A Two-Stage Stochastic Programming Approach for Location-Allocation Models in Uncertain Environments

Markus Kaiser, Kathrin Klamroth

Optimization & Approximation
Department of Mathematics
University of Wuppertal

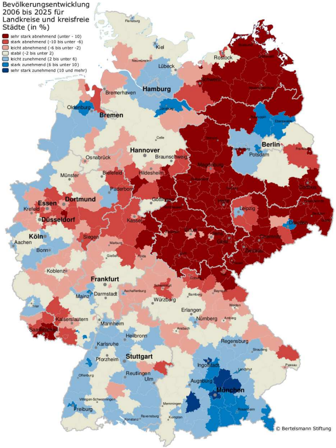
EWGLA XIX
Nantes, October 12-14, 2011

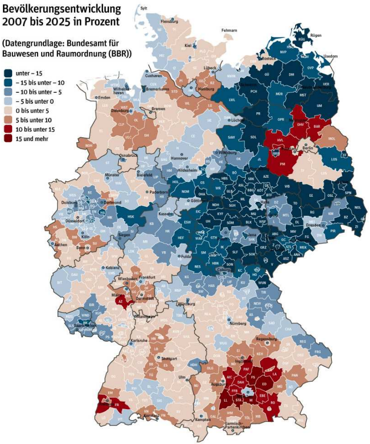
Outline

BERGISCHE UNIVERSITÄT WUPPERTAL

Example for different future scenarios

Two forcasts describing the population development in Germany until 2025:

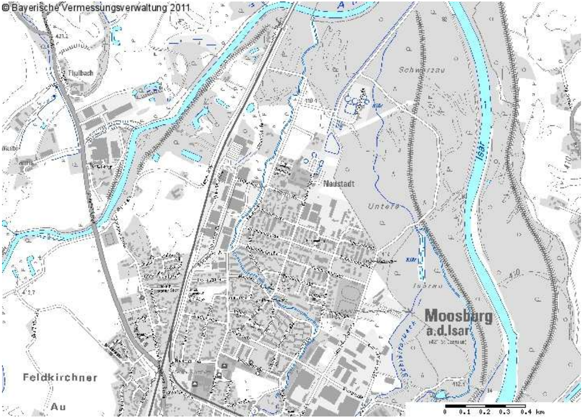




Sources: www.bertelsmann-stiftung.de/cps/rde/xchg/bst/hs.xsl/nachrichten_91824.htm and www.berlin-institut.org/weitere-veroeffentlichungen/demografischer-wandel.html

Example for different future scenarios

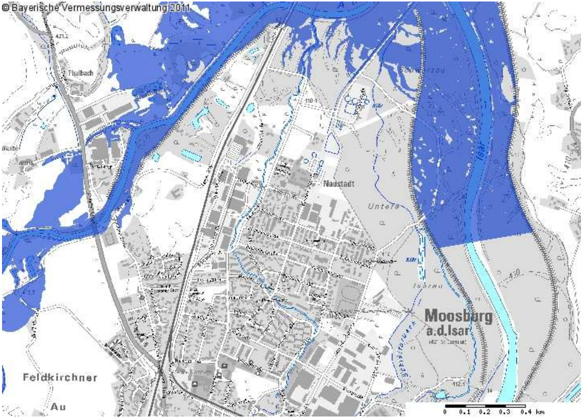
Flood scenarios with various probabilities:



Source: www.lfu.bayern.de/wasser/hw_ue_gebiete/informationsdienst/index.htm

Example for different future scenarios

Flood scenarios with various probabilities:

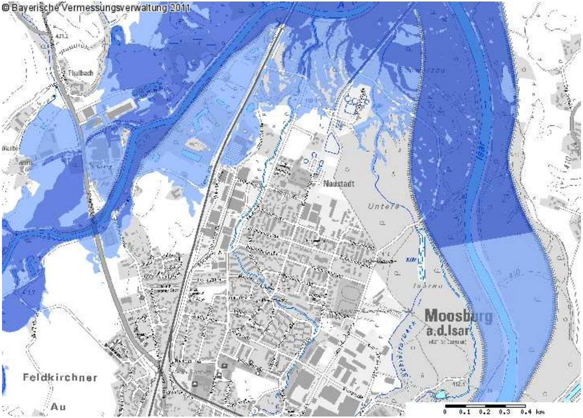


Source: www.lfu.bayern.de/wasser/hw_ue_gebiete/informationsdienst/index.htm

BERGISCHE UNIVERSITÄT WUPPERTAL

Example for different future scenarios

Flood scenarios with various probabilities:



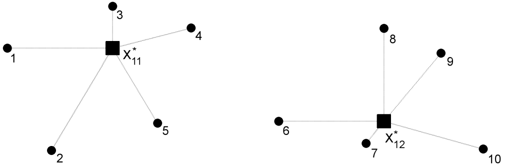
Source: www.lfu.bayern.de/wasser/hw_ue_gebiete/informationsdienst/index.htm

Problem description

Continuous location-allocation problem

Problem description

Continuous location-allocation problem

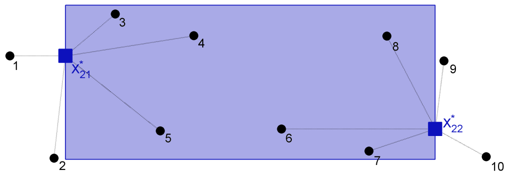


distance measure: Manhattan-Norm
weights: $w^1 = (1, \ldots, 1)$,

Problem description

Continuous location-allocation problem with uncertain development of the according environment:

- various forbidden regions,
- different customer weights.

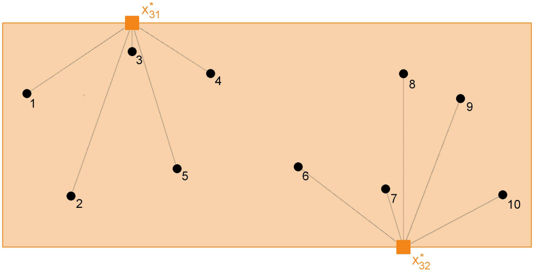


distance measure: Manhattan-Norm
weights: $w^1 = (1, \ldots, 1)$, $w^2 = (1, 1, 0.7, 0.7, 0.7, 0.7, 0.7, 0.7, 1, 1)$,

Problem description

Continuous location-allocation problem with uncertain development of the according environment:

- various forbidden regions,
- different customer weights.

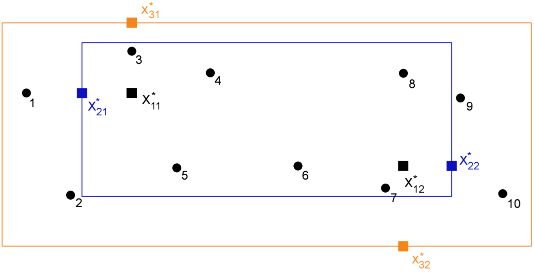


distance measure: Manhattan-Norm
weights: $w^1 = (1, \ldots, 1)$, $w^2 = (1, 1, 0.7, 0.7, 0.7, 0.7, 0.7, 0.7, 1, 1)$,
$w^3 = (0.5, \ldots, 0.5)$

Problem description

Continuous location-allocation problem with uncertain development of the according environment:

- various forbidden regions,
- different customer weights.



distance measure: Manhattan-Norm
weights: $w^1 = (1, \dots, 1)$, $w^2 = (1, 1, 0.7, 0.7, 0.7, 0.7, 0.7, 0.7, 1, 1)$, $w^3 = (0.5, \dots, 0.5)$

Goal: Find a solution, which is „optimal“ for the expectation of the future scenarios and the current situation.

Two-stage stochastic location-allocation model (SLAP)

$$\min \sum_{n=1}^{N} \sum_{m=1}^{M} z_{nm} w_n d(x_m, a_n) + \mathbb{E}(z_s)$$

$$\text{s.t.} \sum_{m=1}^{M} z_{nm} = 1 \qquad \forall n \in \mathcal{N}$$

$$\sum_{m=1}^{M} \bar{z}_{nms} = 1 \qquad \forall n \in \mathcal{N},\ \forall s \in \mathcal{S}$$

$$\bar{z}_{nms} \leq 1 - y_{ms} \qquad \forall n \in \mathcal{N},\ \forall s \in \mathcal{S}$$

$$z_{nm}, \bar{z}_{nms} \in \{0, 1\} \qquad \forall n \in \mathcal{N},\ \forall m \in \mathcal{M},\ \forall s \in \mathcal{S}$$

$$x_m \in \mathbb{R}^2 \qquad \forall m \in \mathcal{M}$$

Two-stage stochastic location-allocation model (SLAP)

$$\min \sum_{n=1}^{N} \sum_{m=1}^{M} z_{nm} w_n d(x_m, a_n) + \mathbb{E}(z_s)$$

$$\text{s.t.} \sum_{m=1}^{M} z_{nm} = 1 \qquad \forall n \in \mathcal{N}$$

$$\sum_{m=1}^{M} \bar{z}_{nms} = 1 \qquad \forall n \in \mathcal{N},\ \forall s \in \mathcal{S}$$

$$\bar{z}_{nms} \leq 1 - y_{ms} \qquad \forall n \in \mathcal{N},\ \forall s \in \mathcal{S}$$

$$z_{nm}, \bar{z}_{nms} \in \{0, 1\} \qquad \forall n \in \mathcal{N},\ \forall m \in \mathcal{M},\ \forall s \in \mathcal{S}$$

$$x_m \in \mathbb{R}^2 \qquad \forall m \in \mathcal{M}$$

with

$$\mathbb{E}(z_s) = \sum_{s \in S} p_s \sum_{n=1}^{N} \sum_{m=1}^{M} \bar{z}_{nms} w_n^s d(x_m, a_n) \quad \text{(expectation of the 2nd stage objective)}$$

Two-stage stochastic location-allocation model (SLAP)

$$\min \sum_{n=1}^{N} \sum_{m=1}^{M} z_{nm} w_n d(x_m, a_n) + \mathbb{E}(z_s)$$

$$\text{s.t.} \sum_{m=1}^{M} z_{nm} = 1 \qquad \forall n \in \mathcal{N}$$

$$\sum_{m=1}^{M} \bar{z}_{nms} = 1 \qquad \forall n \in \mathcal{N},\ \forall s \in \mathcal{S}$$

$$\bar{z}_{nms} \leq 1 - y_{ms} \qquad \forall n \in \mathcal{N},\ \forall s \in \mathcal{S}$$

$$z_{nm}, \bar{z}_{nms} \in \{0, 1\} \qquad \forall n \in \mathcal{N},\ \forall m \in \mathcal{M},\ \forall s \in \mathcal{S}$$

$$x_m \in \mathbb{R}^2 \qquad \forall m \in \mathcal{M}$$

with

$$\mathbb{E}(z_s) = \sum_{s \in S} p_s \sum_{n=1}^{N} \sum_{m=1}^{M} \bar{z}_{nms} w_n^s d(x_m, a_n) \quad \text{(expectation of the 2nd stage objective)}$$

and

$$y_{ms} = \begin{cases} 1 & \text{if } x_m \in \mathcal{R}_s \\ 0 & \text{otherwise} \end{cases} \quad \forall m \in \mathcal{M},\ \forall s \in \mathcal{S}$$

Assumptions

Assumptions on the forbidden regions $R_{si},\ s \in \mathcal{S},\ i \in \mathcal{I}_s$:

- rectangular with boundaries parallel to the coordinate axes,
- possibility of more than one forbidden region in each scenario (i.e. $|\mathcal{I}_s| \in \mathbb{N}$),
- not necessarily disjoint.

Assumptions

Assumptions on the forbidden regions $R_{si},\ s \in \mathcal{S},\ i \in \mathcal{I}_s$:

- rectangular with boundaries parallel to the coordinate axes,
- possibility of more than one forbidden region in each scenario (i.e. $|\mathcal{I}_s| \in \mathbb{N}$),
- not necessarily disjoint.

Assumption on the distance measure:

- block-norm or polyhedral gauge.

Assumptions

Assumptions on the forbidden regions $R_{si},\ s \in \mathcal{S},\ i \in \mathcal{I}_s$:

- rectangular with boundaries parallel to the coordinate axes,
- possibility of more than one forbidden region in each scenario (i.e. $|\mathcal{I}_s| \in \mathbb{N}$),
- not necessarily disjoint.

Assumption on the distance measure:

- block-norm or polyhedral gauge.

Assumption on the representation of the future uncertainty:

- discrete set of scenarios with $|\mathcal{S}| < \infty$,
- probabilities given by p_s $(0 \leq p_s \leq 1, \sum_{s \in \mathcal{S}} p_s = 1)$.

Discretisation

Theorem

$\mathcal{C}_1$: *grid points of the construction grid,*

$\mathcal{C}_2$: *intersection points of the construction grid with the boundaries of forbidden regions* $\partial\mathcal{R}$, $\mathcal{R} = \cup_{s\in\mathcal{S}}\mathcal{R}_s$,

$\mathcal{C}_3$: *intersection points of line segments of boundaries of different forbidden regions.*

Then: there is an optimal solution $x^* = (x_1^*, \dots, x_m^*)$ *with*

$$x^* \in \mathcal{C} = \bigcup_{i\in\{1,2,3\}} \mathcal{C}_i.$$

Discretisation

Theorem

$\mathcal{C}_1$: *grid points of the construction grid,*

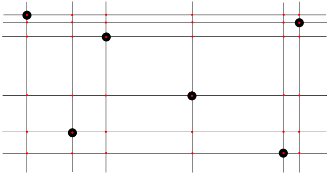
$\mathcal{C}_2$: *intersection points of the construction grid with the boundaries of forbidden regions* $\partial\mathcal{R}$, $\mathcal{R} = \cup_{s \in \mathcal{S}} \mathcal{R}_s$,

$\mathcal{C}_3$: *intersection points of line segments of boundaries of different forbidden regions.*

Then: there is an optimal solution $x^* = (x_1^*, \ldots, x_m^*)$ *with*

$$x^* \in \mathcal{C} = \bigcup_{i \in \{1,2,3\}} \mathcal{C}_i.$$

Example (ℓ_1-norm):



Discretisation

Theorem

$\mathcal{C}_1$: *grid points of the construction grid,*

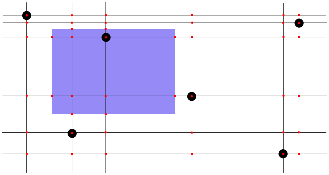
$\mathcal{C}_2$: *intersection points of the construction grid with the boundaries of forbidden regions* $\partial\mathcal{R}$, $\mathcal{R} = \cup_{s\in\mathcal{S}}\mathcal{R}_s$,

$\mathcal{C}_3$: *intersection points of line segments of boundaries of different forbidden regions.*

Then: there is an optimal solution $x^* = (x_1^*, \ldots, x_m^*)$ *with*

$$x^* \in \mathcal{C} = \bigcup_{i\in\{1,2,3\}} \mathcal{C}_i.$$

Example (ℓ_1-norm):



Discretisation

Theorem

$\mathcal{C}_1$: *grid points of the construction grid,*

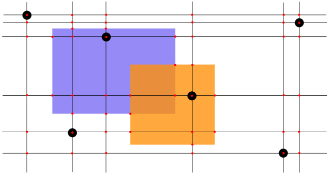
$\mathcal{C}_2$: *intersection points of the construction grid with the boundaries of forbidden regions* $\partial\mathcal{R}$, $\mathcal{R} = \cup_{s\in\mathcal{S}}\mathcal{R}_s$,

$\mathcal{C}_3$: *intersection points of line segments of boundaries of different forbidden regions.*

Then: there is an optimal solution $x^* = (x_1^*, \dots, x_m^*)$ *with*

$$x^* \in \mathcal{C} = \bigcup_{i\in\{1,2,3\}} \mathcal{C}_i.$$

Example (ℓ_1-norm):



Location-allocation heuristic

Alternating method to iteratively improve the location variables x_m of the new facilities and the allocations z_{nm} and $\bar{z}_{nms}$ of the customers.

Input

Iteration loop

Stopping condition

Location-allocation heuristic

Alternating method to iteratively improve the location variables x_m of the new facilities and the allocations z_{nm} and $\bar{z}_{nms}$ of the customers.

Input

Problem data and initial guess for location variables.

Iteration loop

Stopping condition

Location-allocation heuristic

Alternating method to iteratively improve the location variables x_m of the new facilities and the allocations z_{nm} and $\bar{z}_{nms}$ of the customers.

Input

Problem data and initial guess for location variables.

Iteration loop

Sequentially solve the 1-median problems defined by the allocation variables z_{nm} and update the location and the allocation variables.

Stopping condition

Location-allocation heuristic

Alternating method to iteratively improve the location variables x_m of the new facilities and the allocations z_{nm} and $\bar{z}_{nms}$ of the customers.

Input

Problem data and initial guess for location variables.

Iteration loop

Sequentially solve the 1-median problems defined by the allocation variables z_{nm} and update the location and the allocation variables.

Stopping condition

Stop after M successively considered not improving iterations.

Location-allocation heuristic (1-median subproblem)

Consider a mean-problem for the cluster $m' \in \mathcal{M}$ with $\bar{z}_{nm0} = z_{nm},\ p_0 = 1$ and $\bar{\mathcal{N}} := \{n \in \mathcal{N} : \exists\, \bar{z}_{nm's} = 1\}$:

$$\min \sum_{s \in \mathcal{S}} p_s \sum_{n=1}^{\bar{N}} \sum_{m=1}^{M} \bar{z}_{nms} w_n d(x_{m'}, a_n)$$

$$\text{s.t.} \sum_{m=1}^{M} \bar{z}_{nms} = 1 \qquad \forall n \in \bar{\mathcal{N}},\ \forall s \in \mathcal{S}$$

$$\bar{z}_{nms} \leq 1 - y_{ms} \qquad \forall n \in \bar{\mathcal{N}},\ \forall s \in \mathcal{S}$$

$$\bar{z}_{nms} \in \{0, 1\} \qquad \forall n \in \bar{\mathcal{N}},\ \forall m \in \mathcal{M},\ \forall s \in \mathcal{S}$$

$$x_{m'} \in \mathbb{R}^2$$

Location-allocation heuristic (1-median subproblem)

Consider a mean-problem for the cluster $m' \in \mathcal{M}$ with $\bar{z}_{nm0} = z_{nm},\ p_0 = 1$ and $\bar{\mathcal{N}} := \{n \in \mathcal{N} : \exists\, \bar{z}_{nm's} = 1\}$:

$$\min \sum_{s \in \mathcal{S}} p_s \sum_{n=1}^{\bar{N}} \sum_{m=1}^{M} \bar{z}_{nms} w_n d(x_{m'}, a_n)$$

$$\text{s.t.} \sum_{m=1}^{M} \bar{z}_{nms} = 1 \qquad \forall n \in \bar{\mathcal{N}},\ \forall s \in \mathcal{S}$$

$$\bar{z}_{nms} \leq 1 - y_{ms} \qquad \forall n \in \bar{\mathcal{N}},\ \forall s \in \mathcal{S}$$

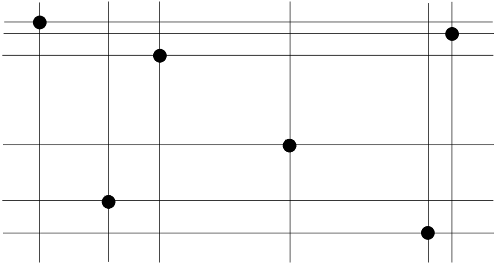
$$\bar{z}_{nms} \in \{0, 1\} \qquad \forall n \in \bar{\mathcal{N}},\ \forall m \in \mathcal{M},\ \forall s \in \mathcal{S}$$

$$x_{m'} \in \mathbb{R}^2$$

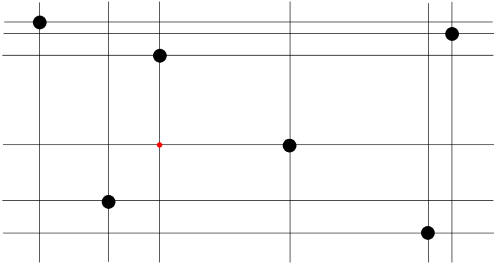
Solution approach:

1) Solve the unrestricted 1-median problem.
2) If $\exists s \in \mathcal{S} :\ x^*_{m'} \in \mathcal{R}_s$ consider additional candidates on the boundaries of the forbidden regions containing $x^*_{m'}$.

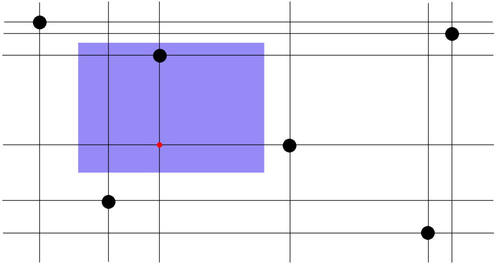
Example for the 1-median subproblem



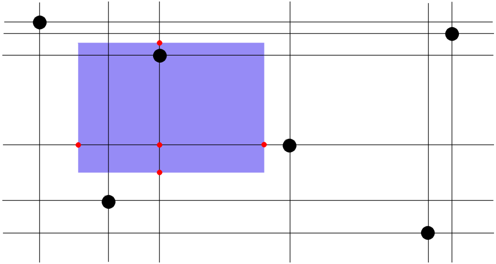
Example for the 1-median subproblem



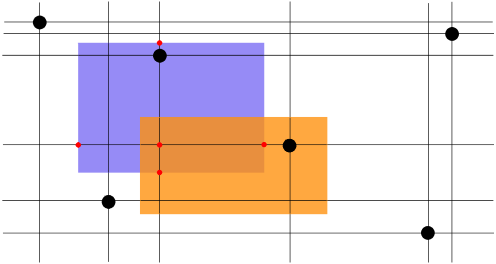
Example for the 1-median subproblem



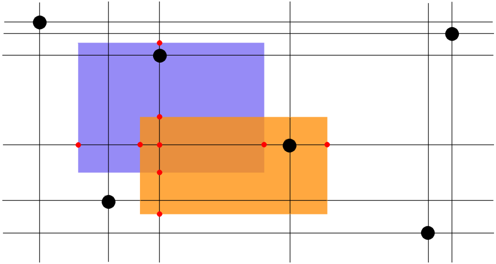
Example for the 1-median subproblem



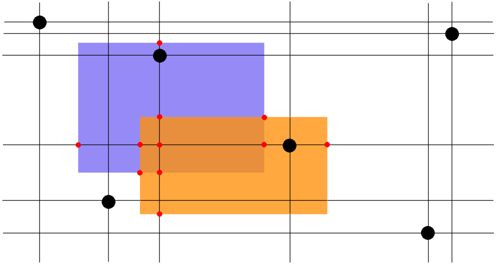
Example for the 1-median subproblem



Example for the 1-median subproblem



Example for the 1-median subproblem



Genetic algorithm (GA) - basic ideas

Design

- individuals: coordinates of the M locations of one particular solution of (SLAP),
- genes: coordinates of one location of one solution of (SLAP),
- fitness function: objective function of (SLAP).

Genetic algorithm (GA) - basic ideas

Design

- individuals: coordinates of the M locations of one particular solution of (SLAP),
- genes: coordinates of one location of one solution of (SLAP),
- fitness function: objective function of (SLAP).

Initial population

Randomly combine all coordinates given by elements of $\mathcal{C}$.

Genetic algorithm (GA) - basic ideas

Design

- individuals: coordinates of the M locations of one particular solution of (SLAP),
- genes: coordinates of one location of one solution of (SLAP),
- fitness function: objective function of (SLAP).

Initial population

Randomly combine all coordinates given by elements of $\mathcal{C}$.

Generating new individuals (crossing-over and mutation)

- combination of single genes of two parent-individuals,
- building pairs (arbitrarily or by bipartite matching) of a gene from each parent, linking the pairs and choosing points on the connection line,
- use all genes of the two parents (infeasible solution) and remove iteratively the worst one until the solution is feasible.

Branch and Bound - basic ideas

Branching

- combinatorial branching based on the allocation variables of the current scenario,
- successively fix the z_{nm} - considering one customer in each level of the branch and bound tree,
- generate M child nodes from one node by realizing every possible allocation of one unconnected customer.

Branch and Bound - basic ideas

Branching

- combinatorial branching based on the allocation variables of the current scenario,
- successively fix the z_{nm} - considering one customer in each level of the branch and bound tree,
- generate M child nodes from one node by realizing every possible allocation of one unconnected customer.

Bounds

- upper bound by LA heuristic (or objectives of completely evaluated nodes),
- lower bound of every node by evaluating the contribution of the partially constructed clusters.

LA heuristic

Computation time with Matlab 7.9, Dual-Core 2.80 GHz CPU and 4 GB memory:

- 2 future scenarios:

$M \setminus N$	10	20	50	100	250	500	1000	2000
2	0.0247	0.0179	0.0267	0.0621	0.0724	0.2610	0.8566	1.0931
3	0.0129	0.0279	0.0312	0.0644	0.1884	0.3793	0.8041	2.1899
5	0.0161	0.0219	0.0677	0.0957	0.2691	0.7110	1.9563	5.3643
10	0.0202	0.0326	0.0946	0.1275	0.4457	1.4313	2.3414	5.4916

- 5 future scenarios:

$M \setminus N$	10	20	50	100	250	500	1000	2000
2	0.0139	0.0743	0.1581	0.1340	0.3052	0.2993	2.0992	2.5409
3	0.0281	0.0786	0.0441	0.2906	0.5216	0.7545	1.0009	6.3129
5	0.0323	0.1117	0.2134	0.2003	0.4455	1.0590	2.6459	8.7888
10	0.0492	0.0691	0.2671	0.2554	0.9458	1.1800	7.5720	32.7529

- 10 future scenarios:

$M \setminus N$	10	20	50	100	250	500	1000	2000
2	0.1685	0.3933	0.2112	0.2706	1.2893	1.8686	3.2462	18.2116
3	0.1851	0.4454	0.4688	0.6975	3.5155	3.3666	4.5294	37.1779
5	0.1373	0.2736	0.6974	3.3337	7.1581	7.7755	38.4467	49.7630
10	0.2508	0.6780	2.0802	3.6927	4.2954	25.5889	38.2305	114.1883

LA heuristic and genetic algorithm

Average improvement (in percent) by metaheuristics compared to average LA heuristic run (considering 5 scenarios):

- LA heuristic with multistart:

$M \setminus N$	100	250	500	1000	2000
2	18.8	19.7	19.2	19.2	18.1
3	19.1	20.4	19,7	19.5	20.4
5	20.1	19.3	18.2	17.5	17.7
10	17.6	18.5	17.3	16.2	16.9

- GA finished by LA heuristic:

$M \setminus N$	100	250	500	1000	2000
2	18.8	19.7	19.1	20.3	20.6
3	19.1	20.5	19.5	20.2	20.4
5	20.2	20.0	18.3	20.4	19.7
10	18.1	19.6	21.5	20.5	20.9

- Hybrid GA combined with LA heuristic:

$M \setminus N$	100	250	500	1000	2000
2	18.8	19.7	19.1	20.2	20.1
3	19.1	20.5	19.4	20.3	20.5
5	20.2	20.0	18.3	19.7	17.9
10	18.3	18.5	20.1	20.5	20.8

Thank you for your attention!

Markus Kaiser
e-Mail: kaiser@math.uni-wuppertal.de