Universität Erlangen-Nürnberg
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Linear and Network Optimization
Exercise 3

Please return your solutions by Tuesday, April 29th, 10:00 a.m., in the mailbox No. 5.

Problem 1 (6 points)
Consider the following constraints:

$$\begin{array}{rcrcl} x_1 & + & x_2 & \leq & 3 \\ -2x_1 & + & x_2 & \leq & 2 \\ x_1 & - & 2x_2 & \leq & 0 \\ & & x_1, x_2 & \geq & 0. \end{array}$$

(a) Draw the feasible region.

(b) Identify the extreme points in the x_1, x_2-space, and at each extreme point identify all possible basic and nonbasic variables.

(c) Suppose that a move is made from the extreme point $(2,1)^T$ to the extreme point $(0,0)^T$ in the x_1, x_2-space. Specify the possible entering and leaving variables.

Problem 2 (3 points)
Does the objective function value get stricly better in each basis exchange? (Proof or counterexample.)

Problem 3 (5 points)
Consider the polyhedral set consisting of all points $\underline{x} \in \mathbb{R}^2$ such that

$$x_1 + x_2 \leq 1.$$

(No nonnegativity constraints given!) Verify geometrically and algebraically that this set (in the x_1, x_2-space) has no extreme points. Formulate an equivalent set in a higher dimension where all variables are restricted to be nonnegative. Show that extreme points of the new set indeed exist.

Problem 4 (8 points)
Let $P = \{\underline{x} \in \mathbb{R}^n : A\underline{x} = \underline{b},\ \underline{x} \geq \underline{0}\}$ where A is an $m \times n$-matrix of rank m.
Let $\underline{x}$ be a feasible solution such that $x_1, \ldots, x_q$ are positive and $x_{q+1}, \ldots, x_n$ are zero. Assuming that the columns $A_1, \ldots, A_q$ of A corresponding to the positive variables are linearly dependent, construct feasible points $\underline{x}'$ and $\underline{x}''$ such that $\underline{x}$ is a convex combination of these points. Hence argue that if $\underline{x}$ is an extreme point of P, it is also a basic feasible solution.
Conversely, suppose that $\underline{x}$ is a basic feasible solution of P with basis B and that $\underline{x} = \lambda\underline{x}' + (1-\lambda)\underline{x}''$ for some $0 < \lambda < 1$ and $\underline{x}', \underline{x}'' \in P$. Denoting $\underline{x}_B$ and $\underline{x}_N$ as the corresponding basic and nonbasic variables, show that $\underline{x}'_N = \underline{x}''_N = \underline{0}$ and $\underline{x}'_B = \underline{x}''_B = A_B^{-1}\underline{b}$. Hence argue that $\underline{x}$ is an extreme point of P.
(Hint: Use Definition 2 on the back of this sheet.)

Definition 1 A set $X \subseteq \mathbb{R}^n$ is *convex* if the linear segment joining any two points in the set lies entirely in the set, i.e.

$$\forall \underline{x}, \underline{y} \in X : \qquad \lambda \underline{x} + (1-\lambda)\underline{y} \in X \qquad \forall \lambda \in [0,1].$$

Definition 2 (Alternative definition of extreme points).
An *extreme point* (or a *vertex*) of a convex set $X \subseteq \mathbb{R}^n$ is a point $\underline{x} \in X$ such that $\underline{x}$ cannot be represented in the form

$$\underline{x} = \lambda \underline{x}_1 + (1-\lambda)\underline{x}_2, \ 0 < \lambda < 1$$

with $\underline{x}_1$, $\underline{x}_2 \in X$, $\underline{x}_1 \neq \underline{x}_2$.

Theorem 1 The feasible set $P := \{\underline{x} \in \mathbb{R}^n : A\underline{x} = \underline{b}, \underline{x} \geq \underline{0}\}$ is convex.