Efficient points for Weber problems with asymmetric mixed distances

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Outline

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Historical roots

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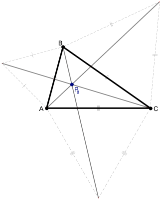
- 

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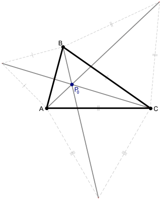
- 

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- Battista Cavalieri (1598-1647): "Exerciones Geometricae"

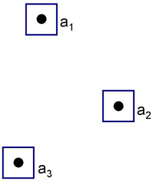
Weber problem with asymmetric mixed distances

Mathematical model of the classical Weber problem :

$$\min_{x \in \mathbb{R}^2} \sum_{n=1}^{N} d(x, a_n),$$

with

- existing facilities $a_n \in A \subset \mathbb{R}^2$, $n \in \mathcal{N} := \{1, \ldots, N\}$
- uniform weights and
- a uniform symmetric distance measure $d(\,\cdot\,,\,\cdot\,) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ induced by a norm



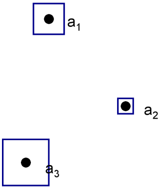
Weber problem with asymmetric mixed distances

Mathematical model of the classical Weber problem :

$$\min_{x \in \mathbb{R}^2} \sum_{n=1}^{N} w_n d(x, a_n),$$

with

- existing facilities $a_n \in A \subset \mathbb{R}^2$, $n \in \mathcal{N} := \{1, \ldots, N\}$
- positive weights $w_n > 0, \forall n \in \mathcal{N}$
- a uniform symmetric distance measure $d(\cdot\,,\,\cdot) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ induced by a norm



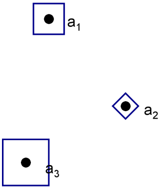
Weber problem with asymmetric mixed distances

Mathematical model of the Weber problem with mixed distances:

$$\min_{x \in \mathbb{R}^2} \sum_{n=1}^{N} w_n d_n(x, a_n),$$

with

- existing facilities $a_n \in A \subset \mathbb{R}^2$, $n \in \mathcal{N} := \{1, \ldots, N\}$
- positive weights $w_n > 0, \forall n \in \mathcal{N}$
- symmetric distance measures $d_n(\cdot, \cdot) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ induced by not necessarily uniform norms



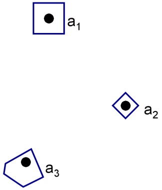
Weber problem with asymmetric mixed distances

Mathematical model of the Weber problem with mixed asymmetric distances:

$$\min_{x \in \mathbb{R}^2} \sum_{n=1}^{N} w_n \gamma_n(x, a_n),$$

with

- existing facilities $a_n \in A \subset \mathbb{R}^2$, $n \in \mathcal{N} := \{1, \ldots, N\}$
- positive weights $w_n > 0, \forall n \in \mathcal{N}$
- asymmetric distance measures induced by (polyhedral) gauges $\gamma_n(x, a) := \inf\{\lambda > 0 : \frac{x}{\lambda} \in B_n + a\}$, with
  - B_n convex, closed and bounded
  - $0 \in \text{int}(B_n)$
  - $\gamma_n(x, y) \geq 0$
  - $\gamma_n(x, y) = 0 \;\Leftrightarrow\; x = y$
  - $\gamma_n(x, y) \leq \gamma_n(x, z) + \gamma_n(z, y)$



Majority principle for the classical Weber problem

Theorem (Witzgall '64)

For existing facilities $a_n \in \mathbb{R}^2$, weights $w_n > 0$ for $n \in \mathcal{N}$ and the classical Weber problem (with uniform symmetric distance measures)

$$\min_{x \in \mathbb{R}^2} \sum_{n=1}^{N} w_n d(x, a_n)$$

an optimal solution can be obtained in the existing facility a_k if

$$w_k \geq \sum_{\substack{j \in \mathcal{N} \\ j \neq k}} w_j .$$

Majority principle for (WAM)

Theorem

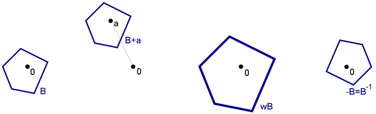
An existing facility a_k, $k \in \mathcal{N}$ is an optimal solution for (WAM) if

a) *all gauges γ_n are symmetrical and $w_k B_k \supseteq \sum_{n \neq k} w_n B_n$ or*

b) $w_k B_k \supseteq \sum_{n \neq k} -w_n B_n =: \sum_{n \neq k} w_n B_n^{-1}$.

Notations and conventions, with $x, a \in \mathbb{R}^2$, $w \in \mathbb{R}^+$ and $B \subset \mathbb{R}^2$:

- $x \in B \Leftrightarrow x + a \in (B + a)$
- $x \in B \Leftrightarrow wx \in wB$
- $x \in B \Leftrightarrow -x \in B^{-1}$



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Proof (sketch).

a) ▶ $w_k B_k \supseteq \sum_{n \neq k} w_n B_n \;\Rightarrow\; w_k \gamma_k(x, a) \geq \sum_{n \neq k} w_n \gamma_n(x, a) \quad (1)$

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▶
$$\sum_{n \in \mathcal{N}} w_n \gamma_n(x, a_n) - \sum_{n \in \mathcal{N}} w_n \gamma_n(a_k, a_n)$$
$$= \sum_{n \neq k} w_n \left(\gamma_n(x, a_n) - \gamma_n(a_k, a_n)\right) + w_k \gamma_k(x, a_k)$$
$$\overset{(1)}{\geq} \sum_{n \neq k} w_n \left(\gamma_n(x, a_n) + \gamma_n(x, a_k) - \gamma_n(a_k, a_n)\right)$$
$$= \sum_{n \neq k} w_n \left(\gamma_n(x, a_n) + \gamma_n(a_k, x) - \gamma_n(a_k, a_n)\right) \geq 0$$

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b) ▶ $w_k B_k \supseteq \sum_{n \neq k} w_n B_n^{-1} \;\Rightarrow\; w_k \gamma_k(x, a) \geq \sum_{n \neq k} w_n \gamma_n(a, x) \quad (2)$

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$$\overset{(2)}{\geq} \sum_{n \neq k} w_n \left(\gamma_n(x, a_n) + \gamma_n(a_k, x) - \gamma_n(a_k, a_n)\right) \geq 0$$

Convex hull properties

Theorem (Wendell '71)

Let $A \subset \mathbb{R}^2$ *be a finite set. Then for every* $x_1 \in \mathbb{R}^2$ *an* $x_2 \in conv\,(A)$ *exists with*

$$\|x_1 - a_n\|_p \geq \|x_2 - a_n\|_p \quad \forall a_n \in A$$

for every $1 \leq p \leq \infty$.

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for every $1 \leq p \leq \infty$.

$\Rightarrow$ At least one optimal solution of the classical Weber problem (with uniform symmetric distance measures)

$$\min_{x \in \mathbb{R}^2} \sum_{n=1}^{N} w_n d(x, a_n)$$

lies within the convex hull of the existing facilities.

Convex hull properties

Theorem (Juel '83)

If x^* *is an optimal solution of*

$$\min_{x \in \mathbb{R}^2} \sum_{n=1}^{N} w_n d(x, a_n),$$

where $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ *is a distance measure induced by an* ℓ_p *norm with* $1 < p < \infty$ *and* $A := \{a_1, \ldots, a_N\}$, *then* $x^* \in conv\,(A)$.

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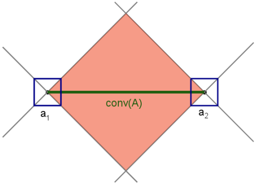
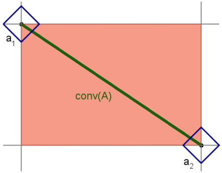
Counterexample for $p = \infty$ and $p = 1$ ($w_1 = w_2$ in both cases):





Counterexample for (WAM)

- For problem (WAM)

$$\min_{x\in\mathbb{R}^2} \sum_{n=1}^{N} w_n \gamma_n(x, a_n)$$

no analog property using conv(A) can be obtained.

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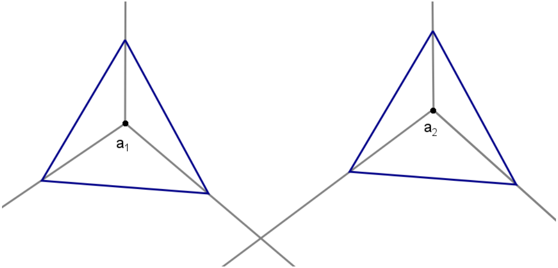
- Example with $w_1 = w_2$ and identical asymmetric distance measures:



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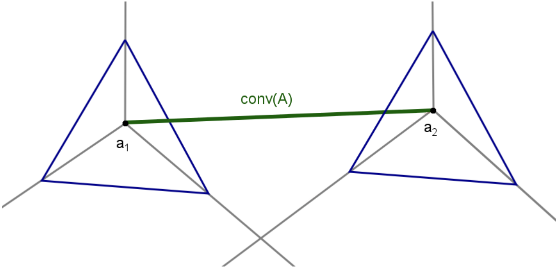

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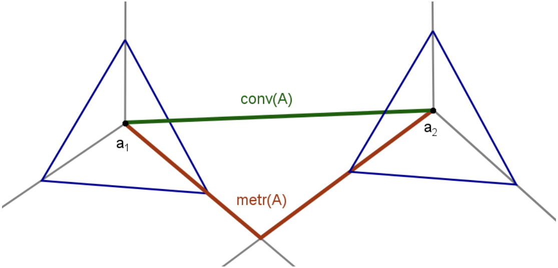

Metric Hull

Definition

The metric hull of the set A of points a_n, $n \in \mathcal{N}$ is defined as

$$\text{metr}(A) = \{x \in \mathbb{R}^2 : \forall y \in \mathbb{R}^2, x \neq y, \\ \exists a_n \in A \ \text{ with } \ \gamma_n(x, a_n) < \gamma_n(y, a_n)\}$$

with respect to the gauges γ_n, $n \in \mathcal{N}$.

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with respect to the gauges γ_n, $n \in \mathcal{N}$.

Basic properties: (Durier '85)

- $A \subset \text{metr}(A)$
- $\exists x^* \in \text{metr}(A)$ such that x^* is optimal for (WAM)

Convex and metric hull

Further properties: (Durier '86)

Let X be a normed space and let the distance measures be induced by uniform ℓ_p norms $1 < p < \infty$, then

$$\text{metr}(A) \subseteq \text{conv}(A).$$

Convex and metric hull

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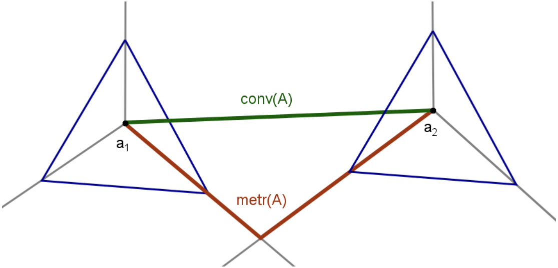
This does not hold for (WAM):



Multicriteria optimization problem

Considering the multicriteria optimization problem

$$\min_{x \in \mathbb{R}^2} \begin{pmatrix} \gamma_1(x, a_1) \\ \vdots \\ \gamma_N(x, a_N) \end{pmatrix} \qquad \text{(MOP)}$$

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the problem (WAM)

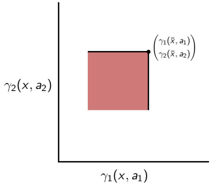
$$\min_{x \in \mathbb{R}^2} \sum_{n=1}^{N} w_n \gamma_n(x, a_n)$$

can be interpreted as weighted sum scalarization with weights according to the customer demand.

Efficient points for (MOP)

Concepts of efficiency in multicriteria optimization:

- $\bar{x}$ is weakly efficient $\Leftrightarrow \nexists x \in \mathbb{R}^2 : \gamma_n(x, a_n) < \gamma_n(\bar{x}, a_n)$ $\forall n \in \mathcal{N}$



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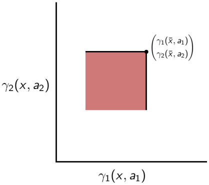
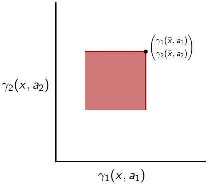


weakly efficient



efficient

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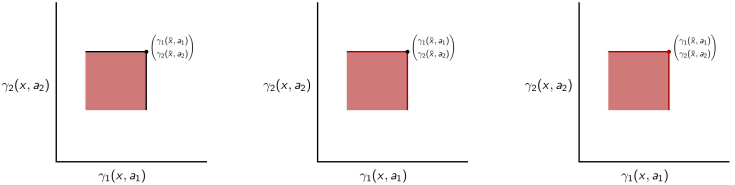
- $\bar{x}$ is strictly efficient $\Leftrightarrow \nexists x \in \mathbb{R}^2, x \neq \bar{x} :$ $\gamma_n(x, a_n) \leq \gamma_n(\bar{x}, a_n)$ $\forall n \in \mathcal{N}$



weakly efficient | efficient | strictly efficient

Boundedness of X_{w-eff}

Theorem

The set of weakly efficient points X_{w-eff} for (WAM) is bounded.

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Proof (sketch).

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- define $\bar{L} := \bigcup_{n \in \mathcal{N}} \bar{L}_{\gamma_n}$, then
  $\forall x \notin \bar{L}$ and $\forall \bar{x} \in \bar{L} : \gamma_n(x, a_n) > \gamma_n(\bar{x}, a_n)$

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  $\forall x \notin \bar{L}$ and $\forall \bar{x} \in \bar{L} : \gamma_n(x, a_n) > \gamma_n(\bar{x}, a_n)$
- $\Rightarrow X_{w-eff} \subseteq \bar{L}$

Algorithm to determine sets of efficient points

Input

Step 1: Grid points

Step 2: Efficient grid points

Step 3: Efficient grid lines

Step 4: Efficient grid cells

Output

Algorithm to determine sets of efficient points

Input

Existing facilities $a_n \in A \subset \mathbb{R}^2$, $n \in \mathcal{N}$ and associated distance measures γ_n induced by polyhedral gauges.

Step 1: Grid points

Step 2: Efficient grid points

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Output

Algorithm to determine sets of efficient points

Input

Step 1: Grid points

Determine the set of grid points $\mathcal{P}_G$ of the construction grid G based on the fundamental directions v_{nj} of the polyhedral gauges γ_n.

Step 2: Efficient grid points

Step 3: Efficient grid lines

Step 4: Efficient grid cells

Output

Algorithm to determine sets of efficient points

Input

Step 1: Grid points

Step 2: Efficient grid points

Test all $x \in \mathcal{P}_G$ if they are (weakly/strictly) efficient.
Set $x \in M_P^{(w/s)eff} \Leftrightarrow x \in \mathcal{P}_G \wedge x \in X_{(w/s)-eff}$.

Step 3: Efficient grid lines

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For all $x \neq y \in M_P^{(w/s)eff}$ determine $t_{xy} = \frac{1}{2}(x + y)$.
If $t \in X_{(w/s)-eff} \wedge \exists n, j : x, y, t \in v_{nj}$ set $\overline{xy} \in M_L^{(w/s)eff}$.

Step 4: Efficient grid cells

Output

Algorithm to determine sets of efficient points

Input

Step 1: Grid points

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If $\overline{x_i x_j} \in M_L^{(w/s)eff}$ yields for all line segments $\overline{x_i x_j}$ that are bounding a construction grid cell D_π, then set $D_\pi \in M_C^{(w/s)eff}$.

Output

Algorithm to determine sets of efficient points

Input

Step 1: Grid points

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Output

Set of (weakly/strictly) efficient points
$X_{(w/s)-eff} = M_P^{(w/s)eff} \cup M_L^{(w/s)eff} \cup M_C^{(w/s)eff}$.

Tests for (weakly/strictly) efficiency

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	$\bar{x} \in X_{w-eff}$
definition	$\nexists x \in \mathbb{R}^2 : \gamma_n(x, a_n) < \gamma_n(\bar{x}, a_n) \ \forall n \in \mathcal{N}$

Tests for (weakly/strictly) efficiency

-

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definition	$\nexists x \in \mathbb{R}^2 : \gamma_n(x, a_n) < \gamma_n(\bar{x}, a_n) \; \forall n \in \mathcal{N}$
hull of $\nabla\gamma_n(\bar{x}, a_n)$	$\bar{x} \in \operatorname{int}\left(\operatorname{conv}\left\{\bar{x} + \nabla\gamma_n(\bar{x}, a_n),\ n \in \mathcal{N}\right\}\right)$

Tests for (weakly/strictly) efficiency

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hull of $\nabla\gamma_n(\bar{x}, a_n)$	$\bar{x} \in \text{int}\left(\text{conv}\left\{\bar{x} + \nabla\gamma_n(\bar{x}, a_n),\; n \in \mathcal{N}\right\}\right)$
angles of $\nabla\gamma_n(\bar{x}, a_n)$	$\max\{\alpha_i + \alpha_j\} \geq 180^\circ$

  where α_i, α_j are the directed angles between two gradients $\nabla\gamma_{n_1}(\bar{x}, a_{n_1}),\ \nabla\gamma_{n_2}(\bar{x}, a_{n_2})$ and $\nabla\gamma_{n_2}(\bar{x}, a_{n_2}),\ \nabla\gamma_{n_3}(\bar{x}, a_{n_3})$ $\forall n_1, n_2, n_3 \in \mathcal{N}$, respectively.

Tests for (weakly/strictly) efficiency

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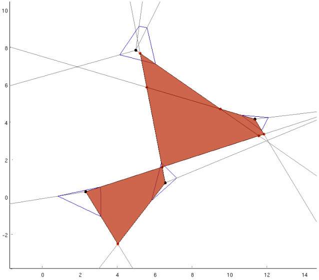
	$\bar{x} \in X_{w-eff}$
definition	$\nexists x \in \mathbb{R}^2 : \gamma_n(x, a_n) < \gamma_n(\bar{x}, a_n) \; \forall n \in \mathcal{N}$
hull of $\nabla\gamma_n(\bar{x}, a_n)$	$\bar{x} \in \text{int}\,(\text{conv}\,\{\bar{x} + \nabla\gamma_n(\bar{x}, a_n),\; n \in \mathcal{N}\})$
angles of $\nabla\gamma_n(\bar{x}, a_n)$	$\max\{\alpha_i + \alpha_j\} \geq 180°$

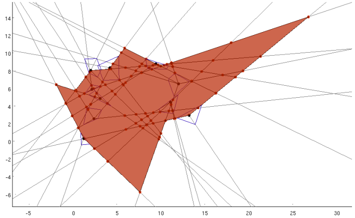
  where α_i, α_j are the directed angles between two gradients $\nabla\gamma_{n_1}(\bar{x}, a_{n_1})$, $\nabla\gamma_{n_2}(\bar{x}, a_{n_2})$ and $\nabla\gamma_{n_2}(\bar{x}, a_{n_2})$, $\nabla\gamma_{n_3}(\bar{x}, a_{n_3})$ $\forall n_1, n_2, n_3 \in \mathcal{N}$, respectively.

- analog conditions can be obtained for strictly efficient and efficient points

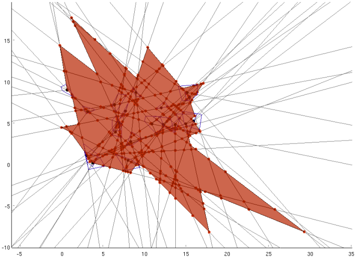
Examples for strictly efficient sets



$N = 4$



$N = 10$



$N = 20$

Thank you for your attention!

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