbf{x}_j > \mathbf{b}_k^+ \mathbf{x}_i$, there must be an intermediate point $\mathbf{x}_l \in P_S(\mathbf{x}_o, \mathbf{x}_d)$ corresponding to a point $t_l < t_j \in [0, 1]$ such that $\mathbf{b}_k^+ \mathbf{x}_j = \mathbf{b}_k^+ \mathbf{x}_l$. By assumption, the line segment connecting $\mathbf{x}_l$ and $\mathbf{x}_j$ is contained in S. Consider the new path $\bar{P}_S(\mathbf{x}_o, \mathbf{x}_l, \mathbf{x}_j, \mathbf{x}_d)$, where the segments $\bar{P}_S(\mathbf{x}_o, \mathbf{x}_l)$, and $\bar{P}_S(\mathbf{x}_j, \mathbf{x}_d)$ coincide with $P_S(\mathbf{x}_o, \mathbf{x}_d)$, and the segment $\bar{P}_S(\mathbf{x}_l, \mathbf{x}_j)$ is the straight line segment connecting $\mathbf{x}_l$ and $\mathbf{x}_j$. If $\mathbf{b}_{k+1}^+ \mathbf{x}_j > \mathbf{b}_{k+1}^+ \mathbf{x}_i$, a new path is constructed by a similar argument. This process is repeated until the constructed path is monotonic so that $\mathbf{x}_o$ and $\mathbf{x}_d$ are d_p visible.

$\square$

Lemma 4 shows that d_p visibility is equivalent to the concept of 'd_p convexity' introduced in Widmayer *et al.* (1987) for distances in fixed orientations and so-called 'd_p polygons', that is, polygons whose boundary consists exclusively of segments parallel to one of the fundamental directions (see also Ottmann *et al.* (1984) for the special case of rectilinear distances). The next lemma shows that d_p visibility is a generalization of convexity in that a convex set is a d_p visible set.

Lemma 5 *A convex set $S \subseteq I\!R^2$ is a d_p visible set.*

Proof: Let $S \subseteq I\!R^2$ be a convex set and let $\mathbf{x}_o, \mathbf{x}_d \in S$. By definition of convexity there exists a straight line path $P_S(\mathbf{x}_o, \mathbf{x}_d) \subseteq S$. Since a straight line path is a d_p shortest path, S is a d_p visible set.

$\square$

The barriers B_1, B_3, B_4, B_5 in the example problem are d_p visible sets but are not convex.

3 Barriers

We will use the following concept of a barrier thoughout the remainder of the paper:

Definition: *A set $B \subseteq I\!R^2$ is a* barrier *if it satisfies the following five conditions: (1) B is d_p visible, (2) the boundary of B is a closed, rectifiable Jordan curve, (3)* $\text{clos}(\text{int}(B)) = B$, *(4)* $\text{int}(B) \cap \text{int}(B') = \emptyset$ *for all other barriers B', and (5)* $\text{int}(B) \cap E = \emptyset$.

Condition (2) ensures that the boundary of a barrier is continuous and of finite length. Condition (3) ensures that a barrier has non empty interior and no protruding line segments. For each barrier B_i, define $2p$ points called *fundamental vertices*, denoted by $\mathbf{v}_{i,k}$ for $k = 1, \ldots, 2p$, as follows:

$$\mathbf{v}_{i,k} = \arg(\max(\mathbf{b}_k^{+}\mathbf{x} : \mathbf{x} \in B_i)).$$

It follows from the definition that a barrier is compact, so that the fundamental vertices exist. If there are alternative solutions to the maximization of $\mathbf{b}_k^+\mathbf{x}$, choose any particular solution so that the designated fundamental vertex $\mathbf{v}_{i,k}$ is unique for each $k = 1, \ldots, 2p$. For notational convenience, let $\mathbf{v}_{i,2p+k} = \mathbf{v}_{i,k}$. Figure 6 shows the barrier B_5 from the example problem with its fundamental vertices.

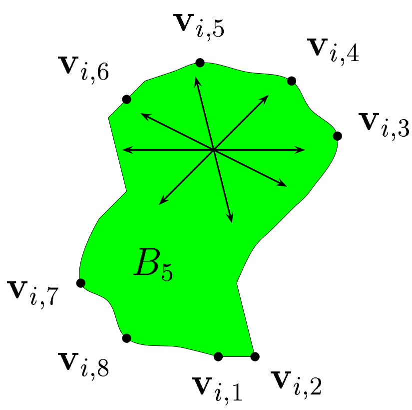

Figure 6: The barrier B_5 and its fundamental vertices

For a barrier B_i, the portion of the boundary between consecutive pairs of fundamental vertices is a feasible path with respect to B_i and is denoted by $P_\partial(\mathbf{v}_{i,k}, \mathbf{v}_{i,k+1})$ for $k = 1, \ldots, 2p$. It follows from the definition of fundamental vertices that $\mathbf{v}_{i,k+1} - \mathbf{v}_{i,k} \in \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$ for $k = 1, \ldots, 2p$.

The following results show that $P_\partial(\mathbf{v}_{i,k}, \mathbf{v}_{i,k+1})$ is a d_p shortest feasible path with respect to B_i.

Theorem 1 *Let B_i be a set satisfying conditions (2), (3), (4), and (5). Then B_i is a barrier if and only if each path $P_\partial(\mathbf{v}_{i,k}, \mathbf{v}_{i,k+1})$, for $k = 1, \ldots, 2p$, is a d_p shortest path.*

Proof: Suppose $P_\partial(\mathbf{v}_{i,k}, \mathbf{v}_{i,k+1})$, for each $k = 1, \ldots, 2p$, is a d_p shortest path. Show that B_i is d_p visible. Choose $\mathbf{x}_1$ and $\mathbf{x}_2 \in B_i$, and suppose $\mathbf{x}_2 - \mathbf{x}_1 \in \Gamma(\mathbf{b}_l, \mathbf{b}_{l+1})$ for some $l = 1, \ldots, 2p$, so that $\mathbf{x}_2 - \mathbf{x}_1 = \mu \mathbf{b}_l + \nu \mathbf{b}_{l+1}$ with $\mu, \nu \geq 0$. Let $\mathbf{x}_s$ be the point such that $\mathbf{x}_2 - \mathbf{x}_s = \mu \mathbf{b}_l$ and $\mathbf{x}_s - \mathbf{x}_1 = \nu \mathbf{b}_{l+1}$, and let $\mathbf{x}_t$ be the point such that $\mathbf{x}_2 - \mathbf{x}_t = \nu \mathbf{b}_{l+1}$ and $\mathbf{x}_t - \mathbf{x}_1 = \mu \mathbf{b}_l$. If $\mathbf{x}_s \in B_i$ then the path $P(\mathbf{x}_1, \mathbf{x}_s, \mathbf{x}_2)$ consisting of the line segment from $\mathbf{x}_1$ to $\mathbf{x}_s$ and the line segment from $\mathbf{x}_s$ to $\mathbf{x}_2$, is a d_p shortest path in B_i so that $\mathbf{x}_1$ and $\mathbf{x}_2$ are d_p visible. If $\mathbf{x}_t \in B_i$, the same argument shows that $\mathbf{x}_1$ and $\mathbf{x}_2$ are d_p visible.

Suppose $\mathbf{x}_t$ is not in B_i. By the definition of fundamental vertices, $\mathbf{b}_l^+ \mathbf{v}_l \geq \mathbf{b}_l^+ \mathbf{x}_1$. By the construction of $\mathbf{x}_t$, $\mathbf{b}_l^+ \mathbf{x}_1 = \mathbf{b}_l^+ \mathbf{x}_t$. Since $\mathbf{x}_t \notin B_i$, $\mathbf{b}_l^+ \mathbf{x}_t > \mathbf{b}_l^+ \mathbf{v}_{l+1}$. Since the path $P(\mathbf{v}_l, \mathbf{v}_{l+1})$ is continuous and monotone, by the Intermediate Value Theorem, it must cross the line $\mathbf{b}_l^+ \mathbf{x} = \mathbf{b}_l^+ \mathbf{x}_1$ exactly once. A segment of the path $P(\mathbf{v}_l, \mathbf{v}_{l+1})$ may coincide with a segment of the line $\mathbf{b}_l^+ \mathbf{x} = \mathbf{b}_l^+ \mathbf{x}_1$. Let $\mathbf{x}_q$ be a point on the line $\mathbf{b}_l^+ \mathbf{x} = \mathbf{b}_l^+ \mathbf{x}_1$ and on the path $P(\mathbf{v}_l, \mathbf{v}_{l+1})$. Similarly, $\mathbf{b}_{l+1}^+ \mathbf{v}_{l+1} \geq \mathbf{b}_{l+1}^+ \mathbf{x}_2 = \mathbf{b}_{l+1}^+ \mathbf{x}_t > \mathbf{b}_{l+1}^+ \mathbf{v}_l$, so that the continuous, monotone path $P(\mathbf{v}_l, \mathbf{v}_{l+1})$ must cross the line $\mathbf{b}_{l+1}^+ \mathbf{x} = \mathbf{b}_{l+1}^+ \mathbf{x}_2$ exactly once. Let $\mathbf{x}_r$ be such a crossing point. Then the path $P(\mathbf{x}_1, \mathbf{x}_q, \mathbf{x}_r, \mathbf{x}_2)$ consisting of the line segment from $\mathbf{x}_1$ to $\mathbf{x}_q$, the path $P_\partial(\mathbf{x}_q, \mathbf{x}_r)$ and the line segment from $\mathbf{x}_r$ to $\mathbf{x}_2$ is a d_p shortest path within $\Gamma(\mathbf{b}_l, \mathbf{b}_{l+1})$, and $\mathbf{x}_1$ and $\mathbf{x}_2$ are d_p visible.

Now suppose that B_i is d_p visible and show that the boundary path $P_\partial(\mathbf{v}_{i,k}, \mathbf{v}_{i,k+1})$, is a d_p shortest path for each $k = 1, \ldots, 2p$. By convention $\mathbf{v}_{i,k+1} - \mathbf{v}_{i,k} \in \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$.

Suppose that $P_\partial(\mathbf{v}_{i,k}, \mathbf{v}_{i,k+1})$, is not a d_p shortest path for some $k =$

$1, \ldots, 2p$. Then for some intermediate points $\mathbf{x}_i$ and $\mathbf{x}_j$, $\mathbf{x}_j - \mathbf{x}_i \notin \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$. Then either $\mathbf{b}_k^+(\mathbf{x}_j - \mathbf{x}_i) > 0$ or $\mathbf{b}_{k+1}^+(\mathbf{x}_j - \mathbf{x}_i) < 0$.

Consider the case $\mathbf{b}_k^+(\mathbf{x}_j - \mathbf{x}_i) > 0$. By definition of $\mathbf{v}_{i,k}$, $\mathbf{b}_k^+\mathbf{v}_{i,k} \geq \mathbf{b}_k^+\mathbf{x}$ for all $\mathbf{x} \in B_i$. Thus $\mathbf{b}_k^+\mathbf{v}_{i,k} \geq \mathbf{b}_k^+\mathbf{x}_j > \mathbf{b}_k^+\mathbf{x}_i$. Consider the ray $\mathbf{x}_j + \lambda(-\mathbf{b}_k)$ for $\lambda \geq 0$, which, by the intermediate value theorem, must intersect the continuous path $P_\partial(\mathbf{v}_{i,k}, \mathbf{x}_j)$ at some point, say $\mathbf{x}_r$. By visibility, the line segment from $\mathbf{x}_j$ to $\mathbf{x}_r$ is contained in B_i, and by visibility, the region bounded by the line segment from $\mathbf{x}_j$ to $\mathbf{x}_r$ and the path $P_\partial(\mathbf{x}_r, \mathbf{x}_j)$ is a subset of B_i. Thus, for some $\epsilon > 0$, the neighborhood $N(\mathbf{x}_i, \epsilon) \subset B_i$. Thus $\mathbf{x}_i$ is not a boundary point, which is a contradiction. Thus, $P_\partial(\mathbf{v}_{i,k}, \mathbf{v}_{i,k+1})$, is a d_p shortest path for each $k = 1, \ldots, 2p$.

□

Corollary 1 *If S is a convex set in $\mathbb{R}^2$ satisfying conditions (2), (3), (4), and (5), then S is a barrier.*

Proof: By Lemma 5, S is d_p visible and thus a barrier.

□

Definition: *Given a block distance d_p, the* block hull *of a compact set S, denoted $H(S)$, is the smallest polytope enclosing S with sides parallel to the fundamental directions of d_p:*

$$H(S) = \{\mathbf{x} : \mathbf{b}_k^+\mathbf{x} \leq \max_{\mathbf{x} \in S} \mathbf{b}_k^+\mathbf{x} \text{ for } k = 1, \ldots, 2p\}.$$

Observe that the fundamental vertices of a barrier B_i are fundamental vertices of $H(B_i)$. The faces of the polytope $H(B_i)$ are denoted by $F_{i,k} = \{\mathbf{x} \in H(B_i) : \mathbf{b}_k^+ \mathbf{x} = \mathbf{b}_k^+ \mathbf{v}_{i,k}\}$ for $k = 1, \ldots, 2p$. Let $\mathbf{h}_{i,k}$ be the point of intersection of faces $F_{i,k}$ and $F_{i,k+1}$ for $k = 1, \ldots, 2p$. Some of the points $\mathbf{h}_{i,k}$ may coincide, but the distinct points $\mathbf{h}_{i,k}$ are extreme points of $H(B_i)$.

There is at least one fundamental vertex of B_i on each face of $H(B_i)$ and one or more fundamental vertices may be coincident with an extreme point of $H(B_i)$. See Figure 7 for an example. Different barriers may have the same block hull. Also, two barriers having the same fundamental vertices will have the same block hull.

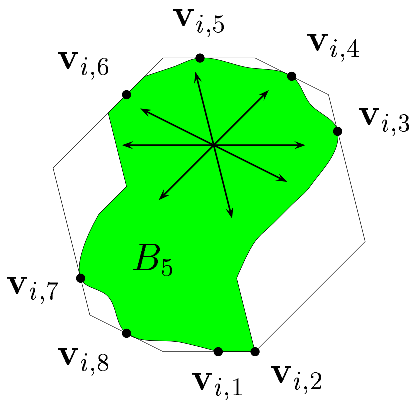

Figure 7: The block hull of the barrier B_5 in the example problem

Define a corner set, $C_{i,k}$ of $H(B_i)$ as the set bordered by the faces $F_{i,k}$, $F_{i,k+1}$, and the path $P_\partial(\mathbf{v}_{i,k}, \mathbf{v}_{i,k+1})$ for each $k = 1, \ldots, 2p$. Alternatively,

$$C_{i,k} = (\mathbf{h}_{i,k} + \Gamma(\mathbf{b}_{k+1}, -\mathbf{b}_k)) \setminus (\text{int}(B_i) + \Gamma(\mathbf{b}_{k+1}, -\mathbf{b}_k)) .$$

Note that $C_{i,k}$ has empty interior if $P_\partial(\mathbf{v}_{i,k}, \mathbf{v}_{i,k+1}) \subseteq F_{i,k} \cup F_{i,k+1}$.

Definition: *For each barrier B_i with fundamental vertices $\mathbf{v}_{i,k}$, $k = 1, \ldots, 2p$, define a modified barrier, denoted B'_i, to be a barrier such that:*

(1) $H(B_i) = H(B'_i)$,

(2) $\mathbf{v}_{i,k}$ *is a fundamental vertex of* B'_i *for each* $k = 1, \ldots, 2p$,

(3) $\text{int}(B'_i) \cap \text{int}(B_j) = \emptyset$ *for* $k = 1, \ldots, 2p$ *and* $j \neq i$,

(4) $\text{int}(B'_i) \cap E = \emptyset$ *for* $i = 1, \ldots, n$.

Alternatively, a barrier B_i is modified by choosing its boundary to be any alternative d_p shortest path between each consecutive pair of fundamental vertices, while maintaining conditions (3) and (4). A modified barrier B'_i may be a subset of B_i, a super set of B_i, or neither. Figure 8 illustrates a modification of the barrier B_5 in the example problem.

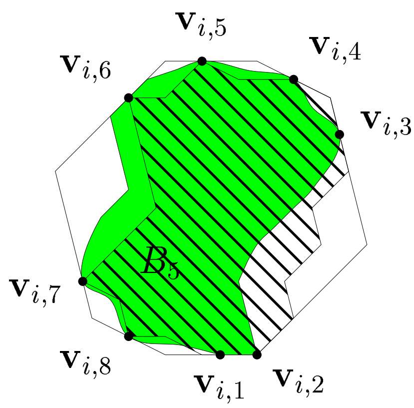

Figure 8: The barrier with diagonal lines is modification of the barrier B_5

4 Equivalence Results

Given a planar location problem with barriers B_i, $i = 1, \ldots, n$, and the feasible set $\mathcal{F} = \mathbb{R}^2 \setminus \mathrm{int}(\mathcal{B})$, where $\mathcal{B} = \bigcup_{i=1}^n B_i$, consider the location problem with modified barriers B'_i, $i = 1, \ldots, n$, and feasible set $\mathcal{F}' = \mathbb{R}^2 \setminus \mathrm{int}(\mathcal{B}')$, where $\mathcal{B}' = \bigcup_{i=1}^n B'_i$. The following lemmas and theorem show that the original problem $\min_{\mathbf{x}\in\mathcal{F}} f\{d_{p,\mathcal{F}}(\mathbf{x}, E)\}$ is equivalent to the modified problem $\min_{\mathbf{x}\in\mathcal{F}'} f\{d_{p,\mathcal{F}'}(\mathbf{x}, E)\}$.

Lemma 6 *Let $\mathbf{x}_o, \mathbf{x}_d \in \mathcal{F} \cap \mathcal{F}'$. Then $d_{p,\mathcal{F}}(\mathbf{x}_o, \mathbf{x}_d) = d_{p,\mathcal{F}'}(\mathbf{x}_o, \mathbf{x}_d)$.*

Proof: For $\mathbf{x}_o, \mathbf{x}_d \in \mathcal{F} \cap \mathcal{F}'$, let $P_{\mathcal{F}}(\mathbf{x}_o, \mathbf{x}_d)$ be a d_p shortest feasible path with respect to $\mathcal{F}$ so that $d_p(P_{\mathcal{F}}(\mathbf{x}_o, \mathbf{x}_d)) = d_{p,\mathcal{F}}(\mathbf{x}_o, \mathbf{x}_d)$. If $P_{\mathcal{F}}(\mathbf{x}_o, \mathbf{x}_d) \subseteq \mathcal{F}'$, then $P_{\mathcal{F}}(\mathbf{x}_o, \mathbf{x}_d)$ is a feasible path with respect to $\mathcal{F}'$ and

$$d_{p,\mathcal{F}'}(\mathbf{x}_o, \mathbf{x}_d) \leq d_p(P_{\mathcal{F}}(\mathbf{x}_o, \mathbf{x}_d)) = d_{p,\mathcal{F}}(\mathbf{x}_o, \mathbf{x}_d).$$

If $P_{\mathcal{F}}(\mathbf{x}_o, \mathbf{x}_d) \not\subseteq \mathcal{F}'$, then $P_{\mathcal{F}}(\mathbf{x}_o, \mathbf{x}_d)$ must contain at least one point, say $\mathbf{x}_t$ such that $\mathbf{x}_t \in \mathcal{F} \setminus \mathcal{F}'$, that is, $\mathbf{x}_t \in B'_i \setminus B_i$ for some i. Thus x_t must be in a corner set $C_{i,k}$ of $H(B_i)$ for some $k = 1, \ldots, 2p$. Therefore $P_{\mathcal{F}}(\mathbf{x}_o, \mathbf{x}_t)$ must intersect the boundary path $P'_\partial(\mathbf{v}_k, \mathbf{v}_{k+1})$ of B'_i at some point, say $\mathbf{x}_q$ and $P_{\mathcal{F}}(\mathbf{x}_t, \mathbf{x}_d)$ must intersect the boundary path $P'_\partial(\mathbf{v}_k, \mathbf{v}_{k+1})$ of B'_i at some point, say $\mathbf{x}_r$, so that the path segment $P'_\partial(\mathbf{x}_q, \mathbf{x}_r) \subseteq \mathcal{F}'$.

Construct a new path $P^*_{\mathcal{F}}(\mathbf{x}_o, \mathbf{x}_q, \mathbf{x}_r, \mathbf{x}_d)$ which consists of the segments $P_{\mathcal{F}}(\mathbf{x}_o, \mathbf{x}_q)$, $P'_\partial(\mathbf{x}_q, \mathbf{x}_r)$ and $P_{\mathcal{F}}(\mathbf{x}_r, \mathbf{x}_d)$.

This process is repeated for each segment of $P^*_{\mathcal{F}}(\mathbf{x}_o, \mathbf{x}_q, \mathbf{x}_r, \mathbf{x}_d)$ that lies in $\mathcal{F} \setminus \mathcal{F}'$ until a new path is constructed, say $P^*_{\mathcal{F}}(\mathbf{x}_o, \mathbf{x}_d)$, that lies in $\mathcal{F}'$. Then the new path $P^*_{\mathcal{F}}(\mathbf{x}_o, \mathbf{x}_d)$ is a feasible path with respect to $\mathcal{F}'$ and

$$d_{p,\mathcal{F}'}(\mathbf{x}_o, \mathbf{x}_d) \leq d_p(P^*_{\mathcal{F}}(\mathbf{x}_o, \mathbf{x}_d)) = d_{p,\mathcal{F}}(\mathbf{x}_o, \mathbf{x}_d).$$

To show $d_{p,\mathcal{F}}(\mathbf{x}_o, \mathbf{x}_d) \leq d_{p,\mathcal{F}'}(\mathbf{x}_o, \mathbf{x}_d)$, the argument is repeated reversing the roles of $\mathcal{F}$ and $\mathcal{F}'$.

□

Lemma 7 *There exists a point $\mathbf{x}' \in \mathcal{F} \cap \mathcal{F}'$ that is an optimal solution to the location problem:* $\min_{\mathbf{x}\in\mathcal{F}} f\{d_{p,\mathcal{F}}(\mathbf{x}, E)\}$.

Proof: Let $\mathbf{x}^*$ be an optimal solution to the location problem $\min_{\mathbf{x}\in\mathcal{F}} f\{d_{p,\mathcal{F}}(\mathbf{x}, E)\}$. Then $\mathbf{x}^* \in \mathcal{F}$. If $\mathbf{x}^* \in \mathcal{F}'$ then let $\mathbf{x}' = \mathbf{x}^*$. If $\mathbf{x}^* \notin \mathcal{F}'$ then $\mathbf{x}^* \in \mathcal{F} \setminus \mathcal{F}'$ and $\mathbf{x}^*$ is located within the interior of some barrier B'_i and within a corner $C_{i,k}$ of $H(B_i)$ for some $k = 1, \ldots, 2p$ and $i = 1, \ldots, n$. Let $\mathbf{x}'$ be the point where the ray $\mathbf{x}^* + \lambda(\mathbf{b}_k - \mathbf{b}_{k+1})$ intersects the boundary path $P'_\partial(\mathbf{v}_{i,k}, \mathbf{v}_{i,k+1})$ of B'_i, so that $\mathbf{x}' = \mathbf{x}^* + \lambda_0(\mathbf{b}_k - \mathbf{b}_{k+1})$ for some $\lambda_0 > 0$ is an intermediate point of $P'_\partial(\mathbf{v}_{i,k}, \mathbf{v}_{i,k+1})$.

For any existing facility $\mathbf{e}_j$, let $P_{\mathcal{F}}(\mathbf{e}_j, \mathbf{x}^*)$ be a d_p shortest feasible path with respect to $\mathcal{F}$. Since $\mathbf{e}_j \notin \mathrm{int}(B'_i)$, $P_{\mathcal{F}}(\mathbf{e}_j, \mathbf{x}^*)$ must intersect the boundary $P'_\partial(\mathbf{v}_{i,k}, \mathbf{v}_{i,k+1})$ of B'_i, at some point, say $\mathbf{x}_q$. Since $\mathbf{x}'$ and $\mathbf{x}_q$ are intermediate points of $P'_\partial(\mathbf{v}_{i,k}, \mathbf{v}_{i,k+1})$, they are ordered either as $P'_\partial(\mathbf{v}_{i,k}, \mathbf{x}_q, \mathbf{x}', \mathbf{v}_{i,k+1})$, or as $P'_\partial(\mathbf{v}_{i,k}, \mathbf{x}', \mathbf{x}_q, \mathbf{v}_{i,k+1})$. Suppose the former. Let $P'_\partial(\mathbf{x}_q, \mathbf{x}')$ be the segment of $P'_\partial(\mathbf{v}_{i,k}, \mathbf{v}_{i,k+1})$ from $\mathbf{x}_q$ to $\mathbf{x}'$ so that $\mathbf{x}' - \mathbf{x}_q \in \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$.

If $\mathbf{x}^* - \mathbf{x}_q \in \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$ then $d_p(\mathbf{x}_q, \mathbf{x}') = d_p(\mathbf{x}_q, \mathbf{x}^*)$ since $\mathbf{x}^*$ and $\mathbf{x}'$ are both on the facet, given by $\mathbf{b}^0_k(\mathbf{x} - \mathbf{x}_q) = \mathbf{b}^0_k(\mathbf{x}' - \mathbf{x}_q)$, of the d_p ball centered at $\mathbf{x}_q$ with radius $\mathbf{b}^0_k(\mathbf{x}' - \mathbf{x}_q)$.

If $\mathbf{x}^* - \mathbf{x}_q \notin \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$ then $d_p(\mathbf{x}_q, \mathbf{x}') < d_p(\mathbf{x}_q, \mathbf{x}^*)$ since $\mathbf{x}^*$ is on the support, given by $\mathbf{b}_k^0(\mathbf{x} - \mathbf{x}_q) = \mathbf{b}_k^0(\mathbf{x}' - \mathbf{x}_q)$, of the d_p ball centered at $\mathbf{x}_q$ with radius $\mathbf{b}_k^0(\mathbf{x}' - \mathbf{x}_q)$. In either case, $d_p(\mathbf{x}_q, \mathbf{x}') \leq d_p(\mathbf{x}_q, \mathbf{x}^*)$.

If the intermediate points $\mathbf{x}_q$ and $\mathbf{x}'$ are ordered as $P'_\partial(\mathbf{v}_{i,k}, \mathbf{x}', \mathbf{x}_q, \mathbf{v}_{i,k+1})$, then $\mathbf{x}_q - \mathbf{x}' \in \Gamma(-\mathbf{b}_k, -\mathbf{b}_{k+1})$, and a similar argument yields $d_p(\mathbf{x}_q, \mathbf{x}') \leq d_p(\mathbf{x}_q, \mathbf{x}^*)$.

Thus $d_p(\mathbf{e}_j, \mathbf{x}') \leq d_p(\mathbf{e}_j, \mathbf{x}^*)$ for all $j = 1, \ldots, m$. Since the objective function f is convex and nondecreasing,

$$f\{d_{p,\mathcal{F}}(\mathbf{x}', E)\} \leq f\{d_{p,\mathcal{F}}(\mathbf{x}^*, E)\}.$$

Thus $\mathbf{x}'$ must also be an optimal solution.

□

Lemma 8 *There exists a point $\mathbf{x}' \in \mathcal{F} \cap \mathcal{F}'$ that is an optimal solution to the location problem:* $\min_{\mathbf{x} \in \mathcal{F}'} f\{d_{p,\mathcal{F}'}(\mathbf{x}, E)\}$.

Proof: The proof is analogous to the proof of Lemma 7.

□

Theorem 2 *There exists a point $\mathbf{x}^* \in \mathcal{F} \cap \mathcal{F}'$ that is an optimal solution to both of the location problems* $\min_{\mathbf{x} \in \mathcal{F}'} f\{d_{p,\mathcal{F}'}(\mathbf{x}, E)\}$ *and* $\min_{\mathbf{x} \in \mathcal{F}} f\{d_{p,\mathcal{F}}(\mathbf{x}, E)\}$.

Proof: By Lemma 7 there exists a point $\mathbf{x}_1 \in \mathcal{F} \cap \mathcal{F}'$ that minimizes $f\{d_{p,\mathcal{F}}(\mathbf{x}, E)\}$, and by Lemma 8 there exists a point $\mathbf{x}_2 \in \mathcal{F} \cap \mathcal{F}'$ that minimizes $f\{d_{p,\mathcal{F}'}(\mathbf{x}, E)\}$. By Lemma 6, $d_{p,\mathcal{F}}(\mathbf{x}_1, E) = d_{p,\mathcal{F}'}(\mathbf{x}_1, E) =$

$d_{p,\mathcal{F}\cap\mathcal{F}'}(\mathbf{x}_1, E)$, and $d_{p,\mathcal{F}}(\mathbf{x}_2, E) = d_{p,\mathcal{F}'}(\mathbf{x}_2, E) = d_{p,\mathcal{F}\cap\mathcal{F}'}(\mathbf{x}_2, E)$. Since $\mathcal{F} \cap \mathcal{F}' \subseteq \mathcal{F}$ and $\mathcal{F} \cap \mathcal{F}' \subseteq \mathcal{F}'$, $\mathbf{x}_1$ and $\mathbf{x}_2$ both minimize $f\{d_{p,\mathcal{F}\cap\mathcal{F}'}(\mathbf{x}, E)\}$. Since $\mathbf{x}_1$ and $\mathbf{x}_2$ are both optimal solutions to the same problem, they must have the same objective function value. Therefore, $\mathbf{x}_1$ and $\mathbf{x}_2$ both minimize $f\{d_{p,\mathcal{F}}(\mathbf{x}, E)\}$ and $f\{d_{p,\mathcal{F}'}(\mathbf{x}, E)\}$.

□

The implication of Theorem 2 is that a solution to a given location problem with barriers B_i and feasible set $\mathcal{F}$ may be obtained by solving a location problem with modified barriers B_i' and feasible set $\mathcal{F}'$. If an optimal solution $\mathbf{x}^*$ to the modified problem exists in $\mathcal{F}' \setminus \mathcal{F}$, then there is an optimal solution $\mathbf{x}'$ to the original problem that is constructed from $\mathbf{x}^*$.

5 An Explicit Modification

A modification is given in this section that decreases the feasible set by modifying each barrier B_i to be as large as possible within its block hull $H(B_i)$, but without enclosing any existing facility or intersecting the interior of another barrier. The resulting modified barrier is denoted by B_i' and is shown to be a (setwise) maximal modified barrier. The maximal modified barrier B_i' is obtained from B_i by reducing each corner $C_{i,k}$ of $H(B_i)$, for $k = 1, \ldots, 2p$, as much as possible. For any point $\mathbf{x}'$ in a corner $C_{i,k}$ of $H(B_i)$, define the restricted hull of $\mathbf{x}'$ and the corner point $\mathbf{h}_{i,k}$, denoted $H_{i,k}(\mathbf{h}_{i,k}, \mathbf{x}')$, as the smallest parallelogram containing $\mathbf{h}_{i,k}$ and $\mathbf{x}'$ with sides parallel to the faces $F_{i,k}$ and $F_{i,k+1}$ of $H(B_i)$. That is,

$$H_{i,k}(\mathbf{h}_{i,k},\mathbf{x}') = \{\mathbf{x} : \mathbf{b}_k^+\mathbf{x}' \le \mathbf{b}_k^+\mathbf{x} \le \mathbf{b}_k^+\mathbf{h}_{i,k}, \mathbf{b}_{k+1}^+\mathbf{x}' \le \mathbf{b}_{k+1}^+\mathbf{x} \le \mathbf{b}_{k+1}^+\mathbf{h}_{i,k}\}.$$

Next define the following sets for each $i = 1,\dots,n$ and $k = 1,\dots,2p$ representing what must remain in each corner.

$$\begin{aligned} U_{i,k} &= \textstyle\bigcup_{\mathbf{e}_j \in C_{i,k}} H_{i,k}(\mathbf{h}_{i,k},\mathbf{e}_j) \\ V_{i,k} &= \textstyle\bigcup_{\mathrm{int}(B_l)\cap C_{i,k}\neq\emptyset, \mathbf{v}_{l,m}\in C_{i,k}} H_{i,k}(\mathbf{h}_{i,k},\mathbf{v}_{l,m}) \\ W_{i,k} &= \textstyle\bigcup_{\mathrm{int}(B_l)\cap C_{i,k}\neq\emptyset} \{B_l \cap C_{i,k}\} \end{aligned}$$

Then the k^{th} corner of the modified barrier is given by $C'_{i,k} = U_{i,k} \cup V_{i,k} \cup W_{i,k}$, and the modified barrier B'_i is obtained by removing from the block hull $H(B_i)$ all of the modified corners $C'_{i,k}$ for $k = 1,\dots,2p$. That is,

$$B'_i = \mathrm{clos}\left\{ H(B_i) \setminus \textstyle\bigcup_{k=1,\dots,2p} C'_{i,k} \right\}.$$

It follows that $B_i \subseteq B'_i$ and that $C'_{i,k} = \emptyset$ if $E \cap C_{i,k} = \emptyset$, and if $\mathrm{int}(B)_l \cap C_{i,k} = \emptyset$ for $l \neq k$. In particular, if $H(B_i)$ contains no existing facilities and does not intersect the interior of any other barrier, then $B'_i = H(B_i)$. The construction of a maximal modified barrier B'_i is illustrated in Figure 9.

Observe that the boundary of B'_i between consecutive pairs of fundamental vertices $\mathbf{v}_{i,k}$ and $\mathbf{v}_{i,k+1}$ is made up of segments that are either a segment of a common boundary with another barrier or a staircase path in the directions of $\mathbf{b}_k$ and $\mathbf{b}_{k+1}$.

Lemma 9 *Each modified barrier B'_i obtained from B_i is set-wise maximal, that is, there is no barrier B^*_i that strictly contains B'_i.*

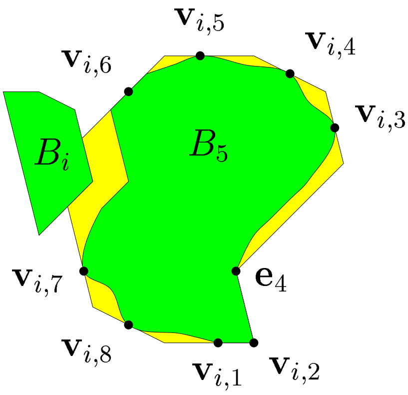

Figure 9: The maximal modification of barrier B_5 with existing facility $\mathbf{e}_4$, and a barrier B_i.

Proof: Observe that the boundary of each set $H_{i,k}(\mathbf{h}_{i,k}, \mathbf{e}_j)$ is a staircase path with intermediate point $\mathbf{e}_j$. If any proper subset of $H_{i,k}(\mathbf{h}_{i,k}, \mathbf{e}_j)$ is removed, then either the existing facility $\mathbf{e}_j$ is interior to the modified barrier, or the boundary path is not a d_p shortest path. Thus $H_{i,k}(\mathbf{h}_{i,k}, \mathbf{e}_j)$ is a minimal subset that can be removed to ensure that $\mathbf{e}_j \cap \text{int} B_i' = \emptyset$ and that B_i' is a barrier. A similar argument implies that each set $H_{i,k}(\mathbf{h}_{i,k}, \mathbf{v}_{l,m})$ is minimal. Clearly, each set $B_l \cap C_{i,k}$ is a minimal set that can be removed to ensure that the modified barriers remain disjoint. The removal of minimal subsets from B_i implies that the resulting set B_i' is maximal.

□

For a given location problem, maximal modified barriers B_i' are constructed sequentially for $i = 1, \ldots, n$ as follows:

(1) For $i = 1, \ldots, n$,

(2) Modify B_i to obtain B'_i.

(3) Reset: $B_i \longleftarrow B'_i$.

The new feasible region is given by $\mathcal{F}' = I\!R^2 \setminus \text{int}(\mathcal{B}')$ where $\mathcal{B}' = \bigcup_{i=1}^n B'_i$. Modifying the barriers B_i in a different order from the index i may result in a different feasible region $\mathcal{F}'$, however, the optimal solutions will not be changed.

A further reduction of the feasible region is given by the following lemma, which says that there is some optimal solution within $H(E, \mathcal{B}')$, the block hull of all existing facilities and all modified barriers.

Lemma 10 *There exists a point* $\mathbf{x}^* \in H(E, \mathcal{B}')$ *that is a minimal solution to* $f\{d_{p,\mathcal{F}'}(\mathbf{x}, E)\}$.

Proof: Denote each face of $H(E, \mathcal{B}')$ by $\mathbf{b}_k^+\mathbf{x} = \mathbf{b}_k^+\mathbf{x}_k^0$ for $k = 1, \ldots, 2p$, where each $\mathbf{x}_k^0$ is either a fundamental vertex $\mathbf{v}_{i,k}$ for some $i = 1, \ldots, n$ and $k = 1, \ldots, 2p$, or an existing facility $\mathbf{e}_j$ for some $j = 1, \ldots, m$. Thus, $\mathbf{b}_k^+\mathbf{x} \leq \mathbf{b}_k^+\mathbf{x}_k^0$ is a supporting hyperplane of $H(E, \mathcal{B}')$ for $k = 1, \ldots, 2p$.

If $\mathbf{x}^* \notin H(E, \mathcal{B}')$, then $\mathbf{b}_k^+\mathbf{x}^* > \mathbf{b}_k^+\mathbf{x}_k^0$ for some $k = 1, \ldots, 2p$. Let $\mathbf{x}_k^1$ be the point of intersection between the ray $\mathbf{x}^* - \gamma\mathbf{b}_k^+$ and the supporting hyperplane.

For each $j = 1, \ldots, m$, let $P_{\mathcal{F}'}(\mathbf{e}_j, \mathbf{x}^*)$ be a d_p shortest feasible path between $\mathbf{x}^*$ and $\mathbf{e}_j$. Then there is some intermediate point $\mathbf{x}'_j$ that is on the supporting hyperplane $\mathbf{b}_k^+\mathbf{x} = \mathbf{b}_k^+\mathbf{x}_k^0$. Since there are no barriers outside $H(E, \mathcal{B}')$, $\mathbf{x}'_j$ and $\mathbf{x}^*$ are d_p visible so that $d_p(P_{\mathcal{F}'}(\mathbf{x}'_j, \mathbf{x}^*)) = d_{p,\mathcal{F}'}(\mathbf{x}'_j, \mathbf{x}^*)$ and $\mathbf{x}^* - \mathbf{x}'_j \in \Gamma(\mathbf{b}_l, \mathbf{b}_{l+1})$ for some $l = 1, \ldots, 2p$. Each cone $\mathbf{x}'_j + \Gamma(\mathbf{b}_l, \mathbf{b}_{l+1})$ lies in the halfspace $\mathbf{b}_k^+\mathbf{x} \geq \mathbf{b}_k^+\mathbf{x}_k^0$, so that $\mathbf{b}_k^+$ is an ascending direction with respect to $\mathbf{b}_l$ and $\mathbf{b}_{l+1}$. Thus, $d_{p,\mathcal{F}'}(\mathbf{x}'_j, \mathbf{x}_k^1) \leq d_{p,\mathcal{F}'}(\mathbf{x}'_j, \mathbf{x}^*)$ for each $j = 1, \ldots, m$.

If $\mathbf{x}_k^1 \in H(E, \mathcal{B}')$, then $d_{p,\mathcal{F}'}(\mathbf{e}_j, \mathbf{x}_k^1) \leq d_{p,\mathcal{F}'}(\mathbf{e}_j, \mathbf{x}^*)$ for all j, so that $\mathbf{x}_k^1$ is an alternative minimum solution.

Otherwise, $\mathbf{x}_k^1$ violates some other supporting hyperplane of $H(E, \mathcal{B}')$, and the argument is repeated, generating a new point that lies on the violated supporting hyperplane and is closer to $\mathbf{e}_j$. For any point $\mathbf{x} \notin H(E, \mathcal{B}')$ there can be at most p supporting hyperplanes violated by $\mathbf{x}$. Thus this process will require at most p repetitions until the point generated is in $H(E, \mathcal{B}')$, and the result follows.

□

Figure 10 illustrates the maximal modified barriers for the example problem and the block hull $H(E, \mathcal{B}')$. The feasible region is the unshaded region inside $H(E, \mathcal{B}')$ and the common boundaries between barriers (bold line segments)

6 A Grid and Cell Structure

The feasible set $\mathcal{F}'$ is now divided into subsets, called cells, that are formed by the intersections of a collection of cones. The set of vertices of these cones is defined below. Each cone is generated by extending rays from a vertex along a pair of consecutive fundamental directions into the feasible set.

For each maximal modified barrier B'_i, let $F'_{i,k}$ be the k^{th} face for $k = 1, \ldots, 2p$, that is, $F'_{i,k} = \{\mathbf{x} \in B'_i : \mathbf{b}_k^+\mathbf{x} = \mathbf{b}_k^+\mathbf{v}_{i,k}\}$. If F'_{ik} is a line segment, let $\mathbf{r}_{i,k}$ and $\mathbf{s}_{i,k}$ be its end points, that is, $\mathbf{b}_k\mathbf{r}_{i,k} = \max_{\mathbf{x} \in F'_{i,k}} \mathbf{b}_k\mathbf{x}$, and $-\mathbf{b}_k\mathbf{s}_{i,k} =$

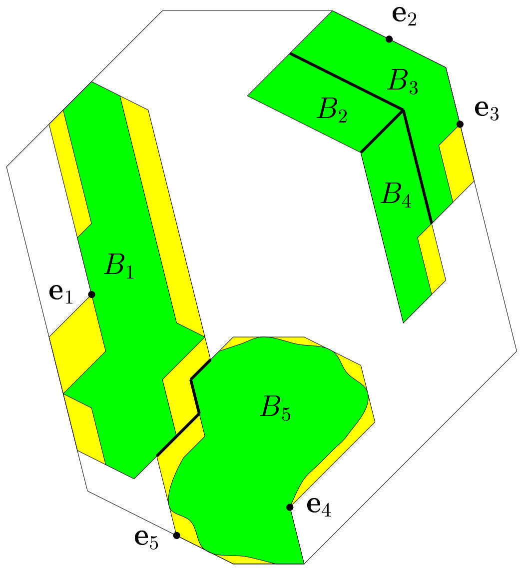

Figure 10: Example problem with maximal modified barriers and the block hull

$\max_{\mathbf{x}\in F'_{i,k}} -\mathbf{b}_k\mathbf{x}$. If $F'_{i,k}$ is a point, then $F'_{i,k} = \{\mathbf{v}_{i,k} = \mathbf{r}_{i,k} = \mathbf{s}_{i,k}\}$. The set of vertices consists of all distinct points that are either existing facilities in E, or end points $\mathbf{r}_{i,k}$ and $\mathbf{s}_{i,k}$ for $i = 1, \ldots, n$ and $k = 1, \ldots, 2p$. Denote the set of vertices by W and let R be the index set of vertices, so that $W = \{\mathbf{w}_r : r \in R\}$. There are at most $4p$ vertices for each barrier, so the number of vertices is of the order $O(np + m)$. Figure 11 illustrates the faces and the end points of the faces of a maximal modified barrier B'_i. Observe that the faces of a barrier may not be the entire boundary of the barrier. The vertices of cones associated with the barrier B'_i in Figure 11 are the points $\{\mathbf{r}_{i,1}, \mathbf{e}_j, \mathbf{v}_{l,5}, \mathbf{v}_{k,5}, \mathbf{v}_{k,6}, \mathbf{s}_{i,2}, \mathbf{r}_{i,2}, \mathbf{r}_{i,4}, \mathbf{r}_{i,6}\}$.

For each $r \in R$ and for each $k = 1, \ldots, 2p$, construct the ray $\mathbf{w}_r + \lambda\mathbf{b}_k$

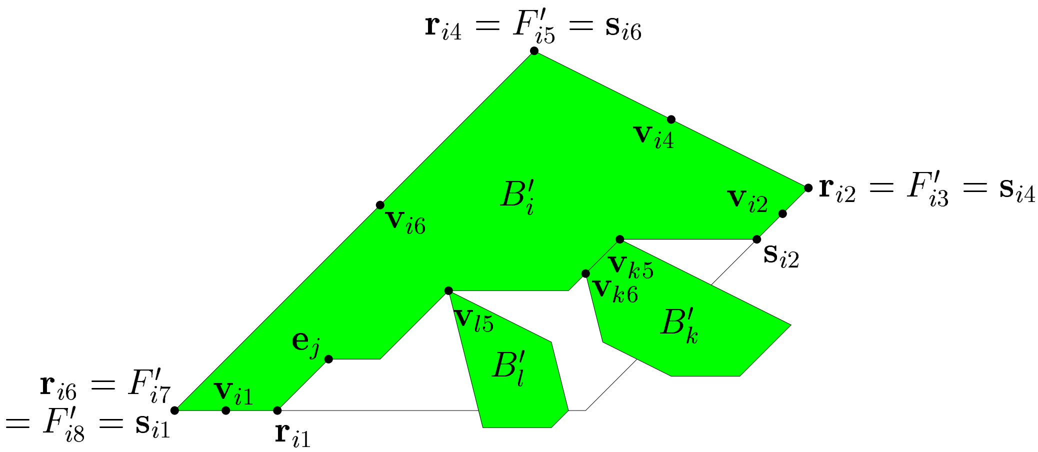

Figure 11: Endpoints of faces of a barrier B'_i and vertices of cones: $\{\mathbf{r}_{i,1}, \mathbf{e}_j, \mathbf{v}_{l,5}, \mathbf{v}_{k,5}, \mathbf{v}_{k,6}, \mathbf{s}_{i,2}, \mathbf{r}_{i,2}, \mathbf{r}_{i,4}, \mathbf{r}_{i,6}\}$

for $\lambda > 0$, if possible, until it intersects the interior of some modified barrier or until it intersects the boundary of the block hull $H(E, \mathcal{B}')$. The collection of rays extending from each vertex is called the *grid structure* for the feasible set $\mathcal{F}'$. Figure 12 illustrates the grid structure for the example problem. The number of rays in the grid is of the order $O(np^2 + mp)$.

A *cell* C is a subset of $\mathcal{F}'$ with nonempty interior that is bounded by the fundamental rays and/or barrier boundaries such that no fundamental ray passes through its interior. In Figure 12 the cells for the example problem are the regions delineated by the barrier boundaries and the rays extending from each vertex.

Lemma 11 *Each point* $\mathbf{x} \in \mathcal{F}'$ *is in some cell* C *or is on a common boundary between two maximal modified barriers.*

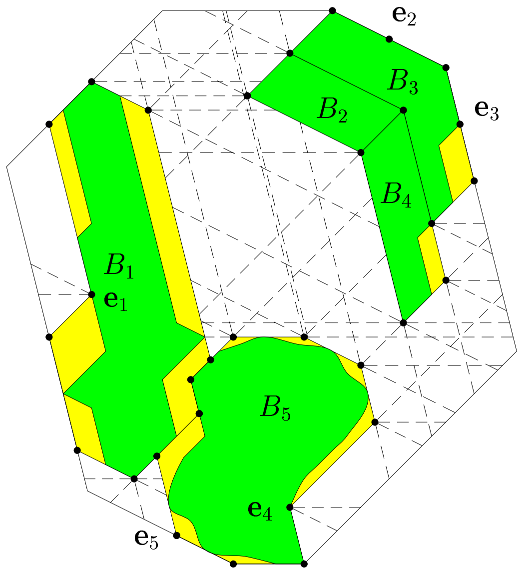

Figure 12: Grid structure of the example problem formed by the vertices (dots) and rays (dashed lines) extending from each vertex

Proof: Suppose $\mathbf{x}' \in \mathcal{F}'$ but is not a point on a common boundary between two maximal modified barriers. The boundary of each maximal barrier is a line segment whose equation is of the form $\mathbf{b}_k^+\mathbf{x} = \mathbf{b}_k^+\mathbf{w}_r$ for some vertex $\mathbf{w}_r \in W$. Since $\mathbf{x}'$ is not interior to a barrier, then either $\mathbf{b}_k^+\mathbf{x}' \geq \mathbf{b}_k^+\mathbf{w}_r$ or $-\mathbf{b}_{k+p}^+\mathbf{x}' \geq -\mathbf{b}_{k+p}^+\mathbf{w}_r$ for some vertex $\mathbf{w}_r \in W$. In either case, $\mathbf{x}'$ is in some cone and hence in some cell.

□

Denote the cones formed by the grid structure as $\Gamma_{r,k} = \mathbf{w}_r + \Gamma(\mathbf{b}_k, \mathbf{b}_{k+1})$ for $(r,k) \in R \times \{1, \ldots, 2p\}$. For each cell C, let $I(C)$ be the set of indices (r,k) such that $C \subseteq \Gamma_{r,k}$, that is, $I(C) = \{(r,k) \in R \times \{1, \ldots, 2p\} : C \subseteq \Gamma_{r,k}\}$.

Thus, C has the polyhedral representation given by

$$C = \{\mathbf{x} : \mathbf{b}_k^+ \mathbf{x} \leq \mathbf{b}_k^+ \mathbf{w}_r, \mathbf{b}_{k+1}^+ \mathbf{x} \geq \mathbf{b}_{k+1}^+ \mathbf{w}_r, \text{for}(r,k) \in I(C)\}.$$

In this representation, some of the inequalities may be redundant.

The problem of identifying the set $I(C)$ and constructing the polyhedral representation of each cell C is the problem in computational geometry of finding an *arrangement* in $\mathbb{R}^2$, given a set of points $\mathbf{w}_r$, $r \in R$, and a set of rays $\mathbf{w}_r + \lambda \mathbf{b}_k$ for $k = 1, \ldots, 2p$. Agarwal and Sharir (2000) present polynomial-time algorithms for finding arrangements.

The grid structure and cells contain all the information needed to solve the 'median' or 'total cost' location problem, that is, when the function f is the sum of nonnegative weighted distances $d_{p,\mathcal{F}'}(\mathbf{e}_j, \mathbf{x})$. Since $d_p(\mathbf{w}_r, \mathbf{x})$ is linear for $\mathbf{x}$ within each cone with vertex $\mathbf{w}_r$, then the distance $d_{p,\mathcal{F}'}(\mathbf{e}_j, \mathbf{x})$ is concave within each cell since it is the minimum of linear functions $d_p(\mathbf{w}_r, \mathbf{x}) + d_{p,\mathcal{F}'}(\mathbf{e}_j, \mathbf{w}_r)$. Thus the sum of nonnegative weighted distances $d_{p,\mathcal{F}'}(\mathbf{e}_j, \mathbf{x})$ is concave over each cell. Also, f is concave over each line segment that is a common boundary between two maximal barriers. Thus the set of optimal solutions consists of either (a) cell corners, (b) cell facets, (c) cells, or (d) the end points of line segments that are common boundaries between two maximal barriers. The median problem is solved by evaluating the objective function at each of the cell corners and at the end points of each common barrier boundary, and choosing the minimum values. A similar result was reported by Larson and Sadiq (1983) for l_1 distances and by Hamacher and Klamroth (2000) for block distances and with barriers that are polytopes.

In general, when f is a nondecreasing, convex function of the distances $d_{p,\mathcal{F}'}(\mathbf{e}_j, \mathbf{x})$, the objective function $f(d_{p,\mathcal{F}'}(\mathbf{x}, E))$ may be non-concave and/or non-convex over each cell. In the general case, the cells are subdivided into convex subsets, called *convex domains*, on which $d_{p,\mathcal{F}'}(\mathbf{e}_j, \mathbf{x})$ is convex for all $\mathbf{e}_j, j = 1, \ldots, m$. Then on each convex domain the objective function $f(d_{p,\mathcal{F}'}(\mathbf{x}, E))$ is convex and may be minimized by methods appropriate for the convex function. The convex domains are constructed so that for each $\mathbf{x}$ in the convex domain and any existing facility $\mathbf{e}_j$, there is a vertex $\mathbf{w}_r$ such that $d_{p,\mathcal{F}'}(\mathbf{e}_j, \mathbf{x}) = d_p(\mathbf{w}_r, \mathbf{x}) + d_{p,\mathcal{F}'}(\mathbf{e}_j, \mathbf{w}_r)$.

For each existing facility $\mathbf{e}_j$, and for each pair of cones $\Gamma_{r,k}$ and $\Gamma_{s,l}$ with nonempty intersection, define the bisector, $b(j, r, k, s, l)$, to be the set of points $\mathbf{x}$ in the intersection of the two cones such that the distance between $\mathbf{x}$ and $\mathbf{e}_j$ through $\mathbf{w}_r$ equals the distance between $\mathbf{x}$ and $\mathbf{e}_j$ through $\mathbf{w}_s$. That is,

$$b(j, r, k, s, l) = \left\{ \begin{array}{l} \mathbf{x} \in \Gamma_{r,k} \cap \Gamma_{s,l} : \\ d_{p,\mathcal{F}'}(\mathbf{w}_r, \mathbf{e}_j) + \mathbf{b}_k^0(\mathbf{x} - \mathbf{w}_r) = d_{p,\mathcal{F}'}(\mathbf{w}_s, \mathbf{e}_j) + \mathbf{b}_l^0(\mathbf{x} - \mathbf{w}_s) \end{array} \right\}$$

or alternatively,

$$b(j, r, k, s, l) = \left\{ \begin{array}{l} \mathbf{x} \in \Gamma_{r,k} \cap \Gamma_{s,l} : \\ (\mathbf{b}_k^0 - \mathbf{b}_l^0)\mathbf{x} = \mathbf{b}_k^0\mathbf{w}_r - \mathbf{b}_l^0\mathbf{w}_s + d_{p,\mathcal{F}'}(\mathbf{w}_s, \mathbf{e}_j) - d_{p,\mathcal{F}'}(\mathbf{w}_r, \mathbf{e}_j) \end{array} \right\}$$

If the bisector $b(j, r, k, s, l)$ exists, it is either a line segment or the set of all points in the intersection of the two cones. Necessary and sufficient conditions for a bisector to exist, to be a line segment, or to be the set of all points in the intersection of the two cones are developed next.

The first step is to determine the shortest path distance $d_{p,\Gamma_{r,k} \cup \Gamma_{s,l}}(\mathbf{w}_r, \mathbf{w}_s)$, that is, the distance between $\mathbf{w}_r$ and $\mathbf{w}_s$ when the path is restricted to the union of the cones $\Gamma_{r,k}$ and $\Gamma_{s,l}$. If $\mathbf{w}_r \in \Gamma_{s,l}$, then $d_{p,\Gamma_{r,k} \cup \Gamma_{s,l}}(\mathbf{w}_r, \mathbf{w}_s) =$

$\mathbf{b}_l^0(\mathbf{w}_s - \mathbf{w}_r)$. Likewise, if $\mathbf{w}_s \in \Gamma_{r,k}$, then $d_{p,\Gamma_{r,k}\cup\Gamma_{s,l}}(\mathbf{w}_r, \mathbf{w}_s) = \mathbf{b}_k^0(\mathbf{w}_r - \mathbf{w}_s)$. If $\mathbf{w}_r \notin \Gamma_{s,l}$, and $\mathbf{w}_s \notin \Gamma_{r,k}$, define the point $\mathbf{x}_{k,l+1}$ to be the intersection of the rays $\mathbf{w}_r + \lambda\mathbf{b}_k$ and $\mathbf{w}_s + \lambda\mathbf{b}_{l+1}$ and the point $\mathbf{x}_{l,k+1}$ to be the intersection of the rays $\mathbf{w}_r + \lambda\mathbf{b}_{k+1}$ and $\mathbf{w}_s + \lambda\mathbf{b}_l$. Then $d_{p,\Gamma_{r,k}\cup\Gamma_{s,l}}(\mathbf{w}_r, \mathbf{w}_s) = min\{\mathbf{b}_k^0(\mathbf{x}_{k,l+1} - \mathbf{w}_r) + \mathbf{b}_l^0(\mathbf{x}_{k,l+1} - \mathbf{w}_s), \mathbf{b}_k^0(\mathbf{x}_{l,k+1} - \mathbf{w}_r) + \mathbf{b}_l^0(\mathbf{x}_{l,k+1} - \mathbf{w}_s)\}$.

The following result has been observed in other settings, but is stated here in the context of the barrier problem with block distance.

Lemma 12 *The bisector $b(j, r, k, s, l)$ exists in $\Gamma_{r,k} \cap \Gamma_{s,l}$, if and only if $|d_{p,\mathcal{F}'}(\mathbf{w}_r, \mathbf{e}_j) - d_{p,\mathcal{F}'}(\mathbf{w}_s, \mathbf{e}_j)| < d_{p,\Gamma_{r,k}\cup\Gamma_{s,l}}(\mathbf{w}_r, \mathbf{w}_s)$.*

If $(\mathbf{b}_k^0 - \mathbf{b}_l^0) \neq \mathbf{0}$, the definition shows that $b(j, r, k, s, l)$ is a line. Alternatively, if $(\mathbf{b}_k^0 - \mathbf{b}_l^0) = \mathbf{0}$ and if $\mathbf{b}_k^0\mathbf{w}_r - \mathbf{b}_l^0\mathbf{w}_s + d_{p,\mathcal{F}'}(\mathbf{w}_s, \mathbf{e}_j) - d_{p,\mathcal{F}'}(\mathbf{w}_r, \mathbf{e}_j) = 0$, then $b(j, r, k, s, l)$ is the set of all points in the intersection of the two cones (called a *degenerate bisector* by Widmayer *et al.* (1987)). Some algebraic manipulation shows that $\mathbf{b}_k^0\mathbf{w}_r - \mathbf{b}_l^0\mathbf{w}_s + d_{p,\mathcal{F}'}(\mathbf{w}_s, \mathbf{e}_j) - d_{p,\mathcal{F}'}(\mathbf{w}_r, \mathbf{e}_j) = 0$ if and only if $|d_{\mathcal{F}'}(\mathbf{w}_r, \mathbf{e}_j) - d_{\mathcal{F}'}(\mathbf{w}_r, \mathbf{e}_j)| = d_{p,\Gamma_{r,k}\cup\Gamma_{s,l}}(\mathbf{w}_r, \mathbf{w}_s)$.

Define $h(j, r, k, s, l)$ to be the set of points $\mathbf{x}$ in the intersection of the two cones so that the distance between $\mathbf{x}$ and $\mathbf{e}_j$ through $\mathbf{w}_r$ is less than or equal to the distance between $\mathbf{x}$ and $\mathbf{e}_j$ through $\mathbf{w}_s$. The set $h(j, r, k, s, l)$ is a half plane intersected with the cones $\Gamma_{r,k}$ and $\Gamma_{s,l}$, that is,

$$h(j, r, k, s, l) = \left\{ \begin{array}{l} \mathbf{x} \in \Gamma_{r,k} \cap \Gamma_{s,l} : \\ d_{p,\mathcal{F}'}(\mathbf{w}_r, \mathbf{e}_j) + \mathbf{b}_k^0(\mathbf{x} - \mathbf{w}_r) \leq d_{p,\mathcal{F}'}(\mathbf{w}_s, \mathbf{e}_j) + \mathbf{b}_l^0(\mathbf{x} - \mathbf{w}_s) \end{array} \right\}$$

or alternatively,

$$h(j, r, k, s, l) = \left\{ \begin{array}{l} \mathbf{x} \in \Gamma_{r,k} \cap \Gamma_{s,l} : \\ (\mathbf{b}_k^0 - \mathbf{b}_l^0)\mathbf{x} \leq \mathbf{b}_k^0\mathbf{w}_r - \mathbf{b}_l^0\mathbf{w}_s + d_{p,\mathcal{F}'}(\mathbf{w}_s, \mathbf{e}_j) - d_{p,\mathcal{F}'}(\mathbf{w}_r, \mathbf{e}_j) \end{array} \right\}$$

Notice that the border between the two sets $h(j,r,k,s,l)$ and $h(j,s,l,r,k)$ is $b(j,r,k,s,l)$.

Next let $h(j,r,s)$ be the set of all points $\mathbf{x} \in \mathcal{F}'$ whose distance to $\mathbf{e}_j$ through the vertex $\mathbf{w}_r$ is less than or equal to the distance to $\mathbf{e}_j$ through the vertex $\mathbf{w}_s$, that is,

$$h(j,r,s) = \bigcup_{l,k:\Gamma_{r,k}\cap\Gamma_{s,l}\neq\emptyset} h(j,r,k,s,l).$$

The Voronoi region $V(j,r)$ is the set of all points $\mathbf{x} \in \mathcal{F}'$ whose distance to $\mathbf{e}_j$ through $\mathbf{w}_r$ is less than or equal to the distance to $\mathbf{e}_j$ through any other vertex, and is defined by

$$V(j,r) = \bigcap_{s\neq r} h(j,r,s).$$

The Voronoi regions partition the set $\mathcal{F}'$, and the borders of the Voronoi regions form a Voronoi diagram of $\mathcal{F}'$ with respect to the existing facility $\mathbf{e}_j$. This is an example of a Voronoi diagram with a fixed distance $d_{p,\mathcal{F}'}(\mathbf{w}_r, \mathbf{e}_j)$ added to $d_p(\mathbf{x}, \mathbf{w}_r)$ for each $r \in R$.

Figure 13 shows the Voronoi diagram and the bisectors with respect to the existing facility $\mathbf{e}_3$ and the vertices $\mathbf{w}_1, \mathbf{w}_2$ and $\mathbf{w}_3$. The line segments A, B, C, D, E, F, G correspond to the bisectors as follows: Line $A \subseteq b(3,1,6,3,4)$, $B \subseteq b(3,1,6,3,3)$, $C \subseteq b(3,1,6,2,5)$, $D = b(3,1,7,2,5)$, $E = b(3,1,7,2,4)$, $F \subseteq b(3,2,5,3,3)$, and $G = b(2,2,6,3,3)$,

Mitchell (2000) gives algorithms that are polynomial in $|R|$ and p for constructing Voronoi diagrams and Voronoi regions in the presence of polygonal barriers using a variety of distances. Although block distances are not explicitly discussed, the results extend directly to block distances. A *shortest*

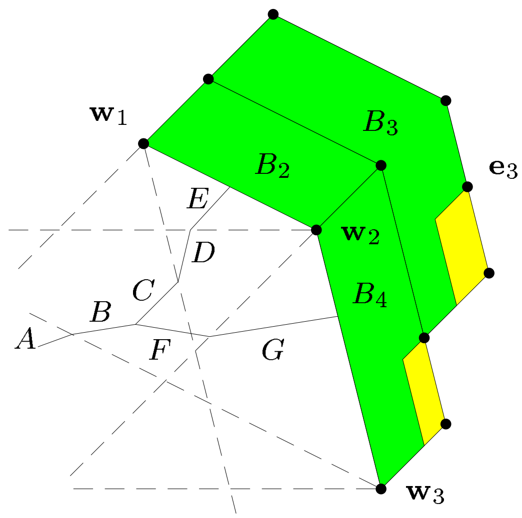

Figure 13: Voronoi diagram with respect to $\mathbf{e}_3$ and vertices $\mathbf{w}_1, \mathbf{w}_2$ and $\mathbf{w}_3$

path map is constructed of vertices $\mathbf{w}_r, r \in R$ that lie on each shortest path rooted at each existing facility $\mathbf{e}_j$. Bisectors exist only for pairs of vertices $\mathbf{w}_r$ and $\mathbf{w}_s$ in the shortest path map for each $\mathbf{e}_j$. Mitchell (2000) gives a wave front algorighm that requires $O(p|R|log^2|R|)$ time for constructing the shortest path map for each $\mathbf{e}_j$.

Convex domains within each cell C are given by the intersection of C with each Voronoi region $V(j,r)$ for $(r,k) \in I(C)$ and for each $\mathbf{e}_j, j = 1, \ldots, m$. Thus each convex domain CD has the polyhedral representation given by

$$CD = \{\mathbf{x} \in C \cap V(j,r) : (r,k) \in I(C), j = 1, \ldots, m\}.$$

The convex domains may be constructed as an arrangement given the set of line segments which are bisectors in the Voronoi Diagram and the rays from all vertices in the grid system. The number of rays and bisector segments is polynomial of order $O(|R|^2 p^2 m^2)$.

Figure 14 shows the bisectors of the example problem as connected line segments, labeled by $b(\mathbf{e}_j)$ for $j = 1, 2, 3, 5$. The convex domains are the regions formed by the rays, barrier boundaries and the bisectors. The results of this section show that solving a planar location problem with block distance and barriers requires at most a polynomial amount of additional time over solving the same problem without barriers.

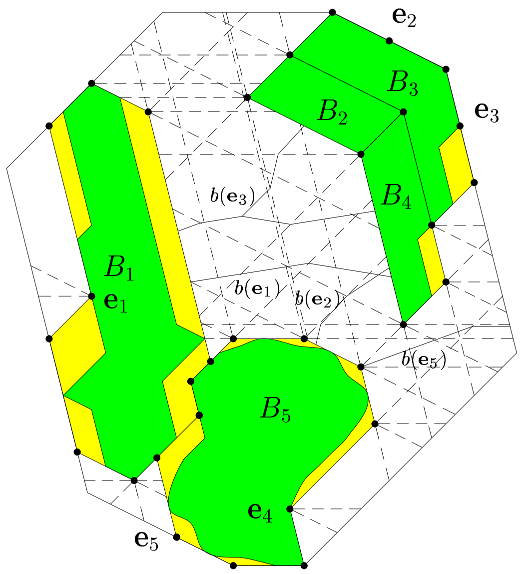

Figure 14: The bisectors, denoted by $b(\mathbf{e}_j)$ and the resulting convex domains

7 Example Problems

The median and center problems are solved to illustrate the use of the grid structure and the convex domains.

First consider the median problem where the objective function is given by $\sum_{j=1}^{n} d_{p,\mathcal{F}'}(\mathbf{x}, \mathbf{e}_j)$. The problem is solved by evaluating the objective function at each intersection point of two grid lines or of a grid line and a barrier boundary, and at the end points of common barrier boundaries. The minimum of these values yields the optimal point. The optimal point for the example problem is denoted by $\mathbf{x}^+$ in Figure 15 and has an objective function value of 29.7.

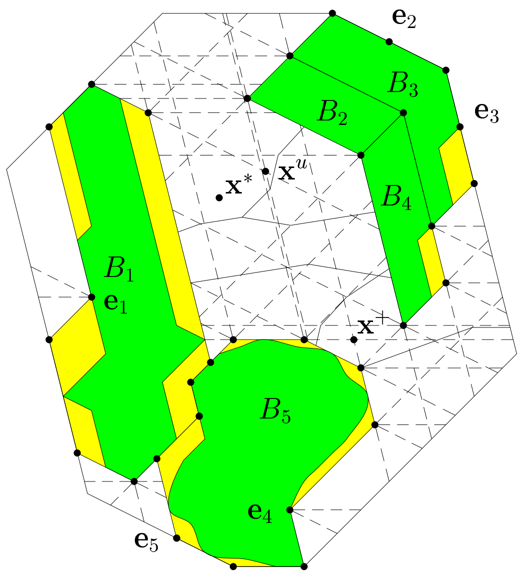

Figure 15: Optimal solution $\mathbf{x}^+$ to the median problem, optimal solution $\mathbf{x}^*$ and upper bound $\mathbf{x}^u$ to the center problem

Next consider the center problem where the objective function is given by $max_{j=1,\ldots,m} d_{p,\mathcal{F}'}(\mathbf{x}, \mathbf{e}_j)$. An upper bound is determined by evaluating the objective function at all intersection points of grid lines, bisectors and barrier boundaries. This yields the point denoted by $\mathbf{x}^u$ in Figure 15 with an objective function value of 7.31.

Lower bounds are next computed for each convex domain. For a convex domain CD, let $C(CD)$ denote the index set of corner points and let $\mathbf{y}_{l,CD}, l \in C(CD)$ denote the corner points of CD. Each $\mathbf{y}_{l,CD}$ is an intersection point of grid lines, bisectors or barrier boundaries. A lower bound is given by

$$lb(CD) = max_{j=1,\ldots,m}\{min_{l\in C(CD)} d_{p,\mathcal{F}'}(\mathbf{y}_{l,CD}, \mathbf{e}_j)\}.$$

A similar lower bound is computed for each common barrier boundary. The lower bound was computed for each convex domain and each common barrier boundary. There were six convex domains whose lower bound was less than the upper bound of 7.31. The convex domain with the smallest lower bound, 6.47, was chosen and the center problem solved over that convex domain, yielding the unique point $\mathbf{x}^*$ in Figure 15, with an objective function value of 7.0. All other convex domains had lower bounds greater than 7.0, so they were eliminated from consideration. Thus $\mathbf{x}^*$ is the unique optimal solution.

8 Extensions

The results of this paper may be extended in two ways. A relatively straight forward extension is to relax the requirement that barriers be d_p visible. For the case of l_1 distances, this relaxation was reported in Dearing and Segars Jr. (2002a). With this relaxation, barrier sets must be partitioned into subsets that are d_p visible, which become the barriers. Figure 16 shows a set that is not d_p visible and a partition into d_p visible subsets B_1, B_2, B_3 that are barriers. (The partition is not unique.) The maximal modification of the barriers is also shown. The boundary between subsets (dashed lines) is excluded from the feasible set.

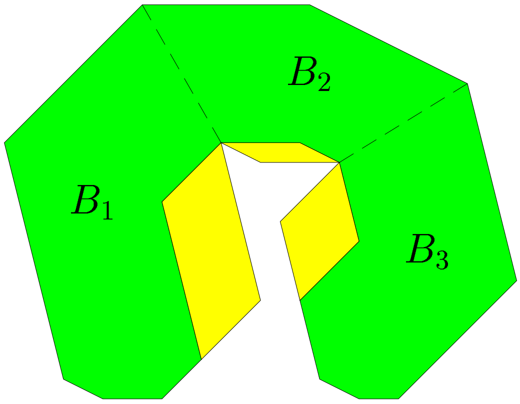

Figure 16: A barrier set that is not d_p visible and a partition into d_p visible subsets B_1, B_2, B_3

The results reported here may also be extended by relaxing the assumption of block distances to *polyhedral gauges*, that is, distances whose unit ball is a polytope that is not necessarily symmetrical. The fundamental directions are the extreme points of the polytope and are not necessarily symmetric. The extension to polyhedral gauges will be reported in a future paper.

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