Exact Solution Approaches for Location-Allocation Problems with Uncertain Environment

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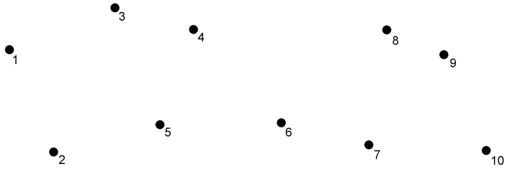
Outline

Problem description

Continuous location-allocation problem



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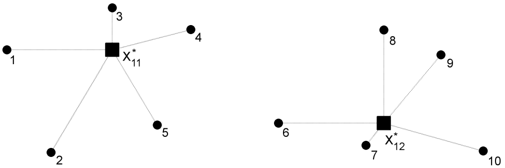

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weights: $w^1 = (1, \ldots, 1)$,

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- various forbidden regions,
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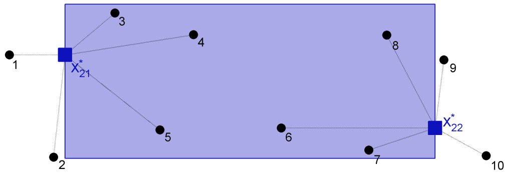

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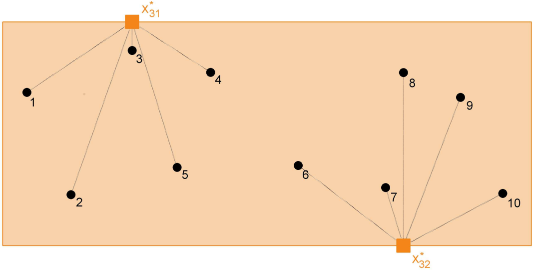

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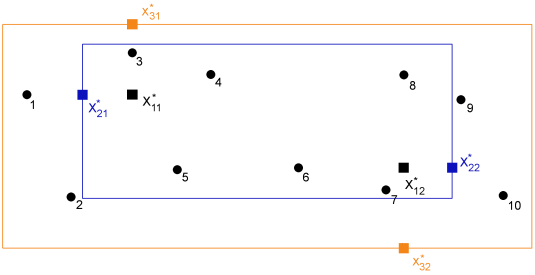

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- locate M new facilities in the plane $\mathbb{R}^2$ such that:
- under each scenario s, at least one facility remains available, i.e. it is not inside a forbidden region
- and the (expected) total weighted transportation cost is minimized.

Assumptions

Assumptions on the forbidden regions $R_{si},\ s \in \mathcal{S},\ i \in \mathcal{I}_s$:

- convex and polyhedral,
- possibility of more than one forbidden region in each scenario (i.e. $|\mathcal{I}_s| \in \mathbb{N}$),
- not necessarily disjoint.

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Assumption on the representation of the future uncertainty:

- discrete set of scenarios with $|\mathcal{S}| < \infty$,
- probabilities given by p_s $(0 \le p_s \le 1, \sum_{s \in \mathcal{S}} p_s = 1)$.

Two-stage stochastic location-allocation model (SLAP)

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\min \sum_{n=1}^{N} \sum_{m=1}^{M} & z_{nm} w_n d(x_m, a_n) + \mathbb{E}(\zeta^s) & \\
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and

$$y_m^s = \begin{cases} 1 & \text{if } x_m \in \mathcal{R}^s \\ 0 & \text{otherwise} \end{cases} \quad \forall m \in \mathcal{M},\ \forall s \in \mathcal{S}$$

Convexity features

- "Standard" location-allocation problem:

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- For fixed z_{nm} $\forall n \in \mathcal{N}$, $\forall m \in \mathcal{M}$ and $x_m \in \mathbb{R}^2$ $\forall m \in \mathcal{M} \setminus \{\bar{m}\}$ the objective function is convex in $x_{\bar{m}}$.

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- Can an analog property be found for (SLAP)?

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- Location-allocation problem with demands w_n:

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Convexity Features

- Location-allocation problem with future scenarios (without forbidden regions):

$$\min \sum_{n=1}^{N}\sum_{m=1}^{M} z_{nm} w_n d(x_m, a_n) + \sum_{s\in\mathcal{S}} p^s \sum_{n=1}^{N}\sum_{m=1}^{M} z_{nm}^s w_n^s d(x_m, a_n)$$

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- (SLAP):

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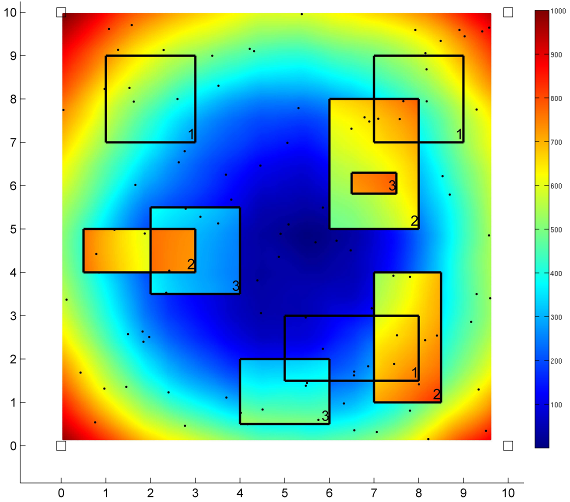
Examples



- $|\mathcal{N}| = 100$, $|\mathcal{M}| = 5$ and $|\mathcal{S}| = 3$
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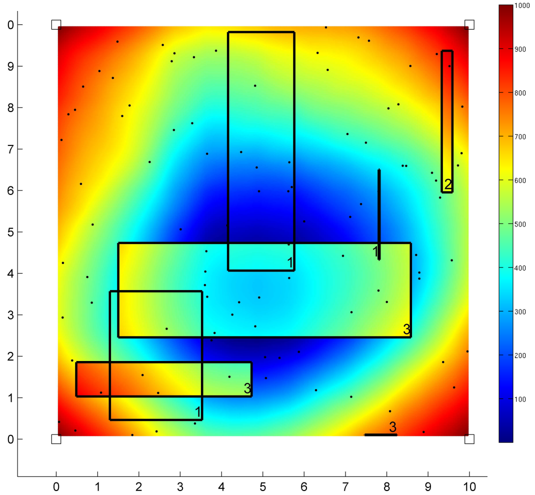

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FDS for location-allocation problems with polyhedral gauges

- A finite dominating set for the location-allocation problem is

$$\mathcal{P}_G := A \cup \left(\bigcup_{\substack{n_1, n_2 \in \mathcal{N} \\ n_1 \neq n_2}} \bigcup_{j_i \in \mathcal{J}_{n_i}} \bigcap_{i \in \{1,2\}} (a_{n_i} + r_{n_i j_i}) \right).$$

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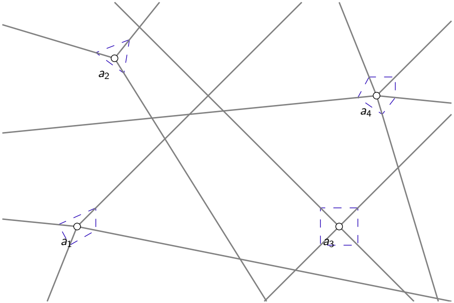
- Construction grid G for mixed polyhedral gauges:



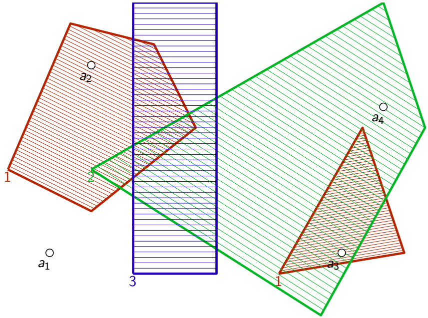
Extensions for (SLAP)

- Additional grid given by a partition

$$D_{\bar{\mathcal{S}}} := \bigcap_{s \in \bar{\mathcal{S}}} \Big(\bigcup_{q \in \mathcal{Q}^s} R_q^s \Big) \quad \text{with} \quad \bar{\mathcal{S}} \subseteq \mathcal{S},\ \bar{\mathcal{S}} = \emptyset$$

$$\text{and} \quad D_\emptyset := \mathbb{R}^2 \setminus R^{\max} = \mathbb{R}^2 \setminus \bigcup_{\substack{\bar{\mathcal{S}} \subseteq \mathcal{S} \\ |\bar{\mathcal{S}}| \geq 1}} D_{\bar{\mathcal{S}}}$$

of $\mathbb{R}^2$ due to the forbidden regions:



Finite dominating set for (SLAP)

Theorem

$\mathcal{P} := \mathcal{P}_G \cup \mathcal{P}_D \cup \mathcal{P}_{GD}$ with

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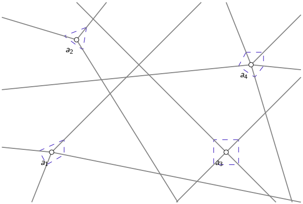
$$\mathcal{P}_D := \bigcup_{s_1, s_2 \in \mathcal{S}} \bigcup_{\substack{q_1 \in \mathcal{Q}^{s_1} \\ q_2 \in \mathcal{Q}^{s_2} \\ (s_1, q_1) \neq (s_2, q_2)}} \left(\partial R_{q_1}^{s_1} \cup \partial R_{q_2}^{s_2} \right) \quad \textit{and}$$

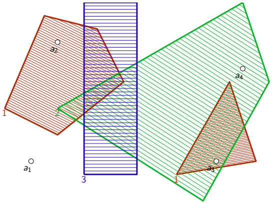
$$\mathcal{P}_{GD} := \Big(\bigcup_{n \in \mathcal{N}} \bigcup_{j \in \mathcal{J}_n} (a_n + r_{nj}) \Big) \cap \Big(\bigcup_{s \in \mathcal{S}} \bigcup_{q \in \mathcal{Q}^s} \partial R_q^s \Big)$$

is a FDS for (SLAP).

Construction grid for (SLAP)



Construction grid for mixed polyhedral gauges



Partition of $\mathbb{R}^2$ by forbidden regions of different scenarios

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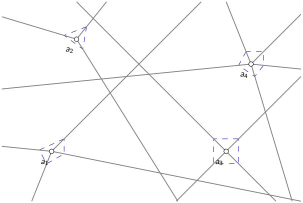

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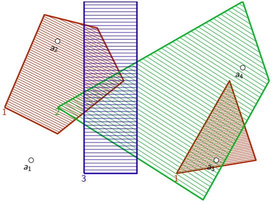

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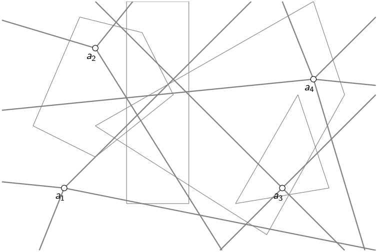

Construction grid for (SLAP)

Step 1: Global upper bound by location-allocation heuristic

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Alternating method to iteratively improve the location variables x_m of the new facilities and the allocations z_{nm} and z^s_{nm} of the customers.

Input

Iteration loop

Stopping condition

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Sequentially solve the 1-median problems defined by the allocation variables z_{nm} and update the location and the allocation variables.

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Stop after M successively considered not improving iterations.

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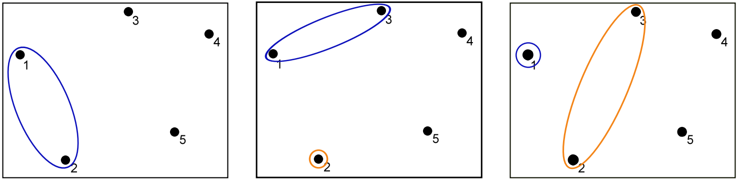
Initializing the B&B algorithm

Fixing z_{nm} until one cluster has more than one element.



Bounding

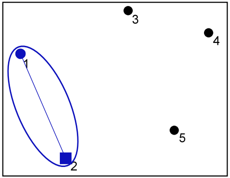
Branching

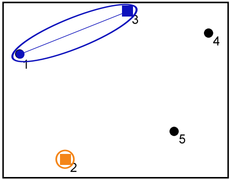
Step 2: Branch & Bound on the allocation variables

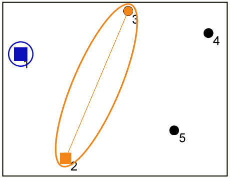
Initializing the B&B algorithm

Bounding

Determine the minimal contribution of the clusters to the objective.







Branching

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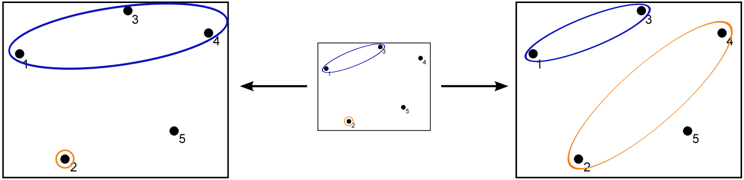
Initializing the B&B algorithm

Bounding

Branching

Fixing one additional allocation variable.



Step 3: Determine the optimal location and second stage allocation variables

For not rejected leaf nodes of the B&B in Step 2

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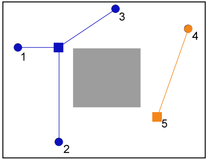

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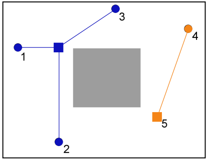
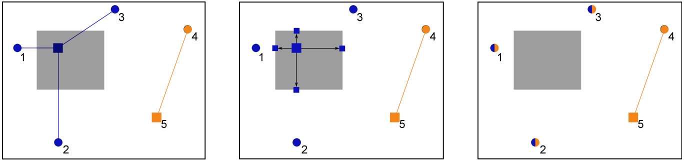


- or the optimal z_{nm}^{s}, x_m, $\forall n \in \mathcal{N}$, $\forall m \in \mathcal{M}$, $\forall s \in \mathcal{S}$ can be found by relocating or reallocation.



Some numerical results

- Computation time for M new facilities, N demand nodes and S scenarios:

$M = 2$:

$N \setminus S$	2	3	5
10	0.4631	3.6366	14.4513
15	1.7145	6.6252	32.4374
25	8.7551	21.0843	74.4513

$M = 3$:

$N \setminus S$	2	3	5
10	0.9904	7.7156	34.3102
15	3.2793	19.1779	71.8686
25	21.5336	54.2462	213.5866

$M = 5$:

$N \setminus S$	2	3	5
10	2.6974	14.1581	87.3337
15	19.3129	98.5155	283.6780
25	132.7755	415.1883	1759.1373

Matlab 7.9, Dual-Core 2.80 GHz CPU and 4 GB memory

Some numerical results

- relative improvement $\frac{f^*_{heur} - f^*_{B\&B}}{f^*_{B\&B}}$ for M new facilities, N demand nodes and S scenarios:

$M = 2$:

$N \setminus S$	2	3	5
10	0.2522	0.2126	0.2040
15	0.1640	0.1847	0.2344
25	0.1916	0.2320	0.1520

$M = 3$:

$N \setminus S$	2	3	5
10	0.4038	0.2862	0.2560
15	0.1834	0.2511	0.2248
25	0.2712	0.2492	0.3157

$M = 5$:

$N \setminus S$	2	3	5
10	0.2081	0.2927	0.2237
15	0.2509	0.1988	0.3075
25	0.2719	0.2682	0.3164

Matlab 7.9, Dual-Core 2.80 GHz CPU and 4 GB memory

Thank you for your attention!

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