Complete the Correlation matrix*

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Abstract

In this article we discuss a method to complete the correlation matrix in a multi-dimensional stochastic volatility model. We concentrate on the construction of a positive definite correlation matrix. Furthermore we present a numerical integration scheme for this system of stochastic differential equations which improves the approximation quality of the standard Euler-Maruyama method with minimal additional computational effort.

1 Introduction

In stochastic models, especially in finance, often only some part of a correlation matrix is given by measured data. This incomplete model may easily be completed by defining the respective correlations in a reasonable way. However, the problem is to guarantee the positive definiteness of the matrix after completion. This paper develops and describes an efficient and feasible algorithm to accomplish this task, which is based on combining Gaussian-elimination with arguments from graph theory.
In comparison to the results of Grone et al. [GJSW84] and Barrett et al. [BJL89] our algorithm shows that it is possible to find a symmetric positive definite completion of the correlation matrix under the additional restriction that all matrix entries have to satisfy $|a_{(i,j)}| < 1$. Moreover we verify that our choice of the unspecified entries leads to the unique determinant maximising completion without the necessity of solving a sequence of optimisation problems.
The paper is organized as follows. Section 1 defines the problem of completing a correlation matrix and introduces the model setup. The basic idea of the algorithm is motivated by inspecting a small 2×2 example in section 2, and is applied to the general multi-dimensional case in the subsequent section. Finally, we show how the completed correlation matrix can be incorporated into numerical integration schemes for multi-dimensional volatility models.

2 Model setup

We consider the following $2n$-dimensional system of stochastic differential equations

$$\mathrm{d}S_i = \mu_i S_i \mathrm{d}t + f_i(V_i) S_i \mathrm{d}W_{(S,i)} \tag{1}$$

$$\mathrm{d}V_j = a_j(V_j)\mathrm{d}t + b_j(V_j)\mathrm{d}W_{(V,j)} \tag{2}$$

with $i, j = 1, \ldots, n$ and Brownian motions $W_{(S,i)}, W_{(V,j)}$. This article focuses on the question how to correlate the $2n$ Brownian motions. Furthermore, we assume that the diffusion of the underlying S_i is only directly coupled to one volatility V_i. If we concentrate on just one volatility-underlying-pair we obtain a typical stochastic volatility model to describe the non-flat implied volatility surface in European option' s

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prices. This problem is intensively discussed in the literature in the recent years. More details can be found in the articles of Heston [Hes93], Scott-Chesney [CS89] and Schöbel-Zhu [SZ99].

We assume that we already know the correlation between the Wiener processes $W_{S,i}$ and $W_{V,i}$

$$\mathrm{d}W_{S,i}\mathrm{d}W_{V,i} = \eta_i \mathrm{d}t \ , \tag{3}$$

which in short-hand may be written as

$$W_{S,i} \cdot W_{V,i} \stackrel{\circ}{=} \eta_i \ . \tag{4}$$

Neglecting the stochastic volatility we also have to decide how to couple the underlyings S_i. This question is quite common in finance and we assume that we know these correlations, too:

$$W_{S,i} \cdot W_{S,j} \stackrel{\circ}{=} \rho_{(i,j)}. \tag{5}$$

This leads to the following structure for the correlation matrix

$$A = a(i,j)_{1\leq i,j,\leq 2n} = \begin{pmatrix} \rho_{(1,1)} & \cdots & \rho_{(1,n)} & \eta_1 & & \mathbf{?} \\ \vdots & & \vdots & & \ddots & \\ \rho_{(n,1)} & \cdots & \rho_{(n,n)} & \mathbf{?} & & \eta_n \\ \eta_1 & & \mathbf{?} & 1 & & \mathbf{?} \\ & \ddots & & & \ddots & \\ \mathbf{?} & & \eta_n & \mathbf{?} & & 1 \end{pmatrix} , \tag{6}$$

with the undefined correlations marked with **?**. The problem of completing a matrix A which is only specified on a given set of positions, the so called *pattern*, is directly related to the structure of the graph $G = (V, E)$ of A. In case of completing symmetric positive definite matrices, Grone et al. [GJSW84, Thm. 7] proved that the partial matrix A is completable if and only if the corresponding graph G is a chordal graph and each principal submatrix is positive semidefinite. Unfortunately this theorem is not applicable in our case as a correlation matrix has the further requirement that all entries are restricted by $|a_{(i,j)}| < 1$. For more informations on matrix completions problems we refer the reader to [Lau01, Joh90].

The problem we have to deal with now is to define the yet unspecified correlations. To make this point clear we will discuss the simplest example of two underlyings and two stochastic volatility processes in the next section and we generalise this to the multi-dimensional case in section 4.

Before we start with the discussion of a two-dimensional example we have to state the following general result which helps us to prove the positive definiteness of the correlation matrix.

Remark 2.1 A square matrix $A \in \mathbb{R}^{n\times n}$ is positive definite if the Gaussian-algorithm[1] can be done with diagonal pivots p_i and if each pivot p_i is greater zero.

3 The 2×2-dimensional example

In this section we discuss how to complete the correlation matrix in the easiest case of two underlyings and two stochastic volatilities. We will refer to this as *two-dimensional* even though, strictly speaking, it would be more correct to call it 2×2-dimensional. The correlation matrix (6) is given by

$$A = \begin{pmatrix} 1 & \rho_{(1,2)} & \eta_1 & \mathbf{?} \\ \rho_{(2,1)} & 1 & \mathbf{?} & \eta_2 \\ \eta_1 & \mathbf{?} & 1 & \mathbf{?} \\ \mathbf{?} & \eta_2 & \mathbf{?} & 1 \end{pmatrix}. \tag{7}$$

We recognize that we have to specify the cross-correlations between $S_1 \sim V_2$ and $S_2 \sim V_1$, as well as the correlation between the different volatilities V_1 and V_2. This problem becomes more clear if we have a look at the corresponding graph[2] of the matrix (7) as shown in figure 1 . One ad-hoc and pragmatic way[3] to

[1] The Gauss-algorithm we are referring to is the standard Gaussian-elimination in *kij-form* as in Duff, Erisman and Reid [DER86, Section 3.3]

[2] Graph theory and sparse matrices are closely linked topics. One can represent a symmetric matrix by an undirected graph. For more information see Golub and van Loan[GvLC96] or Frommer[Fro03].

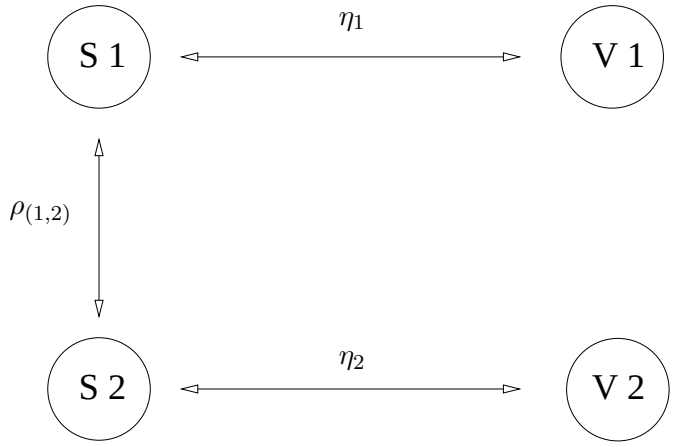


Figure 1: Corresponding graph to matrix (7). The nodes (sometimes also called vertices) are referred by S_1, S_2, V_1 and V_2. This is not entirely consistent with the notation used in the literature as it would be correct to fill these nodes with the diagonal entries $a(i,i) = 1$. In the following we always choose the notation which provides the necessary information. The undirected edges are given by the non-diagonal entries of the matrix (7).

define the correlation between V_1 and S_2 is the product of the correlation between $V_1 \sim S_1$ and $S_1 \sim S_2$

$$a(3,2) \doteq W_{V,1} \cdot W_{S,2} \doteq (W_{V,1} \cdot W_{S,1})\,(W_{S,1} \cdot W_{S,2}) \doteq \eta_1 \cdot \rho_{(1,2)}. \tag{8}$$

In the same way we are able define the correlation of the volatilities as the product of the correlation between $V_1 \sim S_1$, $S_1 \sim S_2$ and $S_2 \sim V_2$

$$a(4,3) \doteq W_{V,1} \cdot W_{V,2} \doteq (W_{V,1} \cdot W_{S,1})\,(W_{S,1} \cdot W_{S,2})\,(W_{S,2} \cdot W_{V,2}) \doteq \eta_1 \cdot \rho_{(1,2)} \cdot \eta_2. \tag{9}$$

On the corresponding graph we just multiply the values of the edges on the path from V_1 to V_2. In the two-dimensional example there is just one possibility for this path but in the multi-dimensional case we have to choose the shortest one. The matrix now looks like

$$A = \begin{pmatrix} 1 & \rho_{(1,2)} & \eta_1 & \eta_2 \cdot \rho_{(1,2)} \\ \rho_{(2,1)} & 1 & \eta_1 \cdot \rho_{(1,2)} & \eta_2 \\ \eta_1 & \eta_1 \cdot \rho_{(2,1)} & 1 & \eta_1 \cdot \rho_{(1,2)} \cdot \eta_2 \\ \eta_2 \cdot \rho_{(2,1)} & \eta_2 & \eta_1 \cdot \rho_{(2,1)} \cdot \eta_2 & 1 \end{pmatrix}. \tag{10}$$

Figure 2 shows the corresponding graph to matrix (10)

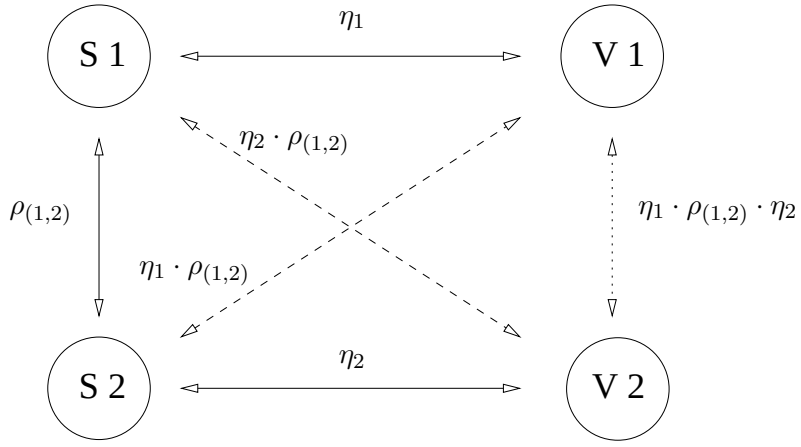


Figure 2: Corresponding graph to matrix (10).

Next we have to verify that this choice of correlations leads to a positive definite matrix. In order to show this we use the Remark 2.1. In the kth step of the Gaussian-elimination we have to choose the diagonal

[3] This idea goes back to P. Jäckel [Jäc05]

element $a(k,k)$ as the pivot p_k and we only operate on the elements $a(i,j)$ with $i,j \geq k$. To indicate the kth elimination step we denote the matrix A as $A^{(k)}$ with entries $a(i,j)^{(k)}$ which will be updated via

$$a(i,j)^{(k+1)} = a(i,j)^{(k)} - \frac{a(i,k)^{(k)} a(k,j)^{(k)}}{a(k,k)^{(k)}}, \qquad i,j > k\ . \tag{11}$$

The first pivot is $p_1 = a(1,1)^{(1)} = 1$ which is indeed greater zero and the remaining matrix looks as follows

$$A^{(2)} = \begin{pmatrix} 1 & \rho_{(1,2)} & \eta_1 & \eta_2 \cdot \rho_{(1,2)} \\ 0 & 1-\rho_{(1,2)}^2 & 0 & \eta_2\left(1-\rho_{(1,2)}^2\right) \\ 0 & 0 & 1-\eta_1^2 & 0 \\ 0 & \eta_2\left(1-\rho_{(1,2)}^2\right) & 0 & 1-\left(\eta_2\rho_{(1,2)}\right)^2 \end{pmatrix}. \tag{12}$$

After the first elimination step we can exclude the first row and first column from further consideration as they do not participate in the following calculations. Thus concentrating on the *active* part of the matrix we recognize that in the third row and third column only the diagonal element $a(3,3)^{(2)}$ is not zero. Therefore this node has lost any connection to other vertices in the corresponding graph, which means that in the following Gaussian-elimination steps this whole row and column stays unmodified. In addition $a(3,3)^{(2)} = 1-\eta_1^2 > 0$ hence we can choose this element as a positive pivot in elimination step $k=3$. The next pivot is $a(2,2)^{(2)} = 1-\rho_{(1,2)}^2 > 0$ and we obtain

$$A^{(3)} = \begin{pmatrix} 1 & \rho_{(1,2)} & \eta_1 & \eta_2 \cdot \rho_{(1,2)} \\ 0 & 1-\rho_{(1,2)}^2 & 0 & \eta_2\left(1-\rho_{(1,2)}^2\right) \\ 0 & 0 & 1-\eta_1^2 & 0 \\ 0 & 0 & 0 & 1-\left(\eta_2\rho_{(1,2)}\right)^2 - \frac{\eta_2^2\left(1-\rho_{(1,2)}^2\right)^2}{1-\rho_{(1,2)}^2} \end{pmatrix} \tag{13}$$

$$= \begin{pmatrix} 1 & \rho_{(1,2)} & \eta_1 & \eta_2 \cdot \rho_{(1,2)} \\ 0 & 1-\rho_{(1,2)}^2 & 0 & \eta_2\left(1-\rho_{(1,2)}^2\right) \\ 0 & 0 & 1-\eta_1^2 & 0 \\ 0 & 0 & 0 & 1-\eta_2^2 \end{pmatrix}. \tag{14}$$

The *active* part is now only the 2×2 submatrix containing all entries $a(i,j)^{(2)}$ with $i,j>2$. In the last two elimination steps we can just choose the elements $a(3,3)^{(3)} = 1-\eta_1^2 > 0$ and $a(4,4)^{(4)} = 1-\eta_2^2 > 0$ as pivots p_3 and p_4 which proves that the original matrix A is positive definite. In the next section we show that the Gaussian-algorithm is quite similar in the multi-dimensional case.

4 Multi-dimensional correlation

In this section we show how it is possible to complete the correlation matrix in the multi-dimensional setting in a very similar way as in the two-dimensional case. Moreover we verify that our choice leads to the determinant maximising completion. To get a first impression we draw the graph of six volatility-underlying-pairs and their corresponding correlations in figure 3. Within this figure we only show the fixed correlation between the underlyings S_i and the volatility-underlying correlation between S_j and V_j. As in the two-dimensional case we define the cross-correlation between the volatility V_i and the underlying S_j by the product of the correlation between $V_i \sim S_i$ and $S_i \sim S_j$

$$a(i+n,j) \doteq W_{V,i} \cdot W_{S,j} \doteq (W_{V,i} \cdot W_{S,i})\,(W_{S,i} \cdot W_{S,j}) \doteq \eta_i \cdot \rho_{(i,j)}. \tag{15}$$

In the same way we define the correlation between two volatilities as

$$a(i+n,j+n) \doteq W_{V,i} \cdot W_{V,j} \doteq (W_{V,i} \cdot W_{S,i})\,(W_{S,i} \cdot W_{S,j})\,(W_{S,j} \cdot W_{V,j}) \doteq \eta_i \cdot \rho_{(i,j)} \cdot \eta_j. \tag{16}$$

This corresponds to the shortest connection[4] in the graph between V_i and V_j. With this choice of the undefined correlations we are able to prove the following:

[4] In the shortest connection we only take into consideration the predefined paths between two underlyings S_i and S_j as well as the fixed underlying-volatility correlation between S_i and V_i

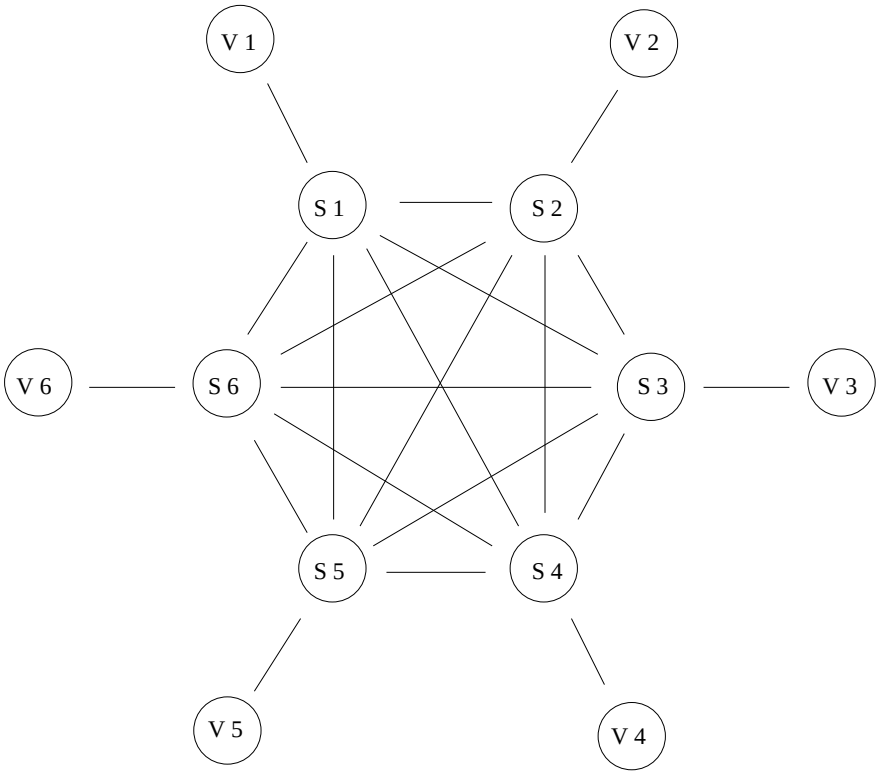


Figure 3: Multi-dimensional correlation graph corresponding to matrix (6) with $n = 6$ where the unspecified correlations are interpreted as zero.

Theorem 4.1 If the correlations between the underlyings

$$W_{S,i} \cdot W_{S,j} = \rho_{(i,j)} \tag{17}$$

are given such that the correlation matrix

$$B = b(i,j) = \rho_{(i,j)} \tag{18}$$

is positive definite and if we choose the cross-correlations due to (15) and (16) then the whole correlation matrix

$$A = a(i,j)_{1\le i,j,\le 2n} = \begin{pmatrix} \rho_{(1,1)} & \cdots & \rho_{(1,n)} & \eta_1 & \cdots & \eta_n \cdot \rho_{(1,n)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \rho_{(n,1)} & \cdots & \rho_{(n,n)} & \eta_1 \cdot \rho_{(n,1)} & \cdots & \eta_n \\ \eta_1 & \cdots & \eta_1 \cdot \rho_{(n,1)} & 1 & \cdots & \eta_1 \cdot \rho_{(1,n)} \cdot \eta_n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \eta_n \cdot \rho_{(1,n)} & \cdots & \eta_n & \eta_1 \cdot \rho_{(n,1)} \cdot \eta_n & \cdots & 1 \end{pmatrix} \tag{19}$$

is positive definite.

Proof: In the two-dimensional setting we observed that the volatility V_1 given by the diagonal entry $a(1+n,1+n)^{(2)}$, where n is the number of volatility-underlying-pairs, lost any connection in the corresponding graph after choosing $S_1 = a(1,1)^{(1)}$ as the first pivot. In the multi-dimensional setting this is equivalent to

$$a(n+1,j)^{(2)} = 0, \qquad j = 2,\ldots,n,n+2,\ldots,2n\ . \tag{20}$$

In general we have to show that after selecting $a(k,k)^{(k)}$ as the kth pivot the corresponding volatility entry $a(k+n,k+n)^{(k+1)}$ is greater zero and has no further connection in the graph of $A^{(k)}$, which means that

$$a(n+k,j)^{(k+1)} = 0, \qquad j = k+1,\ldots,n+k-1,n+k+1,\ldots,2n\ . \tag{21}$$

We will prove the positivity by showing that the following invariant holds

$$a(k+n,k+n)^{(k+1)} - \frac{\left(a(k,k+n)^{(k+1)}\right)^2}{a(k,k)^{(k+1)}} = \tag{22}$$

$$a(k+n,k+n)^{(k)} - \frac{\left(a(k,k+n)^{(k)}\right)^2}{a(k,k)^{(k)}} =$$

$$\dots =$$

$$a(k+n,k+n)^{(1)} - \frac{\left(a(k,k+n)^{(1)}\right)^2}{a(k,k)^{(1)}} = 1-\eta_k^2. \tag{23}$$

We show this by induction. First we verify the following invariants:

$$a(i+n,j)^{(k)} = \frac{a(i+n,i)^{(k)}a(i,j)^{(k)}}{a(i,i)^{(k)}}, \tag{24}$$

$$a(i+n,j+n)^{(k)} = \frac{a(i+n,i)^{(k)}a(i,j)^{(k)}a(j+n,j)^{(k)}}{a(i,i)^{(k)}a(j,j)^{(k)}}, \tag{25}$$

with $i,j>k$. Before we start proving these statements, we explain the origin of the invariants (24) and (25). Having a look at the corresponding graph (see figure 4) in the elimination step k and bearing in mind that $a(i+n,j+n)^{(k)}$ describes the correlation between V_i and V_j then the equation (25) is just the product over the values of the edges on the shortest path from V_i to V_j divided by the values of the vertices $S_i = a(i,i)^{(k)}$ and $S_j = a(j,j)^{(k)}$.

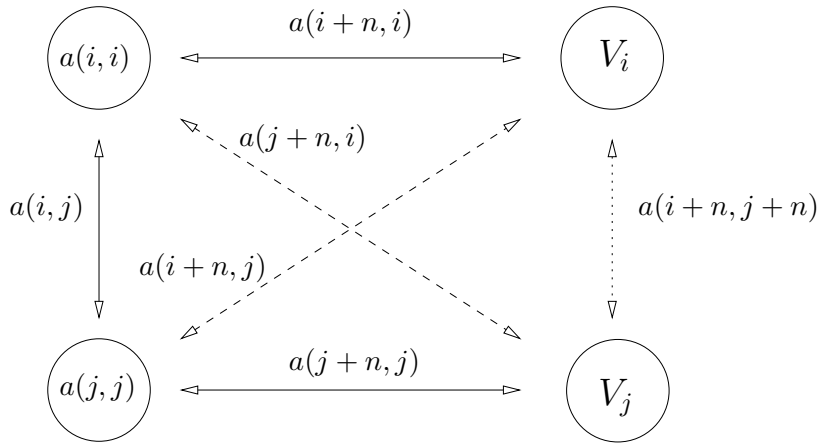


Figure 4: Graph describing the correlation structure of the underlyings S_i and S_j as well as V_i and V_j in the kth step of the Gaussian-algorithm. To simplify the notation we dropped all superscripts (k). The nodes on the left side belong to S_i and S_j which are here filled with the corresponding diagonal entries in the matrix $A^{(k)}$.

First we show by induction that (24) holds. The start of the induction is due to the construction (15)

$$a(i+n,j)^{(1)} = \frac{a(i+n,i)^{(1)}a(i,j)^{(1)}}{a(i,i)^{(1)}} = a(i+n,i)\cdot a(i,j) = \eta_i \cdot \rho_{(i,j)}. \tag{26}$$

Now we assume that (24) is valid until elimination step k and we want to verify that

$$a(i+n,j)^{(k+1)} = \frac{a(i+n,i)^{(k+1)}a(i,j)^{(k+1)}}{a(i,i)^{(k+1)}}\,, \qquad \text{for all } i,j>k+1\,. \tag{27}$$

Due to the Gaussian-algorithm we know that[5]

$$a(i+n,j)^{(k+1)} = a(i+n,j)^{(k)} - \frac{a(i+n,k)^{(k)}a(k,j)^{(k)}}{a(k,k)^{(k)}}\,. \tag{28}$$

[5]In the following calculation we will not mention the index (k) indicating a variable in the elimination step k.

We thus calculate

$$
\begin{aligned}
(29) \qquad a(i+n,j)^{(k+1)} &= a(i+n,j)^{(k)} - \frac{a(i+n,k)^{(k)}a(k,j)^{(k)}}{a(k,k)^{(k)}} \\
&= \frac{1}{a(k,k)}\left(a(k,k)a(i+n,j) - a(i+n,k)a(k,j)\right) \\
&= \frac{1}{a(k,k)}\left(a(k,k)\frac{a(i+n,i)a(i,j)}{a(i,i)} - \frac{a(i+n,i)a(i,k)}{a(i,i)}a(k,j)\right) \\
&= \frac{1}{a(k,k)}\frac{a(i+n,i)}{a(i,i)}\left(a(k,k)a(i,j) - a(i,k)a(k,j)\right) \\
&= \frac{1}{a(k,k)}\left(a(k,k)a(i,j) - a(i,k)a(k,j)\right) \\
&\quad \cdot \frac{\frac{a(i+n,i)}{a(i,i)}\left(a(i,i)a(k,k) - a(i,k)a(i,k)\right)}{\left(a(i,i)a(k,k) - a(i,k)a(i,k)\right)} \\
&= a(i,j)^{(k+1)}\frac{a(k,k)\left(a(i+n,i) - \frac{a(i,k)a(i,k)}{a(k,k)}\right)}{a(k,k)\left(a(i,i) - \frac{a(i,k)a(i,k)}{a(k,k)}\right)} \\
(30) \qquad &= \frac{a(i,j)^{(k+1)}a(i+n,i)^{(k+1)}}{a(i,i)^{(k+1)}} \, .
\end{aligned}
$$

The proof of invariant (25) can be done in the same way. Next we show that equation (23) holds. Again the begin of the induction is valid due to construction (15). Assuming that the equation is valid up to elimination step $l \leq k$, we obtain[6]

$$
\begin{aligned}
a(k+n,k+n)^{(l+1)} - \frac{a(k,k+n)^{(l+1)}a(k,k+n)^{(l+1)}}{a(k,k)^{(l+1)}} &= \\
a(k+n,k+n) - \frac{a(k+n,l)a(k+n,l)}{a(l,l)} - \frac{\left(a(l,l)a(k,k+n) - a(k,l)a(k+n,l)\right)^2}{a(l,l)\left(a(l,l)a(k,k) - a(k,l)a(k,l)\right)} &= \\
a(k+n,k+n) - \frac{\left(a(k+n,l)a(k,l)\right)^2}{a(k,k)^2a(l,l)} - \frac{\left(a(l,l)a(k,k+n) - a(k,l)\frac{a(k+n,k)a(k,l)}{a(k,l)}\right)^2}{a(l,l)\left(a(l,l)a(k,k) - a(k,l)a(k,l)\right)} &= \\
a(k+n,k+n) - \frac{\left(a(k+n,k)a(k,l)\right)^2}{a(k,k)^2a(l,l)} - \frac{a(k+n,k)^2\left(a(l,l)a(k,k) - a(k,l)a(k,l)\right)}{a(k,k)^2a(l,l)} &= \\
a(k+n,k+n)^{(l)} - \frac{a(k+n,k)^{(l)}a(k+n,k)^{(l)}}{a(k,k)^{(l)}} \, .
\end{aligned}
$$

Last we have to prove equation (21). Choosing $a(k,k)^{(k)}$ as the kth pivot leads to

$$
(31) \qquad a(k+n,j)^{(k+1)} = a(k+n,j) - \frac{a(k+n,k)a(k,j)}{a(k,k)} = 0 \, .
$$

The same holds for

$$
\begin{aligned}
(32) \qquad a(k+n,j+n)^{(k+1)} &= a(k+n,j+n) - \frac{a(k+n,k)a(k,j+n)}{a(k,k)} \\
(33) \qquad &= a(k+n,j+n) - \frac{a(k+n,k)a(k,j)a(j,j+n)}{a(j,j)a(k,k)} = 0 \, .
\end{aligned}
$$

Let us summarize this proof. We know that we can choose the n underlying $S_i \sim a(i,i)$ as the first n pivots assuming that the coupling of the underlyings leads to a positive definite matrix. During these n steps all volatilities $V_i \sim a(i+n,i+n)$ are losing their connection in the corresponding graph and furthermore, we verified that $a(k+n,k+n)^{(k+1)} = 1 - \eta_k^2 > 0$. □

[6]The first step is just the calculation rule of the Gaussian-elimination and we drop again the superscript (l).

Remark 4.1 The result can be generalised to the case where one underlying is not necessarily restricted to be directly coupled to one stochastic volatility process. It is also possible to have some underlyings without stochastic volatility and some with two or even more factors. Following the idea of the proof of Theorem 4.1 we just have to make sure that, if one underlying S_i is coupled to more than one volatility processes $V_i^{(m)}$, $m = 1, \ldots, M$, the matrix C_i with

$$c(k,l) \doteq W_{V_i^{(k)}} \cdot W_{V_i^{(l)}}, \qquad c(k, M+1) \doteq W_{V_i^{(k)}} \cdot W_{S,i}, \qquad c(M+1, M+1) \doteq W_{S,i} \cdot W_{S,i} \doteq 1, \tag{34}$$

is positive definite.

Next we verify that the choice of the cross-correlations due to (15) and (16) leads to the unique completion $\tilde{A}$ of the correlation matrix which maximises the determinant. Furthermore we know from [GJSW84, Thm. 2] that this is equivalent, that the inverse (19) contains zero's at each position which was previously unspecified.

Theorem 4.2 Choosing the cross-correlations due to (15) and (16) leads to the unique determinant maximising positive completion.

Proof: First we write the matrix A given by equation (19) as

$$A = \begin{pmatrix} \mathrm{B} & D^T \\ \mathrm{D} & \mathrm{C} \end{pmatrix} \tag{35}$$

with square matrices B, D and C. The entries of B are already specified. Next we introduce the term $\tilde{C} = C - DB^{-1}D^T$ which is the well known Schur-complement. Using this notation we can formally write the inverse of A as

$$A^{-1} = \begin{pmatrix} B^{-1}\left(I + D^T\tilde{C}^{-1}DB^{-1}\right) & -B^{-1}D^T\tilde{C}^{-1} \\ -\tilde{C}^{-1}D^TB^{-1} & \tilde{C}^{-1} \end{pmatrix}. \tag{36}$$

Thus we have to show that $\tilde{C}$ and $B^{-1}D^T$ are diagonal. Since the Gaussian-elimination on B coincides with calculating its inverse, equation (21) verifies that $\tilde{C}$ is diagonal. Moreover (21) also shows that $B^{-1}D^T$ only contains zero's below the diagonal. As this matrix is symmetric, caused by the diagonality of B and D, its diagonal. Hence the inverse of A contains zero's at each previously unspecified position which is equivalent with finding the determinant maximising completion due to Grone et al. [GJSW84]. □

Now we are able to complete the correlation matrix such that we obtain a symmetric positive definite matrix. Next we are confronted with the problem of integrating this system of stochastic differential equations. In case of one volatility-underlying-pair Kahl and Jäckel [KJ05] compared the efficiency of various numerical integration methods. In the next section we show that these results are also applicable in the multidimensional setting.

5 Numerical tests for the multidimensional stochastic volatility model

In this section we discuss suitable numerical integration schemes for the multidimensional stochastic volatility model (1). Without stochastic volatility this problem is comparatively easy to solve as the n underlyings S_i are n-dimensional lognormal distributed. Thus in case of European options we do not have to discretise the time to maturity at all. The situation becomes much more complicated when stochastic volatility comes into play. As we do not know the distribution density we have to apply numerical integration schemes for stochastic differential equations. In the standard model with only one underlying and one stochastic volatility Kahl and Jäckel [KJ05] discussed different integration methods with special regard to the numerical efficiency. It turned out that higher order methods, i.e. the Milstein scheme are inappropriate due to the fact that we have to generate additional random numbers. The finally most efficient integration scheme is referred to *IJK*[7]

$$\begin{aligned} \ln S_{(m+1)} &= \ln S_{(m)} + \left(\mu - \tfrac{1}{4}\left(f^2\left(V_{(m)}\right) + f^2\left(V_{(m+1)}\right)\right)\right)\Delta t + \rho f\left(V_{(m)}\right)\Delta W_{(V,m)} \\ &\quad + \tfrac{1}{2}\left(f\left(V_{(m)}\right) + f\left(V_{(m+1)}\right)\right)\left(\Delta W_{(S,m)} - \rho\Delta W_{(V,m)}\right) \\ &\quad + \tfrac{1}{2}\sqrt{1-\rho^2}f'\left(V_{(m)}\right)b\left(V_{(m)}\right)\left(\left(\Delta W_{(V,m)}\right)^2 - \Delta t\right) \end{aligned} \tag{37}$$

[7] The name *IJK* refers to the originator's of this scheme.

with correlation $\mathrm{d}W_S \cdot \mathrm{d}W_V = \rho \mathrm{d}t$. As one underlying S_i is only directly coupled to one stochastic volatility process V_i we can generalise this integration scheme straightforward to the multidimensional case

$$
\begin{aligned}
\ln S_{(i,m+1)} &= \ln S_{(i,m)} + \left(\mu_i - \tfrac{1}{4}\left(f^2\left(V_{(i,m)}\right) + f^2\left(V_{(i,m+1)}\right)\right)\right)\Delta t + \rho f\left(V_{(i,m)}\right)\Delta W_{(V,i,m)} \\
&\quad + \tfrac{1}{2}\left(f\left(V_{(i,m)}\right) + f\left(V_{(i,m+1)}\right)\right)\left(\Delta W_{(S,i,m)} - \rho \Delta W_{(V,i,m)}\right) \\
&\quad + \tfrac{1}{2}\sqrt{1-\rho^2} f'\left(V_{(i,m)}\right) b\left(V_{(i,m)}\right)\left(\left(\Delta W_{(V,i,m)}\right)^2 - \Delta t\right)
\end{aligned} \tag{38}
$$

where $S_{i,m+1}$ denotes the $(m+1)$th step of the ith underlying. For the *IJK* scheme we assume that we already know the numerical approximation of the whole path of the process V_t. This path has to be computed with a suitable numerical integration scheme depending on the stochastic differential equation for the stochastic volatility. This problem is intensively discussed in [KJ05, Section 3]. The benchmark scheme for the *multidimensional IJK* scheme is the standard *Euler-Maruyama* method

$$
\ln S_{(i,m+1)} = \ln S_{(i,m)} + \left(\mu_i - \tfrac{1}{2} f_i\left(V_{(i,m)}\right)\right)\Delta t_m + f_i\left(V_{(i,m)}\right)\Delta W_{(S,i,m)} \tag{39}
$$

$$
V_{(j,m+1)} = a_j\left(V_{(j,m)}\right)\Delta t_m + b_j\left(V_{(j,m)}\right)\Delta W_{(V,j,m)}. \tag{40}
$$

Here one has to bear in mind that if the stochastic volatility V_j is given by a mean-reverting process the Euler scheme is not able to preserve numerical positivity. Thus if the financial derivative is sensitive to the dynamic of the variance of the underlying we recommend more advanced numerical integration schemes to preserve positivity.

Next we set up a 4×4-dimensional benchmark model to obtain a first impression on the numerical efficiency of both integration schemes. The stochastic volatility $\sigma_t = f(V_t)$ is described by a hyperbolic transformed Ornstein-Uhlenbeck process

$$
\mathrm{d}y_t = -\kappa y_t \mathrm{d}t + \alpha\sqrt{2\kappa}\mathrm{d}W_V \ , \tag{41}
$$

with transformation function $\sigma_t = \sigma_0\left(y_t + \sqrt{y_t^2 + 1}\right)$ which was introduced and discussed in [KJ05]. We choose the following parameter configuration $\kappa = 1$, $\alpha = 0.35$, $\sigma_0 = 0.25$ and $y_0 = 0$ throughout the whole section and for all volatility processes. The initial value of the four underlyings is set to $S_0 = 100$. The decisive point for these tests is the correlation structure of the Wiener processes. The underlying correlation matrix (18) is chosen as follows

$$
B = \begin{pmatrix} 1 & 0.2 & 0 & 0.5 \\ 0.2 & 1 & 0.4 & 0 \\ 0 & 0.4 & 1 & 0.6 \\ 0.5 & 0 & 0.6 & 1 \end{pmatrix}. \tag{42}
$$

For the underlying-volatility correlation we assume a highly negative correlation corresponding to a downward sloping implied volatility surface in European vanilla option markets

$$
v = \begin{pmatrix} \text{-}0.7 \\ \text{-}0.8 \\ \text{-}0.9 \\ \text{-}0.8 \end{pmatrix}. \tag{43}
$$

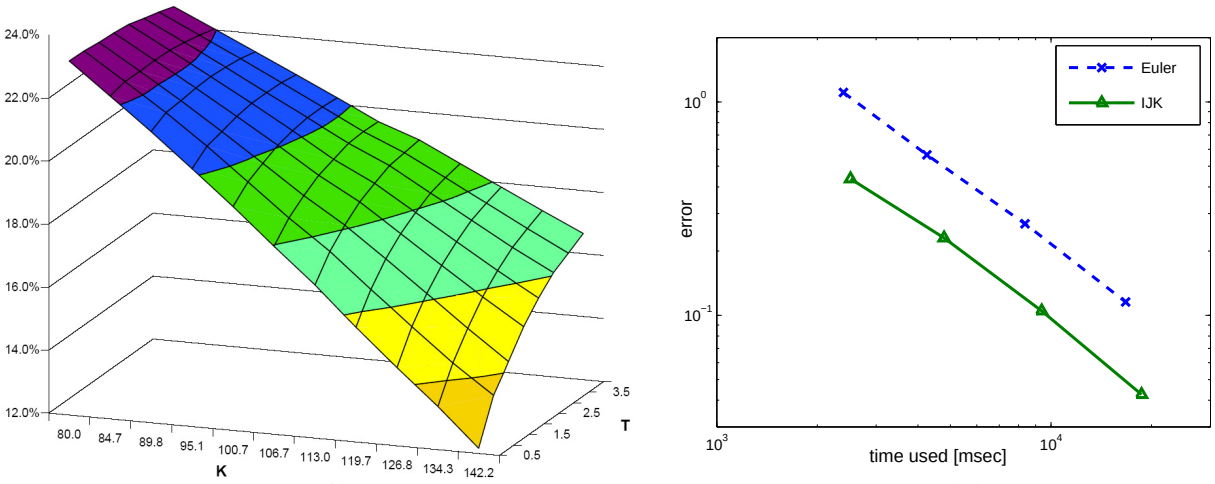
This directly leads to the following correlation matrix completed due to (15) and (16)

$$
A = \begin{pmatrix}
1 & 0.2 & 0 & 0.5 & \text{-}0.7 & \text{-}0.16 & 0 & \text{-}0.4 \\
0.2 & 1 & 0.4 & 0 & \text{-}0.14 & \text{-}0.8 & \text{-}0.36 & 0 \\
0 & 0.4 & 1 & 0.6 & 0 & \text{-}0.32 & \text{-}0.9 & \text{-}0.48 \\
0.5 & 0 & 0.6 & 1 & \text{-}0.35 & 0 & \text{-}0.54 & \text{-}0.8 \\
\text{-}0.7 & \text{-}0.14 & 0 & \text{-}0.35 & 1 & 0.112 & 0 & 0.28 \\
\text{-}0.16 & \text{-}0.8 & \text{-}0.32 & 0 & 0.112 & 1 & 0.288 & 0 \\
0 & \text{-}0.36 & \text{-}0.9 & \text{-}0.54 & 0 & 0.288 & 1 & 0.432 \\
\text{-}0.4 & 0 & \text{-}0.48 & \text{-}0.8 & 0.28 & 0 & 0.432 & 1
\end{pmatrix}. \tag{44}
$$

The first numerical test compares the evaluation of a basket option. Thereby we consider the payoff function to be the mean of the four underlying assets. Thus the fair value of this option is given by

$$C(T,K) = \mathbb{E}\left(\frac{1}{4}\sum_{i=1}^{4} S_i(T) - K\right)^+ . \tag{45}$$

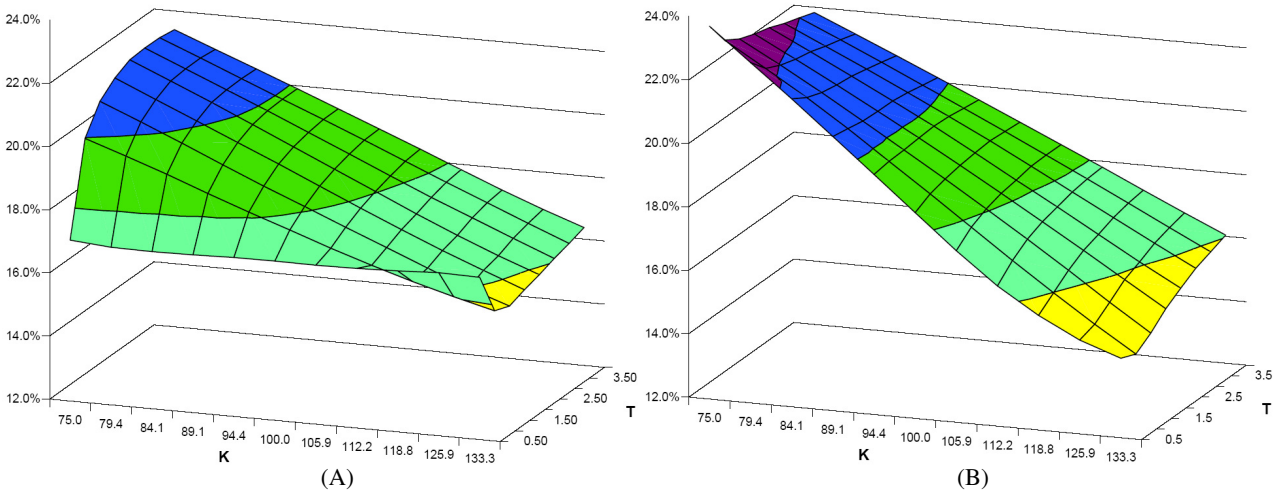
The numerical results are compared with a numerical reference solution computed with the *Euler-Maruyama* scheme (39) and a stepsize of $\Delta t_{\text{exact}} = 2^{-10}$. The prices are calculated for a whole range of strikes $K = \{75, \ldots, 133.3\}$ and a termstructure of maturities $T = \{0.5, 1, 1.5, \ldots, 4\}$. As there is a great difference in prices of at the money options compared to in-the-money and out-of-money options we compute the implied Black volatility, denoted as $IV(C, S, K, r, T)$, where C is the given option price, S the initial value and r the risk free interest rate, to get a fair error measure. The biggest advantage of the implied volatility is that the error-size throughout the whole level of strikes and maturities becomes comparable. The first figure 5 (A) shows the numerical reference solutions where we recognize the strongly downward sloping skew structure of the implied volatility surface as a consequence of the negative correlation between underlyings and volatilities (43) . In the figure 6 we compare the numerical results of the *Euler* scheme with



(A) (B)

Figure 5: (A): Implied volatility surface of the reference solution with a stepsize of $\Delta t_{\text{exact}} = 2^{-10}$ and 32767 paths . (B): Weak approximation error (46) as a function of CPU time [in msec] for the simulation of 32767 paths. The number generator in (A) and (B) was Sobol's method and the paths were constructed via the Brownian bridge.

the *multidimensional IJK* scheme where we integrate along the maturities with a stepsize of $\Delta t = 0.5$. The superiority of the *IJK* scheme is at its most impressive for the first maturity $T = 0.5$ as we obtain a skew within one integration step in comparison to the flat implied-volatility of the *Euler* for this maturity.



(A) (B)

Figure 6: Implied volatility surface calculated with the (A): *Euler* method and the (B): *multidimensional IJK* method. The stepsize was $\Delta t = 2^{-1}$ and the prices were averaged over 32767 paths . The number generator was the Sobol's method and the paths were constructed via the Brownian bridge.

To underscore this result we also compare the error and the computational effort of both integration schemes for different stepsizes. The error is computed as the sum over the absolute value of the differences between the implied volatility of the reference solution $C(t_i, K_j)$ and the numerical approximation $\tilde{C}(t_i, K_j)$

$$\text{Error} = \sum_{i,j} \left| IV(C(t_i, K_j), S, K_j, r, t_i) - IV(\tilde{C}(t_i, K_j), S, K_j, r, t_i) \right| . \tag{46}$$

In figure 5 (B) we see that the *IJK* leads to better results comparing the relation between approximation quality and computational effort. The decisive point is that the computational effort of the *IJK* scheme is only slightly higher than the *Euler* scheme since we do not have to draw any additional random number.

6 Summary

Based on combining Gaussian-elimination and graph theory, we have introduced an algorithm to complete the correlation matrix, if only an incomplete set of measured data is available, which does not allow to define all correlations a unique way. Compared to the results of Grone et al. [GJSW84] and Barrett et al. [BJL89], our algorithm preserves that all entries are bounded by $|a_{(i,j)}| < 1$, and avoids the costly computation of optimisation problems. From an application-oriented point of view, our algorithm can be implemented within pricing tools based on simulating multidimensional stochastic volatility models.

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