Universität Erlangen-Nürnberg
Naturwissenschaftliche Fakultät I
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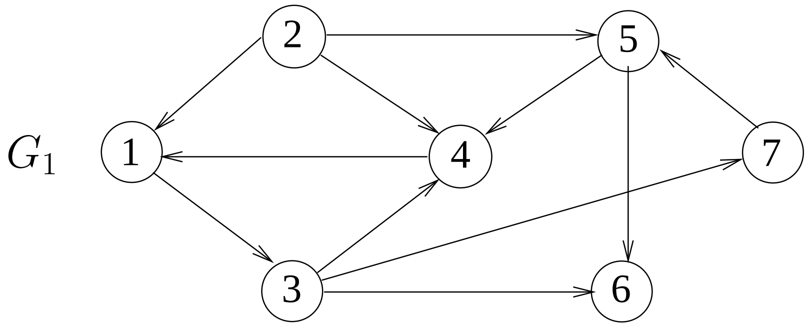
Linear and Network Optimization
Exercise 10

Please return your solutions by Tuesday, June 17th, 10:00 a.m., in the mailbox No. 5.

Problem 1 (4 points)
Consider the digraph G_1:



(a) Determine the incidence matrix of G_1.

(b) Identify two spanning trees T_1 and T_2 of G_1.

(c) How many dicycles can you find in G_1?

(d) Find an s-t-cut with $s = 2$ and $t = 6$.

(e) Let G_2 be the undirected graph obtained from G_1 by making all edges undirected. Determine the adjacency matrix of G_2.

Problem 2 (8 points)
Prove the following:
Theorem 5.8. Let $T = (V, E(T))$ be a spanning tree of $G = (V, E)$. Then:

(1) T has at least 2 leafs.

(2) T has exactly $n - 1$ edges.

(3) Every pair of vertices in T is connected by exactly 1 path.

(4) Let $e \in E \setminus E(T)$. Then $T + e := (V, E(T) \cup \{e\})$ contains exactly 1 cycle.

(5) Let $e \in E \setminus E(T)$ and let C be the uniquely defined cycle in $T + e$. Then $T + e \setminus f := (V, E(T) \cup \{e\} \setminus \{f\})$ is a spanning tree $\forall f \in C$.

(6) Let $T \setminus e := (V, E(T) \setminus \{e\})$. Then $T \setminus e$ is cut into 2 subtrees $(X, E(X))$ and $(\bar{X}, E(\bar{X}))$. Furthermore, $Q := (X, \bar{X}) := \{e = [i, j] \in E \,:\, i \in X,\, j \in \bar{X}\}$ is a cut in G.

(7) Let $e \in E(T)$ and let $Q := (X, \bar{X})$ be the cut in G that is uniquely defined by $T \setminus e$ (c.f. (6)). Then $T \setminus e + f := (V, E(T) \setminus \{e\} \cup \{f\})$ is a spanning tree $\forall f \in (X, \bar{X})$.

Problem 3: Knapsack Problem (5 points)
Consider n objects, each of which has a weight a_j and a value c_j ($a_j, c_j \in \mathbb{N}$). We would like to find a subset of the set of all objects for which the sum of the values is maximal while the sum of the weights does not exceed a given capacity $b \in \mathbb{N}$.
Find a model where this problem is equivalent to finding a longest path in a suitable network. (Hint: Introduce a source node s and a sink node t and, for each object, $b+1$ nodes. Then define an acyclic digraph by adding suitable arcs.)
Under what circumstances is it possible to convert a longest path problem on a network into an equivalent shortest path problem?

Problem 4 (3 points)
Prove the following result:
A dipath P in a digraph G from the source node s to a node $k \neq s$ is a shortest dipath in G if and only if

$$d_j = d_i + c_{ij} \quad \forall (i,j) \in P,$$

where d_i denotes the distance from s to i, $i \in V(P)$.