

# Modelling and simulation of stochastic volatility in finance



## Dissertation

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# Table of Contents

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<b>Mathematical notation</b>	<b>ix</b>
<b>1 Introduction</b>	<b>1</b>
<b>2 Stochastic volatility models</b>	<b>4</b>
2.1 Transformed Ornstein-Uhlenbeck models . . . . .	5
2.2 Affine diffusion stochastic volatility models . . . . .	12
<b>3 Monte Carlo methods</b>	<b>28</b>
3.1 Introduction . . . . .	28
3.2 Quasi-Monte Carlo . . . . .	31
3.3 Path construction methods . . . . .	33
3.4 Fractional Fourier transformation for spectral path construction . . . . .	45
<b>4 European option pricing for transformed Ornstein-Uhlenbeck models</b>	<b>49</b>
4.1 Control variate conditional Monte Carlo . . . . .	49
4.2 Asymptotic approximation using Watanabe's expansion . . . . .	58
<b>5 Optimal Fourier inversion for affine diffusion models</b>	<b>89</b>
5.1 Introduction . . . . .	89
5.2 Integration domain change . . . . .	93
5.3 Optimal choice of alpha . . . . .	95
5.4 Numerical results . . . . .	107
<b>6 Numerical integration schemes for stochastic volatility models</b>	<b>113</b>
6.1 Numerical integration of mean-reverting CEV processes . . . . .	115
6.2 Numerical integration of stochastic volatility models . . . . .	127
6.3 Multidimensional stochastic volatility models . . . . .	146
<b>7 Conclusion</b>	<b>162</b>
<b>A Balanced Milstein Methods for ordinary SDEs</b>	<b>164</b>
A.1 Stability, consistency and convergence . . . . .	166
A.2 Pathwise positivity . . . . .	176
<b>B Proofs</b>	<b>180</b>
<b>Bibliography</b>	<b>189</b>

TABLE OF CONTENTS

---

v

<b>Index</b>	<b>200</b>
--------------	------------

---

## List of Figures

---

1.1	Readers guidance . . . . .	2
2.1	Linear, exponential and hyperbolic transformation functions . . . . .	7
2.2	Densities linear, exponential and hyperbolic transformation . . . . .	8
2.3	Variance and correlation - linear, hyperbolic and exponential transformation .	13
2.4	Trajectory Heston characteristic function in the complex plane . . . . .	21
2.5	Heston's characteristic function . . . . .	22
2.6	Complex discontinuities - counterexample . . . . .	23
3.1	Cumulative <i>variability explained</i> - Brownian motion . . . . .	35
3.2	Brownian bridge . . . . .	37
3.3	Cumulative <i>variability explained</i> - Ornstein-Uhlenbeck bridge . . . . .	39
3.4	Cumulative <i>variability explained</i> - Ornstein-Uhlenbeck process . . . . .	45
3.5	Higher eigenvectors - Ornstein-Uhlenbeck process . . . . .	46
4.1	Control variate and convergence hyperbolic-OU at-the-money . . . . .	55
4.2	Control variate and convergence hyperbolic-OU out-of-the-money . . . . .	56
4.3	Error in implied volatility for the hyperbolic-OU model . . . . .	56
4.4	Watanabe expansion Black-Scholes model . . . . .	62
4.5	Asymptotic implied volatility hyperbolic Ornstein-Uhlenbeck (A) . . . . .	74
4.6	Asymptotic implied volatility hyperbolic Ornstein-Uhlenbeck (B) . . . . .	75
4.7	Asymptotic implied volatility for Jäckel's hyperbolic volatility model . . . . .	78
4.8	Asymptotic implied volatility for the hyperbolic-hyperbolic model . . . . .	82
5.1	Pricing error Heston - different adaptive integration accuracy over varying $\alpha$ .	97
5.2	Integrand of the Carr-Madan representation for different values of $\alpha$ . . . . .	98
5.3	Integrand $\Psi$ (5.49) and its first derivative - Heston model . . . . .	104
5.6	Error in the implied volatility surface for the optimal $\alpha$ . . . . .	109
5.4	Black implied volatilities of the Heston model . . . . .	110
5.5	Error in the implied volatility surface for the saddlepoint approximation . . . .	111
5.7	Function evaluation points for the optimal $\alpha$ . . . . .	112
6.1	Pathwise approximation for the CIR/Heston model . . . . .	128
6.2	Strong convergence CIR/Heston model - different values of volatility . . . . .	129
6.3	Strong convergence CIR/Heston model - different number generators . . . . .	130
6.4	Strong convergence Brennan-Schwartz model . . . . .	131
6.5	Strong convergence exponential/hyperbolic Ornstein-Uhlenbeck: $\rho = 0$ . . . . .	136
6.6	Strong convergence exponential/hyperbolic Ornstein-Uhlenbeck: $\rho = -2/5$ . . . .	137
6.7	Strong convergence exponential/hyperbolic Ornstein-Uhlenbeck: $\rho = -4/5$ . . .	137

6.8	Efficiency exponential/hyperbolic Ornstein-Uhlenbeck: $\rho = 0$	141
6.9	Efficiency exponential/hyperbolic Ornstein-Uhlenbeck: $\rho = -2/5$	141
6.10	Efficiency exponential/hyperbolic Ornstein-Uhlenbeck: $\rho = -4/5$	142
6.11	Efficiency Brennan-Schwartz and Heston model: $\rho = 0$	144
6.12	Efficiency Brennan-Schwartz and Heston model: $\rho = -2/5$	145
6.13	Efficiency Brennan-Schwartz and Heston model: $\rho = -4/5$	145
6.14	2-dimensional example	148
6.15	Completed two dimensional example	149
6.16	Multi-dimensional correlation graph	151
6.17	Graph of the correlation structure during Gaussian-elimination	153
6.18	Multidimensional volatility surface	158
6.19	Multidimensional volatility model convergence	159
6.20	Multidimensional implied volatility surface - Euler vs. IJK	160
6.21	Correlation surface	161

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## List of Tables

---

3.1	CPU-time covariance split . . . . .	42
4.1	Table of conditional expectations for multiple Wiener integrals . . . . .	66
4.2	At-the-money skewness for local and stochastic volatility models . . . . .	81
5.1	Maximum $\alpha$ jump-diffusion - number of Newton-Raphson steps . . . . .	101
5.2	Function evaluations and error for the optimal alpha and standard method .	107
5.3	Tiny option prices . . . . .	107
6.1	Average speed-up <i>IJK</i> compared with <i>log-Euler</i> . . . . .	140
6.2	Average speed-up <i>BMM</i> & <i>IJK</i> compared with <i>Euler</i> & <i>log-Euler</i> . . . . .	143
6.3	Number of non-positive stochastic volatility paths in figure 6.13 (B) . . . . .	144

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## Mathematical notation

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$1_{\{\dots\}}$	indicator function
$\coloneqq$	defined as
$\approx$	approximately
$\mathbf{A}^T$	transpose of the matrix $\mathbf{A} \in \mathbb{R}^{n \times m}$
$\mathbf{x}^T \cdot \mathbf{y}$	inner product, $\mathbf{x}^T \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i$
$\mathbf{x}^T$	transpose of the vector $\mathbf{x} \in \mathbb{R}^n$
$\stackrel{_\circ}{=} \,$	short hand notation for the correlation of Wiener processes, i.e. $W \cdot Z \stackrel{_\circ}{=} \rho$ stands for $dW \cdot dZ = \rho dt$
$\delta(x)$	Dirac delta distribution
$\delta_{i,j}$	Kronecker symbol
e	base of the natural logarithm (Napier's constant)
$\mathbb{E}[X]$	expectation of $X$ , i.e. $\int X d\mathbb{P}_X$
$\hat{f}(z)$	Fourier transform of $f$
$\text{Im}(z)$	imaginary part of the complex variable $z$
$\langle f \rangle_\varphi$	expectation with respect to the distribution density $\varphi$
$\ln(x)$	natural logarithm, i.e. the solution of $x = e^z$
$\mathbb{C}$	complex numbers
$\mathbb{N}$	positive integers
$\mathbb{P}(A)$	probability of $A$
$\mathbb{P}_X$	distribution induced by the random variable $X$
$\mathbb{R}$	real numbers
$\mathbb{R}^+$	positive real numbers

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$\mathbb{Z}$	integers
$\mathcal{C}(U, V)$	continuous functions $f : U \rightarrow V$
$\mathcal{C}^k(U, V)$	functions $f : U \rightarrow V$ , with $k$ -th derivative in $\mathcal{C}(U, V)$
$\mathcal{N}(\mu, \sigma^2)$	normal distribution with mean $\mu$ and standard deviation $\sigma$
$\mathcal{O}(n)$	Landau or asymptotic notation
$\otimes$	tensor product
$\bar{z}$	complex conjugate
$\phi(u, t)$	characteristic function, $\phi(u, t) = \mathbb{E}[e^{iuX_t}]$
$\text{Re}(z)$	real part of the complex variable $z$
$\sigma_X$	standard deviation of $X$ , $\sigma_X = \sqrt{\mathbb{V}[X]}$
$\sim$	distributed according to
$\text{Black}(S, K, \sigma, T)$	Black formula for a European call option, $\text{Black}(S, K, \sigma, T) = S \cdot N\left(\frac{\ln(\frac{S}{K})}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}\right) - K \cdot N\left(\frac{\ln(\frac{S}{K})}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}\right)$
$\text{Cor}[X, Y]$	linear correlation between $X$ and $Y$
$\text{Cov}[X, Y]$	linear covariance between $X$ and $Y$
$\varphi(x)$	standard normal distribution density, $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$
$\mathbb{V}[X]$	variance of $X$ , $\mathbb{V}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$
$f^{(k)}(x)$	short hand notation for the $k$ -th derivative of $f$ , i.e. $\frac{\partial^k f}{\partial x^k}(x)$
$f^{-1}(x)$	inverse function, i.e. $f^{-1}(f(x)) = x$
$h_n$	Hermite polynomial of degree $n$
$i$	$\sqrt{-1}$
$I_{(1)}^{s,t}$	Wiener integral on the interval $[s, t]$ , i.e. $\int_s^t dW_u$
$I_\alpha^{s,t}$	multiple Wiener integral on the interval $[s, t]$ with multi-indices $\alpha$
$N(x)$	cumulative normal distribution function, $N(x) = \int_{-\infty}^x \varphi(z) dz$
$W_t$	Brownian motion
$x \vee y$	maximum of $x$ and $y$
$x \wedge y$	minimum of $x$ and $y$

# CHAPTER I

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## Introduction

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The famous Black-Scholes model [BS73] was the starting point of a new financial industry and has been a very important pillar of all options trading since. One of its core assumptions is that the volatility of the underlying asset is constant. It was realised early that one has to specify a dynamic on the volatility itself to get closer to market behaviour. There are mainly two aspects making this fact apparent. Considering historical evolution of volatility by analysing time series data one observes erratic behaviour over time. Secondly, backing out implied volatility from daily traded plain vanilla options, the volatility changes with strike. The most common realisations of this phenomenon are the implied volatility smile or skew. The natural question arises how to extend the Black-Scholes model appropriately.

Any modelling approach aiming at this problem must fulfil some fundamental requirements. The range of model implied volatilities must be flexible enough to reflect market behaviour. In order to be of practical relevance it must additionally allow the fast and accurate pricing of European options to accomplish calibration within reasonable computation time. The approach of driving volatility itself via a stochastic process turned out to be as intuitive as successful. Depending on the particular choice of the stochastic process for volatility, these models allow different dynamics of implied volatility and thereby mapping of market behaviour.

This thesis focus on the different problems arising within the context of stochastic volatility from a modelling as well as from a numerical point of view. At the very heart of modelling volatility is the choice of a sensible process. Hereby it is of fundamental importance to bear in mind analytical properties such as positivity of volatility. Moreover one usually observes a mean-reverting behaviour of volatility over time and might want to address the analytical tractability, i.e. a closed form solution of the distribution density or its characteristic function. Once a model is chosen one needs to address the problem of calculating or approximating the price of plain vanilla options in order to calibrate the model to market data.

The thesis is divided into six chapters and we illustrate their dependency structure in figure 1.1. The second chapter provides an introduction about the stochastic volatility models

discussed throughout this thesis. We mainly focus on two different modelling approaches. First we introduce stochastic volatility models where volatility is given by a transformed Ornstein-Uhlenbeck process. Here, we concentrate on analysing the analytical properties with respect to different choices of transformation functions. The second class of stochastic volatility models we consider are affine (jump) diffusion models, having the appealing feature that the characteristic function of the underlying asset is known in closed form. In a first step, we discuss a stable numerical implementation of the characteristic function which is a rather challenging problem involving an appropriate handling of multivalued functions such as the complex logarithm. In chapter five, we use the closed form solution of the characteristic function for semi-analytical option pricing of plain vanilla options.

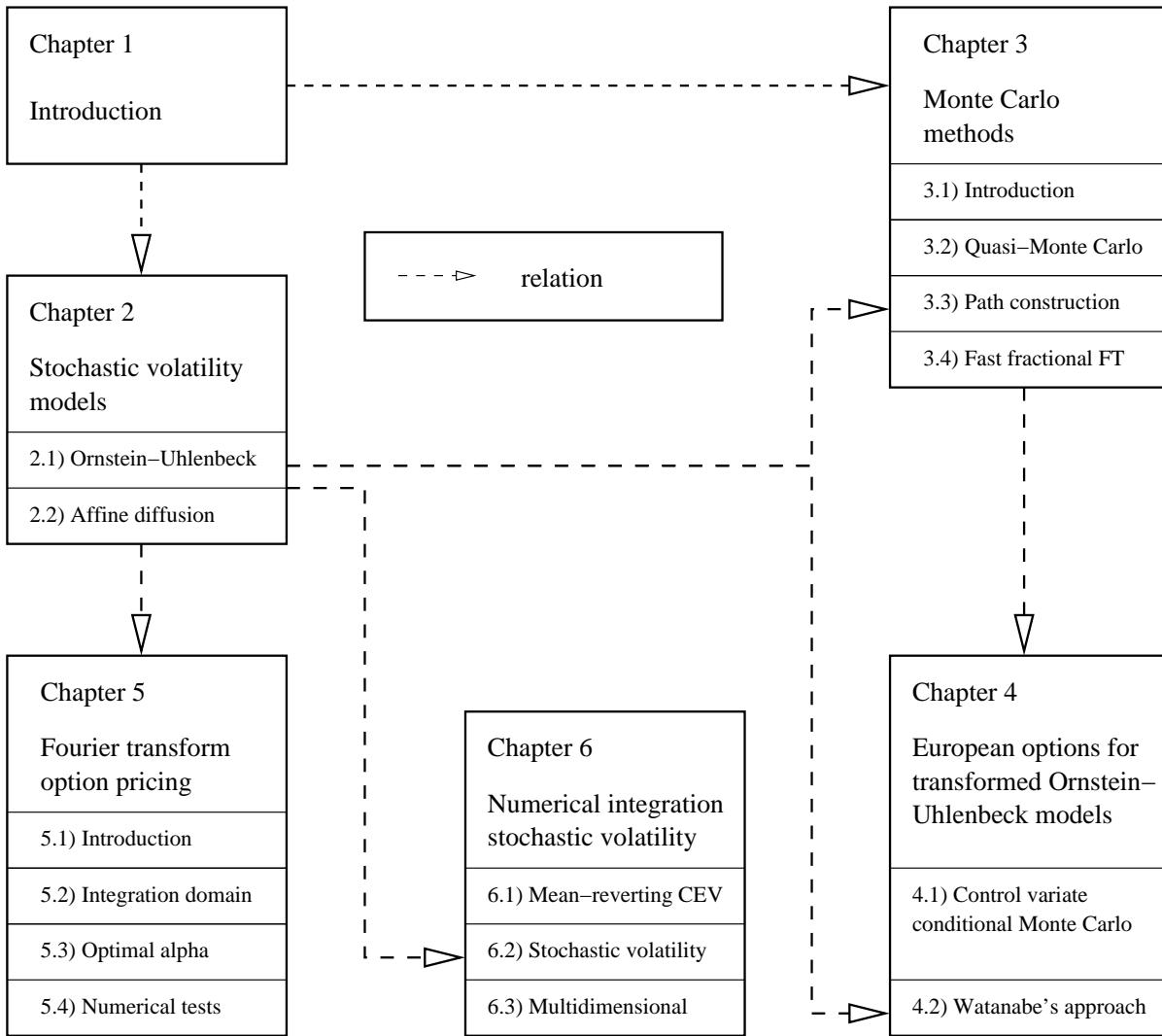


Figure 1.1: Reader’s guidance.

The third chapter deals with Monte Carlo simulation methods. At the very heart of this chapter stands the optimal path construction of Wiener and Ornstein-Uhlenbeck processes within a quasi-Monte Carlo framework. In section 3.3, we compare different path construction

methods with regard to their *variability explained* and their computational complexity. Using a fast fractional Fourier transformation helps to reduce the computational effort for discrete and approximative spectral path construction methods of Wiener and Ornstein-Uhlenbeck processes as outlined in section 3.4.

Next we discuss the computation and approximation of European option prices for transformed Ornstein-Uhlenbeck stochastic volatility models, where we mainly concentrate on two different approaches. On the one hand, we use the efficient path construction of Ornstein-Uhlenbeck processes developed in chapter 3 to derive a fast and accurate control variate conditional Monte Carlo method. On the other hand, we develop asymptotic approximations for European option prices within transformed Ornstein-Uhlenbeck models using Watanabe's approach. Due to the generality of this method we extend the stochastic volatility model by adding a local volatility function which provides greater flexibility when calibrated to market data.

In chapter five, we study the semi-analytical option pricing for affine (jump) diffusion models using Fourier inversion techniques. Fourier inversion allows to price European options solely based on the knowledge of the characteristic function of the underlying asset and the payoff function by the aid of the Plancherel/Parseval equality and applying residual calculus. The numerical implementation of the inverse Fourier integral can be accomplished by transforming the unbounded integration domain to a finite interval using the limiting behaviour of the characteristic function. Furthermore we show that it is of fundamental importance for a fast and robust implementation to choose an appropriate integration contour in the complex plane. Numerical tests provide further evidence for the effectiveness of this approach.

The sixth chapter deals with numerical integration schemes for stochastic volatility models which are required to price path dependent options. Since the volatility process does not explicitly depend on the underlying asset, the integration of a stochastic volatility model can be divided into two separate steps. First we concentrate on the numerical integration of the stochastic volatility process itself. Based on the knowledge of the full volatility path, we focus on the simulation of the underlying asset in section 6.2. Using the strong approximation error as a measure to compare different approximation schemes, we derive a fast and efficient approximation method based on an interpolation of the drift and the decorrelated diffusion. Section 6.3 discusses multidimensional stochastic volatility models, where we particularly focus on how to complete a non-fully specified correlation matrix such that we obtain a symmetric positive semi-definite matrix.

In chapter seven, we summarise the results developed and discussed throughout this thesis and we provide an outlook about further research opportunities. Appendices A and B supplement the previous work.

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# Index

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- adaptive Milstein scheme, 119  
affine diffusion  
    characteristic function, 15  
    models, 13  
    process, 15  
    Riccati equation, 15  
ANOVA, 33  
at-the-money skew, 59
- Balanced implicit method, 118, 163  
Balanced Milstein method, 119, 163  
    pathwise positivity, 175  
Black-Scholes, 1, 16, 52, 61, 106  
Breeden-Litzenberger, 84  
Brennan-Schwartz, 116  
Brownian bridge, 36, 63  
    conditional expectations, 68  
    moments, 37  
Brownian motion  
    approximative spectral path construction,  
        44  
    bridge construction, 37  
    eigencomponents, 42  
    inverse covariance matrix, 42  
Burkholder-Davis-Gundy, 171
- Cauchy-Schwarz inequality, 170  
central limit theorem, 30  
characteristic exponent, 96, 107  
characteristic function, 14, 91  
    affine diffusion, 15  
    stochastic volatility, 16  
Black-Scholes, 16  
forward starting, 27
- Heaviside function, 92  
Heston, 17  
multivariate, 26  
Schöbel-Zhu, 19  
Cholesky decomposition, 35, 52  
complex  
    logarithm, 21  
    variable, 21  
conditional Monte Carlo, 51, 53  
conditional option price, 52  
constant elasticity of variance, 77  
contour integral, 93  
control variate, 51, 55  
    speed-up, 56  
correlation  
    linear and exponential transformation, 13  
    linear and hyperbolic transformation, 12  
    multi-dimensional, 150  
    parametric form, 158  
covariance  
    linear and exponential transformation, 13  
    linear and hyperbolic transformation, 12  
covariance matrix  
    Gaussian-Markov process, 39  
    inverse, 41  
        Brownian motion, 42  
    Ornstein-Uhlenbeck process, 40  
Cox-Ingersoll-Ross process, 115
- diffusion process, 14  
Dirac delta distribution, 92  
discrepancy  
    point set, 32  
    star, 32

- displaced diffusion, 77  
 Doss pathwise approximation, 123  
 effective dimension, 33  
 Euler-Maruyama, 117, 128  
     multi-dimensional, 156  
 European option price  
     optimal contour, 97  
     semi-analytical, 91–93  
 exponential Ornstein-Uhlenbeck model, 7  
 exponential transformation  
     moments, 10  
     stochastic differential equation, 10  
 fast mean-reverting analysis, 87  
 Feynman-Kac, 14  
 forward starting options, 27  
 fractional Fourier transformation, 46  
     Ornstein-Uhlenbeck path construction, 49  
     Wiener path construction, 47  
 Gaussian  
     algorithm, 147  
     elimination, 146  
 Gaussian-Markov  
     bridge construction, 39  
     process, 38  
 Hölder inequality, 169  
 Heaviside function, 92  
 Heston, 5, 17  
 hyperbolic local volatility, 78  
 hyperbolic Ornstein-Uhlenbeck model, 7  
 hyperbolic transformation  
     moments, 11  
     stochastic differential equation, 10  
 hypergeometric function  
     confluent, 11  
     Kummer's, 11  
 IJK scheme, 139  
     multidimensional, 155  
 Jensen's inequality, 52, 114  
 Karhunen-Loëve, 43  
 Koksma-Hlawka, 33  
 Lévy area, 129  
     distribution, 129  
 Lévy-Ciesielski, 44  
 linear Ornstein-Uhlenbeck model, 7  
 linear transformation  
     moments, 11  
     stochastic differential equation, 10  
 low discrepancy sequence, 33  
 mean consistent, 165  
 mean square consistent, 165  
 mean-reverting CEV process, 114  
     boundary classification, 116  
     expectation, 117  
     stationary distribution, 116  
     variance, 117  
 Medvedev-Scailet, 89  
 Milstein scheme, 118  
     adaptive, 119  
     positivity preserving, 119  
 Milstein+ scheme, 120  
 moment matched log-Euler scheme, 122  
 Monte Carlo  
     conditional, 51  
     integration, 30  
     methods, 29  
     quasi, 32  
 multiple Wiener integral, 61  
     conditional expectations, 68  
 noncentral chi-square, 115  
 optimal contour, 97  
 Ornstein-Uhlenbeck, 2  
     approximative spectral path construction, 45  
     covariance matrix, 40  
     distribution, 6  
     exponential volatility transformation, 7  
     hyperbolic volatility transformation, 7  
     linear volatility transformation, 7  
     process, 6  
     stochastic volatility model, 6

- variance, 6
- Parseval's identity, 93
- path construction
  - approximative spectral, 43
  - bridge, 39
  - discrete spectral, 35
  - Gaussian process, 35
  - incremental, 35
- pathwise adapted linearisation, 123
- pathwise approximation scheme, 123
  - Brennan-Schwartz, 124
  - Cox-Ingersoll-Ross, 125
- quadrature
  - convergence, 30
  - error, 29
  - multi-dimensional, 30
  - one-dimensional, 29
- quasi-Monte Carlo, 32
  - convergence, 33
- readers guidance, 2
- residue theorem, 93
- rotation count algorithm, 21, 23
- saddlepoint approximation, 107
  - European option price, 108
  - higher-order, 108
  - simple, 107
- Schöbel-Zhu, 7, 19
- Schauder, 44
- Schur complement, 155
- Scott, 7
- short term asymptotics, 89
- sparse grid, 31
  - convergence, 31
- spectral decomposition, 35
- Stein-Stein, 7
- stochastic differential equation, 113, 163
  - numerical integration scheme, 113
- stochastic process
  - Gaussian-Markov, 38
- stochastic volatility
  - characteristic function, 16
- model, 5
- multidimensional, 146
- Stochastic Volatility Inspired, 86
- strip of regularity, 93
- strong approximation order, 113
- strong convergence, 113
- SVI, 86
  - projection, 86
- time-discrete approximation, 113
  - eternal life time, 175
  - finite life time, 175
- variability explained, 35
- variation
  - Hardy-Krause, 32
  - total, 32, 104
  - Vitali, 32
- Watanabe's expansion, 60
- Black-Scholes, 61
- local and stochastic volatility, 81
  - formula, 82
- local volatility, 77
  - formula, 79
- transformed Ornstein-Uhlenbeck model, 69
  - formula, 75
- weak approximation order, 114
- weak convergence, 114