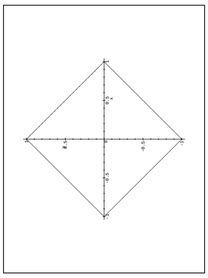
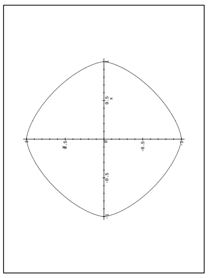
Location Analysis, Handout 1:
Single Facility Weber Problems with l_p Distances, $1/P/\bullet/l_p/\sum$

l_p Distances:

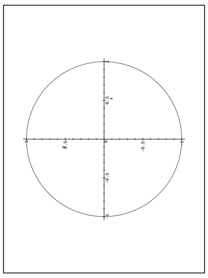
$$
\begin{aligned}
l_p(X,Y) &= (|x_1-y_1|^p + |x_2-y_2|^p)^{\frac{1}{p}}, \qquad X,Y \in \mathbb{R}^2, \quad ,1 \le p < \infty \\
l_\infty(X,Y) &= \max\{|x_1-y_1|, |x_2-y_2|\}, \qquad X,Y \in \mathbb{R}^2
\end{aligned}
$$



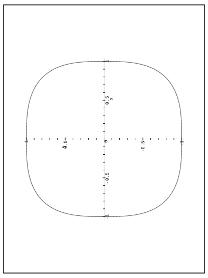
$l_1(X,\underline{0}) = 1$



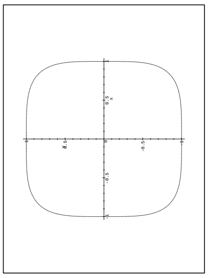
$l_{1.5}(X,\underline{0}) = 1$



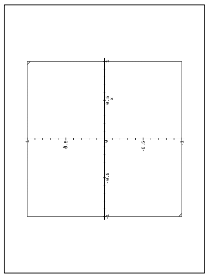
$l_2(X,\underline{0}) = 1$



$l_3(X,\underline{0}) = 1$



$l_4(X,\underline{0}) = 1$



$l_\infty(X,\underline{0}) = 1$

Property 1: $l_p(X,Y)$ decreases as p increases.

Porperty 2: As $p \to \infty$, $l_p(X,Y)$ becomes the larger of $|x_1-y_1|$ and $|x_2-y_2|$.

Weiszfeld Algorithm for $1/P/\bullet/l_2/\sum$

Input: Existing facilities $a_1,\dots,a_n \in \mathbb{R}^2$; weights $w_1,\dots,w_n > 0$.

Step 1: If $CR_r \le w_r$ for $r \in \{1,\dots,n\}$, set $X^* := a_r$, STOP.

Step 2: Select a starting solution $X^{(0)} = (x_1^{(0)}, x_2^{(0)})$; set $l := 0$.
($X^{(0)}$ could be chosen, for example, as the optimal solution of $1/P/\bullet/l_2^2/\sum$.)

Step 3: For $k = 1, 2$ do

$$x_k^{(l+1)} := \frac{\sum_{j=1}^n \frac{w_j a_{jk}}{l_2(X^{(l)},a_j)}}{\sum_{j=1}^n \frac{w_j}{l_2(X^{(l)},a_j)}}$$

Step 4: If $X^{(l+1)}$ satisfies a stopping criterion, set $X^* := X^{(l+1)}$, STOP.
Otherwise, set $l := l+1$ and goto Step 3.

Output: Approximation X^* of an optimal solution for $1/P/\bullet/l_2/\sum$.

Hyperbolic Approximation Algorithm for $1/P/\bullet/l_p/\sum$, $1 < p < \infty$

Input: Existing facilities $a_1,\dots,a_n \in \mathbb{R}^2$; weights $w_1,\dots,w_n > 0$; $p \in \mathbb{R}$ with $1 < p < \infty$.

Step 1: If $CRP_r \le w_r$ for $r \in \{1,\dots,n\}$, set $X^* := a_r$, STOP.

Step 2: Select a starting solution $X^{(0)} = (x_1^{(0)}, x_2^{(0)})$; set $l := 0$.

Step 3: For $k = 1, 2$ do

$$x_k^{(l+1)} := \frac{\sum_{j=1}^n \frac{w_j a_{jk}}{d'(X^{(l)},a_j)\cdot d''(x_k^{(l)},a_{jk})}}{\sum_{j=1}^n \frac{w_j}{d'(X^{(l)},a_j)\cdot d''(x_k^{(l)},a_{jk})}}$$

Step 4: If $X^{(l+1)}$ satisfies a stopping criterion, set $X^* := X^{(l+1)}$, STOP.
Otherwise, set $l := l+1$ and goto Step 3.

Output: Approximation X^* of an optimal solution for $1/P/\bullet/l_p/\sum$.

$$CR_r = \left[\left(\sum_{\substack{j=1\\j\neq r}}^n \frac{w_j(a_{r1}-a_{j1})}{l_2(a_r,a_j)}\right)^2 + \left(\sum_{\substack{j=1\\j\neq r}}^n \frac{w_j(a_{r2}-a_{j2})}{l_2(a_r,a_j)}\right)^2\right]^{\frac{1}{2}}$$

$$CRP_r = \left[\left|\sum_{\substack{j=1\\j\neq r}}^n \frac{w_j \operatorname{sign}(a_{r1}-a_{j1})|a_{r1}-a_{j1}|^{p-1}}{(l_p(a_r,a_j))^{p-1}}\right|^{\frac{p}{p-1}} + \left|\sum_{\substack{j=1\\j\neq r}}^n \frac{w_j \operatorname{sign}(a_{r2}-a_{j2})|a_{r2}-a_{j2}|^{p-1}}{(l_p(a_r,a_j))^{p-1}}\right|^{\frac{p}{p-1}}\right]^{\frac{p-1}{p}}$$

$$d'(X,a_j) = \left(((x_1-a_{j1})^2+\epsilon)^{\frac{p}{2}} + ((x_2-a_{j2})^2+\epsilon)^{\frac{p}{2}}\right)^{1-\frac{1}{p}}$$

$$d''(x_k,a_{jk}) = \left((x_k-a_{jk})^2+\epsilon\right)^{1-\frac{p}{2}}, \qquad k = 1, 2.$$