Universität Erlangen-Nürnberg
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Linear and Network Optimization
Exercise 4

Please return your solutions by Tuesday, May 6^{th}, 10:00 a.m., in the mailbox No. 5.

Problem 1 (3 points)
The following simplex tableau represents a linear program in standard form:

$$T(B) = \begin{array}{|c|cccccc|c|}\hline 1 & 0 & b & e & 0 & 0 & -9 \\ \hline 0 & 1 & c & 1 & 0 & 0 & a \\ 0 & 0 & d & -1 & 1 & 0 & 2 \\ 0 & 0 & -1 & 1 & 0 & 1 & 4 \\ \hline \end{array}$$

Give conditions (for example $b \geq 0$) on the parameters a, b, c, d, and e such that

(a) the tableau is in optimal form,

(b) the tableau is in unbounded form,

(c) the tableau is in infeasible form.

Problem 2 (6 points)
Consider the following linear program:

$$\begin{array}{lrcrcl} \max & 2x_1 & + & x_2 & & \\ \text{s.t.} & 2x_1 & + & x_2 & \leq & 4 \\ & 2x_1 & + & 3x_2 & \leq & 3 \\ & 4x_1 & + & x_2 & \leq & 5 \\ & x_1 & + & 5x_2 & \leq & 1 \\ & & & x_j & \geq & 0, \quad j = 1, 2. \end{array}$$

(a) Transform the given LP into standard form by introducing the slack variables $x_3, \ldots, x_6$.

(b) Determine the simplex tableau $T(B)$ with respect to the basis $B = \{2, 3, 4, 5\}$.

(c) Use the basis $B = \{2, 3, 4, 5\}$ as input of the simplex method and find an optimal solution of the given LP.

Problem 3 (5 points)

(a) Give an example showing that the variable that becomes basic in one iteration of the simplex method can become nonbasic in the next iteration.

(b) Show that the variable that becomes nonbasic in one iteration of the simplex method cannot become basic in the next iteration.

Problem 4 (6 points)
Prove Theorem 3 on the back of this sheet.

Definition 1 A set $X \subseteq \mathbb{R}^n$ is *convex* if the linear segment joining any two points in the set lies entirely in the set, i.e.

$$\forall \underline{x}, \underline{y} \in X: \qquad \lambda\underline{x} + (1-\lambda)\underline{y} \in X \qquad \forall\, \lambda \in [0,1].$$

Definition 2 An *extreme point* (or a *vertex*) of a convex set $X \subseteq \mathbb{R}^n$ is a point $\underline{x} \in X$ such that $\underline{x}$ cannot be represented in the form

$$\underline{x} = \lambda\underline{x}_1 + (1-\lambda)\underline{x}_2,\ 0 < \lambda < 1$$

with $\underline{x}_1$, $\underline{x}_2 \in X, \underline{x}_1 \neq \underline{x}_2$.

Theorem 1 The feasible set $P := \{\underline{x} \in \mathbb{R}^n : A\underline{x} = b, \underline{x} \geq \underline{0}\}$ is convex.

Theorem 2 Any feasible region of an LP can be alternatively viewed as a convex polytope and vice versa.

Definition 3 Let $P \subseteq \mathbb{R}$ be a polytope. The segment $[\hat{\underline{x}}, \hat{\underline{y}}]$ is an *edge* of P if the following is true: For every $\hat{\underline{z}} \in [\hat{\underline{x}}, \hat{\underline{y}}]$: If $\hat{\underline{z}} = \lambda\hat{\underline{z}}' + (1-\lambda)\hat{\underline{z}}''$ with $0 < \lambda < 1$ and $\hat{\underline{z}}', \hat{\underline{z}}'' \in P$, then $\hat{\underline{z}}', \hat{\underline{z}}'' \in [\hat{\underline{x}}, \hat{\underline{y}}]$.

Definition 4 Two vertices $\hat{\underline{x}}$ and $\hat{\underline{y}}$ of a polytope $P \subseteq \mathbb{R}^n$ are called *adjacent* if the line segment $[\hat{\underline{x}}, \hat{\underline{y}}]$ (i.e. the set $L :=\{\underline{x} \in \mathbb{R}^n : \underline{x} = \lambda\hat{\underline{x}} + (1-\lambda)\hat{\underline{y}}, \lambda \in [0,1]\}$) is an edge of the polytope.

Theorem 3 In a non-degenerate pivot operation the simplex algorithm moves from one extreme point of the feasible set to an adjacent extreme point.