Complex logarithms in Heston-like models

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ABSTRACT

The characteristic functions of many affine jump-diffusion models, such as Heston's stochastic volatility model and all of its extensions, involve multivalued functions like the complex logarithm. If we restrict the logarithm to its principal branch, as is done in most software packages, the characteristic function can become discontinuous, leading to completely wrong option prices if options are priced by Fourier inversion. In this paper we prove without any restrictions that there is a formulation of the characteristic function in which the principal branch is the correct one. Seen as this formulation is easier to implement and numerically more stable than the so-called rotation count algorithm of Kahl and Jäckel [2005], we solely focus on its stability in this article. The remainder of this paper shows how complex discontinuities can be avoided in the Variance Gamma and Schöbel-Zhu models, as well as in the exact simulation algorithm of the Heston model, recently proposed by Broadie and Kaya.

Keywords: Complex logarithm, affine jump-diffusion, stochastic volatility, Heston, characteristic function, option pricing, Fourier inversion, Variance Gamma, Schöbel-Zhu, exact simulation.

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1. Introduction

In this paper we will analyse the complex discontinuities one finds when evaluating the characteristic function of several popular option pricing models, in particular the seminal stochastic volatility model of Heston [1993]. Since the initial breakthrough by Heston a whole range of models have appeared that allow for closed-form characteristic functions. In these models, prices of European options can be calculated semi-analytically by means of Fourier inversion. This approach has in recent years been refined by Carr and Madan [1999], Lewis [2001] and Lee [2004] whose combined works show that if a model has a characteristic function, the forward price of e.g. a European call on an underlying asset S can be written as:

$$\mathbb{E}\left[(\mathbf{S}(\mathbf{T}) - \mathbf{K})^{+}\right] = \mathbf{R}_{\alpha}(\mathbf{F}, \mathbf{K}) + \frac{1}{\pi} \operatorname{Re} \int_{0}^{\infty} e^{-i(\mathbf{v} - i\alpha)\mathbf{k}} \frac{\phi(\mathbf{v} - i(\alpha + 1))}{-(\mathbf{v} - i(\alpha + 1))(\mathbf{v} - i\alpha)} d\mathbf{v}$$
(1)

where ϕ is the extended characteristic function of the log-stock price, $\phi(u) = \mathbb{E}\left[e^{iu \ln S(T)}\right]$, and:

$$R_{\alpha}(F,K) = F \cdot \mathbf{1}_{[\alpha \le 0]} - K \cdot \mathbf{1}_{[\alpha \le -1]} - \frac{1}{2} \left(F \cdot \mathbf{1}_{[\alpha = 0]} - K \cdot \mathbf{1}_{[\alpha = -1]} \right)$$
(2)

is a residue term, arising from the poles of the integrand in (1). This representation only holds for values of the damping coefficient α satisfying $\phi(-i(\alpha+1)) < \infty$, i.e. for those values of α where the $(\alpha+1)^{\text{th}}$ moment of S(T) is finite. In theory, the option price is independent of the parameter α ; in practice however, α affects the behaviour of the integrand and choosing the right value for it is crucial. The optimal choice of α is considered in Lee [2004] and Lord and Kahl [2007].

If we use equations (1)-(2) to evaluate the option price, discontinuities in a characteristic function will clearly lead to discontinuities in the integrand of (1). In turn, this will produce completely wrong option prices, as we will see shortly. The problem at hand is not unique to option pricing, but will occur in any application where one requires an evaluation of a characteristic function. Another example of this is the calculation of density functions via inversion of the characteristic function, see the exact simulation algorithm of Broadie and Kaya [2006] for an application of this in the Heston model.

To fix ideas, we will focus on the Heston stochastic volatility model throughout this paper, though we explore other models in the penultimate section. Two formulations of the characteristic function are prevalent in the literature. The first, which is the original formulation of Heston [1993], is known to suffer from discontinuities when the complex logarithm is restricted to its principal branch. Schöbel and Zhu [1999] first mentioned such problems in the literature. They proposed an ad-hoc workaround by letting the integration algorithm used to calculate (1) pick up any discontinuities and correct for them. As this is not foolproof, Kahl and Jäckel [2005] considered the same problem recently and came up with the rotation count algorithm, an easily implementable algorithm that aims to keep the complex logarithm in the Heston model continuous. Though the algorithm seems to work perfectly well, a formal proof is required as one does not want to be caught by a counterexample.

The second and less widespread formulation has to our knowledge first appeared in Bakshi, Cao and Chen [1997, eq. A.11] and later in Duffie, Pan and Singleton [2000] and Gatheral [2006] among others. Interestingly most of the authors using this formulation never mentioned any problems with regard to the complex logarithm, and in fact Duffie et al. specify to use the principal branch of the complex logarithm. In this paper we prove that Duffie et al. were right to do this, as no complex discontinuities arise in this formulation, something that has already been conjectured by Lord and Kahl [2006] and Gatheral [2006]. In Lord and Kahl [2006] we proved this result under a mild constraint on the correlation coefficient ρ . In particular, the result certainly holds true when ρ is non-positive, which seems to be the practically most relevant case. In addition we verified the correctness of the rotation count algorithm under this same constraint. Albrecher et al. [2007] proved the continuity of this second formulation without any restrictions on the parameters. However, they do restrict α in $\phi(v-(\alpha+1)i)$ to be positive. As results in Lee [2004] and Lord and Kahl [2007] demonstratively show that a robust option pricer would need to able to choose α freely, this restriction would hamper the implementation hereof. A final paper to appear on this topic, Fahrner [2007], considers the special case when $\alpha = -1/2$. The restriction on the correlation coefficient is exactly the same as in Lord and Kahl.

As the union of the results from Lord and Kahl and Albrecher et al. do not fully prove the aforementioned conjecture, we set out to finally prove this result without any restrictions on the parameters, and more importantly, without any restrictions on α . The remainder of this paper is structured as follows. In Section 2 we state the two formulations of the Heston characteristic function and show what problems can be caused by using the wrong branch of the complex logarithm. Section 3 proves that the second formulation is continuous when the complex logarithm is restricted to its principal branch. Section 4 considers the rotation count algorithm and analyses in which parameter region we can guarantee that it works. Finally, in section 5 we consider other models with similar problems and show how to avoid the complex discontinuities there. The models considered are the Variance Gamma model of Madan and Seneta [1990] and the stochastic volatility model of Schöbel and Zhu [1999], as well as the recent exact simulation algorithm that Broadie and Kaya [2006] developed for the Heston model and extensions thereof. Section 6 concludes.

2. Complex discontinuities in the Heston model

In this section we will first derive the characteristic function of the underlying asset in the Heston model. In the second subsection we discuss the potential complex discontinuities that are present in the Heston model, and present several examples of the impact this could have on option prices.

2.1. Derivation of the characteristic function

Under the risk-neutral pricing measure the Heston stochastic volatility model is specified by the following set of stochastic differential equations:

$$dS(t) = \mu(t)S(t)dt + \sqrt{v(t)}S(t)dW_{s}(t)$$

$$dv(t) = -\kappa(v(t) - \theta)dt + \omega\sqrt{v(t)}dW_{v}(t)$$
(3)

where the Brownian motions satisfy $dW_s(t) \cdot dW_v(t) = \rho dt$. The underlying asset S has a stochastic variance v, which is modelled as a mean-reverting square root process. The parameter κ is the rate of mean reversion of the variance, θ is the long-term level of variance and ω is the volatility of variance. Finally, the drift $\mu(t)$ is used to fit to the forward curve of the underlying. We make the following assumption on the parameters:

Assumption:

$$\kappa > 0, \, \omega > 0, \, |\rho| < 1 \tag{4}$$

If $\omega = 0$, the model in (3) collapses to the Black-Scholes model with a time-dependent volatility. Similarly, if $|\rho| = 1$, the model is a special case of the local volatility model. The assumption that $\kappa > 0$ is not essential, though it will facilitate the analysis. Though the Heston model as postulated here is typically used for asset classes such as equity and foreign exchange, the mean-reverting square root process can be used as a stochastic volatility driver in any asset class, see e.g. Andersen and Andreasen [2002] and Andersen and Brotherton-Ratcliffe [2005] for applications in an interest rate context, and Mercurio and Moreni [2006] for an inflation context.

Although the model in (3) is not affine in the underlying asset S and the stochastic variance v, it is affine in ln S and v. Furthermore, note that the only time-inhomogeneous part of the model is the forward curve of the underlying asset. Hence, following the analysis of Duffie et al. [2000] we know that the characteristic function of the logarithm of the underlying will be exponentially affine in the logarithm of the forward, and the stochastic variance:

$$\phi(\mathbf{u}) = \mathbb{E}\left[e^{i\mathbf{u}\ln S(T)}\right] = \exp\left(i\mathbf{u}f + A(\mathbf{u},\tau) + B_{\mathbf{v}}(\mathbf{u},\tau) \cdot \mathbf{v}(0)\right)$$
(5)

Here $u \in \mathbb{C}$ and f is shorthand for ln F(T), the logarithm of the forward price. The functions A and B_v satisfy the following system of ODEs:

$$\frac{dB_{v}}{d\tau} = \hat{\alpha}(u) - \beta(u)B_{v} + \gamma B_{v}^{2}$$

$$\frac{dA}{d\tau} = \kappa \theta B_{v}$$
(6)

subject to the initial conditions A(u,0) = 0 and B(u,0) = 0. The auxiliary variables we introduced are $\hat{\alpha}(u) = -\frac{1}{2}u(i+u)$, $\beta(u) = \kappa - \rho\omega ui$ and $\gamma = \frac{1}{2}\omega^2$. Recasting the first Riccati equation as:

$$\frac{dB_v}{d\tau} = \gamma(B_v - a)(B_v - b) \tag{7}$$

immediately leads to the following solution:

$$B_{v}(u,\tau) = ab \frac{1 - e^{(b-a)\gamma\tau}}{a - be^{(b-a)\gamma\tau}}$$
(8)

The roots of the Riccati equation for B_v are $a = (\beta+D)/\omega^2$ and $b = (\beta-D)/\omega^2$ with $D(u) = \sqrt{\beta(u)^2 - 4\hat{\alpha}(u)\gamma}$. The solution for B_v thus equals:

$$B_{v}(u,\tau) = \frac{\beta(u) - D(u)}{\omega^{2}} \frac{1 - e^{-D(u)\tau}}{1 - G(u)e^{-D(u)\tau}}$$
(9)

where $G(u) = \frac{\beta(u) - D(u)}{\beta(u) + D(u)}$ represents the ratio of the two roots b and a. This quantity will play a great role in the remainder of the paper. The solution to A now follows from:

$$\int_{0}^{\tau} B_{v}(s) ds = \frac{\beta - D}{\omega^{2}} \int_{0}^{\tau} \frac{1 - e^{-Ds}}{1 - Ge^{-Ds}} ds = \frac{\beta - D}{\omega^{2} D} \int_{G}^{Ge^{-D\tau}} \frac{z/G - 1}{z(1 - z)} dz$$

$$= \frac{\beta - D}{\omega^{2} D} \cdot \left[\frac{(G - 1)\ln(z - 1) - G\ln z}{G} \right]_{z=G}^{z=Ge^{-D\tau}}$$

$$= \omega^{-2} \left((\beta - D)\tau - 2\ln\left(\frac{Ge^{-D\tau} - 1}{G - 1}\right) \right)$$
(10)

Clearly there are numerous ways of writing the characteristic function, and over the years many different formulations have been used. It turns out to be fundamentally important which formulation one uses for the function $A(u,\tau)$, in particular what one keeps under the logarithm in (10) and what one takes out of it. The formulation we have now derived is what we will refer to as the second formulation, as it is different to the original formulation in Heston [1993]. To the best of our knowledge this formulation has first appeared in Bakshi, Cao and Chen [1997]³, and later in e.g. Duffie, Pan and Singleton [2000] and Gatheral [2006]:

Formulation 2:

$$A(u,\tau) = \kappa \theta \omega^{-2} \left((\beta(u) - D(u))\tau - 2\ln\psi_2(u,\tau) \right), \quad \psi_2(u,\tau) = \frac{G(u)e^{-D(u)\tau} - 1}{G(u) - 1}$$
(11)

The first formulation is the original one used by Heston, and also appears in e.g. the articles of Lee [2004] and Kahl and Jäckel [2005]:

Formulation 1:

$$A(u,\tau) = \kappa \theta \omega^{-2} \left((\beta(u) + D(u))\tau - 2\ln\psi_1(u,\tau) \right), \quad \psi_1(u,\tau) = \frac{c(u)e^{D(u)\tau} - 1}{c(u) - 1}$$
(12)

where we introduced c(u) = 1/G(u). Though both formulations are algebraically equivalent, it is well-known that formulation 1 causes discontinuities when the principal branch of the complex logarithm is used, whereas for formulation 2 this turns out not to be the case. As an example of yet another formulation we mention Zhu [2000]. As this formulation also causes discontinuities when the principal branch of the logarithm is used, we restrict ourselves to these two formulations. The next subsection discusses the complex discontinuities caused by formulation 1.

2.2. Complex discontinuities

An easy way to avoid any complex discontinuities is to integrate the ODEs in (6) numerically, as this would automatically lead to the correct and continuous solution. A comparative advantage of the Heston model is however that it has a closed-form characteristic function, something which significantly reduces the computational effort and would be forsaken if we proceeded in this way. From a computational point of view it is therefore certainly worthwhile to investigate how to avoid discontinuities in the closed-form solution.

³ Although our formulation appears slightly different than that of Bakshi et al. and Duffie et al., the term under the logarithm is actually equivalent.

By taking a closer look at the characteristic function, it is clear that two multivalued functions are present, both of which could cause complex discontinuities. The first candidate is the square root used in D(u). It turns out that the characteristic function is even in D, so that we will from hereon use the common convention that the real part of the square root is nonnegative. The complex logarithm used in (11) and (12) is the second candidate. Let us recall that the logarithm of a complex variable z can be written as:

$$\ln z = \ln |z| + i(\arg(z) + 2\pi n) \tag{13}$$

where arg(z) is the argument of the complex number, |z| is its radius and $n \in \mathbb{Z}$. A typical choice used by most software packages is to restrict the logarithm to its principal branch, by letting $\arg(z)^4$ be the principal argument, $\arg(z) \in [-\pi,\pi)$, and setting n = 0. In this case the branch cut of the complex logarithm is $(-\infty, 0]$, and the complex logarithm is discontinuous along it. It is welldocumented that by restricting the complex logarithm to its principal branch formulation 1 will vield discontinuities in the characteristic function and hence in the option price. For any set of parameters, unless of course $2\kappa\theta/\omega^2 \in \mathbb{N}$, problems will arise if τ is sufficiently large, and in fact the number of discontinuities will grow with the time to maturity. A first mention of this problem was made by Schöbel and Zhu [1999], who encountered the same problem in the implementation of their own stochastic volatility model. They mention: "Therefore we implemented our formula carefully keeping track of the complex logarithm along the integration path. This leads to a smooth CF...". It seems that over the past years this approach was, and perhaps still is, best market practice when it comes to the implementation of stochastic volatility models. The problem with this approach is that it requires a very fine integration grid in order to be sure that no discontinuities can arise. Even then, one is not sure that the integration routine has singled out and corrected for all discontinuities, as one does not know a priori how many discontinuities there are.

A significant improvement on this ad-hoc approach has been made by Kahl and Jäckel [2005], who came up with the rotation count algorithm. This algorithm aims to ensure that the principal argument of ψ_1 in (12) is continuous. Most importantly, this algorithm is easily implementable and allows for any numerical integration scheme to be used to evaluate the option price in (1), hereby opening up myriads of possibilities for improving the efficiency of implementations of stochastic volatility models. We return to this algorithm in section 4.

Though Kahl and Jäckel have in their paper documented extensively what can and will go wrong if one uses the principal argument of the complex logarithm in conjunction with formulation 1, it is good to stress this again in a realistic example. For this we take the parameters that Duffie et al. [2000] implied from market data of S&P500 index options. The left panel of Figure 1 shows the discontinuities we would get in the argument or phase of $\psi_1(v - i(\alpha + 1))$ if we would use the principal branch only, compared with the argument obtained by applying the rotation count algorithm. In the right panel the impact of these discontinuities on the integrand in equation (1) are shown. Here we used the optimal α and transformed the integrand to the unit interval with a logarithmic transformation, as outlined in Lord and Kahl [2007]. Note that the values in the graphs do not include the scaling by $1/\pi$.

The chosen example is a 10 year at-the-forward European call, whose true price is 0.1676. The option price found from the green dashed line would have been 0.0119, a marked difference that will certainly not go unnoticed. At smaller maturities however, the differences may not stand out so clearly. With a maturity of 2.5 years there is only one discontinuity; the true option price is 0.0816 whereas the option price found by bluntly using the principal argument in formulation 1 yields 0.0839.

⁴ In the remainder arg(z) will denote the principal argument of z.



Figure 1: Complex discontinuities in the Heston model

Parameters: $\kappa = 6.21$, $\omega = 0.61$, $\rho = -0.7$, $\theta = 0.019$, v(0) = 0.010201, F = K = 1, $\tau = 10$, $\alpha = 3.35861$ (A) The principal argument/phase of ψ_1 with (red solid line) and without (green dashed line) correction (B) Integrand in equation (1) in formulation 1 (green dashed line) vs. formulation 2 (red solid line)

3. Why the principal branch can be used

In this section we prove the conjecture of Lord and Kahl [2006] that formulation 2 in (11) remains continuous when the complex logarithm is restricted to its principal branch. As mentioned, both Lord and Kahl [2006] and Albrecher et al. [2007] have provided partial proofs for this conjecture, though the combination of their results involves restrictions on both the parameters as well as on Im(u).

In order to discover which part of the puzzle is still missing, we will need to analyse the proofs used in both papers. Let us first introduce some notation. Throughout this paper we will write u = x + yi for $u \in \mathbb{C}$, with x = Re(u) and y = Im(u). For any complex valued number, say $z \in \mathbb{C}$, we will sometimes use the shorthand notation $z_r = \text{Re}(z)$ and $z_i = \text{Im}(z)$.

We only need to consider $u \in \Lambda_x$, the strip of regularity of the characteristic function for which $|\phi(u)| < \infty$. Analysing the strip of regularity, or moment stability, entails analysing the range of $\zeta \in \mathbb{R}$ for which $\phi(-\zeta i) < \infty$. Clearly this range will be of the form (ζ_-, ζ_+) . It can be seen that $\Lambda_x = \{u \in \mathbb{C} \mid -\text{Im}(u) \in (\zeta_-, \zeta_+)\}$, as we then immediately have:

$$\left|\phi(u)\right| = \left|\mathbb{E}[e^{iu\ln S(T)}]\right| \le \mathbb{E}[\left|e^{iu\ln S(T)}\right|] = \phi\left(-\operatorname{Im}(u)i\right) < \infty$$
(14)

Though moment stability in the Heston model is dealt with in great detail in Andersen and Piterbarg [2007], we will require the following result in the proof of our conjecture.

Theorem 1

The characteristic function of the Heston model is analytic for $u \in \Lambda \subset \mathbb{C}$, where $u \notin \Lambda$ if $\zeta = -\text{Im}(u)$ satisfies one of the two following conditions:

1.
$$G(-\zeta i)e^{-D(-\zeta i)\tau} = 1 \wedge D(-\zeta i) \neq 0$$
 (15)

2.
$$D(-\zeta i) = 0 \land \beta(-\zeta i) \neq 0 \land \tau = -2\beta(-\zeta i)^{-1}$$
 (16)

Proof:

Analysing the strip of regularity boils down to analysing the stability of the system of ODEs in equation (6) for $u = -\zeta i$. As the function A will simply be an integral over B_v , the stability of B_v is what matters. The solution in (9) is clearly stable if and only if the denominator is not equal to zero, modulo those cases where both the numerator and the denominator are zero and (9) remains well-defined. Let us first check those cases when the numerator equals zero, namely $\beta = D$ or D = 0. In the first case, the denominator will be equal to 1 and we have $B_v(u,\tau) = 0$. In the last case we need one application of l'Hôpital's rule to find:

$$\lim_{D(u)\to 0} B_{v}(u,\tau) = \lim_{D\to 0} \frac{e^{-D\tau}\tau(\beta - D) - (1 - e^{-D\tau})}{\omega^{2}e^{-D\tau}\frac{2\beta + \beta^{2}\tau - \tau D^{2}}{(\beta + D)^{2}}} = \frac{\beta^{2}\tau}{2\gamma(2 + \beta\tau)}$$
(17)

so that (9) remains well-defined, provided that $\tau \neq -2\beta(u)^{-1}$ whenever D(u) = 0. Note that this can only happen when Re(u) = 0, the situation we have here. Combined with the condition following from the denominator of (9) we arrive at the conditions (15)-(16). Note that $\Lambda_x \subset \Lambda$. \Box

Remark 1

For $\zeta \in \Lambda_z$ where $D(-\zeta i) = 0$ and $\beta(-\zeta i) < 0$ we have $\tau \le -2\beta(-\zeta i)^{-1}$. When $\tau = -2\beta(-\zeta i)^{-1}$, Theorem 1 states that $\zeta \notin \Lambda$, so that certainly $\phi(-\zeta i)$ is infinite for $\tau > -2\beta(-\zeta i)^{-1}$. \Box

Remark 2

Condition (15) is sufficient when $\rho \le \kappa/\omega$, as is proven in Lord and Kahl [2006]. We note that this condition, together with the restriction that $\rho \le \kappa/\omega$ also appears in Lee [2004, Appendix A.2], however without any hint as to how it can be derived. \Box

Finally, we note that solving ζ_+ and ζ_- from (15) is not a well-posed problem, as there are an infinite number of solutions. To this end it is convenient to use the critical time analysed in Andersen and Piterbarg [2007]. In Lord and Kahl [2007] good starting solutions are provided.

3.1. The proof of Lord and Kahl

The proof used by Lord and Kahl [2006] heavily hinges on the observation that $|G| \le 1$ for a large range of parameter values. If this is the case, it is not difficult to show that ψ_2 cannot cross the negative real line. Before turning to the main result, we state the following lemmas.

Lemma 1

When G(u) = 1, ψ_1 and ψ_2 will never cross the negative real line.

Proof:

When G(u) = 1, we have D(u) = 0 and hence:

$$\lim_{D(u)\to 0} \Psi_1(u,\tau) = \lim_{D(u)\to 0} \Psi_2(u,\tau) = 1 + \frac{1}{2}\beta(u)\tau$$
(18)

One can check that D(u) = 0 can only occur when Re(u) = 0. If $\beta(u) \ge 0$ the result is immediately clear. For $\beta(u) < 0$ we can appeal to Remark 1 to conclude that $\tau \le -2\beta(-\zeta i)^{-1}$ which precludes ψ_1 and ψ_2 from crossing the negative real line. \Box

Lemma 2

If $\rho \le \kappa/\omega$, or $\text{Im}(u) \ge y_2$ and $\kappa/\omega \le \rho \le 2\kappa/\omega$, we have:

$$\left| \mathbf{G}(\mathbf{u}) \right| = \left| \frac{\beta(\mathbf{u}) - \mathbf{D}(\mathbf{u})}{\beta(\mathbf{u}) + \mathbf{D}(\mathbf{u})} \right| \le 1$$
(19)

Proof: The proof for $\rho \le 0$ is stated in the appendix. For a full proof we refer the interested reader to Lord and Kahl [2006]. \Box

The main result from Lord and Kahl now follows. It uses the result from Lemma 2 and proves that under these conditions ψ_2 can never cross the negative real line.

Theorem 2

Let $u \in \Lambda_x$, $\rho \le \kappa/\omega$, or $\text{Im}(u) \ge -\kappa/(\rho\omega)$ and $\kappa/\omega \le \rho < 2\kappa/\omega$. If we are evaluating the characteristic function by means of formulation 2 in (11), the principal branch of the complex logarithm is the right one.

Proof:

We have to prove that $\psi_2(u,\tau)$, defined in (11), never crosses the negative real line for the parameter combinations under consideration. Suppose that it does, i.e. that:

$$\Psi_2(\mathbf{u},\tau) = \frac{\mathbf{G}(\mathbf{u})e^{-\mathbf{D}(\mathbf{u})\tau} - 1}{\mathbf{G}(\mathbf{u}) - 1} = -\xi$$
(20)

for some $\xi \ge 0$. We can assume that $G \ne 1$ as this case is covered by Lemma 1. Also, as $u \in \Lambda_x$ the numerator cannot equal zero when $D(u) \ne 0$, due to condition (15) in Theorem 1. Hence, we can assume that ξ is strictly larger than zero. Rearranging (20) yields:

$$G(\xi + e^{-D\tau}) = \xi + 1$$
(21)

In view of ξ , τ , $D_r \ge 0$ and $|G| \le 1$ we can take the modulus of the left-hand side:

$$|G(\xi + e^{-D\tau})| \le |G| \cdot (\xi + e^{-D_{\tau}\tau}) \le \xi + 1$$
(22)

The first inequality is strict unless $D_i \tau = 2n\pi$ with $n \in \mathbb{Z}$, and the second inequality is strict unless |G| = 1 and $D_r = 0$. Equality in (21) can therefore only occur if |G| = 1, $D_r = 0$ and $D_i \tau = 2n\pi$. First, one can check that |G| = 1 implies $\beta_i = 0$ or n = 0. If n = 0, G = 1, which we excluded a priori. Second, D_r can only be zero when u = yi. This shows that $\psi_2 = 1$, contradicting (20). \Box

The restrictions on the parameters in Lemma 2 and Theorem 2 are mainly formulated in terms of the correlation coefficient ρ as it characterises the main difference between stochastic volatility models among various asset classes. Whereas in an equity or FX context ρ is used to fit to the skew or smile present in that market, the correlation parameter ρ is often set to zero when used in a term structure context. Though there is empirical evidence for this, see references in e.g. Andersen and Brotherton-Ratcliffe [2005], the main reason is of a practical nature. If ρ would be unequal to zero, a change of probability measure would change the structure of the stochastic volatility driver and in a term structure context would cause forward rates to appear in its drift. By

setting ρ equal to zero this is avoided. A displacement coefficient is often added to the underlying order to be able to fit the implied volatility skew.

Negative values of ρ are typically required to be able to fit to a skew. Moreover, an implied calibration usually yields $\kappa > \omega$ in an implied calibration, implying that $\rho \le \kappa/\omega$ holds. For evidence of this in the literature we refer the reader to e.g. Bakshi, Cao and Chen [1997], Duffie, Pan and Singleton [2000] and Jäckel [2004]. Nevertheless, there are conceivable situations where this condition may not hold true, spurring us to prove Theorem 2 without restrictions.

3.2. Filling in the missing gaps

Whereas the proof of Lord and Kahl uses the observation that $|G| \le 1$ for a large range of parameter values, Albrecher et al. [2007] assume that the characteristic function $\phi(u)$ is being evaluated in $u \in \mathbb{C}$ with u = x + yi and y < -1, corresponding to using a positive α in (1). Not being able to choose Im(u) freely when valuing options via Fourier inversion would restrict the implementation of a robust option pricer, see Lee [2004] and Lord and Kahl [2007].

With $y_2 = -\kappa/(\rho\omega)$ and $D(x + yi) = \sqrt{p(x, y) + q(x, y) \cdot i}$, where both p and q are functions on the real line, it can be shown that the proof of Albrecher et al. is split into five cases:

 $\begin{aligned} 1. \ \rho &\leq 0 \ \text{and} \ q^{(1,0)}(x,y) \geq 0 \\ 2. \ \rho &\leq 0 \ \text{and} \ q^{(1,0)}(x,y) < 0 \\ 3. \ \rho &> 0 \ \text{and} \ y \geq y_2 \\ 4. \ \rho &> 0, \ y < y_2 \ \text{and} \ q^{(1,0)}(x,y) < 0 \\ 5. \ \rho &> 0, \ y < y_2 \ \text{and} \ q^{(1,0)}(x,y) \geq 0 \end{aligned}$

The proofs of Cases 1 and 3 use similar arguments to those used in Lemma 2. In the remaining cases it is proven that ψ_2 cannot be in either the second or third quadrant, and henceforth can never cross the negative real line.

Though the authors assume that y < -1, this assumption is not explicitly made clear in their proofs of the above cases. In fact, a closer look at their proof reveals that only Cases 1 and 3 use this assumption. First, in the proof of Case 1 it is stated that $\rho \le 0$ and $q^{(1,0)}(x,y) \ge 0$ imply $y \le y_2$. Though this is certainly true when y < -1, it is not true in general. Since the case where $\rho \le 0$ is dealt with in full generality by our Theorem 2, encompassing Cases 1 and 2, we need not worry about this.

Second, in the proof of Case 3 the authors state that $\rho > 0$ and $y \ge y_2$ implies $q^{(1,0)}(x,y) \le 0$. One can check that this is only true when $y \le -1/2$. Hence we have $y_2 \le y \le -1/2$, implying that $\rho \le 2\kappa/\omega$. Once again this case is dealt with in our Theorem 2. The case where $q^{(1,0)}(x,y)$ is larger than zero is an open problem, and is dealt with in the following lemma.

Lemma 3

When $\rho > 0$, $y \ge y_2$ and $q^{(1,0)}(x,y) > 0$, ψ_2 does not cross the negative real line.

Proof: See the appendix. \Box

Finally, though Case 4 has a minor overlap with Theorem 2, both Cases 4 and 5 deal with the situation where $\rho > 2\kappa/\omega$, which is not at all covered by our previous work. Since the only open problem has been dealt with in Lemma 3, we are ready to prove the main result of this paper, that formulation 2 should be used when restricting the complex logarithm to its principal branch.

Theorem 3

Evaluating the Heston characteristic function by means of formulation 2, where we restrict the complex logarithm to its principal branch, ensures that the characteristic function of the Heston model remains continuous.

Proof:

This immediately follows by combining the results of Theorem 2, Lemma 3 and Cases 4 and 5 of Albrecher et al. [2007]. \Box

4. Why the rotation count algorithm works

Having proven that no branch switching of the complex logarithm is required within formulation 2 of the Heston characteristic function, it seems that there is no longer a need for the rotation count algorithm of Kahl and Jäckel [2005]. In addition to its easier implementation, a further distinct advantage of formulation 2 is its numerical stability. Since D(x+yi) tends to $\omega\sqrt{1-\rho^2}x$ as x tends to infinity, its real part becomes quite large when calculating option prices. While this leads to numerical instabilities in the calculation of ψ_1 (formulation 1), as mentioned in Kahl and Jäckel's paper, ψ_2 (formulation 2) is much better behaved.

Nevertheless, from a theoretical perspective it is worthwhile to take a closer look at the rotation count algorithm, and show why it works. The rotation count algorithm automatically adapts the branch of the complex logarithm such that the characteristic function in formulation 1 is continuous, as it should be. For a, $b \in \mathbb{C}$ and $c \in \mathbb{R}$ the algorithm is basically concerned with the evaluation of d in:

$$\mathbf{d} = \mathbf{a}\mathbf{e}^{\mathbf{b}} + \mathbf{c} \tag{23}$$

such that its argument is kept continuous if both the complex function b and the argument of a are continuous functions. In the algorithm we will use both the classical and polar representations of a complex number, i.e. for $z \in \mathbb{C}$ we will write $z = z_r + z_i i$ with $z_r = \text{Re}(z)$ and $z_i = \text{Im}(z)$, as well as $z = |z| e^{iz_{\theta}}$ with |z| being its radius and z_{θ} its argument⁵. The algorithm follows as:

1. Calculate the phase interval of
$$ae^{b}$$
 as $n = \lfloor \frac{1}{2\pi} (a_{\theta} + b_{i} + \pi) \rfloor$, equal to 0 if it is in $[-\pi, \pi)$;
2. $|d| = |ae^{b} + c|$
3. $d_{\theta} = arg(ae^{b} + c) + 2\pi n$ where arg denotes the principal argument.

Algorithm 1: The rotation count algorithm of Kahl and Jäckel

The premise under which algorithm 1 is valid is that the addition of the real number c does not change the phase interval of the resulting complex number. If this is indeed the case, we can use two successive applications of the rotation count algorithm to evaluate the ratio ψ_1 in formulation 1, see equation (12). First we evaluate $d_1(u, \tau) = c(u)e^{D(u)\tau} - 1$, subsequently $d_2(u) = c(u) - 1$, and finally we write:

$$\Psi_{1}(\mathbf{u},\tau) = \frac{|\mathbf{d}_{1}(\mathbf{u},\tau)|}{|\mathbf{d}_{2}(\mathbf{u})|} e^{i(\mathbf{d}_{1\theta}(\mathbf{u},\tau) - \mathbf{d}_{2\theta}(\mathbf{u}))}$$
(24)

⁵ Note that this is not necessarily the principal argument.

The following lemma shows that evaluating ψ_1 in this way leads to a continuous characteristic function, if the premise of the rotation count algorithm is valid.

Lemma 4

Assume that the premise of the rotation count algorithm is valid and that u: $\mathbb{R} \to \Lambda_x$ describes a continuous path in the complex plane. Under these conditions applying the rotation count algorithm to evaluate the argument of $\psi_1(u,\tau)$ as $d_{1\theta}(u,\tau) - d_{2\theta}(u)$ yields a continuous characteristic function.

Proof:

Though the arguments $d_{1\theta}$ and $d_{2\theta}$ are not necessarily continuous, any discontinuities caused by arg(c) are cancelled out as they appear in both terms. The remaining term that could cause a discontinuity is D(u). If we write u = x + yi, one can check there are values of y where $\lim_{x \uparrow 0} D(x + yi) = \overline{\lim_{x \downarrow 0} D(x + yi)}$. Nevertheless this is not a problem either, as for $u \in \Lambda_x$ the characteristic function is real on the imaginary axis, so that this discontinuity in D(u) does not have an impact on the characteristic function. Under the premise of the rotation count algorithm the assertion is thus true. \Box

The remaining problem is now to check that the premise of the rotation count algorithm is true. Consider the following complex function:

$$f(x) = (1-x) - (\frac{1}{2} - x)i$$
(25)

where $x \in \mathbb{R}$. In the following figure we have drawn f(x)+1, f(x), f(x)-1 and f(x)-2 in the complex plane, as well as their principal arguments:



Figure 2: Addition of a real number to a complex number can change the phase interval (A) f(x)+1, f(x), f(x)-1 and f(x)-2 in the complex plane (B) Argument of all four functions as a function of x

While f(x)+1 and f(x) do have a continuous principal argument, f(x)-1 and f(x)-2 clearly do not. The discontinuities here are caused by the fact that these functions make a transition from the second to the third quadrant, implying that the principal argument changes from the interval $[\frac{1}{2}\pi, \pi)$ to $[-\pi, -\frac{1}{2}\pi)$, clearly causing a jump. The following lemma formalises this observation.

Lemma 5

Consider a continuous function $z: \mathbb{R} \to \mathbb{C}$, where both the real and complex part of z are strictly monotone. Assume that z never passes through the origin⁶. Adding $y \in \mathbb{R}$ where $y \neq 0$ to z(x) does not make the principal argument of z(x) + y discontinuous as a function of x when compared to the principal argument of z(x), if and only if:

- Re(z(x)) \notin (-y,0) for y > 0 whenever Im(z(x)) changes sign;
- $\operatorname{Re}(z(x)) \notin (0,-y)$ for y < 0 whenever $\operatorname{Im}(z(x))$ changes sign.

Proof: See the appendix. \Box

Lemma 5 immediately gives necessary and sufficient conditions under which algorithm 1 will work. Unfortunately, proving the rotation count algorithm works is more involved than the proof of Theorem 3. Whereas in the proof of Theorem 3 it was sufficient to check that ψ_2 never crossed the negative real line, we here have to simultaneously check that:

- whenever the imaginary parts of c(u) or $c(u)e^{D(u)\tau}$ are zero, their real parts are not in [0,1];
- if the imaginary part of $c(u)e^{D(u)\tau}$ is zero and its real part is in [0,1], this must also be the case for c(u), and vice versa.

Given that formulation 2 should in any case be preferred over the rotation count algorithm, we do not venture to provide a full proof. Nevertheless, it turns out that Lemmas 1 and 2 have given us enough machinery to prove that the rotation count algorithm works for almost all relevant parameter values. The following theorem is akin to Theorem 2.

Theorem 4

The rotation count algorithm can safely be applied to the Heston model as long as $u \in \Lambda_x$, $\rho \leq \kappa/\omega$, or $Im(u) \geq -\kappa/(\rho\omega)$ and $\kappa/\omega \leq \rho < 2\kappa/\omega$.

Proof:

First note that c = 1/G. For the parameter combinations considered here we have $|c| \ge 1$ by virtue of Lemma 2. Furthermore:

$$|\mathbf{c}\mathbf{e}^{\mathrm{D}\tau}| = |\mathbf{c}|\mathbf{e}^{\mathrm{D}_{\mathrm{r}}\tau} \ge |\mathbf{c}| \ge 1 \tag{26}$$

since we use the convention that the real part of the square root is nonnegative and $\tau \ge 0$. If the inequality is strict, we can immediately conclude that whenever the imaginary parts of c(u) or $c(u)e^{D(u)\tau}$ are zero their real parts are not in the interval [0,1]. When the inequality is an equality, we need only worry about the cases where G = -1 or G = 1. The first case is not a problem, whereas in the second case we need to evaluate ψ_1 as indicated in Lemma 1. \Box

5. Related issues in other models

Having analysed the Heston characteristic function in great detail, it is time to turn to other models. Firstly, we show that the Variance Gamma model does not suffer from any complex

⁶ Note that the principal argument of zero is undefined. As we only consider the characteristic function on its strip of regularity, it will always be well-defined and finite by virtue of (14).

discontinuities, even though it contains the multivalued complex power function. Secondly, using our results from the Heston model, we show how to avoid any complex discontinuities in both the the Schöbel-Zhu model and the exact simulation algorithm of the Heston model. Potential issues in other extensions of the Heston model are deferred till Section 6.

5.1. The Variance Gamma model

To demonstrate that other models besides stochastic volatility models may have complex discontinuities, we turn to a model of the exponential Lévy class, the Variance Gamma (VG) model, first introduced by Madan and Seneta [1990]. Here the underlying asset is modelled as:

$$S(t) = F(t)\exp(\omega t + \theta G(t) + \sigma W(G(t)))$$
(27)

where W(t) is a standard Brownian motion, G(t) is a Gamma process with parameter v > 0 and F(t) is the forward price of the underlying stock at time t. Without loss of generality we assume that σ is strictly positive. The parameter ω is chosen such that the expectation of (27) is F(t):

$$\omega = \frac{1}{\nu} \ln(1 - \theta \nu - \frac{1}{2} \sigma^2 \nu) \tag{28}$$

To simplify notation we introduce $f(t) = \ln F(t)$ and $\tilde{f}(t) = f(t) + \omega t$. If we denote τ as the time to maturity, the conditional characteristic function of the VG model is specified as:

$$\phi(\mathbf{u}) = \mathbb{E}\left[e^{i\mathbf{u}\ln S(T)}\right] = \frac{\exp\left(i\mathbf{u}\,\widetilde{\mathbf{f}}\,(T)\right)}{\left(1 - i\mathbf{u}(\theta + \frac{1}{2}\,i\sigma^{2}\mathbf{u})\mathbf{v}\right)^{\tau/\mathbf{v}}}$$
(29)

The ζ^{th} moment of the underlying asset exists as long as $\zeta \in (\zeta_{-}, \zeta_{+})$, defined by:

$$\zeta_{\pm} = -\frac{\theta}{\sigma^2} \pm \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\nu\sigma^2}}$$
(30)

so that the extended characteristic function in (29) is well-defined for $u \in \Lambda_x$, its strip of regularity. As the VG model is fully time-homogeneous, the maximum and minimum allowed moments do not depend on the maturity T, in contrast with the situation in the Heston model. In the denominator of (29) we are using the complex power function, again a multivalued function. The complex discontinuities in the Heston model were in fact also caused by the branch switching of the complex power function. Although the characteristic exponent of the Heston model contains a complex logarithm, this term is multiplied by $-2\kappa\theta/\omega^2$ and subsequently its exponent is taken. In essence we are thus raising ψ_1 or ψ_2 (depending on which formulation we use) to the power of $-2\kappa\theta/\omega^2$, so that the branch switching of the complex power function is the of our complex discontinuities. As mentioned in Section 2.2, if $2\kappa\theta/\omega^2 \in \mathbb{N}$, there will be no discontinuities in either formulation if we restrict the logarithm to its principal branch.

If we here restrict the complex power function to its principal branch, i.e. if we evaluate z^{α} for $z \in \mathbb{C}$ and $\alpha \in \mathbb{R}$ as $|z|^{\alpha} e^{i\alpha \arg(z)}$, with $\arg(z)$ being the principal argument of z, the characteristic function of the VG model will only be continuous if $1 - iu(\theta + \frac{1}{2}i\sigma^2 u)v$ does not cross the negative real line. In the following theorem it is proven that this never occurs.

Theorem 5

When evaluating the characteristic function of the VG model in $u \in \Lambda_x$, we can safely restrict the complex power function to its principal branch.

Proof:

If we write u = x + yi, the imaginary part of $1 - iu(\theta + \frac{1}{2}i\sigma^2 u)v$ can only equal zero when either x = 0 or $y = \theta/\sigma^2$. For x = 0 we have:

$$1 - iu(\theta + \frac{1}{2}i\sigma^2 u)v = 1 + v\theta y - \frac{1}{2}v\sigma^2 y^2$$
(31)

which for $u \in \Lambda_x$, or here $-y \in (\zeta_-, \zeta_+)$, is strictly positive. For $y = \theta/\sigma^2$ we find:

$$1 - iu(\theta + \frac{1}{2}i\sigma^{2}u)v = 1 + \frac{v\theta^{2}}{2\sigma^{2}} + \frac{1}{2}v\sigma^{2}x^{2} \ge 1$$
(32)

so that $1-iu(\theta + \frac{1}{2}i\sigma^2 u)v$ can clearly never cross the negative real line. The principal branch of the complex power function is the correct one, as this is the only one that leads to real values for the moment generating function. \Box

One can similarly check that the popular CGMY model, also known as the KoBoL or generalised tempered stable model, which contains the Variance Gamma model as a special case, also does not suffer from complex discontinuities in its original formulation.

5.2. The Schöbel-Zhu model

The first mention of discontinuities in characteristic functions caused by the branch switching of the complex logarithm or indeed the complex power function was in the article of Schöbel and Zhu [1999], who encountered these problems when implementing their extension of the Stein and Stein model to allow for non-zero correlation between the underlying asset and the stochastic volatility process. We will investigate whether, as in the Heston model, we can recast its characteristic function into a form suitable for the principal branch of the complex logarithm.

Under the risk-neutral pricing measure the underlying asset in the Schöbel-Zhu model evolves according to the following set of SDEs:

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW_{s}(t)$$

$$d\sigma(t) = -\kappa(\sigma(t) - \theta)dt + \omega dW_{\sigma}(t)$$
(33)

where the Brownian motions satisfy $dW_s(t) \cdot dW_{\sigma}(t) = \rho dt$. The difference with the Heston model is that instead of the stochastic variance, now the stochastic volatility itself follows an Ornstein-Uhlenbeck process. A problem with the Schöbel-Zhu model, as noted by e.g. Jäckel [2004], is that when the volatility process becomes negative, the sign of the instantaneous correlation between S and σ effectively changes. This is economically implausible.

Using the classification of Gaspar [2004] and Cheng and Scaillet [2007] one can conclude that the Schöbel-Zhu model is a linear-quadratic model in ln S and σ , and by the latter paper therefore equivalent to an affine model once we add the coordinate v(t) = $\sigma^2(t)$:

$$dv(t) = 2\sigma(t)d\sigma(t) + \omega^2 dt = \left(-2\kappa v(t) + 2\kappa\theta\sigma(t) + \omega^2\right)dt + 2\omega\sigma(t)dW_{\sigma}(t)$$
(34)

One can check that the model is certainly affine in ln S, σ and v, and its characteristic function will therefore have the same exponentially affine form as (5):

$$\phi(\mathbf{u}) = \mathbb{E}\left[e^{i\mathbf{u}\ln S(T)}\right] = \exp\left(i\mathbf{u}f + A(\mathbf{u},\tau) + B_{\sigma}(\mathbf{u},\tau) \cdot \sigma(0) + B_{v}(\mathbf{u},\tau) \cdot v(0)\right)$$
(35)

where A, B_{σ} and B_{ν} can be solved from the following system of ODEs:

$$\frac{dB_{v}}{d\tau} = \hat{\alpha}(u) - \beta(u)B_{v} + \gamma B_{v}^{2}$$

$$\frac{dB_{\sigma}}{d\tau} = 2\kappa\theta B_{v} + (-\frac{1}{2}\beta(u) + \gamma B_{v}) \cdot B_{\sigma}$$

$$\frac{dA}{d\tau} = \kappa\theta B_{\sigma} + \frac{1}{2}\omega^{2}B_{\sigma}^{2} + \omega^{2}B_{v}$$
(36)

subject to the initial conditions $B_v(u,0) = B_\sigma(u,0) = A(u,0) = 0$. The auxiliary variables are similar to the ones defined in the Heston model, namely $\hat{\alpha}(u) = -\frac{1}{2}u(i+u)$, $\beta(u) = 2(\kappa - \rho\omega ui)$ and $\gamma = 2\omega^2$. Indeed, there are more similarities with the Heston model. Following remarks by both Heston and Schöbel-Zhu, we know that when $\theta = 0$, the Schöbel-Zhu model collapses to a particular instance of the Heston model as can be seen from equation (34) – the variance then has a mean-reversion speed of 2κ , a volatility of variance equal to 2ω and a mean-reversion level of $\omega^2/2\kappa$. If we denote the Heston characteristic function as $\phi_H(u, S(0), v(0), \kappa, \omega, \theta, \rho, \tau)$, the Schöbel-Zhu characteristic function becomes:

$$\phi_{SZ}(u, S(0), \sigma(0), \kappa, \omega, \theta, \rho, \tau) = \phi_{H}(u, S(0), \sigma(0)^{2}, 2\kappa, 2\omega, \omega^{2} / 2\kappa, \rho, \tau)$$

$$\cdot \exp(A_{\sigma}(\tau) + B_{\sigma}(\tau) \cdot \sigma(0))$$
(37)

where A_{σ} follows the ODE $\frac{dA_{\sigma}}{d\tau} = \kappa \Theta B_{\sigma} + \frac{1}{2}\omega^2 B_{\sigma}^2$. By recognising that the characteristic function of the Schöbel-Zhu model can be expressed as an add-on on top of a special case of the Heston model, it is immediately clear that the discontinuities can be avoided in the same way as in the Heston model. Note that Schöbel and Zhu's original formulation of the characteristic function is different: the term under the complex logarithm is different to that found by using (37) in conjunction with either formulation 1 or 2. This explains why they would have had to correct for complex discontinuities when restricting the complex logarithm to its principal branch.

For completeness we provide the remainder of the characteristic function here. Tedious though straightforward manipulations show that A_{σ} and B_{σ} can be solved in closed-form as:

$$B_{\sigma}(u,\tau) = \kappa \theta \frac{\beta - D}{D\omega^{2}} \frac{(1 - e^{-\frac{1}{2}D\tau})^{2}}{1 - Ge^{-D\tau}}$$

$$A_{\sigma}(\tau) = \frac{(\beta - D)\kappa^{2}\theta^{2}}{2D^{3}\omega^{2}} \left(\beta(D\tau - 4) + D(D\tau - 2) + \frac{4e^{-\frac{1}{2}D\tau}\left(\frac{D^{2} - 2\beta^{2}}{\beta + D}e^{-\frac{1}{2}D\tau} + 2\beta\right)}{1 - Ge^{-D\tau}}\right)$$
(38)

with D defined as before in the Heston model.

5.3. The exact simulation algorithm of the Heston model

Though the Heston model was originally proposed in 1993, an exact simulation algorithm for the SDEs in (3) was not published until recently by Broadie and Kaya [2006]. It goes too far to outline the full algorithm in this paper. The crucial step of the algorithm is the simulation of the integrated square root process conditional upon its start and endpoint. As this distribution is not known in closed-form, Broadie and Kaya chose to simulate from it by inverting its cumulative distribution function, which is itself found by inversion of the characteristic function:

$$\begin{split} \phi(\mathbf{u}) &= \mathbb{E}\bigg[\exp\bigg(\mathrm{iu}\int_{s}^{t} \mathbf{v}(\mathbf{u}) \, d\mathbf{u}\bigg) \big| \ \mathbf{v}(s), \mathbf{v}(t)\bigg] = \frac{\mathrm{D}(\mathbf{u})e^{-\frac{1}{2}(\mathrm{D}(\mathbf{u})-\kappa)\tau}(1-e^{-\kappa\tau})}{\kappa(1-e^{-\kappa\tau})} \\ &\quad \cdot \exp\bigg(\frac{\mathbf{v}(s)+\mathbf{v}(t)}{\omega^{2}} \cdot \bigg[\frac{\kappa(1+e^{-\kappa\tau})}{1-e^{-\kappa\tau}} - \frac{\mathrm{D}(\mathbf{u})(1+e^{-\mathrm{D}(\mathbf{u})\tau})}{1-e^{-\mathrm{D}(\mathbf{u})\tau}}\bigg]\bigg) \\ &\quad \cdot \frac{\mathrm{I}_{v}\bigg(\sqrt{\mathbf{v}(s)\,\mathbf{v}(t)} \cdot \frac{4\mathrm{D}(\mathbf{u})e^{-\frac{1}{2}\mathrm{D}(\mathbf{u})\tau}}{\omega^{2}(1-e^{-\kappa\tau})}\bigg)}{\mathrm{I}_{v}\bigg(\sqrt{\mathbf{v}(s)\,\mathbf{v}(t)} \cdot \frac{4\kappa e^{-\frac{1}{2}\kappa\tau}}{\omega^{2}(1-e^{-\kappa\tau})}\bigg)} \end{split}$$
(39)

with $D(u) = \sqrt{\kappa^2 - 2\omega^2 iu}$, the degrees of freedom $\nu = 2\kappa\theta/\omega^2 - 1$ and I_ν representing the modified Bessel function of the first kind. Finally, τ equals t - s. As the characteristic function in (39) depends non-trivially on the two realisations v(s) and v(t), it is not an easy task to precompute major parts of the calculations. As a result this step of the algorithm will be highly time-consuming. It is therefore not surprising that Lord, Koekkoek and Van Dijk [2006] find that several biased simulation schemes, among which their own full truncation scheme and the scheme proposed by Kahl and Jäckel [2006], outperform the exact simulation scheme in terms of both speed and accuracy, even when the asset value is only required at one time instance. Nevertheless, the exact simulation method can be very useful as a benchmark.

Turning to (39), we note that its numerator contains a complex-valued modified Bessel function. Broadie and Kaya carefully tracked arg(z) when evaluating $I_v(z)$ in (39), and changed the branch when necessary by means of the following continuation formula, cf. Abramowitz and Stegun [1972]:

$$I_{v}(ze^{m\pi i}) = e^{m\nu\pi i}I_{v}(z)$$

$$\tag{40}$$

with m an integer value. To demonstrate it really is necessary to track the branch of I_{v} , we define:

$$z(u) = \frac{D(u)e^{-\frac{1}{2}D(u)\tau}}{1 - e^{-D(u)\tau}}$$
(41)

which up to a scaling by $\frac{4}{\omega^2}\sqrt{v(s)v(t)}$ is the complex-valued argument of I_v in (39). In Figure 3 we take the second parameter set Broadie and Kaya considered, and graph $\gamma(u) = z(u) / |z(u)|^{9/10}$ as a function of u. From Figure 3 it is clear that $\arg(z(u))$ is discontinuous, as (the rescaled version of) z(u) repeatedly crosses the negative real line in this plot. The key issue is therefore to find a way to keep track of the correct branch of z(u). The insights of the Heston model allow us to do exactly this, as the following lemma shows.



Figure 3: Plot of $\gamma(u) = z(u) / |z(u)|^{9/10}$ with hue function $\log_{10} (u+1)$ for $u \in [0, 100]$ Parameters from Broadie and Kaya [2006]: $\kappa = \omega = 1, \theta = 0.09, \tau = 5$

Lemma 6

The function $-\frac{1}{2}\tau \operatorname{Im}(D(u)) + \operatorname{arg}(f(u))$ with $f(u) = \frac{D(u)}{1 - e^{-D(u)\tau}}$, corresponding to the argument of z(u) in (41), is continuous for $u \in \Lambda_x = \{u \in \mathbb{C} \mid -\operatorname{Im}(u) \in (-\infty, (\kappa^2 + 4\pi^2)/2\omega^2) \}$.

Proof:

The strip of regularity can easily be checked from (39). That Im(D(u)) is continuous should be clear, so that it is sufficient to prove that f(u) never crosses the negative real line. We once again write u = x + yi. As $f(-\overline{u}) = \overline{f(u)}$ and f(yi) > 0 since $u \in \Lambda_x$, it is sufficient to focus on the case where x > 0. We will prove that the imaginary part of f(u) can never be positive, so that it will never cross the negative real line. First of all note that the sign of Im(D) coincides with the sign of $Im(D^2) = -2\omega^2 x < 0$. To show that $Im(f) \le 0$ is equivalent to proving:

$$\mathbf{d}_{i}\tau \mathbf{e}^{\mathbf{d}_{r}\tau} - \mathbf{d}_{i}\tau\cos(\mathbf{d}_{i}\tau) - \mathbf{d}_{r}\tau\sin(\mathbf{d}_{i}\tau) \le 0$$

$$\tag{42}$$

Replacing $d_r \tau$ by $a \ge 0$ and $d_i \tau$ by $-b \le 0$ for notational convenience we find:

$$-be^{a} + b\cos(b) + a\sin(b) \le b(\cos(b) - 1) + a(\sin(b) - b) \le 0$$
(43)

where the first inequality followed by a first order expansion of the exponent, and the second inequality follows by noting that $\cos(b) - 1 \le 0$ and $\sin(b) - b \le 0$. Finally, note that when D(u) = 0, which happens only when Re(u) = 0 and $\text{Im}(u) = -\kappa^2/2\omega^2$, we have $f(u) = \exp(-\tau)$. \Box

This is sufficient information if we use the power series to evaluate the modified Bessel function, as we can then evaluate $z(u)^k$ as $exp(k \ln z(u))$, where the logarithm is kept continuous if it is evaluated as $\ln z(u) = -\frac{1}{2}D(u)\tau + \ln f(u)$, and $\ln f(u)$ is restricted to its principal branch. Nevertheless, there are alternative numerical methods available for evaluating the modified Bessel function, and we need to come up with a formulation that is independent of the chosen numerical algorithm. Theorem 6 provides us with such a formulation.

Theorem 6

The characteristic function is continuous for $u \in \Lambda_x$ if we evaluate it as:

$$\phi(\mathbf{u}) \cdot \frac{\exp(\mathbf{v} \ln z(\mathbf{u}))}{z(\mathbf{u})^{\mathbf{v}}} \tag{44}$$

where $\phi(u)$ is evaluated by using the principal branch for the modified Bessel function, ln z(u) in the numerator is evaluated as sketched above, and the denominator uses the principal branch of the complex power function.

Proof:

Immediately follows by realising that the additional term in (44) is exactly the correction term in the continuation formula (40). \Box

To monitor the discontinuity of the characteristic function Broadie and Kaya would have had to use a very fine discretisation of the Fourier integral leading to the cumulative density function. The method we propose opens up the possibility of using arbitrary quadrature schemes, hereby speeding up their exact simulation algorithm considerably. Nonetheless, we expect that biased simulation schemes will remain the simulation schemes of choice, certainly after the arrival of the highly accurate schemes recently introduced by Andersen [2007].

6. Conclusions

In this paper we have analysed the complex discontinuities which are found when evaluating the closed-form characteristic function of several popular option pricing models. Such discontinuities have first been documented in option pricing by Schöbel and Zhu [1999], and are, at least in the Heston and Schöbel-Zhu stochastic volatility models and their extensions, caused by the branch switching of the complex logarithm. Being unaware of these issues can lead to completely wrong option prices if we price European options by means of Fourier inversion.

When pricing options via Fourier inversion, the method which most practitioners seem to use to correct for these discontinuities is to carefully monitor the imaginary part of the complex logarithm and change its branch if a discontinuity is detected. Clearly this method is not foolproof, and moreover, highly inefficient. The only method to this date to guarantee a continuous characteristic function is to bypass the closed-form solution and numerically integrate the ordinary differential equation that gives rise to the complex logarithm. Unfortunately this approach forsakes the comparative advantage of these option pricing models, precisely the fact that their characteristic function can be calculated in closed-form.

As a foolproof alternative, Kahl and Jäckel [2005] recently proposed their rotation count algorithm, which is an easily implementable algorithm that claims to be able to keep the complex logarithm in the Heston model continuous. Lord and Kahl [2006] proved, under relatively mild conditions on the parameters, that this is indeed the case. Under the same conditions we had already verified in the same paper that in an alternative formulation of the Heston characteristic function, which has appeared in e.g. Bakshi, Cao and Chen [1997], Duffie, Pan and Singleton [2000] and Gatheral [2006], the principal branch of the complex logarithm is the correct one. Other papers have appeared on this issue. Most notably Albrecher et al. [2007] have considered the second formulation under the restriction that the imaginary argument of the characteristic function is smaller than minus 1, corresponding to positive values of the damping coefficient α in Carr and Madan's option pricing formula, and have proven that in this case the principal branch is the correct one. While our proof and theirs do not overlap entirely, the union of both proofs does

not cover all possible configurations of the Heston model. In this paper we analysed the cases that were still open, and filled in the missing gaps. This proves that with the second formulation we do not have to worry about the branch switching of the complex logarithm, and can stick with its principal branch. As this formulation is easier to implement than the rotation count algorithm, and in addition more numerically stable, it should be the preferred formulation.

With the lessons from the Heston model in hand, the remainder of this article investigates the complex discontinuities that arise in a selection of other models. First of all we show that although the Variance Gamma model involves the complex power function, its characteristic function can be evaluated by restricting the complex power function to its principal branch. Secondly, we have shown how to avoid complex discontinuities in both the Schöbel-Zhu model and the exact simulation algorithm of the Heston model proposed by Broadie and Kaya [2006].

Many other models may suffer from similar problems, but it is clearly beyond the scope of this paper to consider them all here. In conclusion we will merely mention several extensions of Heston's model that are practically relevant. The first example we mention is the pricing of forward starting options in Heston's model, not via the bivariate integral of Kruse and Nögel [2005], but via the equivalent univariate integral of Hong [2004] and Lucic [2004], whose work demonstrates that we can use Carr-Madan's pricing formula to price these options, since the characteristic function of $\ln S(T) / S(t)$ for $t \le T$ can be derived in closed-form. It appears that one can use our findings from Section 3 to construct a formulation in which we can restrict the complex logarithm to its principal branch. The remaining examples we discuss appear to be harder to analyse. One such example is, strangely enough, Heston's model with piecewise constant parameters, considered in e.g. Mikhailov and Nögel [2004]. Though it allows for a much greater flexibility when calibrating to market data, the fact that its characteristic function is solved by repeated application of the tower law of conditional expectation does not facilitate the analysis. Other examples are Matytsin's [1999] model, which we considered in Lord and Kahl [2006], and Duffie, Pan and Singleton's [2000] SVJJ model, which allows for correlated jumps in the asset and stochastic variance. Finally, in the joint characteristic function in the Heston model we have so far not been able to find a formulation in which we can safely restrict the complex logarithm to its principal branch. Though there are no options which directly depend on the latent stochastic volatility, the joint characteristic function may be of importance when pricing exotic options via lattice-based algorithms such as the CONV algorithm, see Lord, Fang, Bervoets and Oosterlee [2008].

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Appendix – Proofs

In this appendix we provide the proofs for lemma's 2, 3 and 5.

Lemma A.1

If x, $y \in \mathbb{C}$ and $y = \sqrt{x}$, with $y_r > 0$, we can write:

$$y_r = \sqrt{\frac{1}{2}x_r + \frac{1}{2}\sqrt{x_r^2 + x_i^2}} \qquad y_i = \frac{x_i}{2y_r}$$
 (A.1)

if we use the convention that the real part of the square root is positive. \Box

Lemma A.2

For x, $y \in \mathbb{C}$ we have that if $\operatorname{Re}(x)\operatorname{Re}(\sqrt{y}) - \operatorname{Im}(x)\operatorname{Im}(\sqrt{y}) \ge 0$, then $x\sqrt{y} = \sqrt{x^2y}$. If the condition is not satisfied, we have $x\sqrt{y} = -\sqrt{x^2y}$.

Lemma A.3

For x, $y \in \mathbb{C}$ consider the following complex number:

$$z = \frac{x - \sqrt{x^2 + y}}{x + \sqrt{x^2 + y}}$$
(A.2)

Its modulus is equal to 1 if x = 0 or $\text{Re}(y/x^2) \le -1$ and $\text{Im}(y/x^2) = 0$, and less than 1 if:

$$\operatorname{Re}(\frac{1}{x})\operatorname{Re}(\sqrt{x^{2}+y}) - \operatorname{Im}(\frac{1}{x})\operatorname{Im}(\sqrt{x^{2}+y}) \ge 0$$
(A.3)

Proof:

If x = 0 we immediately have z = -1. Assume that (A.3) holds true and that $x \neq 0$. Then:

$$z = \frac{x - \sqrt{x^2 + y}}{x + \sqrt{x^2 + y}} = \frac{1 - \frac{1}{x}\sqrt{x^2 + y}}{1 + \frac{1}{x}\sqrt{x^2 + y}} = \frac{1 - \sqrt{1 + \frac{y}{x^2}}}{1 + \sqrt{1 + \frac{y}{x^2}}}$$
(A.4)

by virtue of lemma A.2. We can write this as z = (1-u)/(1+u) with $u_r \ge 0$. Its modulus satisfies:

$$|z|^{2} = \left|\frac{1-u}{1+u}\right|^{2} = \frac{(1-u_{r})^{2} + u_{i}^{2}}{(1+u_{r})^{2} + u_{i}^{2}} \le 1$$
(A.5)

with equality attained only when $u_r = 0$, i.e. if and only if $\text{Re}(y/x^2) \le -1$ and $\text{Im}(y/x^2) = 0$. Clearly the modulus of z is larger than or equal to 1 if (A.3) does not hold true. \Box

The following proposition collects many properties of some functions that we require hereafter. All properties can be proven by using basic algebra, so that we omit the proof.

Proposition

Consider the following two functions:

$$p(x, y) = \operatorname{Re}(D(u)^{2}) = \kappa^{2} + \omega^{2}(1 - \rho^{2})x^{2} - \omega(\omega - 2\kappa\rho)y - \omega^{2}(1 - \rho^{2})y^{2}$$

$$q(x, y) = \operatorname{Im}(D(u)^{2}) = \omega(\omega - 2\kappa\rho)x + 2\omega^{2}(1 - \rho^{2})xy$$
(A.6)

where we have introduced the convention that u = x + yi, for $x, y \in \mathbb{R}$. Furthermore, define:

$$y_1 = -\frac{\omega - 2\kappa\rho}{2\omega(1 - \rho^2)} \qquad y_2 = -\frac{\kappa}{\rho\omega}$$
(A.7)

Note that:

- when $\rho < 0$ or $\rho \ge 2\kappa/\omega$, $y_2 \ge y_1$;
- when $0 < \rho \le 2\kappa/\omega$, $y_2 \le y_1$.

For $x \ge 0$ the function p has the following properties:

- p is maximal w.r.t. y in y_1 ;
- For $y < y_1$, p is strictly increasing in y, for $y > y_1$, p is strictly decreasing in y;
- p is always strictly increasing in x;

Similarly, we can show that for $x \ge 0$ the function q has the following properties:

- $q(x,y_1) = 0;$
- q is positive and strictly increasing in x for $y > y_1$;
- q is negative and strictly decreasing in x for $y < y_1$;
- q is strictly increasing in y. \Box

The following lemma is key to proving lemma 2. Its proof uses many of the previous properties.

Lemma A.4

For $x \ge 0$ the functions p and q defined in the previous proposition satisfy:

$$(\kappa + \omega \rho y) \left(p(x, y) + \sqrt{p(x, y)^2 + q(x, y)^2} \right) - \omega \rho x q(x, y) \ge 0$$
(A.8)

if in addition $\rho \le \kappa/\omega$, or $y \ge y_2$ and $\kappa/\omega \le \rho \le 2\kappa/\omega$.

Proof:

The full proof is given in Lord and Kahl [2006], we only provide the proof for $\rho < 0$ and $\rho = 0$ here. Each case is divided into several sub cases, based on ranges for the variable y.

<u>Case 1: ρ < 0</u>

1a) $y \le y_1 \le y_2$ We can reshuffle:

$$(\kappa + \omega \rho y) \sqrt{p(x, y)^2 + q(x, y)^2} \ge -(\kappa + \omega \rho y) p(x, y) + \omega \rho x q(x, y)$$
(A.9)

Since $y \le y_2$ implies $\kappa + \omega \rho y \ge 0$, it is sufficient to square both sides and prove the resulting inequality. We obtain:

$$\omega^{2} xq(x, y) f(x, y) \ge 0$$

$$f(x, y) = \omega \rho (2\kappa - \omega \rho) (x^{2} + y^{2}) + \kappa^{2} (1 + 2y)$$
(A.10)

Since q is negative for $y \le y_1$, we have to prove that $f(x,y) \le 0$. The function f is maximal with respect to x for x = 0, and maximal w.r.t. y in:

$$y_3 \equiv \frac{\kappa^2}{-\omega\rho(2\kappa - \omega\rho)} \tag{A.11}$$

Here we have $y_3 \ge 0$. Since $y_1 \le 0$, it suffices to show that $f(0,y_1) < 0$. We have:

$$\frac{4\omega(1-\rho^2)^2}{\rho}f(0,y_1) = (2\kappa - \omega\rho)((2\kappa - \omega\rho)^2 + \omega^2(1-\rho^2)) \equiv g(\rho)$$
(A.12)

Clearly, $g(\rho) > 0$ for $\rho < \min(2\kappa/\omega, 1)$. Since the left-hand side is negative for $\rho < 0$, $f(0,y_1) < 0$.

1b) $y_1 \le y \le y_2$

We still have $\kappa + \omega \rho y \ge 0$. If p is positive, the inequality is immediately seen to be true. If p is negative, we have $\sqrt{p(x, y)^2 + q(x, y)^2} \ge -p(x, y)$, so that the inequality clearly also holds.

1c) $y_1 \le y_2 \le y$

Here $\kappa + \omega \rho y \leq 0$. First note that in this region:

$$(\kappa + \omega \rho y)p(x, y) - \omega \rho xq(x, y) \ge 0$$
(A.13)

If p is negative, the proof is easy, since $\rho \le 0$ and q is nonnegative. So let us assume that p is positive. Working out the function shows that x^2 is the only power of x in it, and its coefficient is:

$$\omega^{2} \left(\kappa (1 + \rho^{2}) - \omega \rho (1 + (1 - \rho^{2}) y) \right)$$
(A.14)

which is increasing in y, and positive for $y \ge 0$. Since $y_2 \ge 0$ in this region, the coefficient of x^2 is positive. Because p too is strictly increasing in x, it suffices to check the inequality for that x where p(x,y) = 0. But then the remaining inequality is $-\omega pxq(x,y) \ge 0$ which is immediately seen to be true. Hence, if we can prove the inequality from 1a) but now in reverse, we are done:

$$\omega^2 xq(x,y)f(x,y) \le 0 \tag{A.15}$$

Since q is here positive, it remains to show that f(x,y) is negative. As in 1a), f is maximal w.r.t. x in x = 0 and maximal w.r.t. y in y₃. In this region $y_2 \ge y_3$, so that it is sufficient to check that $f(0,y_2) \le 0$. It turns out that $f(0,y_2) = 0$, which concludes the proof of case 1.

Case 2: $\rho = 0$

When $\rho = 0$, the inequality can be reduced to $p(x, y) + \sqrt{p(x, y)^2 + q(x, y)^2} \ge 0$. Using the rationale of 1b) it is clear that this is true. \Box

We now have the necessary machinery to prove lemma 2.

Lemma 2

If $\rho \le \kappa/\omega$, or $\text{Im}(u) \ge y_2$ and $\kappa/\omega \le \rho \le 2\kappa/\omega$, we have:

$$\left| \mathbf{G}(\mathbf{u}) \right| = \left| \frac{\beta(\mathbf{u}) - \mathbf{D}(\mathbf{u})}{\beta(\mathbf{u}) + \mathbf{D}(\mathbf{u})} \right| \le 1$$
(A.16)

Proof:

As before, we will write u = x + yi here. It is fairly easy to show that $G(-\overline{u}) = G(u)$, so that it suffices to focus on the case $x \ge 0$. Since $D^2 = \beta^2 + \omega^2 u(i+u)$, G(u) is of the form treated in lemma A.3. Let us therefore first assume that $\beta \ne 0$. The condition from lemma A.3 which guarantees that (A.16) holds, is then:

$$\operatorname{Re}(\beta)\operatorname{Re}(D) + \operatorname{Im}(\beta)\operatorname{Im}(D) \ge 0 \tag{A.17}$$

which we obtain from (A.3) by multiplying with $|\beta|^2$.

Case 1: $x \ge 0$, $\beta \ne 0$, Re(D) = 0

Lemma A.1 shows us that:

$$\operatorname{Re}(D(u)) = \sqrt{\frac{1}{2}p(x,y)^{2} + \frac{1}{2}\sqrt{p(x,y)^{2} + q(x,y)^{2}}}$$
(A.18)

If $\operatorname{Re}(D) = 0$ we must therefore have $q = \operatorname{Im}(D^2) = 0$ and $p = \operatorname{Re}(D^2) \le 0$. Since $D = \sqrt{p + qi}$, it is clear that we then have $D = i\sqrt{-p}$, and (A.17) becomes:

$$-\omega\rho x \sqrt{-p(x,y)} \ge 0 \tag{A.19}$$

Clearly q can only be zero if x = 0 or if $y = y_1$. Tedious but straightforward algebra shows that p is strictly positive when $y = y_1$ and x > 0, so that we can conclude that x = 0, and that (A.17) always holds in case 1.

Case 2: $x \ge 0$, $\beta \ne 0$, Re(D) > 0

Since Re(D) > 0, we can multiply (A.17) by 2Re(D) and apply lemma A.1 to obtain:

$$(\kappa + \omega \rho y) \left(p(x, y) + \sqrt{p(x, y)^2 + q(x, y)^2} \right) - \omega \rho x q(x, y) \ge 0$$
(A.20)

This inequality is proven in lemma A.4 under the conditions which we impose on ρ and y.

Case 3: $x \ge 0, \beta = 0$

When $\beta = 0$, lemma A.3 states that |G| = 1. We can only have $\beta = 0$ when $\rho \neq 0$, x = 0 and $y = y_2$.

This concludes the proof. \Box

Lemma 3

When $\rho > 0$, $y \ge y_2$ and $q^{(1,0)}(x,y) > 0$, ψ_2 does not cross the negative real line.

Proof:

We must have $x + yi \in \Lambda_x$, so that $\psi_2(yi)$ will not lie on the negative real line by construction. Since $\psi_2(-\overline{u}) = \overline{\psi_2(u)}$, it suffices to prove that $\psi_2(x+yi)$ will not cross the negative real line for values of $x \ge 0$. In addition we can impose |G| > 1 w.l.o.g., as for $|G| \le 1$ we already know that ψ_2 can never cross the negative real line from Theorem 2. Since $q^{(1,0)}(x,y) > 0$ we have $y \ge y_1$, and one can check that both d_r and d_i are strictly positive. Also, $y \ge y_2$ implies $\kappa + \omega \rho y \ge 0$. Note that:

$$2\psi_2(u) = \beta \cdot \frac{1 - e^{-d\tau}}{d} + 1 + e^{-d\tau}$$
(A.21)

We will prove that ψ_2 cannot lie in the second quadrant, which implies the negative real line can never be crossed. Let us define the following positive constants:

$$A = d_{i}\omega\rho x - d_{r}(\kappa + \omega\rho y)$$

$$B = d_{r}^{2} + d_{i}^{2}$$
(A.22)

$$C = d_{r}\omega\rho x + d_{i}(\kappa + \omega\rho y)$$

Positivity of A follows from the fact that |G| > 1, so that the reverse of (A.17) is true. First of all note that the imaginary part of ψ_2 equals (up to a positive scaling, namely $2|d|^2 \exp(d_r \tau)$):

$$\operatorname{Im}(\Psi_2) \propto -\sin(d_i\tau) \cdot (A+B) + C \cdot \left(\cos(d_i\tau) - e^{d_r\tau}\right)$$
(A.23)

If the sine would be positive, the whole term would obviously be negative, and we would be finished. Hence, we can assume that the sine is negative. Let us suppose that ψ_2 lies in the second quadrant, implying that (A.23) is positive, and hence:

$$B > \frac{C \cdot \left(\cos(d_{i}\tau) - e^{d_{i}\tau}\right)}{\sin(d_{i}\tau)} - A$$
(A.24)

We will try to arrive at a contradiction by proving that $\text{Re}(\psi_2) > 0$. Up to the same scaling:

$$\operatorname{Re}(\Psi_{2}) \propto \sin(d_{i}\tau) \cdot C + A\left(\cos(d_{i}\tau) - e^{d_{r}\tau}\right) + B\left(e^{d_{r}\tau} + \cos(d_{i}\tau)\right)$$
(A.25)

Since the coefficient of B is positive, we can invoke (A.24) to bound this from below by:

$$\frac{-2\operatorname{Ae}^{d_{r}\tau}\sin(d_{i}\tau) - \operatorname{C}(e^{2d_{r}\tau} - 1)}{\sin(d_{i}\tau)}$$
(A.26)

We claim that this is positive, which amounts to proving that:

$$2A\sin(d_{i}\tau) + C(e^{d_{r}\tau} - e^{-d_{r}\tau}) \ge 0$$
(A.27)

The first derivative of the left-hand side w.r.t. τ is:

$$2Ad_{i}\cos(d_{i}\tau) + Cd_{r}(e^{d_{r}\tau} + e^{-d_{r}\tau}) > Cd_{r}(e^{d_{r}\tau} + e^{-d_{r}\tau}) - 2Ad_{i} > 2Cd_{r} - 2Ad_{i}$$
(A.28)

so that it is sufficient to prove that:

$$Cd_{r} - Ad_{i} = (d_{r}^{2} - d_{i}^{2})\omega\rho x + 2d_{i}d_{r}(\kappa + \rho\omega y) > 0$$
(A.29)

as then the left-hand side of (A.27) is increasing in τ , and it is zero for $\tau = 0$, so that we can conclude that (A.27) is true. When $d_r \ge d_i$ (A.29) is obviously true. Dividing (A.29) by $d_i^2 > 0$:

$$\frac{\operatorname{Cd}_{\mathrm{r}} - \operatorname{Ad}_{\mathrm{i}}}{d_{\mathrm{i}}^{2}} = \left(\frac{d_{\mathrm{r}}^{2}}{d_{\mathrm{i}}^{2}} - 1\right)\omega\rho x + 2\frac{d_{\mathrm{r}}}{d_{\mathrm{i}}}(\kappa + \rho\omega y) > 0$$
(A.30)

Now, note that since q(x,y) > 0 we can write:

$$\frac{\mathbf{d}_{\mathrm{r}}}{\mathbf{d}_{\mathrm{i}}} = \frac{\mathbf{p}(\mathbf{x}, \mathbf{y})}{\mathbf{q}(\mathbf{x}, \mathbf{y})} + \sqrt{\frac{\mathbf{p}(\mathbf{x}, \mathbf{y})^{2}}{\mathbf{q}(\mathbf{x}, \mathbf{y})^{2}}} + 1 \Longrightarrow \frac{\mathbf{d}_{\mathrm{r}}}{\mathbf{d}_{\mathrm{i}}} = \mathbf{z}_{\mathrm{d}} \iff \frac{\mathbf{p}(\mathbf{x}, \mathbf{y})}{\mathbf{q}(\mathbf{x}, \mathbf{y})} = \frac{\mathbf{z}_{\mathrm{d}}^{2} - 1}{2\mathbf{z}_{\mathrm{d}}}$$
(A.31)

In order for d_r to be smaller than d_i, p/q must be negative. As for $y > y_1 q$ is positive, this implies p must be negative. Now, let us solve the equation $p/q = z_{pq}$ for x. Note that $z_{pq} < 0$ whereas $0 \le z_d < 1$. This yields two solutions:

$$x_{1,2} = z_{pq}(y - y_1) \pm \frac{\sqrt{h(y)}}{2\omega(1 - \rho^2)}$$
(A.32)

with:

$$h(y) \equiv \left(2z_{pq}\omega(1-\rho^2)(y-y_1)\right)^2 - 4(1-\rho^2)p(0,y)$$
(A.33)

As $p(0,y) = p(x,y) - \omega^2(1-\rho^2)x^2$ and p(x,y) < 0, we must have p(0,y) < 0, so that h(y) is always positive. Note that in (A.32) $z_{pq} < 0$ and $y > y_1$, implying that the first term is negative. Secondly, it is obvious that the smallest solution (x_1) is negative, whereas the largest solution (x_2) is positive. Disregarding the negative solution, we can rewrite inequality (A.30) as:

$$(z_d^2 - 1)\omega\rho x_2 + 2z_d(\kappa + \rho\omega y) > 0$$
(A.34)

Now, consider $p(x_2, y)$. For which values of y is this negative? We have:

$$p(x_2, y) = p(0, y) + \frac{\left(2z_{pq}\omega(1-\rho^2)(y-y_1) + \sqrt{h(y)}\right)^2}{4(1-\rho^2)}$$
(A.35)

Tedious algebra shows that the only two real zeroes of this equation coincide with those of p(0,y):

$$p(0, y) = 0 \Leftrightarrow y = y_1 \pm \frac{\sqrt{(2\kappa - \omega\rho)^2 + \omega^2(1 - \rho^2)}}{2\omega(1 - \rho^2)}$$
(A.36)

Obviously only the positive zero has to be considered here. Let us call this zero y_3 . It can be shown that $y_3 > y_2$. Also note that h(y) is minimal in y_1 , and since $h(y_3) > 0$ and $y_3 > y_1$, h(y) is positive for $y > y_3$. Finally, since $\lim_{y \to \pm \infty} p(x_2, y) = \mp \infty$, we conclude that $p(x_2, y)$ is decreasing and

negative on this domain.

We now have to investigate (A.34) for $y \ge y_3$ and $0 < z_d < 1$ (or $z_{pq} < 0$). It turns out that (A.34) is increasing in y. First of all, note that to show that (A.34) is increasing in y is equivalent to proving that:

$$x'_{2}(y) < -z_{pq}^{-1}$$
 (A.37)

Expanding the left-hand side and rearranging yields:

$$\frac{2\omega(1-\rho^2)(1+z_{pq}^2)(y-y_1)}{\sqrt{h(y)}} < -z_{pq} - \frac{1}{z_{pq}}$$
(A.38)

Since both sides are positive and h(y) > 0, we can bound h(y) in (A.33) as:

$$h(y) > (2z_{pq}\omega(1-\rho^2)(y-y_1))^2$$
(A.39)

and hence:

$$\frac{2\omega(1-\rho^2)(1+z_{pq}^2)(y-y_1)}{\sqrt{h(y)}} < \frac{2\omega(1-\rho^2)(1+z_{pq}^2)(y-y_1)}{-2z_{pq}\omega(1-\rho^2)(y-y_1)} = -z_{pq} - \frac{1}{z_{pq}}$$
(A.40)

as we wanted to prove. Hence, it is sufficient to prove (A.34) for $y = y_3$. Equating y to y_3 and multiplying by $(1-\rho^2)/z_d > 0$ yields:

$$\rho\sqrt{(2\kappa-\omega\rho)^2+\omega^2(1-\rho^2)} > -(2\kappa-\rho\omega)$$
(A.41)

Clearly, if $\rho < 2\kappa/\omega$ this is true. If the reverse is true, we can square both sides and from the assumption that $\rho > 2\kappa/\omega$ it follows that (A.41) is true. Finally, we can conclude that (A.34) and hence (A.29) holds true for $d_r < d_i$ as well. This shows that if $Im(\psi_2) > 0$, ψ_2 must lie in the first quadrant, so that ψ_2 can indeed never lie in the second quadrant. The negative real line can therefore never be crossed. \Box

Lemma 5

Consider a continuous function $z: \mathbb{R} \to \mathbb{C}$, where both the real and complex part of z are strictly monotone. Assume that z never passes through the origin. Adding $y \in \mathbb{R}$ where $y \neq 0$ to z(x) does not add any discontinuities to the principal argument of z(x) + y when compared to the principal argument of z(x), if and only if:

- $\operatorname{Re}(z(x)) \notin (-y,0)$ for y > 0 whenever $\operatorname{Im}(z(x))$ changes sign;
- $\operatorname{Re}(z(x)) \notin (0,-y)$ for y < 0 whenever $\operatorname{Im}(z(x))$ changes sign.

Proof:

Let us first define the difference of the principal arguments of z(x) and z(x) + y as f(x):

$$f(x) = \arg(z(x)) - \arg(z(x) + y)$$
(A.42)

Let a trajectory from quadrant i to quadrant j, without crossing any quadrants inbetween, be denoted as a tuple (i,j). The direction in which the trajectory is traversed does not matter here. Trajectories of z that do not cause any discontinuities are (1,2), (1,4), (3,4). A trajectory of z that does cause discontinuities is (2,3). Clearly the horizontal trajectories (1,2), (3,4) and vice versa can be neglected here, as Im(z(x)) does not change sign here. The diagonal trajectories (1,3) and (2,4) can also be excluded, as we assumed that z never passes through the origin. Finally, let x* be that x on the trajectory such that Im(z(x*)) = 0.

Let us start with the trajectory (1,4). Evidently the trajectory of z(x) + y remains (1,4), provided that $\text{Re}(z(x^*)) + y > 0$. However, if $\text{Re}(z(x^*)) + y < 0$, the trajectory will pass through the origin in an infinitesimal neighbourhood of x^* . This will lead to a discontinuity in the principal argument, so we have to exclude this case. The same happens when $\text{Re}(z(x^*)) + y = 0$.

If we start out with (2,3) as the trajectory of z(x), the same analysis leads to the requirement that $\text{Re}(z(x^*)) + y < 0$, if we want to keep f(x) continuous. Collecting the results we find that f(x) remains continuous provided that:

- If $\operatorname{Re}(z(x^*)) < 0$, y must satisfy $\operatorname{Re}(z(x^*)) + y < 0$;
- If $\operatorname{Re}(z(x^*)) > 0$, y must satisfy $\operatorname{Re}(z(x^*)) + y > 0$.

This result is slightly rephrased in the lemma, so this concludes the proof. \Box