DIPLOMARBEIT

Ein Vergleich von Reeller, reeller und komplexer K-Theorie und ihrer Atiyah-Hirzebruch Spektralsequenzen (A comparison of Real, real and complex K-theory and their Atiyah-Hirzebruch spectral sequences)

Angefertigt am
Mathematischen Institut

Vorgelegt der
Mathematisch-Naturwissenschaftlichen Fakultät der
Rheinischen Friedrich-Wilhelms-Universität Bonn

März 2011

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0 Introduction

0.1 Deutsche Einleitung

In den letzten fünfzig Jahren etablierten sich eine Vielzahl verschiedener K-Theorien. Zum ersten Mal wurde der Begriff K-Theorie von Grothendieck eingeführt, um sein *Grothendieck-Riemann-Roch-Theorem* zu formulieren, welches in einer Arbeit von Borel und Serre veröffentlicht wurde [BS58]. Für jede projektive Varietät X konstruierte er eine Gruppe $K(X)$ aus der Kategorie der kohärenten Garben über X und wies viele schöne Eigenschaften nach.
Die frühen Arbeiten zu topologischer K-Theorie gehen auf Atiyah und Hirzebruch zurück. Sie wendeten Grothendiecks Konstruktion auf die Kategorien komplexer, reeller und quaternionischer Vektorbündel über einem beliebigen parakompakten Hausdorffraum an [AH59]. In dieser und der darauf folgenden Arbeit [AH61] untersuchten sie wichtige Eigenschaften der resultierenden K-Theorien (KU für komplexe, KO für reelle und KSp für quaternionische Vektorbündel). Mit Hilfe der Bott-Periodizität erweiterten sie die K-Gruppen zu Kohomologietheorien.
In seiner Arbeit *K-theory and Reality* [Ati66] führte Atiyah die Reelle K-Theorie KR ein. Er begründete die Kategorie Reeller Vektorbündel, welche komplexe Vektorbündel über einem äquivarianten $\mathbb{Z}/2$-Raum und mit zusätzlichen Eigenschaften sind. Die resultierende *Reelle* K-Theorie kann als gemeinsame Verallgemeinerung von reeller und komplexer K-Theorie verstanden werden.
Neben diesen wohlbekannten Theorien gibt es vielfältige Möglichkeiten, Vektorbündel mit verschiedenen Arten von Formen auszustatten, was in verschiedenen K-Theorien resultiert. Teil I der vorliegenden Arbeit gibt einen Überblick über diese verschiedenen (topologischen) K-Theorien. In Abschnitt 1 werden wir die grundlegenden Definitionen bereitstellen und eine konsistente Notation einführen. Anschließend werden wir die Theorien, vor allem auf der Ebene der Vektorbündel, vergleichen (Abschnitt 2).
Wir fahren fort, die klassifizierenden Räume für Vektorbündel (Abschnitt 3) und für K-Theorien (Abschnitt 4) zu berechnen. Da Reelle Vektorbündel über Reellen Räumen, d.h. Räumen mit einer Aktion von $\mathbb{Z}/2$, definiert sind, ist Reelle K-Theorie eine $\mathbb{Z}/2$-äquivariante Kohomologietheorie. Darum ist auch der klassifizierende Raum ein $\mathbb{Z}/2$-Raum. Tatsächlich handelt es sich bei Reeller K-Theorie um ein Beispiel einer $RO(G)$-graduierten K-Theorie.
Äquivariante Kohomologietheorien wurden zum ersten Mal von Bredon eingeführt [Bre67]. Eine der ersten Arbeiten, die sich ausschließlich mit dem Begriff der äquivarianten Kohomologietheorie beschäftigten, stammt von Lewis, May und McClure [LMM81]. Allerdings waren schon einige Beispiele für äquivarianten Kohomologietheorien vorher bekannt, und manchmal fällt es schwer, *die* grundlegende Arbeit zu benennen. In der vorliegenden Arbeit werden wir immer auf das Buch von May et al. [M^+96] verweisen, das eine hervorragende Referenz für alle Fragen bezüglich äquivarianter Homotopie- und Kohomologietheorie ist. Es sei darauf hingewiesen, dass dieses Buch viele Artikel verschiedener Autoren beeinhaltet und dass jeder Bezug auf dieses Buch als Bezug auf das jeweilige Originalwerk verstanden werden kann.
Nachdem wir die benötigte Notation für äquivariante Homotopietheorie in Teil II eingeführt haben, geben wir eine Einführung in äquivariante Kohomologietheorien (Teil III). Wir beginnen mit einer axiomatischen Charakterisierung

äquivarianter Kohomologietheorien (Abschnitt 10) und fahren in Abschnitt 11 mit der Beschreibung einer wichtigen Klasse solcher Kohomologietheorien, namentlich *gewöhnlicher* äquivarianter Kohomologietheorien, fort.

Eines der wichtigsten Werkzeuge zur Berechnung von K-Theorien ist die Atiyah-Hirzebruch Spektralsequenz, die zum ersten Mal in der Arbeit *Vector bundles and homogeneous spaces* [AH61] auftrat und seitdem vielfältig genutzt wird. Obwohl sie dazu entwickelt wurde, komplexe K-Theorie auszurechnen, beschränkt sich die Atiyah-Hirzebruch Spektralsequenz nicht auf K-Theorie, sondern kann vielmehr für beliebige Kohomologietheorien verwendet werden.
Es gibt zwei verschiedene Konstruktionen für die nicht-äquivariante Atiyah-Hirzebruch Spektralsequenz, deren Äquivalenz von Maunder bewiesen wurde [Mau63]. Diese Spektralsequenz hat die Form

$$H^p(X, h^q(*)) \Rightarrow h^{p+q}(X), \tag{1}$$

wobei h^* eine verallgemeinerte Kohomologietheorie bezeichnet und es sich bei H^* um singuläre Kohomologie handelt.
Im ersten Zugang filtrieren wir den CW-Komplex X durch Unterkomplexe, während wir im zweiten Zugang das Spektrum filtrieren, das die Kohomologietheorie h^* repräsentiert. Für die Filtrierung des Spektrums werden Postnikov-Schnittfunktoren verwendet.
In Teil IV übertragen wir diese Konstruktionen in die äquivariante Welt. Wir benutzen Duggers Notation [Dug05], um in Abschnitt 9 äquivariante Postnikov-Schnittfunktoren zu beschreiben. Nach diesem vorwiegend technischen Abschnitt geben wir zwei explizite Konstruktionen für die klassische äquivariante Atiyah-Hirzebruch Spektralsequenz an und zeigen deren Äquivalenz (Abschnitte 12-14).
Um die Eigenschaften dieser Spektralsequenz zu untersuchen, erweist sich eine formalere Betrachtungsweise als hilfreich. Viele der bekannten Spektralsequenzen lassen sich als Homotopiespektralsequenz eines Turms von Homotopie(ko-)faserungen konstruieren. Eine großartige einführende Arbeit stammt von Dugger [Dug03]. Die Hauptresultate sind allerdings schon seit über vierzig Jahren wohlbekannt. Eine Aufstellung von Referenzen findet sich in der Einleitung dieser Arbeit. Es ist zu beachten, dass [Dug03] aus zwei Teilen besteht, die als einer zitiert werden. Wir werden nur spezifisch auf einen der beiden Teile verweisen, wenn wir explizite Seitenangaben machen.
In Abschnitt 15 erinnern wir an die Konstruktion der Homotopiespektralsequenz und zeigen anschließend, dass wir die Spektralsequenz, die wir in Teil IV konstruiert haben, in dieser Sprache ausdrücken können.
In den letzten Jahren erregte eine gewisse Spektralsequenz großes Aufsehen, die motivische Kohomologie mit algebraischer K-Theorie in Beziehung setzt. In seiner Dissertation [Dug05] führte Dugger eine weitere Atiyah-Hirzebruch Spektralsequenz für KR-Theorie ein, die sich von den anderen, die wir konstruiert haben, unterscheidet, aber sich analog zu dieser neuen verhält. Wir stellen diese so genannte Slice-Spektralsequenz in Abschnitt 16 vor, welche die Form

$$E_2^{p,q} \cong H^{p,-\frac{q}{2}}(X, \underline{\mathbb{Z}}) \Rightarrow KR^{p+q}(X) \tag{2}$$

hat. Hierbei bezeichnet $H^{*,*}(X, \underline{\mathbb{Z}})$ gewöhnliche äquvariante Kohomologie mit Koeffizienten im konstanten Mackeyfunktor $\underline{\mathbb{Z}}$.

Für Reelle Räume $X = X^{\mathbb{Z}/2}$ mit trivialer Wirkung von $\mathbb{Z}/2$ idenifizieren wir die äquivarianten Kohomologiegruppen $H^{p,-\frac{q}{2}}(X, \underline{\mathbb{Z}})$ mit Summen von nicht-äquivarianten Kohomologiegruppen. Dazu analyiseren wir die Fixpunktmengen des darstellenden Eilenberg-MacLane-Spektrums $H\underline{\mathbb{Z}}$ (Section 18).
Im letzten Teil (VI) dieser Arbeit berechnen wir die Spektralsequenz für die projektiven Räume, also $\mathbb{R}P^n$ und $\mathbb{C}P^n$. Diese werden als Reelle Räume verstanden, wobei die Aktion auf den $\mathbb{R}P^n$ trivial ist und die Aktion auf den $\mathbb{C}P^n$ von der komplexen Konjugation induziert wird. Wir berechnen die Reelle K-Theorie für diese Räume indem wir den E_∞-Term der Spektralsequenz benutzen. Die Ergebnisse waren schon vorher dank Fujii [Fuj67] für reelle projektive Räume und dank Atiyah [Ati66] für komplexe projektive Räume bekannt. Unsere Techniken sind jedoch verschieden, und wir sind zuversichtlich, dass diese für spätere Betrachtungen hilfreich sein werden.

0.2 English introduction

In the last fifty years a multiplicity of different topological K-theories has been established. The first notion of K-theory has been introduced by Grothendieck to formulate his *Grothendieck-Riemann-Roch-therorem*, which was published in a paper by Borel and Serre [BS58]. For each projective variety X, he constructed a group $K(X)$ from the category of coherent sheaves on X and showed that it has many nice properties.
In topology, the early work on K-Theory is due to Atiyah and Hirzebruch. For each paracompact Hausdorff space, they applied Grothendieck's construction to the categories of complex, real and quaternionic vector bundles over this space [AH59]. In this and the subsequent paper [AH61], they investigated important properties of the resulting K-theories (KU for complex, KO for real, and KSp for quaternionic vector bundles). Using the Bott periodicity, they extended the K-groups to cohomology theories.
In his paper *K-theory and Reality* [Ati66], Atiyah introduced Real K-theory KR. He established the category of Real vector bundles, which are complex vector bundles over an equivariant $\mathbb{Z}/2$-space, with additional properties. The resulting *Real* K-theory may be understood as a common generalization of the real and the complex theory.
Beside these well-known theories, there are various possibilities to equip the vector bundles with different kinds of forms, resulting in different K-theories.
Part I of this paper gives an overview of all these different (topological) theories. In Section 1 we will recall the basic definitions and establish a consistent notation. Afterwards we will compare these theories, merely on the level of vector bundles (Section 2).
We proceed to compute the classifying spaces for vector bundles (Section 3) and K-theory (Section 4). Since Real vector bundles are defined over Real spaces, i.e. spaces with an action of $\mathbb{Z}/2$, Real K-theory is a $\mathbb{Z}/2$-equivariant cohomology theory. Hence, its classifying space is a $\mathbb{Z}/2$-space. In fact Real K-Theory is an example of an $RO(G)$-graded cohomology theory.
Equivariant cohomology theories have first been introduced by Bredon [Bre67]. One of the first papers working exclusively with the notion of a $RO(G)$-graded cohomology theory is due to Lewis, May and McClure [LMM81]. However some examples of $RO(G)$-graded cohomology theories have been known earlier, and it is sometimes hard to name *the* foundational paper. During the present paper, we will always refer to the conference book by May et al. [$\mathrm{M}^{+}96$] which is an excellent reference for all questions concerning equivariant homotopy and cohomology theory. Note that this book comprises articles by many different authors and any reference to this book may be understood as a reference to the respective original papers.
After setting up the required notation for equivariant homotopy theory in Part II, we give an introduction to equivariant cohomology theories (Part III). We start with an axiomatic characterization of equivariant cohomology theories (Section10) and proceed in Section 11 with a description of an important class of such cohomology theories namely *ordinary* equivariant cohomology theories.

One of the most important tools to compute K-theories is the Atiyah-Hirzebruch spectral sequence, which first appeared in the paper *Vector bundles and homogeneous spaces* [AH61] and has since been used in many different forms. Although

it was invented to compute complex K-theory, the Atiyah-Hirzebruch spectral sequence does not restrict to K-theory, but can be used for arbitrary cohomology theories.
Non-equivariantly there are two different constructions, which are equivalent as it was proven by Maunder [Mau63]. The spectral sequence takes on the form

$$E_2^{p,q} \cong H^p(X, h^q(*)) \Rightarrow h^{p+q}(X), \tag{3}$$

where h^* denotes a generalized cohomology theory and H^* denotes singular cohomology.
In the first approach, we filter the CW-complex X by subcomplexes, while we "filter" the spectrum representing h^* in the second approach. Filtering the spectrum E is done by applying Postnikov section functors.
In Part IV we transfer these constructions to the equivariant world. In Section 9 we use Dugger's notation [Dug05] to describe equivariant Postnikov section functors. After this rather techniqual section, we give two explicit constructions for the classical equivariant Atiyah-Hirzebruch spectral sequence and show their equivalence (Sections 12-14).
To deal with properties of these spectral sequences, a more formal point of view comes in handy. Many of the familiar spectral sequences can be constructed as the homotopy spectral sequence of a tower of homotopy (co-)fiber sequences. A great expository paper is due to Dugger [Dug03]. However the main results have been well-known for about fourty years. A list of references can be found in the introduction of that paper. Note that [Dug03] consists of two parts, which we will cite as one. We only specify to one of them, if we explicitly give a reference on certain pages.
We recall the construction of the homotopy spectral sequence in Section 15 and show that we can express the sequence constructed in Part IV in this language.
In recent years much attention has been given to a certain spectral sequence relating motivic cohomology to algebraic K-theory. In his dissertation Dugger [Dug05] established another Atiyah-Hirzebruch spectral sequence for KR-theory, which is different to the others constructed in the present paper, but behaves analogously to this new one. We compute this so-called slice spectral sequence in Section 16, which takes on the form

$$E_2^{p,q} \cong H^{p,-\frac{q}{2}}(X, \underline{\mathbb{Z}}) \Rightarrow KR^{p+q}(X), \tag{4}$$

where $H^{*,*}(X, \underline{\mathbb{Z}})$ denotes ordinary equivariant cohomology with coefficients in the constant Mackey functor $\underline{\mathbb{Z}}$.
For Real spaces $X = X^{\mathbb{Z}/2}$ with a trivial action of $\mathbb{Z}/2$, we identify the equivariant cohomology groups $H^{p,-\frac{q}{2}}(X, \underline{\mathbb{Z}})$ with sums of non-equivarant cohomology groups. To this end we analyse the fixed point set of the representing Eilenberg-MacLane spectrum $H\underline{\mathbb{Z}}$ (Section 18).
In the last part (VI) of this paper we will calculate this spectral sequence for the projective spaces, namely $\mathbb{R}P^n$ and $\mathbb{C}P^n$. They are considered as Real spaces, where the action on $\mathbb{R}P^n$ is trivial and the action on $\mathbb{C}P^n$ is induced by complex conjugation. We calculate the Real K-theory for these spaces using the E_∞-terms of the spectral sequences. The results have been known due to Fujii [Fuj67] for the real projective space and to Atiyah [Ati66] for the complex projective space with conjugation. However our techniques are different, and we are confident that they become useful for further considerations.

Part I

Comparison of topological K-theories

Each kind of topological K-theory is constructed similarly. Starting with vector bundles over a paracompact Hausdorff space X, we apply the Grothendieck construction to obtain an abelian group. Afterwards we use multiple suspensions of X to extend it to a cohomology theory. The vector bundles come in various different shapes. The base field can be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ [1]. Moreover they can be equipped with different kind of forms. Somehow an extra role is played by Real K-theory, since we can deduce many of the others from it, and the space X has to come with an action of $\mathbb{Z}/2$ [2]. The first chapter will present a number of these theories and compare them on a categorical level. Finally we will characterize the spectra by which they are represented.

1 Definitions and notations

Most of the notation is taken from [Kar78] I.8. We recall some standard results, which we use frequently.

Definition 1.1. Let $\lambda \mapsto \bar{\lambda}$ denote a continuous involution of $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. We require further that $\bar{1} = 1$. Hence the only involution on the real numbers is the identity, on the complex numbers we can have either the identity or complex conjugation, while there are more possible involutions on the quaternions. In fact we are only interested in the identity and conjugation of quaternionic numbers.

Definition 1.2. Let E be a $\mathbb{K}$-vector bundle over a topological space X. A sesquilinear form on E is a continuous map $\phi : E \times_X E \to \mathbb{K}$ which has the property that the form ϕ_x induced on each fiber E_x, $x \in X$ is sesquilinear. In other words we have $\lambda\phi_x(e, e') = \phi_x(\lambda e, e') = \phi_x(e, \bar{\lambda}e')$ for $e, e' \in E_x$.

If E is an arbitrary vector bundle, let E^t denote either its dual bundle E^* if the involution is trivial, or its antidual $\bar{E}^*$ if $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$ and the involution is conjugation. Each sesquilinear form induces a linear map $\psi : E \to E^t$ and an anti-linear map $\tilde{\psi} : E \to E^*$ in the following way. On each fiber we set $\psi_x(e)(e') = \phi_x(e, e')$ and $\tilde{\psi}_x(e)(e') = \phi_x(e', e)$. Standard results show that these specifications define continuous maps on vector bundles. The sesquilinear form is called non-degenerate, if the morphisms are (antilinear) isomorphisms.

Remark 1.3. *Antilinear morphisms differ from linear morphisms only if we take $\mathbb{K}$ to be $\mathbb{C}$ or $\mathbb{H}$ with conjugation. However in this case $\tilde{\phi}$ can be considered as an $\mathbb{R}$-linear morphism of the underlying real vector bundles $E_{\mathbb{R}} \to E^*_{\mathbb{R}}$.*

[1] Even though the quaternions are no field we will use the terms vector bundle and base field. Moreover we call $\mathbb{H}^n$ a vector space rather then a module. We hope the reader will forgive this inaccuracy.

[2] An action of $\mathbb{Z}/2$ is often called involution.

Definition 1.4. Let $\epsilon = \pm 1$. A sesquilinear form is calles ϵ-symmetric if $\phi_x(e', e) = \overline{\epsilon\phi_x(e', e)}$. If $\epsilon = 1$ or $\epsilon = -1$ and the involution is trivial, such forms will be called symmetric or skew-symmetric, respectively. If $\epsilon = 1$ and the involution is conjugation, the form will be called (quaternionic) Hermitian.

Definition 1.5. Let E be a real, complex or quaternionic vector bundle. A symmetric resp. (quaternionic) Hermitian form is called metric, if it has the additional property $\phi_x(e, e) > 0$ for all $e \neq 0$. Two metrics ϕ_0 and ϕ_1 are called isomorphic, if there exists an automorphism f of the vector bundle E such that $\phi_1(f(e), f(e')) = \phi_0(e, e')$.

Let ϕ be a metric on a complex bundle. If we put

$$\phi(e, e') = h(e, e') + i\omega(e, e'), \tag{5}$$

then h is a real metric and ω a bilinear skew-symmetric form on the underlying real bundle. Similarly, if ϕ is a quaternionic metric and we put

$$\phi(e, e') = h(e, e') + j\omega(e, e'), \tag{6}$$

then h is a complex metric and ω a complex bilinear skew-symmetric form on the underlying complex bundle.

Remark 1.6. *We will always write a quaternionic number in the form* $\lambda = \lambda_1 + \lambda_2 j$, *where* $\lambda_i \in \mathbb{C}$ *and* $j \in \mathbb{H}$ *is an element with* $j^2 = -1$ *and* $ij = -ji$.

Proposition 1.7. *Let* E *be a real, complex or quaternionic vector bundle over a paracompact space* X. *Then there exists a metric on* E, *and it is unique up to ismorphism.*

Proof. [Kar78] I.8.7 and I.8.8. Karoubi proves this result only for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For $\mathbb{K} = \mathbb{H}$ the proof goes through unchanged. □

Remark 1.8. *The importance of the uniqueness is that we now may pick a metric on* isomorphism classes *of vector bundles. If* E *and* F *are isomorphic vector bundles with the isomorphism* $f : E \to F$, *we pick metrics* ϕ_E *and* ϕ_F, *respectively. We obtain a second metric on* E *defined by* $\phi'_E(e, e') := \phi_F(f(e), f(e'))$. *But now, following Proposition 1.7,* ϕ'_E *and* ϕ_E *are isomorphic via an automorphism* $g : E \to E$. *Combining these two results we see that* (E, ϕ_E) *and* (F, ϕ_F) *are isometric via* $f \circ g$. *In the following we will frequently use metrics for various contructions. It is crucial that the choice of a metric doesn't influence the outcome.*

Theorem 1.9. *Let* X *be a compact space,* $I = [0, 1]$, *and* E *be a vector bundle on* $X \times I$, *then* $E|_{X\times\{0\}}$ *and* $E|_{X\times\{1\}}$ *are isomorphic.*

Proof. [Kar78] I.7.1 □

Definition 1.10. Let $\mathbb{K}$ be either $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, X a compact space and $n \in \mathbb{N}_0$. Denote by

- $\Phi_n^{\mathbb{K}}(X)$ the set of isomorphism classes of n-dimensional $\mathbb{K}$-vector bundles over the X,

- $Sym^{\mathbb{K}}_{+,n}(X)$ the set of isomorphism classes of n-dimensional $\mathbb{K}$-vector bundles over X provided with a non-degenerate bilinear symmetric form,
- $Sym^{\mathbb{K}}_{-,n}(X)$ denote the set of isomorphism classes of $2n$-dimensional $\mathbb{K}$-vector bundles over X provided with a non-degenerate bilinear skew-symmetric form, and
- $Herm^{\mathbb{C}(X)}_n$ the set of isomorphism classes of n-dimensional complex vector bundles provided with a non-degenerate hermitian form.

Remark 1.11. *Any vector bundle with a non-degenerate bilinear skew-symmetric form is even dimensional, so it makes sense to denote by $Sym^{\mathbb{K}}_{-,n}(X)$ the set of classes of $2n$-dimensional bundles.*

Definition 1.12. Let $\mathbb{K}$ be either $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and X a compact space. Denote by

- $\Phi^{\mathbb{K}}(X)$ the set of isomorphism classes of $\mathbb{K}$-vector bundles over X,
- $Sym^{\mathbb{K}}_{+}(X)$ the set of isomorphism classes of $\mathbb{K}$-vector bundles over X provided with a non-degenerate bilinear symmetric form,
- $Sym^{\mathbb{K}}_{-}(X)$ the set of isomorphism classes of $\mathbb{K}$-vector bundles over X provided with a non-degenerate bilinear skew-symmetric form, and
- $Herm^{\mathbb{C}}(X)$ the set of isomorphism classes of complex vector bundles provided with a non-degenerate hermitian form.

To be precise: We do not take isomorphism classes and then assign forms. Instead we begin with vector bundles equipped with forms and then take isomorphism classes. But now the isomorphisms are required to respect the forms. All these sets turn out to be monoids, where the bifunctor is given by the Whitney sum, which is just the pointwise sum of vector spaces.
As indicated above, there is another important class of topological vector bundles, namely Real vector bundles:

Definition 1.13. A Real space X is a space together with an involution, i.e. a self-inverse automorphism. Sometimes Real spaces are called $\mathbb{Z}/2$-spaces.

Definition 1.14. By a Real vector bundle over the Real space X we mean a complex vector bundle E over X which is also a Real space such that

- the projection $E \to X$ is Real, i.e. it commutes with the involutions on E and X, and
- the map $E_x \to E_{\bar{x}}$ is anti-linear.

Remark 1.15. *Real vector bundles are defined to be complex vector bundles with additional structure. So a priori a Real bundle is locally trivial in the category of complex vector bundles. In fact, it is also trivial in the category of Real vector bundles (cf. [Ati66] p. 374). To find a trivializing neighbourhood around a point x, we have to consider the two cases $x = \bar{x}$ and $x \neq \bar{x}$. Either we have a Real isomorphism $E_x \cong \{x\} \times \mathbb{C}^n$ at once or we pick a complex isomorphism $E_x \cong \{x\} \times \mathbb{C}^n$ and let it induce a Real isomorphism $E|_{\{x,\bar{x}\}} \cong \{x, \bar{x}\} \times \mathbb{C}^n$. Afterwards we can extend the isomorphism to an open neighbourhood of either x or $\{x, \bar{x}\}$: If s is an extension in the category of complex bundles, $\frac{1}{2}(s + \bar{s})$ is an extension in the category of Real bundles.*

Definition 1.16. Let $\bar{\Phi}^{\mathbb{R}}(X)$ denote the set of isomorphism classes of Real vector bundles over the compact Real space X.

Remark 1.17. *The theory of Real vector bundles is an equivariant theory, i.e. all objects are equipped with an action of the group $\mathbb{Z}/2$ and all maps are required to respect this structure. Hence we have to carry out all important constructions equivariantly and give a meaning to $\mathbb{Z}/2$-equivariant homotopies, weak equivalences, (co-)fibrations et cetera. We use some parts of these definitions, when we deal with Real bundles in the following sections, but we would like to postpone any further discussion to the first section of part II. There we will fully engage in the equivariant theory and give an introduction, which is more comprehensive than it would be useful at this point. The reader who is unfamiliar with the used notations is referred to the Sections 5 and 6.*

2 Comparison of vector bundles

Theorem 2.1. *There is an isomorphism*

$$Sym_+^{\mathbb{C}}(X) \to \Phi^{\mathbb{R}}(X), \tag{7}$$

which is natural in X.

Proof. Let E be a $\mathbb{C}$-vector bundle with a non-degenerate symmetric bilinear form θ. Choose a metric ϕ on E. The form θ induces a linear isomorphism $\chi : E \to E^*$ with $\chi(e)(e') = \theta(e', e)$, while the metric induces an anti-linear isomorphism $\tilde{\psi} : E \to E^*$ with $\tilde{\psi}(e)(e') = \tilde{\phi}(e', e)$. Set $\omega := \tilde{\psi}^{-1}\chi : E \to E$ which is anti-linear. As indicated above, we can consider it as a linear isomorphism $\omega : E_{\mathbb{R}} \to E_{\mathbb{R}}$ with the additional property that $\omega i = -i\omega$.
In other words, ω is the unique isomorphism fulfilling

$$\theta(e, e') = \phi(e, \omega e'). \tag{8}$$

Moreover ω^2 is positive and self-adjoint with respect to ϕ, since

$$\phi(e, \omega e') = \theta(e, e') = \theta(e', e) = \phi(e', \omega e) = \overline{\phi(\omega e, e')}, \tag{9}$$

and consequently

$$\phi(e, \omega^2 e') = \phi(\omega^2 e, e'). \tag{10}$$

Hence ω may be written as $h \cdot u$ where $h = \sqrt{\omega^2}$ (which is positive) and $u = h^{-1}\omega$. We have $u^2 = h^{-1}\omega h^{-1}\omega = 1$. Since $p := \frac{1-u}{2}$ is a projection operator, we may write $E_{\mathbb{R}} \cong E_{\mathbb{R}}^+ \oplus E_{\mathbb{R}}^-$, where $E_{\mathbb{R}}^+ = Ker(p)$ and $E_{\mathbb{R}}^- = Ker(1-p)$. Furthermore $E_{\mathbb{R}}^+$ is the 1-eigenspace of u and hence $E \cong E_{\mathbb{R}}^+ \otimes \mathbb{C}$. For $e, e' \in E_{\mathbb{R}}^+$ the following equation holds:

$$\theta(e, e') = \phi(e, he') = \overline{\phi(he', e)} = \overline{\phi(e', he)} = \overline{\theta(e', e)} = \overline{\theta(e, e')} \tag{11}$$

Furthermore,

$$\theta(e, e) = \phi(e, he) \geq 0. \tag{12}$$

Hence θ is a real metric, when restricted to $E_{\mathbb{R}}^+$.

This assignment does not depend on the choice of ϕ. Let ϕ_0 and ϕ_1 be two different metrics on E. There exists a vector bundle F on $X \times I$ with the metric ϕ defined by $\phi_t = t\phi_1 + (1-t)\phi_0$. As above we can use ϕ to decompose $F_{\mathbb{R}} \cong F_{\mathbb{R}}^+ \oplus F_{\mathbb{R}}^-$. Now $F_{\mathbb{R}}^+$ is a bundle on $X \times I$, with $F_{\mathbb{R}}^+|_{X\times\{0\}} \cong E_{\mathbb{R},0}^+$ and $F_{\mathbb{R}}^+|_{X\times\{1\}} \cong E_{\mathbb{R},1}^+$. By Theorem 1.9, $E_{\mathbb{R},0}^+ \cong E_{\mathbb{R},1}^+$.

Let $f : (E,\theta) \to (F,\sigma)$ be an isomorphism of vector bundles with symmetric bilinear forms. For any $e \in E_{\mathbb{R}}^+$ we have

$$\sigma(f(e), f(e)) = \theta(e,e) \geq 0.$$

Hence $f|_{E_{\mathbb{R}}^+} : E_{\mathbb{R}}^+ \to F_{\mathbb{R}}^+$ is an isomorphism of real vector bundles. This means we obtained a well defined map $Sym_+^{\mathbb{C}}(X) \to \Phi^{\mathbb{R}}(X)$.
To obtain an inverse map $Sym_+^{\mathbb{C}}(X) \leftarrow \Phi^{\mathbb{R}}(X)$, we proceed as follows:
Let F be a real vector bundle. Choose a metric σ on F. Define $E = F \otimes \mathbb{C}$ and a form θ on E by

$$\theta(a+bi, c+di) = \sigma(a,c) - \sigma(b,d) + i(\sigma(a,d) + \sigma(b,c)). \tag{13}$$

If σ' is another metric on F, then σ and σ' are isomorphic, and so are the induced forms. Now let $g : E \to F$ be an isomorphism of real vector bundles. It induces a map $g : E \otimes \mathbb{C} \to F \otimes \mathbb{C}$ via $g(e \otimes z) = g(e) \otimes z$, which respects the symmetric forms by definition. It is immediate that the maps $Sym_+^{\mathbb{C}}(X) \to \Phi^{\mathbb{R}}(X)$ and $Sym_+^{\mathbb{C}}(X) \leftarrow \Phi^{\mathbb{R}}(X)$ are inverse to each other.

□

Theorem 2.2. *There is an isomorphism*

$$F : Sym_-^{\mathbb{C}}(X) \to \Phi^{\mathbb{H}}(X), \tag{14}$$

which is natural in X.

The main ingredient for the proof will be

Lemma 2.3. *Let E be a complex vector bundle with compact base, provided with a non-degenerate skew-symmetric form θ. There exists a pair (J, ϕ), which is unique up to isomorphism, where*

(1) J is an automorphism on $E_{\mathbb{R}}$, such that $J^2 = -1$ and $iJ = -Ji$, and

(2) ϕ is a metric on E with $\phi(e, Je') = \theta(e,e')$.

Proof. We begin just as in the proof of Theorem 2.1.
We choose a metric $\tilde{\phi}$ on E. The form θ induces a linear isomorphism $\chi : E \to E^*$ with $\chi(e)(e') = \theta(e', e)$, while the metric induces an anti-linear isomorphism $\tilde{\psi} : E \to E^*$ with $\tilde{\psi}(e)(e') = \tilde{\phi}(e', e)$. Set $\omega := \tilde{\psi}^{-1}\chi : E \to E$ which is anti-linear. Consider it as a linear isomorphism $\omega : E_{\mathbb{R}} \to E_{\mathbb{R}}$ with the additional property that $\omega i = -i\omega$.
ω is the unique isomorphism fulfilling

$$\theta(e,e') = \phi(e, \omega e'). \tag{15}$$

But now ω^2 is negative and self-adjoint with respect to ϕ, since

$$\phi(e, \omega e') = \theta(e,e') = -\theta(e', e) = -\phi(e', \omega e) = -\overline{\phi(\omega e, e')}, \tag{16}$$

and consequently

$$\phi(e, \omega^2 e') = \phi(\omega^2 e, e'). \tag{17}$$

Thus we may define $h = \sqrt{-\omega^2}$ which is itself self-adjoint, but now positive, and commutes with ω. We further set $J := h^{-1}\omega$ and $\psi = \tilde{\psi}h$. The form ϕ induced by ψ is a metric on E.
By definition we get

$$\phi(e, Je') = \tilde{\phi}(e, hh^{-1}\omega e') = \tilde{\phi}(e, \omega e') = \theta(e, e') \tag{18}$$

and finally $J^2 = -1$ as well as $iJ = -Ji$. We call J a quaternionic structure on E, which will be justified later. To what extent is the pair (ϕ, J) unique? The procedure to assign this pair to the chosen metric $\tilde{\phi}$ is canonical and obviously unique. Let (ϕ_0, J_0) and (ϕ_1, J_1) be two pairs fulfilling the conditions of the lemma. Then the J_i are given by $J_i = \psi_i^{-1}\chi$. The metrics are isomorphic via a self-adjoint automorphism f with $f^2 = \psi_1^{-1}\psi_0$ (cf. [Kar78] I.8.8). Hence we obtain $J_1 = f^2 J_0$.

□

Proof of Theorem 2.2. Let E be a complex vector bundle provided with a non-degenerate bilinear skew-symmetric form θ. Pick a pair (ϕ, J) as in Lemma 2.3. Consider E as an $\mathbb{H}$-vector bundle via $\lambda e := \lambda_1 e + \lambda_2 Je$ for all $\lambda = (\lambda_1, \lambda_2) = \lambda_1 + \lambda_2 j \in \mathbb{H}$.
If we replace (ϕ, J) by say (ϕ', J'), the quaternionic vector bundles are clearly isomorphic. If we set $h(e, e') = \phi(e, e') + \omega(Je, e')$, we obtain a quaternionic metric.
If E is a quaternionic bundle, we may pick a quaternionic metric h. We can decompose $h = \phi + j\omega$ where ω is a bilinear skew-symmetric form on the underlying complex bundle. If we choose another metric h', then it is isomorphic to h and so is ω'. Moreover we observe

$$\phi(e, e') + j\omega(e, e') = h(e, e') = jh(e, je') = j\phi(e, je') - \omega(e, je'). \tag{19}$$

Hence $\phi(e, je') = \omega(e, e')$. Consequently the map which maps E to $(E_{\mathbb{C}}, \omega)$ is an inverse to the map above. □

Theorem 2.4. *There are natural isomorphisms*

$$Sym_+^{\mathbb{R}}(X) \cong \Phi^{\mathbb{R}}(X) \times \Phi^{\mathbb{R}}(X), \tag{20}$$

$$Sym_-^{\mathbb{R}}(X) \cong \Phi^{\mathbb{C}}(X), \tag{21}$$

and

$$Herm^{\mathbb{C}}(X) \cong \Phi^{\mathbb{C}}(X) \times \Phi^{\mathbb{C}}(X). \tag{22}$$

Proof. The isomorphisms are constructed analogously to the isomorphisms of Theorem 2.1 and Theorem 2.2, respectively. □

We can now ask ourselves, to what extent we can generalize the results of this sections to categories of vector bundles. The most natural choice for morphisms of vector bundles endowed with forms are maps, which respect these forms.

Definition 2.5. Denote by $\mathscr{S}ym_{+}^{\mathbb{K}}(X)$, $\mathscr{S}ym_{-}^{\mathbb{K}}(X)$ and $\mathscr{H}erm^{\mathbb{C}}(X)$ the categories of isomorphism classes of vector bundles equipped with their respective forms (cf. Definition 1.12) and maps compatible with the forms.

Note that this compatibility implies that the maps are necessarily injective. In order to take the Theorems 2.1, 2.2 and 2.4 over to equivalences of categories we hence have to define suitable morphisms for a category which has the set of objects $\Phi^{\mathbb{K}}(X)$. Recall that the set of isomorphism classes of vector bundles is isomorphic to the set of isomorphism classes of vector bundles, equipped with a metric. Indeed, there exists a metric on any vector bundle and this metric is unique up to isomorphism (cf. Proposition 1.7). Conversely, if a vector bundle E carries a metric and F is an isomorphic vector bundle, we can use the isomorphism to pull back the metric to F.

Definition 2.6. Denote by $\Psi^{\mathbb{K}}(X)$ the category of $\mathbb{K}$-vector bundles equipped with a metric and morphisms, which respect this metric.

Theorem 2.7. *The isomorphisms in Theorems 2.1, 2.2 and 2.4 induce equivalences of categories*

$$\mathscr{S}ym_{+}^{\mathbb{C}}(X) \cong \Psi^{\mathbb{R}}(X), \tag{23}$$

$$\mathscr{S}ym_{-}^{\mathbb{C}}(X) \cong \Psi^{\mathbb{H}}(X), \tag{24}$$

$$\mathscr{S}ym_{+}^{\mathbb{R}}(X) \cong \Psi^{\mathbb{R}}(X) \times \Psi^{\mathbb{R}}(X), \tag{25}$$

$$\mathscr{S}ym_{-}^{\mathbb{R}}(X) \cong \Psi^{\mathbb{C}}(X), \tag{26}$$

and

$$\mathscr{H}erm^{\mathbb{C}}(X) \cong \Psi^{\mathbb{C}}(X) \times \Psi^{\mathbb{C}}(X). \tag{27}$$

Proof. In view of what we have done so far, it remains to define a functorial correspondence of morphisms. We will do this exemplarily for (23) and (24). The other cases work analogously.

- (23)
  We will give an explicit construction of functors
  $$\Gamma : \mathscr{S}ym_{+}^{\mathbb{C}}(X) \leftrightarrows \Psi^{\mathbb{R}}(X) : \Delta, \tag{28}$$
  inducing the equivalence.
  The underlying real bundles of an object $(E, \theta) \in \mathscr{S}ym_{+}^{\mathbb{C}}(X)$ can be decomposed as $E_{\mathbb{R}} \cong E_{\mathbb{R}}^{+} \oplus E_{\mathbb{R}}^{-}$, where the form θ on E restricts to a real metric on $E_{\mathbb{R}}^{+}$. Hence, $\Gamma((E, \theta)) = (E_{\mathbb{R}}^{+}, \theta|_{E_{\mathbb{R}}^{+}})$. Let now (E', θ') be another object and $f : (E, \theta) \to (E', \theta')$ a morphism in $\mathscr{S}ym_{+}^{\mathbb{C}}(X)$. Since we have $0 \leq \theta(e, e) = \theta'(f(e), f(e))$ for $e \in E_{\mathbb{R}}^{+}$, f restricts obviously to a morphism in $\Psi^{\mathbb{R}}(X)$. Thus we set $\Gamma(f) = f|_{E_{\mathbb{R}}^{+}}$.
  Conversely, let (F, ϕ) and (F', ϕ') be objects in $\Psi^{\mathbb{R}}(X)$ and $g : (F, \phi) \to (F', \phi')$ a morphism between them. We know that $\Delta((F, \phi)) = (F \otimes \mathbb{C}, \sigma)$, where
  $$\sigma(a + bi, c + di) = \phi(a, c) - \phi(b, d) + i(\phi(a, d) + \phi(b, c)). \tag{29}$$
  Further, we set $\Delta(g)(a + bi) = g(a) + ig(b)$, which is clearly functorial. It is immediate that $\Gamma \circ \Delta$ and $\Delta \circ \Gamma$ are the identity.

- (24)
As above, we will give an explicit construction of functors

$$\Gamma : \mathscr{S}ym^{\mathbb{C}}_{-}(X) \leftrightarrows \Psi^{\mathbb{H}}(X) : \Delta, \tag{30}$$

inducing the equivalence.
Recall that, given an object $(E, \theta) \in Sym^{\mathbb{C}}_{-}(X)$, there exists a pair (J, ϕ), such that we may consider E as a quaternionic bundle equipped with a metric h, which is given by

$$h(e, e') = \phi(e, e') + \theta(Je, e'). \tag{31}$$

Alternatively, we can define h by

$$h(e, Je') = \theta(e, e') + \overline{\theta(e, e')}, \tag{32}$$

which shows that any map respecting the forms, also respects metrics. Thus, we define $\Gamma((E, \theta)) = (E, h)$ and $\Gamma(f)(a + Jb) = f(a) + J'f(b)$.
Conversely, given an object $(F, h) \in \Psi^{\mathbb{H}}(X)$, we have $\Delta(F, h) = (F_{\mathbb{C}}, \theta)$, where h decomposes as $h = \phi + j\theta$, for a complex metric ϕ and a non-degenerate bilinear skew-symmetric form θ. Clearly, any map compatible with quaternionic metrics, will be compatible with forms alone.

□

3 Classifying spaces

Classifying spaces have been used frequently over the years. We will just give a survey over the general theory and pay a bit more attention to the classifying space for Real K-theory. It is a matter of personal taste how to present this topic. Hence the results can be found with different notations at many different places. We will mainly follow [Ste51], [Hus93] and [Ros94].

3.1 Fibre bundles and structure groups

Definition 3.1. A fiber bundle E over X with fiber F and structure group G consists of a continuous map $p : E \to X$ of topological spaces and a space F where G is a topological group acting as a group of homeomorphisms on F such that

- The map p is locally trivial, i.e. each point $x \in X$ has an open neighbourhood U and a homeomorphism $\phi_U : U \times F \to p^{-1}(U)$ such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[d, "p"] & U \times F \arrow[l, "\phi_U"'] \arrow[dl, "\pi_1"] \\ U & \end{tikzcd} \tag{33}$$

- If U_i and U_j are two open neighbourhoods of x, then there exists a unique map $g_{i,j} : U_i \cap U_j \to G$ such that $\phi_{U_j}(x, y) = \phi_{U_i}(x, g_{i,j}(x)y)$ for all $y \in F$.

If F is homeomorphic to G and G acts on itself by left translation, the bundle is called a principal bundle.

Remark 3.2. *We obtain the same definition if we only require the existence of an open covering of trivializing neighbourhoods. This covering together with the trivializing homeomorphisms is called an atlas. If we have two atlases for the same given (E, B, p, F), then the bundles are isomorphic if the union of these atlases is again an atlas.*

Proposition 3.3. *Let E be a fiber bundle over X with structure group G. Let U_i, U_j and U_k be trivializing neighbourhoods of $x \in X$. Then the $g_{i,j}$ stated in Definition 3.1 above have the following properties*

(1) $g_{i,j}(x)g_{j,k}(x) = g_{i,k}(x)$ *on* $U_i \cap U_j \cap U_k$,

(2) $g_{i,j}(x) = (g_{j,i})(x)^{-1}$ *and*

(3) $g_{i,i}(x) = 1$.

Proof. This is an immediate consequence of the property

$$\phi_{U_j}(x, y) = \phi_{U_i}(x, g_{i,j}(x)y). \tag{34}$$

□

Definition 3.4. Let X be a topological space and $\{U_i\}$ an open covering of X. A set of maps $g_{i,j} : U_i \cap U_j \to G$ fulfilling the properties given in Proposition 3.3 is called a system of transition functions associated with $\{U_i\}$.

The main reason why we are interested in fiber bundles is

Theorem 3.5. *Let $\{U_i\}$ be an open covering of a space X, let G be a topological group acting as a group of homeomorphisms on a space F, and let $\{g_{i,j}\}$ be a set of transformation functions associated with the covering $\{U_i\}$. Then there exist a fibre bundle η, unique up to isomorphism, and an atlas $\{(U_i, \phi_{U_i})\}$ for η such that the set of transition functions for this atlas is $\{g_{i,j}\}$. Moreover, if $F = \mathbb{K}^n$, for $n \in \mathbb{N}$ and $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and if G is a closed subgroup of $GL(n, \mathbb{K})$, then η admits the structure of a vector bundle.*

Proof. [Hus93] Theorem 5.3.2 □

The crucial point is that the structure group $GL(n, \mathbb{K})$ forces the trivializations to be linear on fibers. We have already seen, that each vector bundle gives rise to a set of transformation functions associated to a covering of the base space. Hence we may identify the set of isomorphism classes of vector bundles with isomorphism classes of fibre bundles with structure group $GL(n, \mathbb{K})$. If we pick different subgroups of $GL(n, \mathbb{K})$, we obtain vector bundles which admit extra structure.

Theorem 3.6. *Let E be a vector bundle over X with a metric h (real, complex, or quaternionic). Then E has an atlas $\{(U_i, \phi_{U_i})\}$ such that*

$$(e, e') = h(\phi_{U_i}(b, e), \phi_{U_i}(x, e')), \tag{35}$$

where $(-, -)$ denotes the standard metric on $\mathbb{K}^n$, $n \in \mathbb{N}$. The transition functions of this atlas have their values in $O(n), U(n)$ or $Sp(n)$, respectively.

Proof. [Hus93] Theorem 5.7.4 □

Hence vector bundles with metrics (that is all vector bundles, cf. Proposition 1.7) may be identified with fibre bundles with fibre $\mathbb{K}^n$ and structure group $O(n), U(n)$ or $Sp(n)$ respectively.

Definition 3.7.

- Let $O_{n,p}(\mathbb{K})$ denote the subgroup of $GL_{n+p}(\mathbb{K})$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$), which consists of isometries of $\mathbb{K}^{n+p}$ provided with the form $\sum_{i=1}^{n} x_i\bar{y}_i - \sum_{i=n+1}^{n+p} x_i\bar{y}_i$.
- Let $Sp_{2n}(\mathbb{K})$ denote the subgroup of $GL_{n2n}(\mathbb{K})$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$), which consists of isometries of $\mathbb{K}^{2n}$ provided with the form $\sum_{i=1}^{n} x_iy_{i+n} - \sum_{i=1}^{n} x_{i+n}y_i$.
- Let $O_n(\mathbb{C})$ denote the subgroup of $GL(n, \mathbb{C})$, which consists of isometries of $\mathbb{C}^n$ provided with the quadratic from $\sum_{i=1}^{n}(x_i)^2$.

As we have already seen, a vector bundle E carrying a bilinear non-degenerate symmetric form ω (in the real case) or a hermitian form ω (in the complex case) can be written as an orthogonal sum $E_+ \oplus E_-$, where ω is positive definite on E_+ and negative definite on E_- (cf. Theorem 2.4). We can apply Theorem 3.6 on both summands seperately and obtain

Theorem 3.8. *Let E be a vector bundle over X with a bilinear non-degenerate symmetric resp. hermitian form ω. Then E has an atlas $\{(U_i, \phi_{U_i})\}$ such that $(e, e') = \omega(\phi_{U_i}(x, e), \phi_{U_i}(x, e'))$, where $(-,-)$ denotes the form on $\mathbb{K}^{n+p}$ defined by $(e', e) = \sum_{i=1}^{n} e'_i\bar{e}_i - \sum_{i=n+1}^{n+p} e'_i\bar{e}_i$. The transition functions of this atlas have their values in $O_{n,p}(\mathbb{K})$.*

In fact we lose some information in the second statement. From $(e, e') = \omega(\phi_{U_j}(x, e), \phi_{U_j}(x, e')) = \omega(\phi_{U_i}(x, g_{i,j}(x)e), \phi_{U_i}(x, g_{i,j}(x)e')) = (g_{i,j}(x)e, g_{i,j}(x)e')$ we see that the $g_{i,j}$ respect the form $(-,-)$. Since we used the splitting $E = E_+ \oplus E_-$ in the proof, we could have obtained that the transition functions have their values in $O(n) \times O(p)$ or $U(n) \times U(p)$, respectively. This is not surprising as it is nothing more than the isomorphism in Theorem 2.4.
We can do an analogous construction for vector bundles with bilinear non-degenerate skew-symmetric forms. We use the isomorphisms of Theorem 2.2 and Theorem 2.4 and the induced metrics to obtain the desired atlases.

Theorem 3.9. *Let E be a vector bundle over X with a a bilinear non-degenerate skew-symmetric form ω. Then E has an atlas $\{(U_i, \phi_{U_i})\}$ such that $(e, e') = \omega(\phi_{U_i}(x, e), \phi_{U_i}(x, e'))$, where $(-,-)$ denotes the form on $\mathbb{K}^{2n}$ defined by $(e, e') = \sum_{i=1}^{n} e_ie'_{i+n} - \sum_{i=1}^{n} e_{i+n}e'_i$. The transition functions of this atlas have their values in $Sp_{2n}(\mathbb{K})$.*

Finally we deduce the result for complex vector bundles with bilinear symmetric forms, or equivalently with quadratic forms. (By standard linear algebra each quadratic form q induces a bilinear symmetric form by $\omega(e, e') := \frac{1}{2}(q(e + e') - q(e) - q(e'))$, and vice versa any bilinear symmetric form ω induces a quadratic form by $q(e) := \omega(e, e)$).

Theorem 3.10. *Let E be a complex vector bundle over X provided with a non-degenerate quadratic form q. Then E has an atlas $\{(U_i, \phi_{U_i})\}$ such that $\tilde{q}(e) = q(\phi_{U_i}(x, e))$, where $\tilde{q}$ denotes the quadratic from on $\mathbb{C}^n$ defined by $\tilde{q}(e) = \sum_{i=1}^{n}(e_i)^2$. The transition functions of this atlas have their values in $O_n(\mathbb{C})$.*

We now have computed the structure groups of all types of vector bundles we defined in Definition 1.12 (that is all types we have considered, except Real bundles). In view of the isomorphisms given in the Theorems 2.1, 2.2 and 2.4 it looks like as if there were different choices:

- $O_{n,p}(\mathbb{K})$ and $O(n) \times O(p)$ or $U(n) \times U(p)$ for bundles with non-degenerate symmetric bilinear resp. hermitian forms,
- $Sp_{2n}(\mathbb{K})$ and $U(n)$ resp. $Sp(n)$ for bundles with non-degenerate bilinear skew-symmetric forms, and
- $O_n(\mathbb{C})$ and $O(n)$ for complex bundles with quadratic resp. non-degenerate bilinear symmetric forms.

However, the following theorem solves this ambiguity:

Theorem 3.11. *There exist deformation retractions of*

- *$O_{n,p}(\mathbb{K})$ onto $O(n) \times O(p)$ if $\mathbb{K} = \mathbb{R}$ and onto $U(n) \times U(p)$ if $\mathbb{K} = \mathbb{C}$,*
- *$Sp_{2n}(\mathbb{K})$ onto $U(n)$ if $\mathbb{K} = \mathbb{R}$ and onto $Sp(n)$ if $\mathbb{K} = \mathbb{C}$, and of*
- *$O_n(\mathbb{C})$ onto $O(n)$.*

For the proof we need some general statements on Lie-groups [HN91]:

Remark 3.12. *(Notation) If G is a Lie-group, we use the letter $\mathfrak{g}$ to denote the Lie-algebra $Lie(G)$.*

Theorem 3.13. *Let G be a connected Lie-group, $\mathfrak{k} \subseteq \mathfrak{g}$ a maximal compactly embedded subalgebra, $K := \langle exp(\mathfrak{k})\rangle$ and $T \subseteq K$ a maximal compact subgroup. Then T is maximal compact in G, and there exists a closed submanifold $M \cong \mathbb{R}^n \subseteq G$, such that the mapping*

$$M \times T \to G, \quad (m, t) \mapsto mt \tag{36}$$

is a diffeomorphism.

Proof. [HN91] p.286 $\square$

Theorem 3.14. *Let $G \subseteq GL_n(\mathbb{C})$ be a subgroup, which is the zero set of a set of polynomials in the $2n^2$ entries of the matrix and which is invariant under the transposition $g \mapsto g^* = \bar{g}^T$. Set*

$$\mathfrak{k} := \{X \in \mathfrak{g} : X^* = -X\} \quad \mathfrak{p} := \{X \in \mathfrak{g} : X^* = X\} \tag{37}$$

and $K := G \cap U(n)$. Then $Lie(K) = \mathfrak{k}$, K is a maximal compact subgroup of G and the mapping

$$\Phi : K \times \mathfrak{p} \to G, \quad (k, P) \mapsto k \exp(P) \tag{38}$$

is a diffeomorphism.

Proof. [HN91] p. 321 □

Proof of Theorem 3.11. We apply the Theorems 3.13 and 3.14 to our setting. We deal with all different situations analogously. To prove that X is a deformation retract of A we consider X as a subgroup of A with inclusions we specify below.

- Via the inclusion
$$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \tag{39}$$
we regard $O(n) \times O(p)$ and $U(n) \times U(p)$ as subgroups of $O(n,p)(\mathbb{R})$ and $O(n,p)(\mathbb{C})$, respectively.

- $U(n)$ is a subgroup of $Sp_{2n}(\mathbb{R})$ via the inclusion
$$Z = X + iY \mapsto \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}, \tag{40}$$
and $Sp(n)$ is a subgroup of $Sp_{2n}(\mathbb{C})$ via the inclusion
$$Z = X + jY \mapsto \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}. \tag{41}$$

- $O(n)$ is clearly a subgroup of $O_n(\mathbb{C})$, since the elements of $O(n)$ respect the quadratic form $\sum_{i=1}^n (x_i)^2$ by definition.

All groups in place of A fulfill the conditions of G in Theorem 3.14. We only have to show that $A \cap U(n) = B$ which is basic linear algebra. This proves that X is a maximal compact subgroup of A. Afterwards we apply Theorem 3.14 to each component, seperately. Hence A is diffeomorphic to $B \times M$, where M is contractible. This finishes the proof. □

3.2 Classifying spaces for classical vector bundles

Once we have identified the structure groups, we finally can deduce classifying spaces for vector bundles.

Theorem 3.15. *Let G be a group. There exists a contractible CW-complex X with a free and cellular action of G, such that X/G is a CW-complex. We write $X = EG$ and $X/G = BG$. $EG \to BG$ is a G principal bundle. BG is called a classifying space for fiber bundles with structure group G.*

Proof. [Ros94] Theorem 5.1.15. □

Proposition 3.16. *The classifying space construction yields a functor X from the category of groups with group homomorphisms to the category of topological spaces with homotopy classes of maps.*

Proof. [Ros94] Proposition 5.1.18. □

Theorem 3.17. *Let G be a group and X a CW-complex. Then the pullback induces a 1-1 correspondence between homotopy classes of maps $X \to BG$ on the one hand, and G principal bundles over X on the other hand.*

Proof. [Ste51] Theorem 19.3. □

We are looking for a correspondence of homotopy classes of maps $X \to BG$ and vector bundles over X, which have the structure group G. Luckily we can construct it easily:
As we recall from Theorem 3.5, we can think of vector bundles as atlases together with transition functions associated to this atlases. From this point of view the fiber becomes an independent extra datum. Hence, if we start with a vector bundle and forget the fibre, we can replace it by G and use the transition functions and Theorem 3.5 to obtain a G principal bundle carrying all the information except the datum of the fibre. This new bundle is called the associated G principal bundle. Conversely, if we start with a G principal bundle and a space F with a suitable action of G, we can use this data to construct a fibre bundle with structure group G.

Theorem 3.18. *Two bundles having the same fibre, base space, and structure group are equivalent, if and only if their associated principal bundles are equivalent.*

Proof. [Ste51] Theorem 8.2. □

Corollary 3.19. *There are functor isomorphisms*

- $\Phi_n^{\mathbb{R}}(X) \cong [X, BO(n)]$,
- $\Phi_n^{\mathbb{C}}(X) \cong Sym_{-,n}^{\mathbb{R}}(X) \cong [X, BU(n)]$,
- $\Phi_n^{\mathbb{H}}(X) \cong Sym_{-,n}^{\mathbb{C}}(X) \cong [X, BSp(n)]$,
- $Sym_{+,n,p}^{\mathbb{K}}(X) \cong [X, BO(n,p)]$, *and*
- $Herm_{n,p}^{\mathbb{C}}(X) \cong [X, BU(n,p)]$,

where the $Sym_{+,n,p}^{\mathbb{K}}(X)$ *and* $Herm_{n,p}^{\mathbb{C}}(X)$ *denote the sets of vector bundles, which decompose into sums of two real resp. complex vector bundles of dimensions* n *and* p.

3.3 The classifying space for Real vector bundles

Recall from Definition 1.14 that a Real vector bundle over a Real space X is a complex bundle, which is also a Real space, such that the projection is equivariant, i.e. compatible with the action of $\mathbb{Z}/2$. Consequently a classifying space for Real vector bundles has to carry a $\mathbb{Z}/2$-action as well. Instead of developing the construction described in the last section in an equivariant setting, we use this opportunity, to compute a classifying space directly. In fact, this explicit construction works also non-equivariantly, so can be seen as an important example for the computation of classifying spaces. As soon as we are dealing with G-spaces, all maps are required to be equivariant.
It is well known that a model for BU(n) is given by Grassmannians (cf. for instance [Hat03] Theorem 1.6). The classifying space for Real vector bundles is also denoted by BU, but now carries a Real structure. We will repeat the proof of [Hat03] carefully and add all equivariant details.
First we define the Grassmann manifold $G_n(\mathbb{C}^k)$ for nonnegative integers $n \leq k$.

Here, $\mathbb{C}^k$ is the Real space, where the involution is given by complex conjugation. As a set, $G_n(\mathbb{C}^k)$ is the collection of all n-dimensional subspaces of $\mathbb{C}^k$, and this set is topologized as a quotient of a subspace of the n-fold product of spheres in $\mathbb{C}^k$.
The canonical vector bundle on $G_n(\mathbb{C}^k)$ is defined to be $E_n(\mathbb{C}^k) = \{(l,v) \in G_n(\mathbb{C}^k) \times \mathbb{C}^k | v \in l\}$ with the projection $p : E_n(\mathbb{C}^k) \to G_n(\mathbb{C}^k)$, $(l,v) \mapsto v$. We already know that this is a complex vector bundle and it is straighforward to check, that it is a Real bundle. As usual, we set $G_n = \cup_k G_n(\mathbb{C}^k)$ and $E_n = \cup_k E_n(\mathbb{C}^k)$.

Theorem 3.20. *For paracompact spaces X, the map $[X, G_n]_G \to \bar{\Phi}^{\mathbb{R}}(X)$, $[f] \mapsto f^*(E_n)$ is a bijection.*

Proof. The main ingredient is the following: For an n-dimensional Real vector bundle $p : E \to X$, an isomorphism $E \cong f^*(E_n)$ is equivalent to an equivariant map $g : E \to C^\infty$, which is a linear injection on each fiber.
Indeed, if we start with an equivariant map $f : X \to G_n$ and an isomorphism $E \cong f^*(E_n)$, we have a commutative diagram

$$\begin{tikzcd} E \arrow[r, "\cong"] \arrow[dr, "p"'] & f^*(E_n) \arrow[r, "\tilde{f}"] \arrow[d] & E_n \arrow[r, "\pi"] \arrow[d] & \mathbb{C}^\infty \\ & X \arrow[r, "f"] & G_n & \end{tikzcd} \tag{42}$$

where $\pi(l,v) = v$. The composition across the top row is a map $g : E \to \mathbb{C}^\infty$ with the desired properties. Conversely a linear equivariant fiberwise injective map $g : E \to \mathbb{C}^\infty$ allows us to define $f : X \to G_n$, $x \mapsto g(p^{-1}(x))$. This yields a diagram as above.
To show surjectivity of the map $[X, G_n] \to \bar{\Phi}^{\mathbb{R}}(X)$, we suppose $p : E \to X$ is an n-dimensional Real vector bundle. Let $\{U_\alpha\}$ be an open cover of X such that E is trivial over each U_α. Note that the bundle is indeed locally trivial as a *Real* bundle. Since X is paracompact, $\{U_\alpha\}$ can be assumed to be countable and admit an equivariant partition of unity $\{\phi_\alpha\}$[3], where ϕ_α is supported in U_α. Now, let $g_\alpha : p^{-1}(U_\alpha) \to \mathbb{C}^n$ be the composition of the trivialization $p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^n$ with the projection onto $\mathbb{C}^n$. The maps $(\phi_\alpha p)g_\alpha$ extend to maps $E \to \mathbb{C}^n$, which are zero outside $p^{-1}(U_\alpha)$, and near each point of X, finitely many ϕ_α are nonzero. Hence, the $(\phi_\alpha p)g_\alpha$ are the coordinates of a map $g : E \to (\mathbb{C}^n)^\infty = \mathbb{C}^\infty$, which is an injection on each fiber.
For injectivity we have to show the following:
If $E \cong f_0^*(E_n)$ and $E \cong f_1^*(E_n)$ are two isomorphisms for maps $f_0, f_1 : X \to G_n$, then they induce maps $g_0, g_1 : E \to \mathbb{C}^\infty$ as in the first paragraph of this proof. These maps are $\mathbb{Z}/2$-homotopic and thus f_0 and f_1 are $\mathbb{Z}/2$-homotopic as well via $f_t(x) = g_t(p^{-1}(x))$.
We define $L_t : \mathbb{C}^\infty \to \mathbb{C}^\infty$ by $L_t(x_1, x_2, ...) = (1-t)(x_1, x_2) + t(x_1, 0, x_2, 0, ...)$. If we compose g_0 with L_t, we obtain a $\mathbb{Z}/2$-homotopy of g_0 onto a map which we also denote by g_0. Now the image of g_0 is moved into the odd-numbered coordinates. Similarly we can move the image of g_1 into the even-numbered

[3]If p' is any partition of unity, $p(x) = \frac{p'(x)+p'(\bar{x})}{2}$ is obviously an equivariant partition of unity.

coordinates. Let $g_t = (1-t)g_0 + tg_1$. This is a $\mathbb{Z}/2$-homotopy of g_0 onto g_1, which is linear and injective on fibers.

□

4 Stably isomorphic vector bundles and K-theory

Proposition 4.1. *Let M be a monoid, i.e. a set with an operation satisfying all axioms of a group, possibly without existence of inverses. There exists an abelian group $S(M)$ and a homomorphism of the underlying monoids $s : M \to S(M)$ with the following universal property:*
For any abelian group G and any homomorphism of the underlying monoids $f : M \to G$, there is a unique group homomorphism $\tilde{f} : S(M) \to G$ which makes the following diagram commute:

$$\begin{tikzcd} M \arrow[rr, "s"] \arrow[dr, "f"'] & & S(M) \arrow[dl, "\tilde{f}"] \\ & G & \end{tikzcd} \tag{43}$$

Proof. Define $S(M) = M \times M/\sim$, where $(m,n) \sim (m',n')$ if, and only if there exists a $p \in M$ such that $m + n' + p = n + m' + p$. Further set $s(m) = (m, 0)$.

□

Remark 4.2.

- *We will write the elements of $S(M)$ as $(m,n) = m - n$. The procedure of assigning the group $S(M)$ to the monoid M is referred to as the Grothendieck construction, and $S(M)$ is called the Grothendieck group associated to M.*
- *The group $S(M)$ depends functorially on M in an obvious way.*

If we apply the Grothendieck construction to the monoids given by isomorphism classes of vector bundles introduced in Definition 1.12, we obtain the different types of topological K-theories. We will denote the Grothendieck groups $S(M)$ for these monoids by $K(X)$. Usually $K(X)$ is only used for the Grothendieck group of the monoid of complex vector bundles. We will find distinguished names for the different theories later, but right now all situations can be treated similarly. Likewise, we use $\Phi(X)$ as a common name for all different sets of isomorphism classes of vector bundles. If we want to specify the dimension of the vector bundles we write $\Phi_n(X)$ or $\Phi_{n,p}(x)$. Recall that for bundles with non-degenerate bilinear skew-symmetric forms, the dimension of bundles in $\Phi_n(X)$ is $2n$ rather than n, while for real bundles with symmetric bilinear forms and complex bundles with hermitian forms the index (n,p) indicates bundles with an n-dimensional subbundle carrying a positiv definite form, and a p-dimensional subbundle carrying a negative definite form. The trivial bundles over X will generally be denoted by θ_n or $\theta_{n,p}$ where the dimension is again n, $2n$ or $n+p$. The forms are given by $\begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}$ or $\begin{pmatrix} E_n & 0 \\ 0 & -E_p \end{pmatrix}$, respectively. The structure group for $\Phi(X)$ is denoted either by $G(n)$ or $G(n,p)$, the classifying space by $BG(n)$ or $BG(n,p)$.

Definition 4.3. If X is a pointed space, the inclusion $* \hookrightarrow X$ induces a homomorphism $K(X) \to K(*)$. Denote the kernel of this map by $\tilde{K}(X)$, the reduced K-group.

Definition 4.4. By adding trivial 1-dimensional (resp. 2-dimensional) bundles we obtain inclusions $\Phi_n(X) \to \Phi_{n+1}(X)$, $\Phi_{n,p}(X) \to \Phi_{n+1,p}(X)$ and $\Phi_{n,p}(X) \to \Phi_{n,p+1}(X)$. Let $\Phi'(X)$ be the colimit over all these maps. Similarly we have inclusions $G(n) \to G(n+1)$, $G(n,p) \to G(n+1,p)$ and $G(n,p) \to G(n,p+1)$ which induce maps on the classiyfing spaces. Denote the colimit of the given diagram by BG. The space BG turns out to be a classifying space in the sense of Theorem 3.15.

Proposition 4.5. *For every compact space X, there are natural isomorphisms $\tilde{K}(X) \cong \Phi'(X) \cong [X, BG]$.*

Proof. For the first part we only need the following observation: Any element in $K(X)$ can be written as $E - \theta$ for some trivial bundle θ. For $E - \theta$ to live in $\tilde{K}(X)$, the dimensions of E and θ have to be equal. Hence there are surjective homomorphisms from Φ_n resp. $\Phi_{n,p}$ onto $\tilde{K}(X)$. Two elements $E - \theta$ and $F - \sigma$ are equal in $\tilde{K}(X)$ if $E \oplus \sigma \cong F \oplus \theta$. This is the same as being equal in $\Phi'(X)$. For the second part observe that the diagrams

$$\begin{tikzcd} \Phi_n(X) \arrow[r] \arrow[d] & {[X, BG(n)]} \arrow[d] \\ \Phi_{n+1}(X) \arrow[r] & {[X, BG(n+1)]} \end{tikzcd} \tag{44}$$

and

$$\begin{tikzcd} \Phi_{n,p}(X) \arrow[r] \arrow[d] & {[X, BG(n,p)]} \arrow[d] \\ \Phi_{n+1,p}(X) \arrow[r] & {[X, BG(n+1,p)]} \end{tikzcd} \qquad \begin{tikzcd} \Phi_{n,p}(X) \arrow[r] \arrow[d] & {[X, BG(n,p)]} \arrow[d] \\ \Phi_{n,p+1}(X) \arrow[r] & {[X, BG(n,p+1)]} \end{tikzcd} \tag{45}$$

commute and hence induce isomorphisms on the colimits. Finally, $colim[X, BG(n)] = [X, BG]$ and $colim[X, BG(n,p)] = [X, BG]$.

$\square$

Part II

Equivariant homotopy theory

5 The category of G-spaces

Throughout the remainder of this paper we work in the category of topological G-spaces. Even though we encountered $\mathbb{Z}/2$-spaces earlier when we dealt with Real vector bundles, we now give a more complete introduction to the equivariant world. Since we barely use *non*-equivariant objects anymore we often just write space (or the name of any other object), instead of equivariant space et cetera. If it is important to distinguish between objects, we rather indicate the non-equivariant counterpart than the equivariant one. Most of the notation is taken from May [M$^+$96].

Definition 5.1. A G-space is a topological space together with a continuous action of a topological group G. If we don't need to specify the group G, we sometimes use the term *equivariant* space.

Definition 5.2. A map $f : X \to Y$ between two G-spaces is called equivariant or G-map if $f(gx) = gf(x)$.

Definition 5.3. Denote by $\mathscr{U}$ the category of compactly generated and weak Hausdorff spaces and by $G\mathscr{U}$ the category of G-equivariant compactly generated and weak Hausdorff spaces.

The categories $\mathscr{U}$ and $G\mathscr{U}$ are our objects of study, i.e. we assume every space to be in either of these categories.
The usual constructions on spaces apply in the category $G\mathscr{U}$ of G-spaces and G-maps. For example, G acts diagonally on Cartesian products and by conjugation on mapping spaces.
Moreover, we assume subgroups $H \subset G$ to be closed.

Definition 5.4. For a subgroup $H \subset G$, X^H denotes the set of points which are fixed under the action of H, and X/H is the orbit space, obtained by identifying points in the same orbit.

Definition 5.5. A based G-space is a G-space with a distinguished basepoint, which is fixed under the action of G. We write X_+ for the union of a G-space X and a disjoint basepoint.

Definition 5.6. A G-CW complex is a G-space with a decomposition $X = colim_{p\geq 0} X^p$, where X^0 is a disjoint union of orbits G/H and X^{p+1} is obtained from X^p by attaching G-cells $G/H \times D^{p+1}$ along G-maps $G/H \times S^p \to X^n$.

Definition 5.7. A based homotopy between two based G-maps $X \to Y$ is given by a based G-map $X \wedge I_+ \to Y$, where G acts trivially on the unit interval I. We denote the set of homotopy classes of G-maps between two G-spaces X and Y by $[X,Y]$. For a subgroup $H \subset G$, $[X,Y]^H$ denotes the set of H-equivariant homotopy classes of H-maps. In particular $[X,Y]^e$ denotes the set of non-equivariant homotopy classes. The set of based homotopy classes of based G-maps is denoted by $[X,Y]_*$. Two G-spaces X and Y are equivariantly weakly equivalent if there are G-maps $f : X \to Y$ and $g : Y \to X$ such that fg and gf are G-homotopic to the identity on X and Y, respectively.

Remark 5.8. *Let X and Y be equivariantly weakly equivalent via maps f and g. Then the restrictions of f and g to the sets of fixed points X^H and Y^H for an arbitrary subgroup $H \subset G$ induce maps $f^H : X^H \to Y^H$ and $g^H : Y^H \to X^H$. These maps give an ordinary weak equivalence between X^H and Y^H. Conversely, given two maps f and g, which induce ordinary weak equivalences between X^H and Y^H, they give an equivariant weak equivalence. This statement might not be obvious at this point, but will be proven in the next paragraph.This leads to an equivalent definition of equivariant weak equivalence.*

Definition 5.9. Two G-spaces X and Y are equivariantly weakly equivalent if there is a map $X \to Y$ such that the restrictions $X^H \to Y^H$ are ordinary weak equivalences for every subgroup $H \subset G$.

Definition 5.10. Denote the homotopy category of G-spaces by $hG\mathscr{U}$ and let $\bar{h}G\mathscr{U}$ denote the category constructed from $hG\mathscr{U}$ by formally inverting the weak equivalences.

Remark 5.11. *In the literature the term* homotopy category *is sometimes used differently. Given a model category $\mathscr{C}$, the homotopy category is then what we here call $\bar{h}\mathscr{C}$. In the present paper the homotopy category $h\mathscr{C}$ has the same objects as $\mathscr{C}$, and homotopy classes of maps as morphisms. This is also the notation used by May et. al. [M+96].*

6 The model structure

There are two ways to look at fibrations and cofibrations. Non-equivariantly they are either defined explicitly as Serre (co-)fibrations or, more abstractly, as part of a model structure on the category of topological spaces. If we want to define equivariant fibrations and cofibrations, both approaches are possible. In chapter I.2 of [M+96] it reads

> In either the based or unbased context, fibrations and cofibrations are defined exactly as in the non-equivariant setting by the covering homotopy property and the homotopy extension property, except that all maps in sight are equivariant. The theory goes through unchanged.

It is not hard to believe, that most proofs apply unchanged. While this might be sufficient for many purposes, it would certainly be nice to have a more conceptual definition.
Thus, we will introduce a model structure on the category of G spaces using a beautiful approach by Robert Piacenza. In section VI.5 of [M+96] he developed a closed model structure on the category of diagrams of spaces indexed on a small indexing category J. The key observation is then that the category of G-CW-complexes and the category of diagrams of CW-complexes, indexed on the orbit category of G, are equivalent. Moreover this equivalence will respect weak equivalences and finally induces an equivalence between the homotopy categories.

Definition 6.1. Let G be a topological group. Denote by $\mathscr{G}$ the orbit category of G, i.e. the objects are orbits G/H for all subgroups $H \subset G$. For another subgroup $K \subset G$ the space of morphisms is the space of G-maps $G/H \to G/K$. Define a $\mathscr{G}$-space to be a functor $\mathscr{G}^{op} \to \mathscr{U}$. A map of $\mathscr{G}$-spaces is a natural transformation, and we write $\mathscr{G}\mathscr{U}$ for the category of $\mathscr{G}$-spaces.

Definition 6.2. Let T be a $\mathscr{G}$-space and K a G-space. Then we are able to form the product $T \times K$ by setting $(T \times K)(G/H) := T(G/H) \times K$. In particular, $T \times I$ is defined, where I is the unit interval. Thus, we have a notion of homotopy between maps of $\mathscr{G}$-spaces. Write $[T, T']_{\mathscr{G}}$ for the set of homotopy classes of maps $T \to T'$.

There are two functors $\Phi : G\mathscr{U} \leftrightarrows \mathscr{G}\mathscr{U} : \Psi$ relating the categories $G\mathscr{U}$ and $\mathscr{G}\mathscr{U}$:

Definition 6.3. Let $X \in G\mathscr{U}$. Then $\Phi(X)(G/H) := X^H$.

The functor Ψ is *not* given by the naive assignment which maps a $\mathscr{G}$-space T to the G-space $T(G/e)$ [4]. Instead Ψ is given by Elmendorf's Theorem.

Theorem 6.4. *There are a functor $\Psi : \mathscr{G}\mathscr{U} \to G\mathscr{U}$ and a natural transformation $\epsilon : \Phi\Psi \to id$ such that each $\epsilon : \Phi\Psi T(G/H) \cong (\Psi T)^H \to T(G/H)$ is a homotopy equivalence. If X has the homotopy type of a G-CW complex, then there is a natural bijection*

$$[X, \Psi T]^G \cong [\Phi X, T]^{\mathscr{G}} \tag{46}$$

Proof. [M+96] V.3.2. □

We can now define a (closed) model structure on the category of diagrams.

Definition 6.5. Let J be a small topological category over $\mathscr{U}$ with discrete object space. Define $\mathscr{U}^J$ to be the category of continuous functors $J^{op} \to \mathscr{U}$. Its objects are called either diagrams or J-spaces, and its morphisms which are natural transformations, are called J-maps.

Remark 6.6. *The category $\mathscr{G}\mathscr{U}$ is just the category $\mathscr{U}^{\mathscr{G}}$. The category $\mathscr{U}^J$ for the trivial indexing category J is just $\mathscr{U}$.*

Definition 6.7. Let $j \in J$. Then $\underline{j}$ denotes the element of $\mathscr{U}^J$ which maps $k \in J$ to the mapping space $J(k, j)$.

Definition 6.8. A J-complex is an object $X \in \mathscr{U}^J$ with a decomposition $X = colim_{p \geq 0} X^p$, where

$$X^0 = \coprod_{\alpha \in A_0} D^{n_\alpha} \times \underline{j}_\alpha \tag{47}$$

and, inductively,

$$X^p = X^{p-1} \bigcup_f \Big(\coprod_{\alpha \in A_p} D^{n_\alpha} \times \underline{j}_\alpha \Big) \tag{48}$$

for some attaching J-map $f : \coprod_{\alpha \in A_p} S^{n_\alpha - 1} \times \underline{j}_\alpha \to X^{p-1}$ and indexing sets A_p. We call X a J-CW complex, if X is a J-complex such that $n_\alpha = p$ for all $p \geq 0$ and $\alpha \in A_p$.

[4] In fact this is a functor $\mathscr{G}\mathscr{U} \to G\mathscr{U}$. It is simply not the one we are looking for.

As in Definition 6.2 we can form the product of a diagram T and a space X by setting $(T \times X)(j) := T(j) \times X$ for all $j \in J$. For $X = I$ this gives the notion of a homotopy of J-maps and hence of a J-weak equivalence.

Definition 6.9. A J-map $f : X \to Y$ is a weak equivalence, if there is a J-map $g : Y \to X$ such that fg and gf are J-homotopic to the identity. This is the same as a levelwise homotopy $f(j) : Y(j) \to X(j)$ for all $j \in J$.

Definition 6.10. A J-map $f : X \to Y$ is a J-fibration, if $f(j) : Y(j) \to X(j)$ is a Serre fibration for all $j \in J$. An acyclic J-fibration is a J-fibration, which is also a J-weak equivalence. A J-map $g : A \to B$ is a J-cofibration, if it has the left lifting property with respect to all acyclic J-fibrations. Explicitly, if the outer square of the following diagram commutes, where f is an acyclic J-fibration, then there exists a diagonal map, completing the diagram.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd} \tag{49}$$

Theorem 6.11. *With the structure just defined, $\mathscr{U}^J$ is a model category.*

Proof. [M+96] VI.5.2. ☐

Corollary 6.12. *A J-map $g : A \to B$ is a J-cofibration, if and only if it is a retract of the inclusion $i : A' \to B'$ of a relative J-complex. Explicitly, g is a retract of i, if there exist maps completing the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"] \arrow[rr, bend left, "id"] & A' \arrow[r] \arrow[d, "i"] & A \arrow[d, "g"] \\ B \arrow[r] \arrow[rr, bend right, "id"'] & B' \arrow[r] & B \end{tikzcd} \tag{50}$$

Proof. [M+96] VI.5.3. ☐

Corollary 6.13. *A J-map $f : X \to Y$ is an acyclic J-fibration, if and only if it has the right lifting property with respect to each J-cofibration $S^n \times \underline{j} \to D^{n+1} \times \underline{j}$.*

Proof. [M+96] VI.5.4. ☐

Finally we can interpret the model structure on $\mathscr{U}^{\mathscr{G}}$ as a model structure on $G\mathscr{U}$ thanks to

Theorem 6.14. *Each $\mathscr{G}$-(CW-)complex is isomorphic to $\Phi(X)$ for some G-(CW-)complex X. Therefore Φ is an equivalence between the category of $\mathscr{G}$-(CW-)complexes and the category of G-(CW-)complexes.*

Proof. [M+96] VI.6.2. ☐

The Theorems 6.4 and 6.14 now lead to

Proposition 6.15. *The functors Φ and Ψ induce an equivalence of categories between $\bar{h}\mathscr{G}\mathscr{U}$ and $\bar{h}G\mathscr{U}$.*

Definition 6.16. For a G-map $f : X \to Y$ of G-CW-complexes, we call the sequence

$$X \xrightarrow{f} Y \xrightarrow{j} Y \cup_f CX \tag{51}$$

a special cofibre sequence. A (general) cofiber sequence is any sequence

$$E \to F \to G \tag{52}$$

for which there is a homotopy commutative diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\simeq"] & F \arrow[r] \arrow[d, "\simeq"] & G \arrow[d, "\simeq"] \\ X \arrow[r, "f"] & Y \arrow[r, "j"] & Y \cup_f CX \end{tikzcd} \tag{53}$$

for some special cofibre sequence

$$X \xrightarrow{f} Y \xrightarrow{j} Y \cup_f CX. \tag{54}$$

Remark 6.17. *A priori the notion of a cofiber sequence is independent of the notion of a cofibration. But every cofibration is the retract of a G-CW-inclusion $A \xrightarrow{i} X$, with the quotient $X/A \simeq X \cup_i A$. Hence, every cofibration fits into a cofiber sequence. Conversely, any map $f : X \to Y$ can be turned into a cofibration $X \to M_f \simeq Y$, where M_f denotes the mapping cylinder. Then $M_f/X \simeq Y \cup_f CX$.*

Lemma 6.18. *Given a homotopy commutative diagram of the form*

$$\begin{tikzcd} A \arrow[r, hook, "f"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\ C \arrow[r, "g"] & D \end{tikzcd} \tag{55}$$

of G-CW complexes, there exists a G-map $\beta' \simeq \beta$, such that

$$\beta' f = g\alpha. \tag{56}$$

Proof. Consider the diagram

$$\begin{tikzcd} B \wedge \{0\}^+ \cup A \wedge I^+ \arrow[r, hook] \arrow[d, "H"] & B \wedge I^+ \arrow[dl, dotted, "\tilde{H}"] \\ D & \end{tikzcd} \tag{57}$$

where $H|_{A\wedge I^+}$ is a homotopy from βf to $g\alpha$ and $H|_{B\wedge\{0\}^+} = \beta$. The map $\tilde{H}$ is defined by the homotopy extension property of the CW-inclusion

$$B \wedge \{0\}^+ \cup A \wedge I^+ \hookrightarrow B \wedge I^+. \tag{58}$$

The map β is then homotopic via $\tilde{H}$ to a map which we denote by β'. By definition

$$\beta' f = \tilde{H}|_{A\wedge\{1\}^+} = H|_{A\wedge\{1\}^+} = g\alpha. \tag{59}$$

□

Lemma 6.19. *Given a homotopy commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, "h"] \arrow[d, dotted, "\gamma"] & X \wedge S^1 \arrow[d, "\alpha\wedge 1"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X' \wedge S^1 \end{tikzcd} \tag{60}$$

in which the rows are cofiber sequences, there exists a map $\gamma : Z \to Z'$ *making the resulting diagram homotopy commutative.*

Proof. (cf. [Swi02], Lemma 8.31.) We may assume that both rows are special cofiber sequences and that f is an inclusion. Thus, the diagram is of the form

$$\begin{tikzcd} X \arrow[r, hook, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Y \cup_f CX \arrow[r, "h"] \arrow[d, dotted, "\gamma"] & X \wedge S^1 \arrow[d, "\alpha\wedge 1"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Y' \cup_{f'} CX' \arrow[r, "h'"] & X' \wedge S^1 \end{tikzcd} \tag{61}$$

By the previous lemma, we may replace β by a homotopic map β', such that the diagram commutes strictly. We define γ by $\gamma|_B = \beta'$ and $\gamma|_{CA} = C\alpha$. Then the resulting diagram commutes strictly for β', and thus up to homotopy for β. □

7 Representations

Definition 7.1. Let G be a Lie-group. A representation V of G is a finite dimensional real inner product space, i.e. a finite dimensional real vector space, with a given smooth action of G through linear isometries. We may think of a Lie- group homomorphism $\rho : G \to O(V)$. If V is a representation of G, S^V denotes the one-point compactification of V. For a G-space X, we define the suspension $\Sigma^V X := S^V \wedge X$ and the loop space $\Omega^V X := Maps_G(S^V, X)$ with respect to V. If $V \subset W$, then $W - V$ denotes the orthogonal complement of V in W.

Definition 7.2. A G-universe U is a countable direct sum of representations, such that U contains a trivial representation and contains each of its subrepresentations infinitely often. Thus U can be written as a direct sum of subspaces $(V_i)^\infty$, where $\{V_i\}$ runs over a set of distinct irreducible representations of G. We call a universe complete if it, up to isomorphism, contains every irreducible representation of G. If G is finite, one example is V^∞, where V is the regular representation of G. A finite dimensional sub G-space of a universe U is said to be an indexing space in U.

Definition 7.3. Let U be a complete universe. For a finite G-CW complex X and any based G-space Y, define

$$\{X, Y\}^G := colim_V [\Sigma^V X, \Sigma^V Y]^G_*. \tag{62}$$

Here V runs through all indexing spaces in U. The colimit is taken over the functions

$$[\Sigma^V X, \Sigma^V Y]^G_* \to [\Sigma^W X, \Sigma^W Y]^G_*, \quad V \subset W, \tag{63}$$

obtained by sending a map $\Sigma^V X \to \Sigma^V Y$ to its smash product with the identity on S^{W-V}.

This definition only works properly for finite G-CW complexes.

Definition 7.4. Let U be a complete G-universe. For a based G-space X define

$$QX := colim_V \Omega^V \Sigma^V X, \tag{64}$$

where V runs over the indexing spaces in U and the colimit is taken over the maps

$$\Omega^V \Sigma^V X \to \Omega^W \Sigma^W X, \quad V \subset W, \tag{65}$$

that are obtained by sending a map $S^V \to X \wedge S^V$ to its smash product with the identity of S^{W-V}.

Lemma 7.5. *If X is a finite G-CW complex, then*

$$\{X, Y\}^G \cong [X, QY]^G_* \tag{66}$$

Proof. This is immediate from the compactness of X, which ensures that

$$[X, QY]^G \cong colim_V [X, \Omega^V \Sigma^V Y]^G. \tag{67}$$

□

We take $\{X, Y\}^G := [X, QY]^G_*$ as the definition of stable maps for infinite complexes X.

Definition 7.6. Let U be a universe. Denote by $\mathscr{R}O(G;U)$ the category whose objects are the representations embeddable in U and whose morphisms $V \to W$ are G-linear isometries[5]. Say that two such maps are homotopic if their associated based G-maps $S^V \to S^W$ are stably homotopic and let $h\mathscr{R}O(G;U)$ be the resulting homotopy category. The representation group $RO(G;U)$ relative to the given universe U is obtained by applying the Grothendieck construction to the category $\mathscr{R}O(G;U)$. Explicitly, the elements are equivalence classes of formal differences $V \ominus W$, where $V \ominus W$ is equivalent to $V' \ominus W'$, if there is a G-linear isometry $\alpha : V \oplus W' \to V' \oplus W$. $RO(G;U)$ is a ring if tensor products of representations, which are embeddable in U, are embeddable in U themself.

Definition 7.7. let U be a universe. A G-prespectrum E indexed on U is a family of based spaces EV, one for each indexing space V in U, together with structure maps $\sigma_{V,W} : \Sigma^{W-V} EV \to EW$ for all $V \subset W$. We require $\sigma_{V,V} = id$, and we require the following diagram to commute for $V \subset W \subset Z$:

$$\begin{tikzcd}
\Sigma^{Z-W}\Sigma^{W-V}EV \arrow[r] \arrow[d, "\cong"] & \Sigma^{Z-W}EW \arrow[d] \\
\Sigma^{Z-V}EV \arrow[r] & EZ.
\end{tikzcd} \tag{68}$$

We call E a G-spectrum indexed on U if E is a G-prespectrum such that the adjoints $\tilde{\sigma} : EV \to \Omega^{W-V} EW$ are weak equivalences.
A map $f : E \to F$ of G-spectra is a weak equivalence if its component maps $f : EV \to FV$ are weak equivalences.

[5]In our notation isometries are already required to be isomorphisms. Other authors explicitly call them isometric isomorphisms.

For many purposes it is sufficient to index a G-spectrum not on all indexing spaces in a given universe U, but only on a subset of representations in U, such that any representation in U is contained in one of these distinguished representations. We will illustrate this by an example.
Let $G = \mathbb{Z}/2$. Any representation of $\mathbb{Z}/2$ may uniquely be written as $\mathbb{R}^p \oplus (\mathbb{R}_-)^q$ for some p and q, where $\mathbb{R}$ and $\mathbb{R}_-$ denote the one-dimensional real vector spaces with trivial and sign involution, respectively. If V is as above, we will use the notation

$$S^V = S^{p+q,q}. \tag{69}$$

The sphere associated with the regular representation of $\mathbb{Z}/2$ is $S^{2,1}$. We will sometimes write $(p+q,q)$ for the representation $V = \mathbb{R}^p \oplus (\mathbb{R}_-)^q$ itself.

Definition 7.8. A (naive) $\mathbb{Z}/2$-prespectrum is a sequence of $\mathbb{Z}/2$-spaces $E_0, E_1, E_2, ...$ together with maps $\Sigma^{2,1}E_n \to E_{n+1}$. An $\mathbb{Z}/2$-spectrum is one for which the adjoints $E_n \to \Omega^{2,1}E_{n+1}$ are weak equivalences.

Remark 7.9. *This is how Dugger [Dug05] defines* naive *$\mathbb{Z}/2$-spectra. In the notation of May [M+ 96], a naive spectrum is a G-spectrum indexed on the trivial universe U, containing only the trivial representation.*

So we have indexed the $\mathbb{Z}/2$-spectrum on the representations $V = \mathbb{C}^n$ (regarded as a real vector space with the conjugation action). Let E be a $\mathbb{Z}/2$-spectrum in the original sense (sometimes called genuine spectrum). Then, we obtain a naive $\mathbb{Z}/2$-spectrum by setting $E_n = E\mathbb{C}^n$. Conversely, let F be a naive $\mathbb{Z}/2$-spectrum. Any $\mathbb{Z}/2$ representation V is contained in a $\mathbb{C}^n$ for a big enough value of n. Thus, we gain, up to weak equivalence, a genuine $\mathbb{Z}/2$-spectrum if we set $EV = \Omega^{\mathbb{C}^n - V}E_n$ for a suitable n. This is well defined, since the adjoints $\tilde{\sigma}$ were required to be weak equivalences. Hence we have a one-to-one correspondence between equivalence classes of naive and genuine $\mathbb{Z}/2$-spectra.

Example 7.10. *Recall from Section 3.2, that $\mathbb{Z} \times BU$ is a classiyfing space for Real K-theory. The reduced canonical line bundle over $\mathbb{C}P^1$ is classified by an equivariant map $S^{2,1} = \mathbb{C}P^1 \to Z \times BU$, and so one gets a map $S^{2,1} \wedge (\mathbb{Z} \times BU) \to \mathbb{Z} \times BU$ by using the multiplication in $\mathbb{Z} \times BU$. So we have a $\mathbb{Z}/2$-prespectrum, called the KR-(pre-)spectrum, in which every term is $\mathbb{Z} \times BU$. Equivariant Bott-periodicity [Ati66] shows that $\mathbb{Z} \times BU \to \Omega^{2,1}(\mathbb{Z} \times BU)$ is a weak equivalence. Hence, KR is a $\mathbb{Z}/2$-spectrum.*

Let V be a fixed given representation of G and write $dimV = |V|$. For $H \subseteq G$, $V(H)$ denotes the orthogonal complement of V^H in V. If W is a representation of H, we let $D(W)$ and $S(W)$ denote the unit disc and sphere in W.
Using these notations we can generalize the notion of connectivity. The key observation is that we can make sense of $[S^{V+k} \wedge G/H_+, X]_*$ for $k \geq -|V^H|$. Consider the chain of isomorphisms

$$[S^{V+k} \wedge G/H_+, X]_* \cong [S^{V_H+k}, X]_*^H \cong [S^{V(H)+|V^H|+k}, X]_*^H \tag{70}$$

Since the right-hand side makes sense for $k \geq -|V^H|$ we can define

Definition 7.11. A pointed G-space X is called V-connected if

$$[S^{V+k} \wedge G/H_+, X]_* = 0 \tag{71}$$

for all subgroups H and all $0 \geq k \geq -|V^H|$.

Waner showed that this is equivalent to requiring that X^H is $|V^H|$-connected for all subgroups $H \subset G$. This result eventually appeared, in expanded form, in [Lew92b] Lemma 1.2.

Lemma 7.12. *Suppose $V \supseteq 1$, and let X and Y be pointed G-spaces, which are both $(V-1)$-connected. Then a map $X \to Y$ is a weak equivalence, iff it induces an isomorphism $[S^{V+k} \wedge G/H_+, X]_* \cong [S^{V+k} \wedge G/H_+, Y]_*$ for every $k \geq 0$ and every subgroup H .*

Proof. [Lew92a] Lemma 3.7 □

Definition 7.13. A G-CW(V) complex X is a G-space with a decomposition $X = colim_{n\in\mathbb{N}_0} X^n$ such that X^0 is a disjoint union of orbits G/H, where H acts trivially on V and X^{n+1} is obtained from X^n by attaching cells $G \times_H D(V(H) \oplus \mathbb{R}^t)$, where $|V(H)| + t = n$, along G-maps $G \times_H S(V(H) + \mathbb{R}^t) \to X^{n-1}$.

Remark 7.14. *Note that for $V = 0$ we obtain the classical case of a G-CW complex.*

Proposition 7.15. *For any G-space X, there is a G-CW(V) complex ΓX and a weak equivalence $\gamma : \Gamma X \to X$.*

Proof. [M+96] X.3.5. □

Proposition 7.16. *Every G-map $f : X \to Y$ of G-CW(V) complexes is G-homotopic to a cellular map.*

Proof. [M+96] X.3.4. □

8 Mackey functors

Mackey functors will play the role of coefficient systems for equivariant cohomology theories. There is one short definition, which will be sufficient for basic constructions. Moreover, there is a more subtle definition which is equivalent for finite groups. This definition will become useful, when we construct multiplicative structures on equivariant cohomology theories.

Definition 8.1. Define the Burnside category $\mathscr{B}_G$ to have orbit spaces G/H as objects and to have morphisms

$$\mathscr{B}_G(G/H, G/K) = \{G/H_+, G/K_+\}_G = colim_V [\Sigma^V G/H_+, \Sigma^V G/K_+]^G_* \tag{72}$$

Definition 8.2. A Mackey functor is an additive functor $\mathscr{B}_G^{op} \to \mathscr{A}b$. A co-Mackey functor is an additiv functor $\mathscr{B}_G \to \mathscr{A}b$.

Definition 8.3. The Burnside Mackey functor $\underline{A}$ is defined on objects by $\underline{A}(G/H) = \{G/H_+, S^0\}^G \cong \{S^0, S^0\}^H$. Its contravariant functoriality is clear from the description. It is a fundamental insight by Segal ([Seg70], p.60), that $\underline{A}(G/H)$ is isomorphic to the Burnside ring $A(H)$. Here $A(H)$ is defined to be the Grothendieck ring of isomorphism classes of finite G-sets with addition and multiplication given by disjoint union and Cartesian product, respectively.

Definition 8.4. Let X and Y be based G-spaces. Denote by $\underline{\{X,Y\}^G}$ the Mackey functor defined by $\underline{\{X,Y\}^G}(G/H) = \{G/H_+ \wedge X, Y\}^G$. The functoriality is clear. If $X = S^V$, denote this Mackey functor by $\underline{\pi}_V(Y)$. In particular $\underline{\pi}_0(S^0)$ is the Burnside Mackey functor.

For finite G there is an equivalent description of Mackey functors.The key observation is the following:

Proposition 8.5. *Let G be a finite group and H_1, H_2 subgroups. Then the free abelian group generated by equivalence classes of diagrams*

$$G/H_1 \xleftarrow{\phi} G/K \xrightarrow{\chi} G/H_2 \tag{73}$$

for $K \subseteq H_1, H_2$ is isomorphic to the group

$$\{(G/H_1)_+, (G/H_2)_+\}_G. \tag{74}$$

Here, two diagrams are considered to be equivalent, if there is a G-homoemorphism $\xi : G/K \to G/K'$ such that the following combined diagram is commutative:

$$\begin{tikzcd}
& G/K \arrow[dl, "\phi"'] \arrow[dr, "\chi"] \arrow[dd, "\xi"] & \\
G/H_1 & & G/H_2 \\
& G/K' \arrow[ul, "\chi'"] \arrow[ur, "\phi'"'] &
\end{tikzcd} \tag{75}$$

May ([M$^+$96] XIX.3.) proves this in a way more general setting. We will concentrate on the special case considered here and outline the main ideas of the approach.

Any diagram of the form $G/H_1 \xleftarrow{\phi} G/K \xrightarrow{\chi} G/H_2$ can be used to determine a stable map in $\{(G/H_1)_+, (G/H_2)_+\}_G$. Obviously χ induces a stable map in $\{(G/K)_+, (G/H_2)_+\}_G$. The hard part is to construct the so called transfer map in $\{(G/H_1)_+, (G/K)_+\}_G$ induced by ϕ.

This is done in two steps. First we construct an element in $\{S^0, (G/K)_+\}_G$ for any group $K \subseteq G$. We start with a G-representation V, in which we may embed G/H. Afterwards we choose a tubular neighbourhood of G/H in V. If we now identify everything outside this neighbourhood to one point, we obtain a projection

$$S^V \to G_+ \wedge_H S^V \cong (G/H)_+ \wedge S^V. \tag{76}$$

This is a special case of the Pontrjagin-Thom map. Since we made a choice when we picked V, it is natural to consider this map as a stable map. We denote the transfer map by $\tau(G/H)$.

More generally for subgroups $H \subset K$of G there is a stable transfer G-map $\tau(\pi) \in \{(G/K)_+, (G/H)_+\}_G$, associated with the projection $\pi : G/H \to G/K$. It is the composition

$$(G/K)_+ \wedge S^V \cong G_+ \wedge_K S^V \to G_+ \wedge_K ((K/H)_+ \wedge S^V) \cong (G/H)_+ \wedge S^V \tag{77}$$

where the arrow denotes the extension of $\tau(K/H)$ to a G-map. Note that we considered V as a K-representation by restriction.
But everey map $f : G/H_+ \to G/K_+$ is the composition of a conjugation isomorphism $c_g : G/H \to G/g^{-1}Hg$ and the projection induced by the inclusion $g^{-1}Hg \subseteq K$. So the map associated with f is just $\tau(f) = c_g^{-1} \circ \tau(\pi)$.

Putting all steps together, we have constructed a map, associating a stable G-map with every diagram of the form $G/H_1 \xleftarrow{\phi} G/K \xrightarrow{\chi} G/H_2$. This map turns out to induce the desired isomorphism. The reader is referred to [M+96] for a full proof.

The main advantage of this new view on stable maps is that we can express the composition of stable maps in a purely algebraic way. Given $G/H_1 \xleftarrow{\phi'} G/K' \xrightarrow{\chi'} G/H_2$ and $G/H_2 \xleftarrow{\phi''} G/K'' \xrightarrow{\chi''} G/H_3$, the composition is

$$\begin{tikzcd} & & G/K \arrow[dl] \arrow[dr] & & \\ & G/K' \arrow[dl, "\phi'"'] \arrow[dr, "\chi'"] & & G/K'' \arrow[dl, "\phi''"'] \arrow[dr, "\chi''"] & \\ G/H_1 & & G/H_2 & & G/H_3 \end{tikzcd} \tag{78}$$

where the upper square is a pullback.
This finally leads to a new perspective on Mackey functors.

Definition 8.6. A Mackey functor M consists of two additive functors $M^* : \mathscr{B}_G^{op} \to \mathscr{A}b$ and $M_* : \mathscr{B}_G \to \mathscr{A}b$ which have the same object function and the property that $M^*(\alpha) \circ M_*(\beta) = M_*(\delta) \circ M^*(\gamma)$ for pullback squares of orbits

$$\begin{tikzcd} G/H_1 \arrow[r, "\delta"] \arrow[d, "\gamma"] & G/H_2 \arrow[d, "\alpha"] \\ G/H_3 \arrow[r, "\beta"] & G/H_4 \end{tikzcd} . \tag{79}$$

This definition is indeed equivalent to the notion of Mackey functors in the old sense: Let M denote such a Mackey functor. We easily obtain a Mackey functor in the new sense,which we will now denote by $[M_*, M^*]$.
Set $M_* = M^* = M$ on objects and let $M_*(f) = M(\phi, 1)$ and $M^*(f) = M(1, \chi)$ for a morphism $f : G/H_1 \xleftarrow{\phi} G/K \xrightarrow{\chi} G/H_2$, which we abbreviate with $f = [\phi, \chi]$. Since we can write $[\phi, \chi] = [\phi, 1] \circ [1, \chi]$, this works conversely if we set $M(f) = M^*(f) \circ M_*(f)$.

We can now formulate the definition of a pairing of Mackey functors:

Definition 8.7. Let L, M, and N be Mackey functors. A pairing of Mackey functors $\mu : L \times M \to N$ consists of functorially (co- and contravariantly) maps $L(X) \times M(X) \to N(X)$, one for each finite G-set X, such that the following diagram commutes for a G-map $f : X \to Y$:

$$\begin{tikzcd} L(X) \otimes M(Y) \arrow[rr, "L_*(f)\otimes id"] \arrow[d, "id\otimes M^*(f)"] & & L(Y) \otimes M(Y) \arrow[d, "\mu"] \\ L(X) \otimes M(X) \arrow[r, "\mu"] & N(X) \arrow[r, "N_*(f)"] & N(Y) \end{tikzcd} \tag{80}$$

Definition 8.8. A Mackey functor is a Green functor, if it is equipped with a pairing $\mu : M \times M \to M$, that makes each $M(X)$ a commutative and associative unital ring.

Definition 8.9. Let $\tau(G/H) : S^V \to G/H_+ \wedge S^V$ be the transfer map and $\xi : S^V \wedge G/H_+ \to S^V$ the projection. The Euler characteristic of G/H is the map $\chi(G/H) \in \{S^0, S^0\}^G$, obtained by stabilizing the composition $\xi \circ \tau(G/H)$.

Proposition 8.10. *Let G be any compact Lie group. There is a unique Mackey functor $\underline{\mathbb{Z}} : \mathscr{B}_G^{op} \to \mathscr{A}b$ such that*

- *the underlying coefficient system is constant at $\mathbb{Z}$,*
- *if $G/K_+ \to G/H_+$ is the stable transfer map associated to an inclusion $H \subset K$, then the induced homomorphism $\mathbb{Z} \to \mathbb{Z}$, is multiplication by the Euler characteristic $\chi(K/H)$.*

Proof. [M$^+$96] IX.4.3. □

9 Equivariant Postnikov-section functors

Definition 9.1. Let $\mathcal{A}$ be a set of well-pointed spaces (i.e $* \to A \in \mathcal{A}$ are cofibrations), all of which are compact Hausdorff. A space Z is called $\mathcal{A}$-null, if it has the property that the maps $[*, Z] \to [\Sigma^n A, Z]$ are isomorphisms for $n \in \mathbb{N}_0$ and all $A \in \mathcal{A}$.

Proposition 9.2. *For a given space X there exists a new space $P_{\mathcal{A}}(X)$ with the properties*

- *There is a natural map $X \to P_{\mathcal{A}}(X)$;*
- *$P_{\mathcal{A}}(X)$ is $\mathcal{A}$-null;*
- *If Z is an $\mathcal{A}$-null space, then for any map $X \to Z$ there is a lifting*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z \\ P_{\mathcal{A}}(X) \arrow[ur, dotted] & \end{tikzcd} \tag{81}$$

Proof.
We construct the space $P_{\mathcal{A}}(X)$ as follows:
For any space Y, let $F_{\mathcal{A}}(Y)$ be defined by the pushout square

$$\begin{tikzcd} \coprod_\sigma \Sigma^n A \arrow[r] \arrow[d, hook] & Y \arrow[d, hook, dotted] \\ \coprod_\sigma C(\Sigma^n A) \arrow[r, dotted] & F_{\mathcal{A}}(Y) \end{tikzcd} \tag{82}$$

where $C(\Sigma^n A)$ is the cone over $\Sigma^n A$, and σ runs over all maps $\Sigma^n A \to Y$. Now we consider the sequence of closed inclusions

$$X \hookrightarrow F_{\mathcal{A}}X \hookrightarrow F_{\mathcal{A}}F_{\mathcal{A}}X \hookrightarrow F_{\mathcal{A}}F_{\mathcal{A}}F_{\mathcal{A}}X \hookrightarrow ... \tag{83}$$

and define $P_{\mathcal{A}}(X)$ as the colimit. □

Remark 9.3. *For a non-equivariant space X we can take $\mathcal{A}_n = \{S^{n+1}, S^{n+2}, ...\}$. Then $P_{\mathcal{A}}$ is the classical Postnikov-section functor.*

Definition 9.4. Let X be an equivariant space and V a representation of G. We define the functors $\mathbb{P}_V := P_{\tilde{\mathcal{A}}_V}$ and $P_V := P_{\mathcal{A}_V}$ with

$$\tilde{\mathcal{A}}_V = \{S^W \wedge G/H_+ : W \supseteq V+1,\ H \leq G\} \tag{84}$$

$$\mathcal{A}_V = \{S^W \wedge G/H_+ : W \supset V,\ H \leq G\} \tag{85}$$

Let us gather some of the most important properties of these two functors:

Proposition 9.5. *Let X be a pointed G-space and V a G-representation. Then $\mathbb{P}$ satisfies the following properties:*

(1) The map $X \to \mathbb{P}_V X$ induces an isomorphism of the sets $[S^k \wedge G/H_+, -]_*$ for $0 \leq k \leq dimV^H$, and an epimorphism for $k = dimV^H + 1$.

(2) If W is a G-representation, for which $dimW^H \leq dimV^H$ for all subgroups $H \subseteq G$, then $[S^W, X]_* \to [S^W, \mathbb{P}_V X]_*$ is an isomorphism.

(3) The homotopy fiber of $\mathbb{P}_{V+1}X \to \mathbb{P}_V X$ is an Eilenberg- MacLane space of type $K(\underline{\pi}_{V+1}(X), V+1)$ [6]. This is an anticipation of Definition 11.6, where we define equivariant Eilenberg-MacLane spaces. For completeness this property is stated here.

(4) The homotopy limit of the sequence

$$... \to \mathbb{P}_{V+2}X \to \mathbb{P}_{V+1}X \to \mathbb{P}_V X \tag{86}$$

is weakly equivalent to X.

(5) If V contains the regular representation of G, then $\mathbb{P}_V(S^V)$ is an Eilenberg-MacLane space of type $K(\underline{A}, V)$, where $\underline{A}$ is the Burnside-ring Mackey functor.

Proof. [Dug05] Proposition 3.6. Dugger's proof uses standard techniques, which he sketches briefly in his paper. □

The properties of P are weaker than those of $\mathbb{P}$.

Proposition 9.6. *Let X be a pointed G-space and V a G-representation. Then P satisfies the following properties:*

(1) The map $X \to P_V X$ induces an isomorphism of the sets $[S^k \wedge G/H_+, -]_*$ for $0 \leq k < dimV^H$, and an epimorphism for $k = dimV^H$.

[6] Note that there is a typing error in the original paper [Dug05].

(2) *If W is a G-representation, for which $dim W^H < dim V^H$ for all subgroups $H \subseteq G$, then $[S^W, X]_* \to [S^W, P_V X]_*$ is an isomorphism.*

Proof. [Dug05] Proposition 3.7. □

Remark 9.7. *The two preceding propositions show that $\mathbb{P}_V$ is better behaved than P_V. This is caused by the following consideration.*
From the non-equivariant situation we are used to the fact, that there are now non-trivial maps $S^n \to S^m$ when $n < m$. In the equivariant situation this is not true and there can be non-trivial maps $S^V \to S^W$, even though $V \subset W$. However, if we require $V+1 \subseteq W$, everything is well behaved and all equivariant maps $S^V \to S^W$ are null-homotopic. This leads to the definition of $\mathbb{P}_V$, which is therefore denoted as the *equivariant Postnikov section functor (cf. [M+96] II.1). Despite that, we have reason to care about the functor P_V. We will return to this matter in section 16.*

Part III
RO(G)-graded cohomology theories

10 Axioms for $RO(G)$-graded cohomology theories

Definition 10.1. A $RO(G)$-graded cohomology theory is a functor

$$E_G^* : h\mathscr{R}O(G;U) \times (\bar{h}G\mathscr{T})^{op} \to \mathscr{A}b \tag{87}$$
$$(V, X) \mapsto E_G^V(X) \tag{88}$$

together with natural isomorphisms (contravariantly in X and covariantly in V)

$$\sigma^W : E_G^V(X) \to E_G^{V\oplus W}(\Sigma^W X), \tag{89}$$

such that the following axioms are satisfied:

(1) For each representation V, the functor E_G^V is exact on cofiber sequences and sends wedges to products.

(2) If $\alpha : W \to W'$ is a map in $h\mathscr{R}O(G)$, then the following diagram commutes:

$$\begin{tikzcd} E_G^V(X) \arrow[r, "\sigma^W"] \arrow[d, "\sigma^{W'}"] & E_G^{V\oplus W}(\Sigma^W X) \arrow[d, "E_G^{id\oplus\alpha}(id)"] \\ E_G^{V\oplus W'}(\Sigma^{W'} X) \arrow[r, "(\Sigma^\alpha id)^*"] & E_G^{V\oplus W'}(\Sigma^W X) \end{tikzcd} \tag{90}$$

(3) $\sigma^0 = id$ and the isomorphisms σ are transitive in the sense that the following diagram commutes for each pair of representations (W, Z):

$$\begin{tikzcd} E_G^V(X) \arrow[rr, "\sigma^W"] \arrow[dr, "\sigma^{W\oplus Z}"'] & & E_G^{V\oplus W}(\Sigma^W X) \arrow[dl, "\sigma^Z"] \\ & E_G^{V\oplus W\oplus Z}(\Sigma^{W\oplus Z} X) & \end{tikzcd} \tag{91}$$

We extend a theory so defined to formal differences $V \ominus W$ for any pair of representations by setting

$$E_G^{V\ominus W}(X) = E_G^V(\Sigma^W X). \tag{92}$$

Remark 10.2. *Of course there is an axiomatic description of $RO(G)$-graded homology theory. The only point that needs to be mentioned is that homology theories must be given by contravariant functors on $h\mathscr{R}O(G)$ in order to make sense of the homological counterpart of Axiom (2). Since we will usually encounter cohomology theories in this paper, we decided to stick to the corresponding axioms.*

Definition 10.3. Let $\mathscr{I}O(G;U)$ and $h\mathscr{I}O(G;U)$ be the full subcategories of $\mathscr{R}O(G;U)$ and $h\mathscr{R}O(G;U)$ whose objects are the indexing spaces in U. Let $\Psi : \mathscr{I}O(G;U) \to \mathscr{R}O(G;U)$ be the inclusion, and also write Ψ for the inclusion of the homotopy categories.
For each representation V which is embeddable in U, choose an indexing space ΦV in U and a G-linear isomorphism $\phi_V : V \to \Phi V$. If V is itself an indexing space in U, choose $\Phi V = V$ and $\phi_V = id$. Extend Φ to a functor by sending a map $\alpha : V \to W$ to the composite

$$\Phi V \xrightarrow{\phi_V^{-1}} V \xrightarrow{\alpha} W \xrightarrow{\phi_W} \Phi W. \tag{93}$$

Then the following lemma is immediate.

Lemma 10.4. *The isomorphisms ϕ_V define a natural transformation $Id \to \Psi \circ \Phi$, and the composition $\Phi \circ \Psi$ is equal to the identity. Thus, the functors Φ and Ψ give an equivalence between the categories $\mathscr{I}O(G;U)$ and $\mathscr{R}O(G;U)$, and they induce an equivalence of categories between $h\mathscr{I}O(G;U)$ and $h\mathscr{R}O(G;U)$.*

Proposition 10.5. *A G-spectrum indexed on a universe U represents a $RO(G;U)$-graded cohomology theory E_G^* on based G-spaces by*

$$E_G^V(X) = [X, E\Phi V]^G \textit{ and} \tag{94}$$

$$E_G^\alpha(X) = [X, E\Phi\alpha]^G \tag{95}$$

for each $\alpha : V \to W$.

Proof. [M+96] XIII.2.2. □

The converse is also true.

Proposition 10.6. *A $RO(G;U)$-graded cohomology theory E_G^* on based G-spaces is represented by a spectrum indexed on U.*

Proof. [M+96] XIII.3.2. □

11 Ordinary $RO(G)$-graded cohomology

Definition 11.1. Let X be a G-CW(V)-complex. Define a chain complex $\underline{C}_*^V(X)$ in the category of Mackey functors as follows:

$$\underline{C}_n^V(X) = \underline{\{S^{V-|V|+n}, X^n/X^{n-1}\}}_G \tag{96}$$

where for any G-spaces X and Y we have a Mackey functor $\underline{\{X,Y\}}_G(G/H) := \{G/H_+ \wedge X, Y\}_G$.
Let

$$d_n : \underline{C}_n^V(X) \to \underline{C}_{n-1}^V(X) \tag{97}$$

be the stable connecting homomorphism of the triple (X^n, X^{n-1}, X^{n-2}).

Remark 11.2. *Since X^n/X^{n-1} is the wedge over cells of the form $G/H_+ \wedge S^{V-|V|+n}$ we have*

$$\underline{C}_*^V(X)(G/H) = \{G/H_+ \wedge S^{V-|V|+n}, \bigvee_\sigma G/H_{\sigma+} \wedge S^{V-|V|+n}\}_G \tag{98}$$

$$= \sum_\sigma \{G/H_+, G/H_{\sigma+}\}_G. \tag{99}$$

Definition 11.3. Let X be a G-CW(V) complex. For a Mackey functor M, define the ordinary cohomology of X with coefficients in M to be

$$H_G^{V+n}(X;M) = H^{|V|+n}(Hom_{\mathscr{B}_G}(\underline{C}_*^V(X), M)). \tag{100}$$

For a coMackey functor N, define the ordinary homology of X with coefficients in N to be

$$H^G_{V+n}(X;N) = H_{|V|+n}(\underline{C}_*^V(X) \otimes_{\mathscr{B}_G} N). \tag{101}$$

The categorical tensor product of functors $- \otimes_{\mathscr{B}_G} -$ is defined to be

$$M \otimes_{\mathscr{B}_G} N = \sum M(G/H) \otimes N(G/H)/(\approx), \tag{102}$$

where the sum runs over all subgroups $H \subseteq G$ and $(f^*m, n) \approx (m, f_*(n))$ for a map $f : G/H \to G/K$ and elements $m \in M(G/K)$ and $n \in M(G/H)$.
Similar definitions apply to give relative (co-)homology groups. For the special case when A is a subcomplex of X, we have $\underline{C}_*^V(X,A) \cong \underline{C}_*^V(X)/\underline{C}_*^V(A)$ and we obtain the expected long exact sequence. For a fixed basepoint $* \in X$ and $(X,*)$ being a relativ G-CW(V) complex, we define the reduced (co-)homology of X by

$$\tilde{H}_G^{V+n}(X;M) = H_G^{V+n}(X,*;M) \quad \text{and} \quad \tilde{H}^G_{V+n}(X;N) = H^G_{V+n}(X,*;N). \tag{103}$$

We can extend the definition to arbitrary G-spaces, approximating them by weakly equivalent G-CW(V) complexes.
Last but not least we have to check that we have the desired suspension isomorphisms.

Proposition 11.4. *Let X be a G-space, V and W representations. Then there are isomorphisms*

$$\tilde{H}_G^{W-V+n}(X;M) \cong \tilde{H}_G^{W+n}(\Sigma^V X;M). \tag{104}$$

Proof. For a relative G-CW(W) complex $(X,*)$ and any representation V, $(\Sigma^V X,*)$ inherits a structure of a relative G-CW($V \oplus W$) complex, such that the W-cellular chain complex of $(X,*)$ is isomorphic to the $(V \oplus W)$-cellular chain complex of $(\Sigma^V X,*)$. □

Remark 11.5. *For $V = 0$ we obtain the usual Bredon (co-)homology. [Bre67] and [M+96] p. 16-17.*

Definition 11.6. The V-th space in the representing spectrum of the ordinary $RO(G)$-graded cohomology theory with coefficients in M is called an Eilenberg-MacLane space of type $K(M,V)$. The spectrum as a whole is called HM.

Proposition 11.7. *Let M be a Mackey functor. An Eilenberg-MacLane space of type $K(M,V)$ is a V-1-connected space such that*

$$[S^{V+k} \wedge G/H_+, K(M,V)]_*^G = \begin{cases} 0 & k \neq 0 \\ M(G/H) & k = 0. \end{cases} \tag{105}$$

Proof. [M+96] XIII.4 □

Part IV
The classical equivariant Atiyah-Hirzebruch spectral sequence

12 The spectral sequence

Definition 12.1. A $RO(G)$-graded spectral sequence is a family of spectral sequences, one for each $\alpha = V \ominus W$, such that V and W contain no trivial summands. The values in gradings $\alpha + n$ form a $\mathbb{Z}$-graded theory. The spectral sequences are related by suspension isomorphisms.

For this whole section, V denotes a G-representation containing no trivial summands.

The non-equivariant Atiyah-Hirzebruch spectral sequence for an extraordinary cohomology theory is well known. This sequence is an important tool to calculate the cohomology groups of a given space X. There are essentially two ways to construct this spectral sequence. In the first approach we filter the space X, usually by its CW-skeletons (Section 13). The alternative is then to use Postnikov functors to obtain a filtration of the spectrum representing the cohomology theory (Section 14). Maunder [Mau63] compares these two approaches and shows their equivalence.
This section will give the equivariant analogues.

Definition 12.2. Let h_G^* be a $RO(G)$-graded cohomology theory. Define a Mackey functor $\underline{h}_G^V(*)$ by $\underline{h}_G^V(*)(G/H) = h_G^V(G/H_+)$.

Theorem 12.3. *Let X be a G-CW-complex . Moreover let h_G^* be a $RO(G)$-graded cohomology theory. There is a $RO(G)$-graded spectral sequence*

$$E_2^{p,q}(V) = H_G^{V+p}(X, \underline{h}_G^q(*)) \Rightarrow h_G^{V+p+q}(X). \tag{106}$$

In this section we choose the direct construction via exact couples. Later (Section 15) we will see that the sequence is the homotopy spectral sequence of a certain Postnikov tower. This will lead to an easy desciption of the convergence (Section 15.1).

Definition 12.4. An exact couple is a pair of bigraded abelian groups (C, A) together with three maps f, g and h, such that the following diagram is commutative and exact.

$$\begin{tikzcd} A \arrow[rr, "f"] & & A \arrow[dl, "g"] \\ & C \arrow[ul, "h"] & \end{tikzcd} \tag{107}$$

Let (C, A) be an exact couple. Define $d = g \circ h : C \to C$. The composition $d^2 = g \circ h \circ g \circ h$ is trivial, since the couple is defined to be exact. We can hence form the quotient

$$C' = Ker(d)/Im(d). \tag{108}$$

The map $h : C \to A$ induces a map $h' : C' \to Im(f) := A'$ by $[x] \mapsto h(x)$. Two representatives x and x' of the class $[x]$ differ by an element $d(y)$ for some $y \in C$. Since we have $Im(d) = Im(gh) \subseteq Im(g) = Ker(h)$, h' is well defined. Moreover, any representative x is in the kernel of d. In particular, $h(x) \in Ker(g) = Im(f)$. Analogously, the map $g : A \to C$ induces a map $g' : A' \to C'$. This is a bit more complicated. Let $x \in A'$, then we may pick a preimage y with $f(y) = x$. $g'(x)$ is then defined to be $[g(y)] \in C'$. If y' is another preimage of x, then $y - y' \in Ker(f) = Im(h)$ and thus $g(y - y') \in Im(gh) = Im(d)$. Hence, the definition of g' is independent of the choice of y. Finally $f : A \to A$ induces a map $f' : A' \to A'$. Here, we simply define $f' = f|_{A'}$.
It is straightforward to check that the diagram

$$\begin{tikzcd} A' \arrow[rr, "f'"] & & A' \arrow[dl, "g'"] \\ & C' \arrow[ul, "h'"] & \end{tikzcd} \tag{109}$$

is again exact and commutative.

Definition 12.5. Let (C, A) be an exact couple. The exact couple $(C', A') =: (C^{(1)}, A^{(1)})$ is called the derived couple of (C, A). Inductively, we define the r-th derived couple by $(C^{(r)}, A^{(r)}) := ((C^{(r-1)})', (A^{(r-1)})')$ for $r \in \mathbb{N}$.

13 Construction via filtering of the space

Theorem 13.1. *Let X be a G-CW(V)-complex and X^n its n-sceleton. Further, let h^*_G be a $RO(G)$-graded cohomology theory. There exists a spectral sequence $E_r^{p,q}(X, V) \Rightarrow h_G^{V+p+q}(X)$, $r \in \mathbb{N}, p, q \in \mathbb{Z}$ with*

$$\begin{aligned} E_1^{p,q}(V) &= h_G^{V+p+q}(X^{|V|+p}, X^{|V|+p-1}) \text{ and} \\ E_2^{p,q}(V) &= H_G^{V+p}(X, \underline{h}^q_G(*)). \end{aligned} \tag{110}$$

Lemma 13.2. *Let M be a Mackey functor and $K \subseteq G$ a subgroup. There exists an isomorphism*

$$\begin{aligned} Hom_{\mathscr{B}_G}(\underline{\{*, G/K_+\}^G}, M) &\cong M(G/K) \\ \phi &\mapsto \phi(1_{G/K}) \end{aligned} \tag{111}$$

Proof. For another subgroup $H \subseteq G$, we have

$$\underline{\{*, G/K_+\}^G}(G/H) = \{G/H_+, G/K_+\}^G. \tag{112}$$

Every element $f \in \{G/H_+, G/K_+\}^G$ can be expressed as $f = f^*(id_{G/K_+})$, where

$$f^* : \{G/K_+, G/K_+\}^G \to \{G/H_+, G/K_+\}^G \tag{113}$$

is induced by f. Thus, $\phi \in Hom_{\mathscr{B}_G}(\underline{\{*, G/K_+\}^G}, M)$ is determined by $\phi(id_{G/K})$ via $\phi(f) = f^*\phi(id_{G/K})$. $\square$

Proof of Theorem 13.1. Define bigraded abelian groups C and A by

$$\begin{aligned} C_{p,q} &:= h_G^{V+p+q}(X^{|V|+p}, X^{|V|+p-1}) \text{ and} \\ A_{p,q} &:= h_G^{V+p+q}(X^{|V|+p}). \end{aligned} \tag{114}$$

The groups $C_{p,q}$ and $A_{p,q}$ fit into a long exact sequence, namely the long exact cohomology sequence for the pair $(X^{|V|+p}, X^{|V|+p-1})$:

$$\begin{aligned} \cdots \to h_G^{V+p+q-1}(X^{|V|+p-1}) &\xrightarrow{\delta} h_G^{V+p+q}(X^{|V|+p}, X^{|V|+p-1}) \\ &\xrightarrow{j^*} h_G^{V+p+q}(X^{|V|+p}) \xrightarrow{i^*} h_G^{V+p+q}(X^{|V|+p-1}) \to \cdots \end{aligned} \tag{115}$$

Here, i and j are the maps in the short exact sequence

$$X^{|V|+p-1} \xrightarrow{i} X^{|V|+p} \xrightarrow{j} X^{|V|+p}/X^{|V|+p-1}; \tag{116}$$

δ is the boundary homomorphism.
We obtain an exact couple

$$\begin{tikzcd} A \arrow[rr, "f"] & & A \arrow[dl, "g"] \\ & C \arrow[ul, "h"] & \end{tikzcd} \tag{117}$$

where $f_{p,q} : A_{p,q} \to A_{p-1,q+1}$, $g_{p,q} : A_{p,q} \to C_{p+1,q}$, and $h_{p,q} : C_{p,q} \to A_{p,q}$ are defined by (115).
Finally we set $E_r^{p,q}(V) = C_{p,q}^{(r-1)}$ and $d_r(V) = g^{(r-1)}h^{(r-1)}$ to define the spectral sequence.
It remains to identify the E_2-term with $H_G^{V+p}(X, \underline{h}_G^q(*))$. Recall that $\underline{C}_*^V(X)$ is the direct sum of coefficient systems of the form $\underline{\{*, G/K_+\}_G}$. For these functors we have by Lemma 13.2

$$Hom_{\mathscr{B}_G}(\underline{\{*, G/K_+\}^G}, \underline{h}_G^q(*)) \cong \underline{h}_G^q(*)(G/K) = h_G^q(G/K_+). \tag{118}$$

The final expression, on the other hand, is isomorphic to $h_G^{V-|V|+n+q}(S^{V-|V|+n} \wedge G/K_+)$ by a suspension isomorphism. Hence,

$$H_G^{V+p}(X, \underline{h}_G^q(*)) = H^{|V|+p}(h_G^{V-|V|+q+*}(X^*, X^{*-1})), \tag{119}$$

which is precisely the derived group C'. Indeed, the differentials for the ordinary cohomology theory and the differentials in the spectral sequence are defined by the same long exact sequences. This concludes the identification of E_2 with $H_G^{V+p}(X, \underline{h}_G^q(*))$. $\square$

Remark 13.3. *In his dissertation, Kronholm [Kro10] constructed a $RO(G)$-graded Serre-spectral sequence for a fibration of G-spaces. Specializing his results to the case of the trivial fibration we obtain $E_2^{p,q}(V) = H_G^p(X, \underline{h}_G^{V+q}(*))$. In fact this is just a different indexing. We would have got the same spectral sequence, if we chose $E_1^{p,q}(V) = h_G^{V+p+q}(X^p, X^{p-1})$.*

14 Construction via filtering of the representing spectrum

Theorem 14.1. *Let X be a G-$CW(V)$-complex. Further, let h^*_G be a $RO(G)$-graded cohomology theory represented by the G-spectrum h_V. There is a spectral sequence $E_r^{p,q}(V) \Rightarrow h_G^{V+p+q}(X)$ with*

$$E_2^{p,q}(V) = H_G^{V+p}(X, \underline{\pi}_{V+p}(h_{V+p+q})). \tag{120}$$

Proof. Define

$$\begin{aligned} \bar{C}_{p,q} &= [X, K(\underline{\pi}_{V+p}(h_{V+p+q}), V+p)] \text{ and} \\ \bar{A}_{p,q} &= [X, \mathbb{P}_{V+p}(h_{V+p+q})]. \end{aligned} \tag{121}$$

Moreover, let $\bar{h}_{p,q} : \bar{C}_{p,q} \to \bar{A}_{p,q}$ and $\bar{f}_{p,q} : \bar{A}_{p,q} \to \bar{A}_{p-1,q+1}$ be the maps induced by the long exact sequence

$$\cdots \to K(\underline{\pi}_{V+p}(h_{V+p+q}), V+p) \to \mathbb{P}_{V+p}(h_{V+p+q}) \to \mathbb{P}_{V+p-1}(h_{V+p+q}) \to \cdots \tag{122}$$

The definition of the map $\bar{g}_{p,q} : \bar{A}_{p,q} \to \bar{C}_{p+2,q-1}$ is a bit more subtle. The space $K(\underline{\pi}_{V+p+2}(h_{V+p+q+1}), V+p+2)$ fits into the long exact sequence

$$\begin{aligned} \cdots \to \Omega\mathbb{P}_{V+p+1}(h_{V+p+q+1}) &\to K(\underline{\pi}_{V+p+2}(h_{V+p+q+1}), V+p+2) \\ &\to \mathbb{P}_{V+p+2}(h_{V+p+q+1}) \to \mathbb{P}_{V+p+1}(h_{V+p+q+1}) \to \cdots \end{aligned} \tag{123}$$

The diagram

$$\begin{tikzcd} h_{V+p+q} \simeq \Omega h_{V+p+q+1} \arrow[r] \arrow[d] & \Omega\mathbb{P}_{V+p+1}(h_{V+p+q+1}) \\ \mathbb{P}_{V+p}(h_{V+p+q}) \arrow[ur, dotted] & \end{tikzcd} \tag{124}$$

shows that there exists a map $\mathbb{P}_{V+p}(h_{V+p+q}) \to \Omega\mathbb{P}_{V+p+1}(h_{V+p+q+1})$. Combining this map with the long exact sequence (123) induces a map $\bar{g}_{p,q} : \bar{A}_{p,q} \to \bar{C}_{p+2,q-1}$.
We take $\bar{C}$ to be the E_2-term of our spectral sequence and get

$$\begin{aligned} \bar{E}_2^{p,q}(V) = \bar{C}_{p,q} =& [X, K(\underline{\pi}_{V+p}(h_{V+p+q}), V+p)] \\ =& H_G^{V+p}(X, \underline{\pi}_{V+p}(h_{V+p+q})). \end{aligned} \tag{125}$$

□

We can identify $\bar{C}_{p,q}$ with $C'_{p,q}$, since

$$\begin{aligned} \underline{h}^q_G(*)(G/H) &= h^q_G(G/H_+) && (126) \\ &= [G/H_+, h(q)] && (127) \\ &= \underline{\pi}_*(h_q)(G/H) && (128) \\ &= \underline{\pi}_{V+p}(h_{V+q+p})(G/H) && (129) \end{aligned}$$

Hence $\underline{h}^q_G(*) = \underline{\pi}_{V+p}(h_{V+p+q})$.

Theorem 14.2. *There exist isomorphisms $\phi_r^{p,q} : E_r^{p,q} \to \bar{E}_r^{p,q}$ for $r \geq 2$ commuting with the differentials.*

So we can identify the two spectral sequences from the E_2-term on. We prove it on the level of exact couples. The Theorem is immediate from the following Lemma.

Lemma 14.3. *There exists a couple map (i.e a map compatible with the maps of the respective couples) $\psi : (C', A') \to (\bar{C}, \bar{A})$ from the first derived exact couple of (C, A) to the couple $(\bar{C}, \bar{A})$, which preserves both gradings and is an isomorphism from C' to $\bar{C}$.*

Proof. We mimic the idea of Maunders non-equivariant proof [Mau63].
By definition $A'_{p,q} = Im[i^* : h_G^{V+p+q}(X^{|V|+p+1}) \to h_G^{V+p+q}(X^{|V|+p})]$. Consider the commutative diagram

$$\begin{tikzcd}
{[X^{|V|+p+1}, h_{V+p+q}]} \arrow[r, "i^*"] \arrow[d, "\theta"] & {[X^{|V|+p}, h_{V+p+q}]} \arrow[d, "\tilde{\theta}"] \\
{[X^{|V|+p+1}, \mathbb{P}_{V+p}(h_{V+p+q})]} \arrow[r, "\tilde{i^*}"] & {[X^{|V|+p}, \mathbb{P}_{V+p}(h_{V+p+q})]}
\end{tikzcd} \tag{130}$$

where i^* and $\tilde{i}^*$ are induced by the inclusion $i : X^{|V|+p} \to X^{|V|+p+1}$ and θ as well as $\tilde{\theta}$ are induced by the natural map $h_{V+p+q} \to \mathbb{P}_{V+p}(h_{V+p+q})$.
So $Im(\tilde{\theta} i^*) \subset Im(\tilde{i^*})$, but $\tilde{i^*}$ is a monomorphism, since its kernel $Ker(\tilde{i^*})$ is given by

$$Im[[X^{|V|+p+1}/X^{|V|+p}, \mathbb{P}_{V+p}(h_{V+p+q})]_* \to [X^{|V|+p+1}, \mathbb{P}_{V+p}(h_{V+p+q})]] \tag{131}$$

and

$$[X^{|V|+p+1}/X^{|V|+p}, \mathbb{P}_{V+p}(h_{V+p+q})]_* \cong [\bigvee G/H_+ \wedge S^{V+p+1}, \mathbb{P}_{V+p}(h_{V+p+q})] = 0. \tag{132}$$

Thus θ induces a map

$$\psi_A : A'_{p,q} \to [X^{|V|+p+1}, \mathbb{P}_{V+p}(h_{V+p+q})] \cong [X, \mathbb{P}_{V+p}(h_{V+p+q})] = \bar{A}_{p,q}. \tag{133}$$

We define $\psi_C : C'_{p,q} \to \bar{C}_{p,q}$ to be the identity.
It remains to check that the map $\Psi : (C', A') \to (\bar{C}, \bar{A})$ is indeed a couple map.

- To prove $\psi_A f'_{p,q} = \bar{f}_{p,q}\psi_A$ we consider the diagram

$$
\begin{tikzcd}
{[X^{|V|+p+1}, h_{V+p+q}]} \arrow[r, "i_1^*"] \arrow[d] & {[X^{|V|+p}, h_{V+p+q}]} \arrow[r, "i_2^*"] \arrow[d] \arrow[dr] & {[X^{|V|+p-1}, h_{V+p+q}]} \arrow[dr] & \\
{[X^{|V|+p+1}, \mathbb{P}_{V+p}(h_{V+p+q})]} \arrow[r, "\iota_1^*"] & {[X^{|V|+p}, \mathbb{P}_{V+p}(h_{V+p+q})]} \arrow[r] & {[X^{|V|+p}, \mathbb{P}_{V+p-1}(h_{V+p+q})]} \arrow[r, "\iota_2^*"] & {[X^{|V|+p-1}, \mathbb{P}_{V+p-1}(h_{V+p+q})]} \\
{[X, \mathbb{P}_{V+p}(h_{V+p+q})]} \arrow[r, "\bar{f}_{p,q}"] \arrow[u, "\cong"] \arrow[ur] & {[X, \mathbb{P}_{V+p-1}(h_{V+p+q})]} \arrow[ur, "\cong"] & &
\end{tikzcd}
\tag{134}
$$

Here, $A'_{p,q} = Im(i_1^*)$, $A'_{p-1,q+1} = Im(i_2^*)$ and $f'_{p,q} = i_2^*|_{Im(i_1^*)}$. Further, ι_1^*

and ι_2^* are monomorphic. Therefore, $f'_{p,q}\psi_A = \psi_A \bar{f}_{p,q}$ is obvious.

- The compatibility $\psi_C g'_{p,q} = \bar{g}_{p,q}\psi_A$ is rather technical and complicated. Consider the diagram

$$\begin{tikzcd}
{[X^{|V|+p+1}, h_{V+p+q}]} \arrow[r, "g_{p+1,q-1}"] \arrow[d] & {[X^{|V|+p+2}/X^{|V|+p+1}, K(\underline{\pi}_{V+p+2}(h_{V+p+q+1}), V+p+2)]_*} \arrow[d] \\
{Im([X^{|V|+p+1}, h_{V+p+q}] \to [X^{|V|+p}, h_{V+p+q}])} \arrow[r, "g'_{p,q}"] \arrow[d] & {[X^{|V|+p+2}, K(\underline{\pi}_{V+p+2}(h_{V+p+q+1}), V+p+2)]} \arrow[d, "="] \\
{[X^{|V|+p+1}, \mathbb{P}_{V+p}(h_{V+p+q})]} \arrow[r, "\bar{g}_{p,q}"] & {[X^{|V|+p+2}, K(\underline{\pi}_{V+p+2}(h_{V+p+q+1}), V+p+2)].}
\end{tikzcd} \tag{135}$$

We want to prove commutativity of the lower square. The upper square shows, how $g'_{p,q}$ is induced by $g_{p+1,q-1}$, and it commutes by definition of $g'_{p,q}$. Hence, it suffices to show commutativity of the outer square.

Since $[X^{|V|+p+1}, h_{V+p+q}] \cong [X^{|V|+p+1}, \mathbb{P}_{V+p+1}(h_{V+p+q})]$ by Proposition 9.5, this can be reduced to the diagram:

$$\begin{tikzcd}
{[X^{|V|+p+1}, \mathbb{P}_{V+p+1}(h_{V+p+q})]} \arrow[r, "g_{p+1,q-1}"] \arrow[d] & {[X^{|V|+p+2}/X^{|V|+p+1}, K(\underline{\pi}_{V+p+2}(h_{V+p+q+1}), V+p+2)]_*} \arrow[d] \\
{[X^{|V|+p+1}, \mathbb{P}_{V+p}(h_{V+p+q})]} \arrow[r, "\bar{g}_{p,q}"] & {[X^{|V|+p+2}, K(\underline{\pi}_{V+p+2}(h_{V+p+q+1}), V+p+2)].}
\end{tikzcd} \tag{136}$$

Note that we supressed an ismorphism

$$[X^{|V|+p+1}, \mathbb{P}_{V+p}(h_{V+p+q})] \cong [X^{|V|+p+2}, \mathbb{P}_{V+p}(h_{V+p+q})] \tag{137}$$

in the lower left corner of both diagrams.

An element $x \in [X^{|V|+p+1}, \mathbb{P}_{V+p+1}(h_{V+p+q})]$ may either be represented by a map $\mu : X^{|V|+p+1} \to \mathbb{P}_{V+p+1}(h_{V+p+q})$ or a map $\mu' : \Sigma X^{|V|+p+1} \to \mathbb{P}_{V+p+2}(h_{V+p+q+1})$:

$$
\begin{tikzcd}
X^{|V|+p+1} \arrow[r, "\mu"] \arrow[d] & \mathbb{P}_{V+p+1}(h_{V+p+q}) \arrow[d] \\
X^{|V|+p+2} \arrow[r, dotted, "\rho"] \arrow[d] & \mathbb{P}_{V+p}(h_{V+p+q}) \arrow[d] \\
X^{|V|+p+2}/X^{|V|+p+1} \arrow[r, dotted, "\nu"] \arrow[d] & K(\underline{\pi}_{V+p+2}(h_{V+p+q+1}), V+p+2) \arrow[d] \\
\Sigma X^{|V|+p+1} \arrow[r, "\mu'"] & \mathbb{P}_{V+p+2}(h_{V+p+q+1})
\end{tikzcd} \tag{138}
$$

The map

$$X^{|V|+p+1} \to \mathbb{P}_{V+p+1}(h_{V+p+q}) \to \mathbb{P}_{V+p}(h_{V+p+q}) \tag{139}$$

extends uniquely to a map

$$\rho : X^{|V|+p+2} \to \mathbb{P}_{V+p}(h_{V+p+q}), \tag{140}$$

since $\mathbb{P}_{V+p}(h_{V+p+q})$ is $\tilde{\mathcal{A}}_{V+p+1}$-null.
By Lemma 6.19 there now exists a unique lift

$$\nu : X^{|V|+p+2}/X^{|V|+p+1} \to K(\underline{\pi}_{V+p+2}(h_{V+p+q+1}), V+p+2). \tag{141}$$

Then the inner square defines two homotopic maps

$$X^{|V|+p+2} \to K(\underline{\pi}_{V+p+2}(h_{V+p+q+1}), V+p+2). \tag{142}$$

On the other hand these two maps also represent the images of $x \in [X^{|V|+p+1}, \mathbb{P}_{V+p+1}(h_{V+p+q})]$ under the maps of Diagram (136). Thus, we have just shown the commutativity of this diagram.

- For $\psi_A h'_{p,q} = \bar{h}_{p,q}\psi_C$ we finally consider the diagram

$$\begin{tikzcd}
{[X^{|V|+p+1}, h_{V+p+q}]} \arrow[r, "i^*"] \arrow[ddd, "\theta"] & {[X^{|V|+p}, h_{V+p+q}]} \arrow[r, "\delta^*"] \arrow[ddd, bend right, "\theta"] & {[X^{|V|+p+1}/X^{|V|+p}, h_{V+p+q+1}]^*} \arrow[ddd, "\theta"] \\
 & {[X^{|V|+p}/X^{|V|+p-1}, h_{V+p+q}]^*} \arrow[u, "h_{p,q}"] \arrow[ur, "d"] \arrow[d, "\cong"] & \\
 & {[X^{|V|+p}/X^{|V|+p-1}, \mathbb{P}_{V+p}(h_{V+p+q})]^*} \arrow[d, "\bar{j}"] \arrow[dr, "d"] & \\
{[X^{|V|+p+1}, \mathbb{P}_{V+p}(h_{V+p+q})]} \arrow[r, "\bar{\iota}"] & {[X^{|V|+p}, \mathbb{P}_{V+p}(h_{V+p+q})]} \arrow[r, "\bar{\delta}"] & {[X^{|V|+p+1}/X^{|V|+p}, \mathbb{P}_{V+p+1}(h_{V+p+q+1})]_*.}
\end{tikzcd} \tag{143}$$

The top and bottom row are exact and the maps denoted by d are the ordinary boundary maps. The map $\psi_A h'_{p,q}$ is induced by $(\bar{\iota})^{-1}\theta h_{p,q}$. This map on the other hand induces the standard map

$$\bar{h}_{p,q} : [X, K(\underline{\pi}_{V+p}(h_{V+p+q}), V+p)] \to [X, \mathbb{P}_{V+p}(h_{V+p+q})]. \tag{144}$$

$\square$

Part V
The slice spectral sequence

15 The homotopy spectral sequence

A more conceptual way to construct spectral sequences is the construction via the homotopy spectral sequence. We will follow the notation of [Dug03].

Definition 15.1. Let $f : A \to B$ be a map between pointed spaces. For $p \geq 1$ define $\pi_p(B, A)$ to be the set of equivalence classes of diagrams Δ of the form

$$\begin{tikzcd} S^{p-1} \arrow[r] \arrow[d, hook] & A \arrow[d] \\ D^p \arrow[r] & B \end{tikzcd} \tag{145}$$

where two diagrams Δ and Δ' are regarded as equivalent if there is a diagram

$$\begin{tikzcd} S^{p-1} \times I \arrow[r] \arrow[d, hook] & A \arrow[d] \\ D^p \times I \arrow[r] & B \end{tikzcd} \tag{146}$$

which restricts to Δ and Δ' under the inclusions $\{0\} \hookrightarrow I$ and $\{1\} \hookrightarrow I$, respectively.

Proposition 15.2. *Let $f : A \to B$ be a map between pointed spaces, and let $hofib(f)$ denote its homotopy fiber. Then there exists an isomorphism*

$$\pi_p(B, A) \cong \pi_{p-1}(hofib(f)). \tag{147}$$

Proof. Consider the diagram

$$\begin{tikzcd} & & hofib(f) \arrow[d] \\ & S^{p-1} \times \{0\} \arrow[r] \arrow[dl, hook'] \arrow[d, hook] & E_f \simeq A \arrow[d] \\ S^{p-1} \times I \arrow[r] \arrow[urr, dotted] & D^p \cong (S^{p-1} \times I)/(S^{p-1} \times \{1\}) \arrow[r] & B \end{tikzcd} \tag{148}$$

where $E_f := \{(a, \gamma) \in A \times B^I | \gamma(0) = f(a)\}$. The dashed map is defined by the homotopy lifting property and induces a map from $S^{p-1} \to hofib(f)$, since the composition of the lower horizontal maps is a null-homotopy. Conversely, any map $g : S^{p-1} \to hofib(f)$ induces a map $S^{p-1} \to A$, which becomes null-homotopic when it is composed with f. Hence, there is an element in $\pi_p(B, A)$ corresponding to g. The maps $\pi_p(B, A) \leftrightarrows \pi_{p-1} hofib(f)$ are easily seen to be inverse to each other.

$\square$

Definition 15.3. A tower is a sequence of pointed spaces W_q, $q \in Z$ with (basepoint preserving) maps

$$... \to W_3 \to W_2 \to W_1 \to W_0 \to W_{-1} \to ... \tag{149}$$

Given such a tower we set

$$A_{p,q} = \pi_p(W_q, *) \text{ and } C_{p,q} = \pi_p(W_q, W_{q+1}) \text{ for } p \geq 1, q \in \mathbb{Z} \tag{150}$$

To get an exact couple we need to define the maps fitting into the diagram

$$\begin{tikzcd} A \arrow[rr, "f"] & & A \arrow[dl, "g"] \\ & C \arrow[ul, "h"] & \end{tikzcd} \tag{151}$$

For any element in the source of the following maps we write a representing diagram on the left hand side and its image on the right hand side. We set

- $f_{p,q} : A_{p,q} = \pi_p(W_q, *) \to \pi_p(W_{q-1}, *) = A_{p,q-1}$

$$S^p \longrightarrow W_q \qquad\qquad S^p \longrightarrow W_q \longrightarrow W_{q-1} \tag{152}$$

- $g_{p,q} : A_{p,q} = \pi_p(W_q, *) \to \pi_p(W_q, W_{q+1}) = C_{p,q}$

$$\begin{tikzcd} S^{p-1} \arrow[r] \arrow[d] & * \arrow[d] \\ D^p \arrow[r] & W_q \end{tikzcd} \qquad\qquad \begin{tikzcd} S^{p-1} \arrow[r, "*"] \arrow[d] & W_{q+1} \arrow[d] \\ D^p \arrow[r] & W_q \end{tikzcd} \tag{153}$$

- $h_{p,q} : C_{p,q} = \pi_p(W_q, W_{q+1}) \to \pi_{p-1}(W_{q+1}, *) = A_{p-1,q+1}$

$$\begin{tikzcd} S^{p-1} \arrow[r] \arrow[d] & W_{q+1} \arrow[d] \\ D^p \arrow[r] & W_q \end{tikzcd} \qquad\qquad S^{p-1} \longrightarrow W_{q+1} \tag{154}$$

Definition 15.4. The spectral sequence obtained from the tower above is called the homotopy spectral sequence of this tower. For later purposes we set $E_2^{p,q} = C_{p,q}$ and more generally $E_r^{p,q} = C_{p,q}^{(r-2)}$ for $r \geq 2$. In other words, the homotopy spectral sequence starts on the E_2-term.

Remark 15.5. *The differential d_r has the form $d_r : E_r^{p,q} \to E_r^{p-1,q+r}$, $r \in \mathbb{N}, p, q \in \mathbb{Z}$. This is called the* Adams indexing.

Remark 15.6. *In general we don't really have an exact couple or an exact sequence. The problem is that π_0 and π_1 need not to be abelian groups. We will assume that we are in a setting where these problems do not arise. For example, the W_i could be connected with abelian fundamental groups.*

Remark 15.7. *There are many ways to index a spectral sequence. We will change the indices whenever it fits in better with our respective situation.*

Remark 15.8. *One can generalize this construction to towers of spectra. We will not dwell on any details, but refer again to [Dug03].*

In this paper the tower will always be a tower of homotopy fibrations arising from Postnikov-filtrations.

Example 15.9. *Consider a spectrum of pointed G-spaces Y and a G-representation V. From Proposition 9.5 we deduce that there exists the Postnikov-tower*

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{P}_{V+2}(Y) \arrow[r] & \mathbb{P}_{V+1}(Y) \arrow[r] & \mathbb{P}_V(Y) \\ & K(\underline{\pi}_{V+2}(Y), V+2) \arrow[u] & K(\underline{\pi}_{V+1}(Y), V+1) \arrow[u] & \end{tikzcd} \tag{155}$$

Taking mapping spaces from another G-space X into each of these spaces gives us another tower of homotopy fibrations.

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{P}_{V+2}(Y)^X \arrow[r] & \mathbb{P}_{V+1}(Y)^X \arrow[r] & \mathbb{P}_V(Y)^X \\ & K(\underline{\pi}_{V+2}(Y), V+2)^X \arrow[u] & K(\underline{\pi}_{V+1}(Y), V+1)^X \arrow[u] & \end{tikzcd} \tag{156}$$

Now take the homotopy spectral sequence of this tower, i.e. the spectral sequence arising from the exact couple with

$$\begin{aligned} C_{p,q} &= \pi_p(K(\underline{\pi}_{V+q+1}(Y_V), V+q+1)^X, *) && (157)\\ &\cong [S^p, K(\underline{\pi}_{V+q+1}(Y_V), V+q+1)^X] && (158)\\ &\cong [S^p \wedge X, K(\underline{\pi}_{V+q+1}(Y_V), V+q+1)] && (159)\\ &\cong [X, K(\underline{\pi}_{V+q+1}(Y_V), V+q+1-p)] && (160)\\ &\cong H^{V-p+q+1}(X, \underline{\pi}_{V+q+1}(Y_V)) \end{aligned}$$

According to Remark 15.7 *we can change the indices. If we shift p to $-p-q$ and q to $-q-1$ we obtain the sequence of Theorem* 14.1.

Example 15.10. *We specialize further and consider the spectrum $\mathbb{Z} \times BU$ and $X = *$. We write down the sequence for $V = 0$*

$$E_2^{p,q} = H_G^p(*, \underline{KR}^q(*)) = \begin{cases} KR^q(*) & \textit{if } p = 0 \\ 0 & \textit{else} \end{cases} \tag{161}$$

For the definition of $\underline{KR}^q(*)$ see Definition 12.2. This sequence collapses right after the E_2-term. We will define another spectral sequence for KR later. It will turn out, that this sequence is highly non-trivial even for the point-space.

15.1 Convergence

Definition 15.11. Let A^s be a sequence of (possibly graded, or bigraded) abelian groups, together with homomorphisms $i : A^{s+1} \to A^s$. Let A^∞ denote the limit of this sequence. The derived limit RA^∞ is defined by the exact sequence

$$0 \to A^\infty \to \Pi_s A^s \xrightarrow{1-i} \Pi_s A^s \to RA^\infty \to 0 \tag{162}$$

(cf. [Mil62] p.338).

The homotopy spectral sequence for a bounded below tower is automatically conditionally convergent ([Boa99] Definition 5.10). So if $RE_\infty = 0$, it converges strongly by [Boa99] Theorem 7.4.

The most common situation in which the condition $RE_\infty = 0$ is satisfied, is when for each p, q there exists an N such that $E_r^{p,q} = E_{r+1}^{p,q}$ for all $r \geq N$.

15.2 Multiplicative structure

Finding a multiplicative structure on the spectral sequence will simplify the computation. However, establishing a multiplicative structure is not easy.

We will first recall how products arise in homotopy spectral sequences. The crucial task is to construct a pairing of towers, i.e. a family of maps $W_m \wedge X_n \to Y_{m+n}$ for three towers W_*, X_* and Y_*. Again we follow [Dug03].

Let $F : A \to B$ and $g : C \to D$ be two maps of pointed G-spaces. Let P be the pushout of $A \wedge D \leftarrow A \wedge C \to B \wedge C$ and note that there is a canonical map $P \to B \wedge D$.

We can construct a natural pairing $\pi_p(B, A) \otimes \pi_q(D, C) \to \pi_{p+q}(B \wedge D, P)$ in the following way. Given two diagrams

$$\begin{tikzcd} S^{p-1} \arrow[r] \arrow[d] & A \arrow[d] \\ D^p \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} S^{q-1} \arrow[r] \arrow[d] & C \arrow[d] \\ D^q \arrow[r] & D \end{tikzcd} \tag{163}$$

we can form the new diagram

$$\begin{tikzcd} S^{p+q-1} \arrow[r, "\cong"] \arrow[d] & (S^{p-1} \times D^q) \amalg_{S^{p-1} \times S^{q-1}} (D^p \times S^{q-1}) \arrow[r] & (A \wedge D) \amalg_{A \wedge C} (B \wedge C) \arrow[d] \\ D^{p+q} \arrow[r, "\cong"] & D^p \times D^q \arrow[r] & B \wedge D \end{tikzcd} \tag{164}$$

which defines an element in $\pi_{p+q}(B \wedge D, P)$. This is well-defined and bilinear.

We apply this construction to a pairing of towers. Let W_*, X_* and Y_* be three towers with the resulting homotopy spectral sequences $E_*(W)$, $E_*(X)$ and $E_*(Y)$, respectively. Assume that there are pairings $W_m \wedge X_n \to Y_{m+n}$ such that the following diagram commutes (on the nose!):

$$\begin{tikzcd} W_{m+1} \wedge X_n \arrow[r] \arrow[d] & Y_{m+n+1} \arrow[d] & W_m \wedge X_{n+1} \arrow[l] \arrow[d] \\ W_m \wedge X_n \arrow[r] & Y_{m+n} & W_m \wedge X_n \arrow[l] \end{tikzcd} \tag{165}$$

From the construction above and by naturality of π_* it follows that there is an induced pairing

$$\pi_k(W_m, W_{m+1}) \otimes \pi_l(X_n, X_{n+1}) \to \pi_{k+l}(Y_{m+n}, Y_{m+n+1}). \tag{166}$$

In other words, we have produced a multiplication on spectral sequences

$$E_2^{p,q}(W) \otimes E_2^{s,t}(X) \to E_2^{p+s,q+t}(Y). \tag{167}$$

The following proposition concludes this discussion.

Proposition 15.12. *The product $E_2(W) \otimes E_2(X) \to E_2(Y)$ descends to pairings of the E_r-terms, satisfying the Leibniz rule $d_r(a \cdot b) = d_r(a) \cdot b + (-1)^p a \cdot d_r(b)$ for $a \in E_r^{p,q}(W)$ and $b \in E_r^{s,t}(W)$.*

Sketch of proof. For details see [D1]. The basic idea is to check the Leibniz rule for each r and conclude that the pairing descends to $r+1$. For $r = 1$ this is a geometric consideration, for bigger r we use the fact that each element in $E_r^{p,q}(W)$ (and $E_r^{p,q}(X)$) can be represented by a square

$$\begin{tikzcd} & S^{p-1} \arrow[r] \arrow[dl, "h"'] \arrow[d] & D^p \arrow[d] \\ W_{q+r} \arrow[r] & W_{q+1} \arrow[r] & W_q \end{tikzcd} \tag{168}$$

and then apply the same argument as for $r = 1$ to the outer square. $\square$

16 The slice spectral sequence

In this section we will consider the objects $P_{\mathbb{C}^n}(\mathbb{Z} \times BU)$, where $\mathbb{C}$ is the regular representation of $\mathbb{Z}/2$. Adopting Dugger's notation, we will write the functor $P_{\mathbb{C}^n}$ as P_{2n}. Be aware that P_{2n} could also denote the Postnikov functor for the $2n$-dimensional trivial representation.

The central result [Dug05] is the following.

Theorem 16.1. *Let $\beta : S^{2,1} \to \mathbb{Z} \times BU$ be the map representing the Bott element in $\tilde{KR}^{0,0}(S^{2,1})$, and let $\beta^n : S^{2n,n} \to \mathbb{Z} \times BU$ denote its n-th power. Then*

$$P_{2n}(S^{2n,n}) \xrightarrow{\beta^n} P_{2n}(\mathbb{Z} \times BU) \to P_{2n-2}(\mathbb{Z} \times BU) \tag{169}$$

is a homotopy fiber sequence.

Proof. [Dug05] Theorem 4.1. $\square$

Lemma 16.2. *Let V be a representation of $G = \mathbb{Z}/2$ containing the trivial one-dimensional representation. Then the space $P_V(S^V)$ has the equivariant weak homotopy type of the Eilenberg-MacLane space $K(\underline{\mathbb{Z}}, V)$.*

Proof. [Dug05] Theorem 3.8. $\square$

Corollary 16.3. *There is a tower of homotopy fiber sequences*

$$\begin{tikzcd} \cdots \arrow[r] & P_4(\mathbb{Z} \times BU) \arrow[r] & P_2(\mathbb{Z} \times BU) \arrow[r] & P_0(\mathbb{Z} \times BU) \arrow[r] & * \\ & K(\mathbb{Z}(2), 4) \arrow[u] & K(\mathbb{Z}(1), 2) \arrow[u] & \mathbb{Z} \arrow[u] & \end{tikzcd} \tag{170}$$

and the homotopy limit of this tower is $\mathbb{Z} \times BU$*.*

Proof. [Dug05] Corollary 4.2. □

It is possible to prove Corollary 16.3 without the knowledge of Theorem 16.1. The main steps are the following two results.

Proposition 16.4. *The homotopy fiber of* $\mathbb{P}_{2n}(\mathbb{Z} \times BU) \to \mathbb{P}_{2n-2}(\mathbb{Z} \times BU)$ *is an Eilenberg-MacLane space of type* $K(\mathbb{Z}(n), 2n)$*.*

Proof. [Dug05] Proposition 4.3. □

Proposition 16.5. *The natural map* $\mathbb{P}_{2n,n}(\mathbb{Z} \times BU) \to P_{2n,n}(\mathbb{Z} \times BU)$ *is a weak equivalence.*

Proof. [Dug05] Lemma 4.5. □

Let us calculate the homotopy spectral sequence for the tower given in Corollary 16.3 (after taking mapping spaces). We choose an indexing which will turn out to be suitable later.

Theorem 16.6. *Let* X *be a* G*-space. The homotopy spectral sequence for the Postnikov tower*

$$\begin{tikzcd} \cdots \arrow[r] & P_4(\mathbb{Z} \times BU)^X \arrow[r] & P_2(\mathbb{Z} \times BU)^X \arrow[r] & P_0(\mathbb{Z} \times BU)^X \arrow[r] & * \\ & K(\mathbb{Z}(2), 4)^X \arrow[u] & K(\mathbb{Z}(1), 2)^X \arrow[u] & \mathbb{Z}^X \arrow[u] & \end{tikzcd} \tag{171}$$

is called the slice spectral sequence for X *and takes on the form*

$$E_2^{p,q} \cong H^{p,-\frac{q}{2}}(X, \underline{\mathbb{Z}}) \Rightarrow KR^{p+q}(X) \tag{172}$$

with the differentials

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+2r-1, q-2r+2} \tag{173}$$

of bidegree $(2r-1, -2r+2)$ *for* $r \geq 2$*.*

Proof. The E_2-term takes on the form:

$$\begin{aligned} E_2^{p,q}(X) &= \pi_{-p-q}(K(\mathbb{Z}(-\frac{q}{2}), -q)^X) \\ &\cong [X \wedge S^{-(p+q)}, K(\mathbb{Z}(-\frac{q}{2}), -q)] \\ &\cong [X, K(\mathbb{Z}(-\frac{q}{2}), p)] \\ &\cong H^{p,-\frac{q}{2}}(X, \underline{\mathbb{Z}}) \end{aligned} \tag{174}$$

Note that the entries of the spectral sequence are interpreted as 0 for odd q. The differentials on the E_2-term have the form:

$$\begin{aligned}
H^{p,-\frac{q}{2}}(X) \cong & [S^0, K(\mathbb{Z}(-\frac{q}{2}), p)^X] \\
\cong & [S^{-p-q}, K(\mathbb{Z}(-\frac{q}{2}), -q)^X] \\
\cong & \pi_{-p-q+1}(P_{-q-2}(\mathbb{Z}\times BU)^X, P_{-q}(\mathbb{Z}\times BU)^X) \\
\xrightarrow{h} & \pi_{-p-q}(P_{-q}(\mathbb{Z}\times BU)^X) \\
\xrightarrow{g} & \pi_{-p-q}(P_{-q}(\mathbb{Z}\times BU)^X, P_{-q+2}(\mathbb{Z}\times BU)^X) \\
\cong & \pi_{-p-q-1}(K(\mathbb{Z}(\frac{-q+2}{2}), -q+2)^X) \\
\cong & [S^0, K(\mathbb{Z}(\frac{-q+2}{2}), p+3)^X] \cong H^{p+3,-\frac{q}{2}+1}(X). \qquad (175)
\end{aligned}$$

Recall from the construction on page 41f. that $d_r : E_r \to E_r$ is given by the composition $d_r = g^{(r-2)} \circ h^{(r-2)}$, where $g^{(r-2)}$ and $h^{(r-2)}$ are induced by g and h, respectively. To obtain $h^{(r-2)}$ from h, we just inductively take the induced map on the successive quotient spaces, and thus the bidegree of $h^{(r-2)}$ coincides with the bidegree of h. To compute $g^{(r-2)}([x])$ for any class $[x] \in E_r$, on the other hand, we first take a preimage of a representative x under the map f^{r-2} and then apply g to it. Hence, the bidegree of $g^{(r-2)}$ differs from the bidegree of g.
All in all, the differential d_r is induced by

$$\begin{aligned}
E_2^{p,q} \quad & \cong \quad \pi_{-p-q+1}(P_{-q-2}(\mathbb{Z}\times BU)^X, P_{-q}(\mathbb{Z}\times BU)^X) \\
& \xrightarrow{h} \quad \pi_{-p-q}(P_{-q}(\mathbb{Z}\times BU)^X) \\
& \xrightarrow{f^{r-2}} \quad \pi_{-p-q}(P_{-q+2r-4}(\mathbb{Z}\times BU)^X) \\
& \xrightarrow{g} \quad \pi_{-p-q}(P_{-q+2r-4}(\mathbb{Z}\times BU)^X, P_{-q+2r-2}(\mathbb{Z}\times BU)^X) \\
& \cong \quad E_2^{p+2r-1,q-2r+2}(X). \qquad (176)
\end{aligned}$$

Hence, we have

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+2r-1,q-2r+2} \qquad (177)$$

The spectral sequence converges conditionally to

$$[S^{-p-q,0} \wedge X_+, \mathbb{Z}\times BU]_* \cong KR^{p+q}(X), \qquad (178)$$

since $\mathbb{Z}\times BU^X$ is the homotopy limit of the Postnikov tower. □

Remark 16.7. *We want to emphasize the importance of the notation used in this equation. We will frequently express the equivariant cohomology groups $H^{p,-\frac{q}{2}}(X)$ by the equivariant homotopy groups*

$$\pi_{-p-q+1}(P_{-q-2}(\mathbb{Z}\times BU)^X, P_{-q}(\mathbb{Z}\times BU)^X), \qquad (179)$$

especially when we compute the differentials in special cases of the slice spectral sequence.

16.1 Multiplicative structure on the slice spectral sequence

Unfortunately the Postnikov tower is not the right one to produce a pairing $P_*(X) \wedge P_*(Y) \to P_*(X \wedge Y)$. There is no suitable pairing even for ordinary, non-equivariant Postnikov towers. There is a way to avoid this obstacle. First we have to define a second tower with an isomorphic spectral sequence.

Let W_n be the homotopy fiber of $\mathbb{Z} \times BU \to P_{2n-2}(\mathbb{Z} \times BU)$. Consider the diagram

$$\begin{tikzcd}
K(\mathbb{Z}(n), 2n) \arrow[r] & P_{2n}(\mathbb{Z} \times BU) \arrow[r] & P_{2n-2}(\mathbb{Z} \times BU) \\
W_n \arrow[u, dotted] \arrow[r] & \mathbb{Z} \times BU \arrow[u] \arrow[r] & P_{2n-2}(\mathbb{Z} \times BU) \arrow[u, "id"] \\
W_{n+1} \arrow[u, dotted] \arrow[r] & \mathbb{Z} \times BU \arrow[u, "id"] \arrow[r] & P_{2n}(\mathbb{Z} \times BU) \arrow[u]
\end{tikzcd} \tag{180}$$

where all horizontal maps are homotopy fibrations. We deduce that $P_{2n}(\mathbb{Z} \times BU) \to P_{2n-2}(\mathbb{Z} \times BU)$ induces a natural map $W_{n+1} \to W_n$ and that $\mathbb{Z} \times BU \to P_{2n-2}(\mathbb{Z} \times BU)$ induces a natural map $W_n \to K(\mathbb{Z}(n), 2n)$. Chasing through the diagram yields that $W_{n+1} \to W_n \to K(\mathbb{Z}(n), 2n)$ is a homotopy fibration. Hence we can identify the homotopy fiber of $W_{n+1} \to W_n$ with $\Omega K(\mathbb{Z}(n), 2n) \cong K(\mathbb{Z}(n), 2n-1)$. To abbreviate the notation we will denote $K(\mathbb{Z}(n), 2n)$ by F_n. Since there are maps $\Omega P_{2n-2}(\mathbb{Z} \times BU) \to W_n$ (well defined up to homotopy), we obtain a map of towers

$$\begin{tikzcd}
 & \Omega F_2 \arrow[d] & \Omega F_1 \arrow[d] & \Omega F_0 \arrow[d] & \\
\arrow[r] & W_3 \arrow[r] & W_2 \arrow[r] & W_1 \arrow[r] & \mathbb{Z} \times BU \\
\arrow[r] & \Omega P_4(\mathbb{Z} \times BU) \arrow[u] \arrow[r] & \Omega P_2(\mathbb{Z} \times BU) \arrow[u] \arrow[r] & \Omega P_0(\mathbb{Z} \times BU) \arrow[u] \arrow[r] & * \arrow[u] \\
 & \Omega F_2 \arrow[u] & \Omega F_1 \arrow[u] & \Omega F_0 \arrow[u] &
\end{tikzcd} \tag{181}$$

inducing isomorphisms on the fibers and hence on the resulting spectral sequences. Finally we have to produce pairings $W_m \wedge W_n \to W_{n+m}$. This is no easy task and not particularly revealing for our purpose. In fact, we have to use a complicated mixture of general arguments and special ones due to our situation. We will only give an idea how it works. For details see [Dug03] part II, pages 8-12.

Consider the diagram

$$
\begin{tikzcd}
W_m \wedge W_n \arrow[r] \arrow[d, dotted, "\lambda"] & \mathbb{Z} \times BU \wedge \mathbb{Z} \times BU \arrow[d] & \\
W_{m+n} \arrow[r] & \mathbb{Z} \times BU \arrow[r] & P_{2(m+n)-2}(\mathbb{Z} \times BU)
\end{tikzcd}
\tag{182}
$$

The dashed map λ exists, since the composition of the three maps (upper horizontal, vertical, right horizontal) is null-homotopic. λ is unique up to homotopy. This pairing doesn't neccesarily commute with the structure maps in the towers. It does, however, commute with the structure maps up to homotopy. Then we have to use obstruction theory to alter these maps, so that the relevant diagrams do indeed commute.

Now we have constructed a multiplicative structure on the spectral sequence. Earlier we have identified the E_2-term with the cohomology theory $H^{*,*}(-;\underline{\mathbb{Z}})$, which carries an own multiplication. It is natural to ask, whether both multiplicative structures coincide.

16.2 Ring G-spectra and products

The following is taken from May [M+96].
Let E and E' be G-spectra, and let X and X' be G-spaces (or rather their suspension spectra). The natural map in $\bar{h}G\mathscr{S}$

$$F(X,E) \wedge F(X',E') \xrightarrow{\wedge} F(X \wedge X', E \wedge E') \tag{183}$$

gives rise to a product in $RO(G)$-graded cohomology, when we pass to homotopy groups

$$E^*_G(X) \otimes E'^*_G(X') \to (E \wedge E')^*_G(X \wedge X'). \tag{184}$$

To obtain an internal product, $E = E'$ needs to be a ring G-spectrum.

Definition 16.8. A ring G-spectrum E is a G-spectrum equipped with a product $\phi : E \wedge E \to E$ and a unit map $\eta : S \to E$, such that the following diagrams commute in $\bar{h}G\mathscr{S}$:

$$
\begin{tikzcd}
S \wedge E \arrow[r, "\eta\wedge 1"] \arrow[dr, "\simeq"'] & E \wedge E \arrow[d, "\phi"] & E \wedge S \arrow[l, "1\wedge\eta"'] \arrow[dl, "\simeq"] \\
& E &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
E \wedge E \wedge E \arrow[r, "1\wedge\phi"] \arrow[d, "\phi\wedge 1"] & E \wedge E \arrow[d, "\phi"] \\
E \wedge E \arrow[r, "\phi"] & E
\end{tikzcd}
\tag{185}
$$

E is commutative if

$$
\begin{tikzcd}
E \wedge E \arrow[rr, "\tau"] \arrow[dr, "\phi"'] & & E \wedge E \arrow[dl, "\phi"] \\
& E &
\end{tikzcd}
\tag{186}
$$

commutes, where τ interchanges the entries of the product.

If E is a G-ring spectrum and $X = X'$ a based G-space (again rather its suspension spectrum) we obtain the cup product

$$\cup : E^*_G(X) \otimes E^*_G(X) \to E^*_G(X) \tag{187}$$

by composing the pairing with the reduced diagonal $\Delta : X \to X \wedge X$.

When is HM a ring G-spectrum? The answer is quite easy: HM is a ring G-spectrum, iff M is a Green-functor. This observation is due to May and Greenlees ([GM95] Paragraph 8).

Remark 16.9. *We are mostly interested in the constant Mackey functor $\underline{\mathbb{Z}}$ for $G = \mathbb{Z}/2$. It is straightforward to check that $\underline{\mathbb{Z}}$ is a Green functor. Hence we get a multiplicative structure on $H^{*,*}(-;\underline{\mathbb{Z}})$.*

For the spaces $K(M, V)$ we can make this more explicit. The map $K(M, V) \wedge K(M, W) \to K(M, V + W)$ is given by coning of the higher homotopy groups. Hence the products on E_2 given by

$$\begin{aligned} &\underbrace{\pi_{-p-q}(W^X_{-q}, W^X_{-q+2})}_{\cong E_2^{p,q}(X)} \times \underbrace{\pi_{-s-t}(W^X_{-t}, W^X_{-t+2})}_{\cong E_2^{s,t}(X)} \\ &\longrightarrow \underbrace{\pi_{-q-p-s-t}(W^X_{-s+t}, W^X_{-s-t+2})}_{\cong E_2^{p+q,s+t}(X)} \end{aligned} \tag{188}$$

and on $H^{*,*}$ given by

$$\begin{aligned} &\underbrace{\pi_{-p-q}(K(\mathbb{Z}(\tfrac{-q}{2}), -q)^X)}_{\cong H^{p,-\frac{q}{2}}(X)} \times \underbrace{\pi_{-s-t}(K(\mathbb{Z}(\tfrac{-t}{2}), -t)^X)}_{\cong H^{s,-\frac{t}{2}}(X)} \\ &\longrightarrow \underbrace{\pi_{-q-p-s-t}(K(\mathbb{Z}(\tfrac{-s-t}{2}), -s-t)^X)}_{\cong H^{-p-q,-\frac{s-t}{2}}(X)} \end{aligned} \tag{189}$$

can readily be identified.

Proposition 16.10. *The mulitiplicative structures on $E_2^{*,*}(X)$ and $H^{*,*}(X, \mathbb{Z})$ defined above are globally isomorphic, possibly up to a sign, i.e. the diagram*

$$\begin{tikzcd} E_2^{p,q}(X) \wedge E_2^{s,t}(X) \arrow[r] \arrow[d, "\cong"] & E_2^{p+s,q+t}(X) \arrow[d, "\cong"] \\ H^{p,-\frac{q}{2}}(X) \wedge H^{s,-\frac{t}{2}}(x) \arrow[r] & H^{p+s,-\frac{q+t}{2}}(X) \end{tikzcd} \tag{190}$$

commutes up to a sign.

Remark 16.11. *There are some reasons to take other sign conventions for the product on $H^{*,*}$ (cf. [Dug03]). Moreover, we have to switch spheres, when we compute the pairings above. This introduces another sign, depending on our conventions of orientation. We didn't discuss sign issues for the cohomology theory, hence we can not be too precise about signs in the spectral sequence. In our sitation this should not bother us, since in the spectral sequences to be dealt with, the products will interest us only when the occuring groups are $\mathbb{Z}/2$. Here, signs don't matter.*

17 Connective KR-theory

Stabilizing the slice spectral sequence for KR-theory is not hard. Recall from Section 16.1 that we defined W_n to be the homotopy fiber of $\mathbb{Z} \times BU \to P_{2n-2}(\mathbb{Z} \times BU)$. The main step is the following proposition (cf. [Dug05], Proposition 6.1.).

Proposition 17.1. *There are weak equivalences $W_n \to \Omega^{2,1}W_{n+1}$, unique up to homotopy, which commute with the Bott map in the following diagram:*

$$\begin{tikzcd} W_n \arrow[r] \arrow[d] & \Omega^{2,1}W_{n+1} \arrow[d] \\ \mathbb{Z} \times BU \arrow[r] & \Omega^{2,1}(\mathbb{Z} \times BU) \end{tikzcd} \tag{191}$$

Proof. Recall that a space Z is called $\mathcal{A}$-null, if it has the property that the maps $[*, Z] \to [\Sigma^n A, Z]$ are isomorphisms for $n \geq 0$ and all $A \in \mathcal{A}$. By construction, $P_{2n}(\mathbb{Z} \times BU)$ is $\mathcal{A}_{2n,n}$-null for $\mathcal{A}_V = \{S^W \wedge G/H_+ : W \supset V,\ H \leq G\}$. Hence, using the adjunction of $\Omega^{2,1}$ and $\Sigma^{2,1}$ we conclude that $\Omega^{2,1}P_{2n}$ is $\mathcal{A}_{2n-2,n-1}$-null. Thus there is a lift (unique up to homotopy) of the form

$$\begin{tikzcd} \Omega^{2,1}(\mathbb{Z} \times BU) \arrow[r] \arrow[d] & \Omega^{2,1}P_{2n}(\mathbb{Z} \times BU) \\ P_{2n-2}(\Omega^{2,1}(\mathbb{Z} \times BU)) \arrow[ur, dotted, "j"] & \end{tikzcd} \tag{192}$$

We obtain the horizontal map by applying $\Omega^{2,1}$ to the natural map $\mathbb{Z} \times BU \to P_{2n}(\mathbb{Z} \times BU)$. For the vertical map we apply $\Omega^{2,1}$ to $\mathbb{Z} \times BU$ first and then take the natural map induced by the Postnikov section functor.

Let $\beta : \mathbb{Z} \times BU \to \Omega^{2,1}(\mathbb{Z} \times BU)$ be the Bott map. We consider the bigger diagram

$$\begin{tikzcd} W_n \arrow[r] \arrow[dd, dotted] & \mathbb{Z} \times BU \arrow[r] \arrow[d, "\beta"] & P_{2n-2}(\mathbb{Z} \times BU) \arrow[d, "P_{2n-2}(\beta)"] \\ & \Omega^{2,1}(\mathbb{Z} \times BU) \arrow[r] \arrow[d, "id"] & P_{2n-2}(\Omega^{2,1}(\mathbb{Z} \times BU)) \arrow[d, "j"] \\ \Omega^{2,1}W_{n+1} \arrow[r] & \Omega^{2,1}(\mathbb{Z} \times BU) \arrow[r] & \Omega^{2,1}P_{2n}(\mathbb{Z} \times BU) \end{tikzcd} \tag{193}$$

In order to show that the induced map is a weak equivalence, we use Lemma 7.12. For the special case that $G = \mathbb{Z}/2$ and $V = (2n, n)$ the lemma may be reformulated as follows (cf. [Dug05], Proof of Proposition 6.1.)

Lemma 17.2. *Let X and Y be pointed $\mathbb{Z}/2$-spaces with the properties that*

(1) $[S^{k,0}, X]_* = [S^{k,0}, Y]_* = 0$ *for* $0 \leq k < n$ *and*

(2) $[\mathbb{Z}/2_+ \wedge S^{k,0}, X]_* = [\mathbb{Z}/2_+ \wedge S^{k,0}, Y]_* = 0$ *for* $0 \leq k < 2n$

Then a map $X \to Y$ is a weak equivalence iff it induces isomorphisms

(1) $[S^{2n+k,n}, X]_* \xrightarrow{\cong} [S^{2n+k,n}, Y]_*$ *and*

(2) $[\mathbb{Z}/2_+ \wedge S^{2n+k,n}, X]_* \xrightarrow{\cong} [\mathbb{Z}/2_+ S^{2n+k,n}, Y]_*$

for every $k \geq 0$.

We take the long exact sequences on homotopy groups and use the Propositions 9.5 and 16.5 to obtain

- $[S^{k,0}, W_n]_* = 0$ for $0 \leq k < n$,
- $[\mathbb{Z}/2_+ \wedge S^{k,0}, W_n]_* = 0$ for $0 \leq k < 2n$ and
- the same identities hold if W_n is replaced by $\Omega^{2,1} W_n$.

By the same argument and the fact that the higher homotopy groups of $P_{2n-2}(\mathbb{Z} \times BU)$ are trivial, we obtain the isomorphisms

- $[S^{2n+k,n}, W_n]_* \to [S^{2n+k,n}, \mathbb{Z} \times BU]_*$ and
- $[\mathbb{Z}/2_+ \wedge S^{2n+k,n}, W_n]_* = [\mathbb{Z}/2_+ \wedge S^{2n+k,n}, \mathbb{Z} \times BU]_*$ for $0 \leq k$.

Finally, the square

$$\begin{array}{ccc} W_n & \longrightarrow & \Omega^{2,1} W_{n+1} \\ \downarrow & & \downarrow \\ \mathbb{Z} \times BU & \xrightarrow[\cong]{\beta} & \Omega^{2,1}(\mathbb{Z} \times BU) \end{array} \qquad (194)$$

implies that $W_n \to \Omega^{2,1} W_{n+1}$ induces an isomorphism on $[S^{2n+k,n}, -]_*$ and $[\mathbb{Z}/2_+ \wedge S^{2n+k,n}, -]_*$ for $k \geq 0$. By Lemma 17.2, $W_n \to \Omega^{2,1} W_{n+1}$ is a weak equivalence. This finishes the proof. $\square$

Definition 17.3. Let kr be the equivariant spectrum consisting of the spaces $\{W_n\}$ and the maps $W_n \to \Omega^{2,1} W_{n+1}$. It is called the connective KR-spectrum.

We may form the tower of homotopy cofiber sequences

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \Sigma^{2,1} kr & \xrightarrow{\beta} & kr & \xrightarrow{\Sigma^{-2,-1}\beta} & \Sigma^{-2,-1} kr & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \Sigma^{2,1} H\underline{\mathbb{Z}} & & H\underline{\mathbb{Z}} & & \Sigma^{-2,-1} H\underline{\mathbb{Z}} & & \end{array} \qquad (195)$$

This gives the stable version we were looking for. For a G-space X we have $H^{p,-\frac{q}{2}}(X) \Rightarrow KR^{p+q,0}(X)$.

18 Connection to KO

Since we know that $KR^{p,0}(X) \cong KO^p(X)$ holds for a $\mathbb{Z}/2$-space X with trivial action, we can use the slice spectral sequence to compute real K-theory.

Proposition 18.1. *(cf. [Dug05], Corollary 2.12.) The set of fixed points $K(\mathbb{Z}(n), 2n)^{\mathbb{Z}/2}$ has the homotopy type of*

$$K(\mathbb{Z}, 2n) \times K(\mathbb{Z}/2, 2n-2) \times K(\mathbb{Z}/2, 2n-4) \times \cdots \times K(\mathbb{Z}/2, n) \qquad \begin{pmatrix} n \geq 0 \\ n \text{ even} \end{pmatrix},$$

$$K(\mathbb{Z}/2, 2n-1) \times K(\mathbb{Z}/2, 2n-3) \times \cdots \times K(\mathbb{Z}/2, n) \qquad \begin{pmatrix} n \geq 0 \\ n \text{ odd} \end{pmatrix},$$

$$K(\mathbb{Z}, 2n) \times K(\mathbb{Z}/2, 2n+1) \times K(\mathbb{Z}/2, 2n+3) \times \cdots \times K(\mathbb{Z}/2, n-3) \qquad \begin{pmatrix} n < 0 \\ n \text{ even} \end{pmatrix},$$

$$K(\mathbb{Z}/2, 2n) \times K(\mathbb{Z}/2, 2n+2) \times \cdots \times K(\mathbb{Z}/2, n-3) \qquad \begin{pmatrix} n < 0 \\ n \text{ odd} \end{pmatrix}.$$

This Proposition is a consequence of the Dold-Thom theorem:

Theorem 18.2. *Let X be a (non-equivariant) pointed CW-complex. Then the functor $X \to \pi_i SP^\infty(X)$ coincides with the functor $X \to H_i(X;\mathbb{Z})$ for $i \geq 1$.*

Proof. [Hat03] Theorem 4K.6. □

Corollary 18.3. *A path-connected, commutative, associative H-space with a strict identity element has the weak homotopy type of a product of Eilenberg-MacLane spaces.*

Proof. For a full proof see [Hat03] Corollary 4K.7. Let $M(G, n)$ denote a Moore space. There exist based maps $M(\pi_n(X), n) \to X$ which induce isomorphisms on π_n for $n > 1$ and on H_1. Now the idea is to use the H-space structure to extend the map $\bigvee_n M(\pi_n(X), n) \to X$ to an isomorphism $SP^\infty(\bigvee_n M(G_n, n)) \to X$. $SP^\infty(\bigvee_n M(\pi_n(X), n))$ can be identified with $\Pi_n SP^\infty(M(\pi_n(X), n))$ and, from the Dold-Thom Theorem, it follows that $SP^\infty(M(G_n, n))$ is a $K(\pi_n(X), n)$. □

Proof of Proposition 18.1. $K(\mathbb{Z}(n), 2n)$ and thus $K(\mathbb{Z}(n), 2n)^{\mathbb{Z}/2}$ are abelian groups. Since

$$\pi_i(K(\mathbb{Z}(n), 2n)^{\mathbb{Z}/2}) \cong [S^i, K(\mathbb{Z}(n), 2n)^{\mathbb{Z}/2}]^e \cong [S^i, K(\mathbb{Z}(n), 2n)] \qquad (196)$$

we can read off the homotopy groups from $H^{*,*}(*; \underline{\mathbb{Z}})$ (cf. Appendix A). □

Corollary 18.4. *If $\mathbb{Z}/2$ acts trivially on X, i.e. $X = X^{\mathbb{Z}/2}$, we can express the E_2-term of the slice spectral sequence by*

$$E_2^{p,q} \cong \begin{cases} 0 & q \ odd \\ H^p(X,\mathbb{Z}) \oplus \bigoplus_{i=1}^{-\frac{q}{4}} H^{p-2i}(X,\mathbb{Z}/2) & \begin{array}{c} q \equiv 0 \ mod \ 4 \\ q < 0 \end{array} \\ \bigoplus_{i=0}^{-\frac{q}{4}-\frac{1}{2}} H^{p-(2i+1)}(X,\mathbb{Z}/2) & \begin{array}{c} q \equiv 2 \ mod \ 4 \\ q < 0 \end{array} \\ H^p(X,\mathbb{Z}) \oplus \bigoplus_{i=0}^{\frac{q}{4}-2} H^{p+2i+1}(X,\mathbb{Z}/2) & \begin{array}{c} q \equiv 0 \ mod \ 4 \\ q \geq 0 \end{array} \\ \bigoplus_{i=0}^{\frac{q}{4}-\frac{3}{2}} H^{p+(2i)}(X,\mathbb{Z}/2) & \begin{array}{c} q \equiv 2 \ mod \ 4 \\ q \geq 0 \end{array} \end{cases} \tag{197}$$

Proof. Recall from Section 16, equation (174), that we identified

$$E_2^{p,q} \cong H^{p,-\frac{q}{2}}(X,\underline{\mathbb{Z}}) \cong [X \wedge S^{-(p+q)}, K(\mathbb{Z}(-\frac{q}{2}),-q)]. \tag{198}$$

Since $X \wedge S^{-(p+q)}$ has a trivial action of $\mathbb{Z}/2$, we obtain

$$[X \wedge S^{-(p+q)}, Y] \cong [X \wedge S^{-(p+q)}, Y^{\mathbb{Z}/2}]^e \tag{199}$$

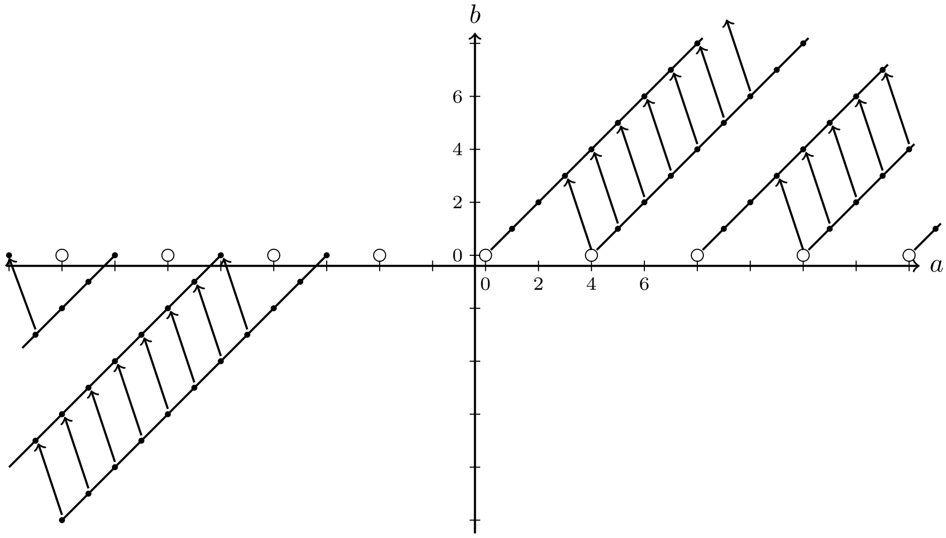
for any space Y. Proposition 18.1 then yields:

$$
\begin{aligned}
E_2^{p,q} &\cong [X \wedge S^{-(p+q)}, K(\mathbb{Z}(-\frac{q}{2}), -q)] \\
&\cong [X \wedge S^{-(p+q)}, K(\mathbb{Z}(-\frac{q}{2}), -q)^{\mathbb{Z}/2}] \\
&\cong \begin{cases}
0 & \text{q odd} \\
\begin{array}{c}[X \wedge S^{-(p+q)}, K(\mathbb{Z}, -q) \times K(\mathbb{Z}/2, -q-2) \\ \times K(\mathbb{Z}/2, -q-4) \times \cdots \times K(\mathbb{Z}/2, -\frac{q}{2})]^e\end{array} & \begin{array}{c} q \equiv 0 \bmod 4 \\ q < 0 \end{array} \\
\begin{array}{c}[X \wedge S^{-(p+q)}, K(\mathbb{Z}/2, -q-1) \\ \times K(\mathbb{Z}/2, -q-3) \times \cdots \times K(\mathbb{Z}/2, -\frac{q}{2})]^e\end{array} & \begin{array}{c} q \equiv 2 \bmod 4 \\ q < 0 \end{array} \\
\begin{array}{c}[X \wedge S^{-(p+q)}, K(\mathbb{Z}, -q) \times K(\mathbb{Z}/2, -q+1) \\ \times K(\mathbb{Z}/2, -q+3) \times \cdots \times K(\mathbb{Z}/2, -\frac{q}{2}-3)]^e\end{array} & \begin{array}{c} q \equiv 0 \bmod 4 \\ q \geq 0 \end{array} \\
\begin{array}{c}[X \wedge S^{-(p+q)}, K(\mathbb{Z}/2, -q) \\ K(\mathbb{Z}/2, -q+2) \times \cdots \times K(\mathbb{Z}/2, -\frac{q}{2}-3)]^e\end{array} & \begin{array}{c} q \equiv 2 \bmod 4 \\ q \geq 0 \end{array}
\end{cases} \qquad (200) \\
&\cong \begin{cases}
0 & \text{q odd} \\
H^p(X, \mathbb{Z}) \oplus \bigoplus_{i=1}^{-\frac{q}{4}} H^{p-2i}(X, \mathbb{Z}/2) & \begin{array}{c} q \equiv 0 \bmod 4 \\ q < 0 \end{array} \\
\bigoplus_{i=0}^{-\frac{q}{4}-\frac{1}{2}} H^{p-(2i+1)}(X, \mathbb{Z}/2) & \begin{array}{c} q \equiv 2 \bmod 4 \\ q < 0 \end{array} \\
H^p(X, \mathbb{Z}) \oplus \bigoplus_{i=0}^{\frac{q}{4}-2} H^{p+2i+1}(X, \mathbb{Z}/2) & \begin{array}{c} q \equiv 0 \bmod 4 \\ q \geq 0 \end{array} \\
\bigoplus_{i=0}^{\frac{q}{4}-\frac{3}{2}} H^{p+(2i)}(X, \mathbb{Z}/2) & \begin{array}{c} q \equiv 2 \bmod 4 \\ q \geq 0 \end{array}
\end{cases}.
\end{aligned}
$$

□

The following picture shows the spectral sequence for the special case $X = *$.

Remark 18.5. *Note that we don't actually use Corollary 18.4 to compute the spectral sequence for $X = *$, but it is the other way round. We used the knowledge of $H^{p,q}(*)$ obtained in Appendix A to compute the homotopy type of the set of fixed points $K(\mathbb{Z}(n), 2n)^{\mathbb{Z}/2}$ and thus to prove Proposition 18.1.*



The hollow circles denote $\mathbb{Z}$, while the solid dots represent $\mathbb{Z}/2$. The indexing is designed such that the a-axis measures $-p-q$, while the b-axis measures p. Observe that we didn't draw all possible differentials, but only the non trivial ones.
Recall the definition of the differentials on the E_2-term:

$$\begin{aligned}
H^{p,-\frac{q}{2}}(*) \cong & [S^0, K(\mathbb{Z}(-\frac{q}{2}), p)] \\
\cong & [S^{-p-q}, K(\mathbb{Z}(-\frac{q}{2}), -q)] \\
\cong & \pi_{-p-q+1}(P_{-q-2}(\mathbb{Z} \times BU), P_{-q}(\mathbb{Z} \times BU)) \\
\to & \pi_{-p-q}(P_{-q}(\mathbb{Z} \times BU)) \\
\to & \pi_{-p-q}(P_{-q}(\mathbb{Z} \times BU), P_{-q+2}(\mathbb{Z} \times BU)) \\
\cong & \pi_{-p-q-1}(K(\mathbb{Z}(\frac{-q+2}{2}), -q+2)) \\
\cong & [S^0, K(\mathbb{Z}(\frac{-q+2}{2}), p+3)] \cong H^{p+3,-\frac{q}{2}+1}(*)
\end{aligned} \tag{201}$$

Thanks to the rich multiplicative structure (and by degree reasons) we only need to compute the differentials for $x \in H^{0,2}(*)$ and $\theta y^{-3} \in H^{-3,-6}(*)$ (cf. Appendix A for notation).
In order to determine the image of x, consider the following diagram, where we use the abbreviation $P_n = P_n(\mathbb{Z} \times BU)$. The rows and columns are given by the long exact homotopy sequences, and we are interested in the dashed map

$$\mathbb{Z} \cong H^{0,2}(*) \cong \pi_5(P_2, P_4) \xrightarrow{d} \pi_4(P_4, P_6) \cong H^{3,3}(*) \cong \mathbb{Z}/2. \tag{202}$$

For the isomorphisms $H^{0,2}(*) \cong \pi_5(P_2, P_4)$ and $\pi_4(P_4, P_6) \cong H^{3,3}(*)$, see equation (175).

$$
\begin{tikzcd}
 & \vdots \arrow[d] & & & \vdots \arrow[d] & \\
 & \pi_6(P_0,P_2) \arrow[d] & & \vdots \arrow[d] & \pi_5(P_0,P_2) \arrow[d] & \\
\cdots \arrow[r] & \pi_5(P_2) \arrow[r] \arrow[d] & \pi_5(P_2,P_4) \arrow[r] \arrow[dr, dotted, "d"] & \pi_4(P_4) \arrow[r] \arrow[d] & \pi_4(P_2) \arrow[r] \arrow[d] & \cdots \\
 & \pi_5(P_0) \arrow[d] & & \pi_4(P_4,P_6) \arrow[d] & \pi_4(P_0) \arrow[d] & \\
 & \vdots & & \pi_3(P_6) \arrow[d] & \vdots & \\
 & & & \vdots & &
\end{tikzcd}
\tag{203}
$$

Several groups are zero:

- $\pi_6(P_0,P_2) \cong H^{-3,1}(*) = 0$ and $\pi_5(P_0,P_2) \cong H^{-2,1}(*) = 0$ (cf. equation (175)),
- $\pi_5(P_0) = \pi_4(P_0) = 0$, since $P_0 = \mathbb{Z}$, and hence
- $\pi_5(P_2) = \pi_4(P_2) = 0$, since the sequences are exact.
- Finally, since $dim(3,0)^H = 3 \leq dim(6,3)^H$, part (2) of Proposition 9.5 implies that $\pi_3(P_6) \cong \pi_3(\mathbb{Z} \times BU) = 0$.

It follows that d is the composition of an isomorphism and a surjective map, and hence d is surjective itself.
To compute the differential $d : H^{-3,-6}(*) \to H^{0,-5}(*)$ we have to use a suitable suspension ismorphism to get a valid expression.

Remark 18.6. *Note that we can make sense of the notion of relative homotopy groups for spheres of the form S^V. For $f : A \to B$, an element of $\pi_V(B,A)$ is represented by a diagram*

$$
\begin{tikzcd}
S(V) \arrow[r] \arrow[d] & A \arrow[d] \\
D(V) \arrow[r] & B
\end{tikzcd}
\tag{204}
$$

If $V \supseteq 1$, the isomorphism $\pi_V(B,A) \cong \pi_{V-1}(hofib(f))$ still holds. The differential is then given by

$$
\begin{aligned}
H^{-3,-6}(*) &\cong \tilde{H}^{8,4}(S^{11,10}) &&\cong \pi_{11,10}(K(\mathbb{Z}(4),8)) \\
&\cong \pi_{12,10}(P_6,P_8) &&\to \pi_{11,10}(P_8) \\
&\to \pi_{11,10}(P_8,P_{10}) &&\cong \pi_{10,10}(K(\mathbb{Z}(5),10)) \\
&\cong \tilde{H}^{10,5}(S^{10,10}) &&\cong H^{0,-5}(*)
\end{aligned} \tag{205}
$$

(cf. equation (175)).

Consider the diagram, where the dashed map indicates the differential. Again, all rows and columns are exact.

$$
\begin{tikzcd}
\vdots \arrow[d] & & & \vdots \arrow[d] & & \\
\pi_{13,10}(P_4,P_6) \arrow[d] & \cdots \arrow[r] & \pi_{11,10}(P_{12}) \arrow[r] & \pi_{11,10}(P_{10}) \arrow[r] \arrow[d] & \pi_{11,10}(P_{10},P_{12}) \arrow[r] & \cdots \\
\pi_{12,10}(P_6) \arrow[rr] \arrow[d] & & \pi_{12,10}(P_6,P_8) \arrow[r] \arrow[dr, dotted] & \pi_{11,10}(P_8) \arrow[rr] \arrow[d] & & \cdots \\
\pi_{12,10}(P_4) \arrow[d] & & & \pi_{11,10}(P_8,P_{10}) \arrow[d] & & \\
\vdots & & & \pi_{10,10}(P_{10}) \arrow[d] & & \\
& & & \vdots & &
\end{tikzcd} \tag{206}
$$

The following groups are zero:

- $\pi_{11,10}(P_{10},P_{12}) \cong H^{2,-4}(*) = 0$ and $\pi_{13,10}(P_4,P_6) \cong H^{-6,-7}(*) = 0$ (cf. equation (175)),
- Part (2) of Proposition 9.5 implies

  $$
  \begin{aligned}
  \pi_{11,10}(P_{12}) &= \pi_{11,10}(\mathbb{Z}\times BU) \cong KR^{-11,-10}(*) \cong KR^{9,0}(*) = 0, \text{and} \\
  \pi_{10,10}(P_{10}) &= \pi_{10,10}(\mathbb{Z}\times BU) \cong KR^{-10,-10}(*) \cong KR^{10,0}(*) = 0,
  \end{aligned} \tag{207}
  $$

  since $dim(11,10)^H \leq dim(12,6)^H$ and $dim(10,10)^H \leq dim(10,5)^H$.
- $\pi_{12,10}(P_4) = 0$ because P_4 is $\mathcal{A}_{(4,2)}$-null, and
- $\pi_{11,10}(P_{10}) = \pi_{12,10}(P_6) = 0$ by exactness.

Hence d is the composition $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/2 \xrightarrow{\cong} \mathbb{Z}/2$, which is an isomorphism.

Part VI

The slice spectral sequence for projective spaces

19 Computation of the KO groups of real projective spaces

We can use the slice spectral sequence to compute the real K-theory of the real projective spaces. This has been done before, but our techniques are different. The notation of the following theorem is taken from Fujii [Fuj67]. But see also the papers cited there. While the method to obtain the filtrations is new, we solve the extension problems by methods due to Fujii [Fuj67] and Adams [Ada61].

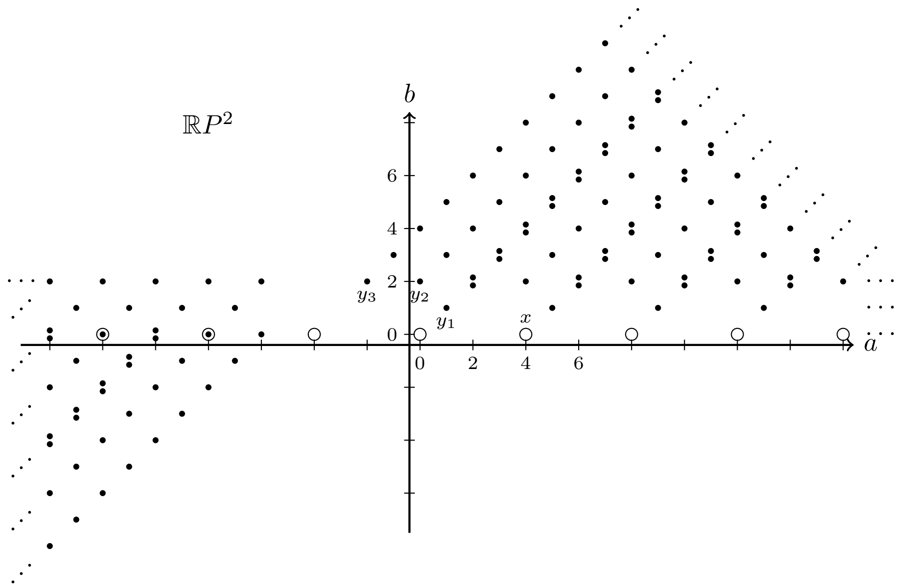
Theorem 19.1. *The groups* $\widetilde{KO}^{-i}(\mathbb{R}P^n)$ *are isomorphic to*

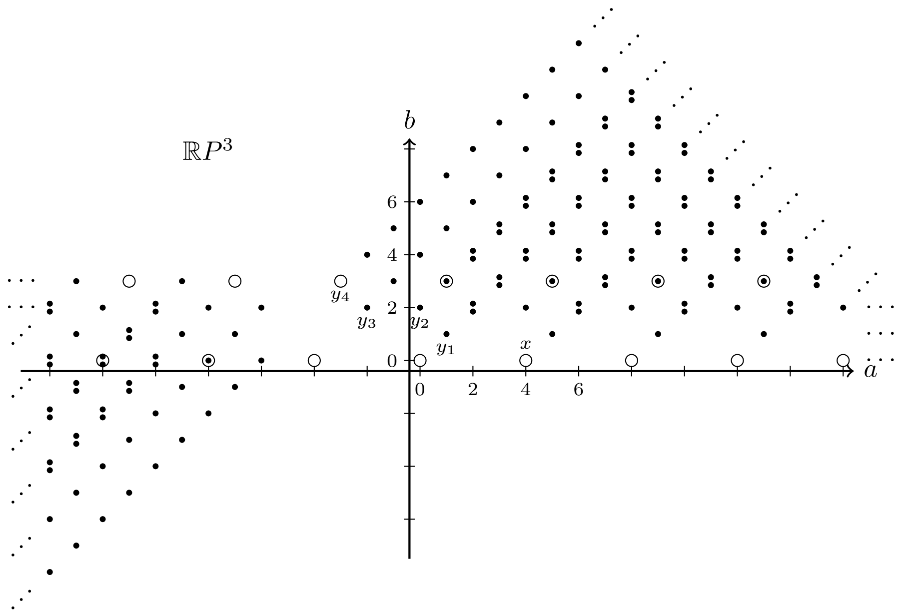
$i\backslash n$	$8r$	$8r+1$	$8r+2$	$8r+3$
0	(2^{4r})	(2^{4r+1})	(2^{4r+2})	(2^{4r+2})
1	${}^{r\neq 0}(2)$	(2)	(2)	$(2)+(\infty)$
2	${}^{r\neq 0}(2)+(2)$	(2)	(2)	(2)
3	${}^{r\neq 0}(2)$	(∞)	0	0
4	(2^{4r})	(2^{4r})	(2^{4r})	(2^{4r})
5	0	0	0	(∞)
6	0	0	(2)	$(2)+(2)$
7	0	(∞)	(2)	$(2)+(2)$
$i\backslash n$	$8r+4$	$8r+5$	$8r+6$	$8r+7$
0	(2^{4r+3})	(2^{4r+3})	(2^{4r+3})	(2^{4r+3})
1	(2)	(2)	(2)	$(2)+(\infty)$
2	(2)	(2)	$(2)+(2)$	$(2)+(2)+(2)$
3	0	(∞)	(2)	$(2)+(2)$
4	(2^{4r+1})	(2^{4r+2})	(2^{4r+3})	(2^{4r+3})
5	0	0	0	(∞)
6	(2)	0	0	0
7	(2)	(∞)	0	0

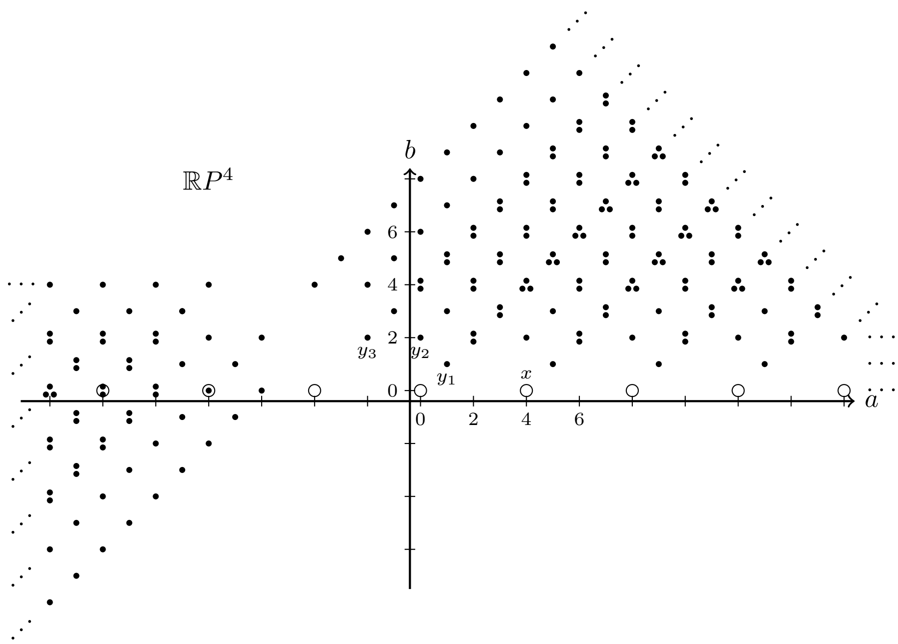
Here, (t) *denotes the cyclic group of order* t*.*

The proof will take most of the remainder of this section. In Theorem 16.6 we computed the slice spectral sequence for a space X, and in Corollary 18.4 we identified the E_2-term of this sequence with sums of non-equivariant singular cohomology groups, if X carries a trivial action of $\mathbb{Z}/2$. In this case $KO^i(X) = KR(X)^i$ and the slice spectral sequence computes real K-theory. As indicated in Section 15.1, the slice spectral sequence for the (finite) real projective spaces converges strongly, if $RE^\infty = 0$. This is obvious, since the differentials exiting or entering the groups $E_r^{p,q}$ for fixed p,q will become trivial eventually when

r increases. Indeed, the differential d_r has bidegree $(2r-1, -2r+2)$. In our indexing where a measures $-p-q$ and b measures p, this bidegree translates into $(2r-1, -1)$. For finite CW-complexes X and sufficiently large values of r the differentials are thus trivial by degree reasons.
The following pictures shows $E_2^{p,q}(\mathbb{R}P^n) \cong H^{p,-\frac{q}{2}}(\mathbb{R}P^n, \underline{\mathbb{Z}})$ for $2 \le n \le 4$. Note that we drew only a cutout and the picture extends to both sides and along the diagonal lines up to the right and down to the left.







Each dot represents a $\mathbb{Z}/2$-summand, and each hollow circle represents a $\mathbb{Z}$-summand.
The generators denoted by x and y_i play an important role. Each element in the first quadrant can be represented by a product of them. For example the group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ at $(3,3)$ has generators y_1^3 and xy_1y_3.
The y_i are the generators for the cohomology rings $H^*(\mathbb{R}P^n, \mathbb{Z})$ and $H^*(\mathbb{R}P^n, \mathbb{Z}/2)$ as well. These rings are the truncated polynomial rings

$$H^*(\mathbb{R}P^{2k}, \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(2\alpha, \alpha^{k+1}), \text{ where } |\alpha| = 2, \tag{208}$$

$$H^*(\mathbb{R}P^{2k+1}, \mathbb{Z}) \cong \mathbb{Z}[\alpha, \beta]/(2\alpha, \alpha^{k+1}, \beta^2, \alpha\beta), \text{ where } |\alpha| = 2,\ |\beta| = 2k+1, \text{ and} \tag{209}$$

$$H^*(\mathbb{R}P^n, \mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]/(\alpha^{n+1}), \text{ where } |\alpha| = 1. \tag{210}$$

For proofs see e.g. [Hat03] pages 212 and 214.

- $y_1 \in H^0(\mathbb{R}P^n, \mathbb{Z}/2)$ is the unit in $H^*(\mathbb{R}P^n, \mathbb{Z}/2)$,
- $y_2 \in H^1(\mathbb{R}P^n, \mathbb{Z}/2)$ is the generator of the cohomology ring with $\mathbb{Z}/2$-coefficients,
- $y_3 \in H^2(\mathbb{R}P^n, \mathbb{Z})$ and $y_4 \in H^3(\mathbb{R}P^n, \mathbb{Z})$ are the two generators of the cohomology ring with $\mathbb{Z}$-coefficients, and
- $x \in H^0(\mathbb{R}P^n, \mathbb{Z})$ is the unit in $H^*(\mathbb{R}P^n, \mathbb{Z})$.

To obtain the E_3-term we therefore only have to compute the value of the differential $d : E_2^{p,q} \to E_2^{p+3,q-2}$ on these generators. The multiplicative structure then determines the differentials on general elements via $d(ab) = d(a)b +$

$(-1)^p ad(b)$. In the indexing we chose for the pictures, the differentials have the degree $(-1,3)$.

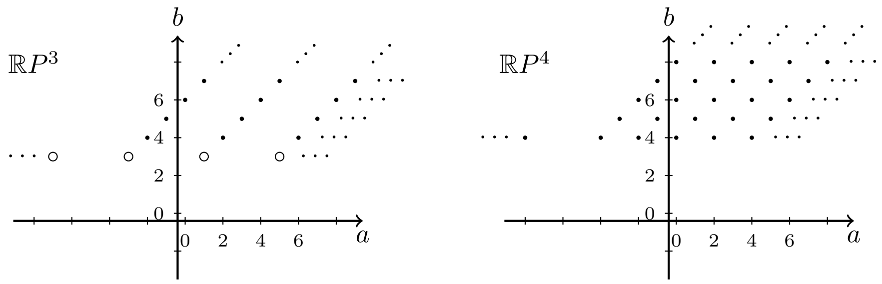
An important consideration is the following: The inclusions $i : \mathbb{R}P^{2n} \hookrightarrow \mathbb{R}P^{2n+1}$ and $j : \mathbb{R}P^{2n} \hookrightarrow \mathbb{R}P^{2n+2}$ induce maps on cohomology. If we look at the cellular chain complexes, we see that these maps are ismorphisms on the cohomology groups up to degree $2n$ and that the groups of degrees $2n+1$ and $2n+2$ are sent to zero. Hence the commutative diagrams

$$\begin{tikzcd} E_2^{p,q}(\mathbb{R}P^{2n+1}) \arrow[r, "d"] \arrow[d, "i^*"] & E_2^{p+3,q-2}(\mathbb{R}P^{2n+1}) \arrow[d, "i^*"] \\ E_2^{p,q}(\mathbb{R}P^{2n}) \arrow[r, "d"] & E_2^{p+3,q-2}(\mathbb{R}P^{2n}) \end{tikzcd} \tag{211}$$

and

$$\begin{tikzcd} E_2^{p,q}(\mathbb{R}P^{2n+2}) \arrow[r, "d"] \arrow[d, "j^*"] & E_2^{p+3,q-2}(\mathbb{R}P^{2n+2}) \arrow[d, "j^*"] \\ E_2^{p,q}(\mathbb{R}P^{2n}) \arrow[r, "d"] & E_2^{p+3,q-2}(\mathbb{R}P^{2n}) \end{tikzcd} \tag{212}$$

show that the differentials in the spectral sequences for $\mathbb{R}P^{2n}$ and $\mathbb{R}P^{2n+1}$ resp. $\mathbb{R}P^{2n+2}$ coincide, if i^* and j^* are isomorphisms.
Indeed, if we pass from one $\mathbb{R}P^{2n}$ to the projective space of the next higher (odd or even) degree, the spectral sequence only changes in dimensions $p \geq 2n+1$ or $p \geq 2n+2$, respectively. The additional elements are of the form $y_4 x^k y_1^l$ resp. $y_3^n x^k y_1^l$ and $y_3^{n-1} y_2 x^k y_1^l$. For the example $n = 1$, these are



(213)

Remark 19.2. *Our center of reference is always a $\mathbb{R}P^n$ with even n. This is simply due to notational reasons. If we would pass from $\mathbb{R}P^n$ to $\mathbb{R}P^{n+1}$, where n is odd we still would add some elements with a structure not very different from the elements in the case that n is even. However, the entries which are given by $\mathbb{Z}$ in dimension n would vanish. This would complexify our presentations, so we rather avoid this obstacle completely.*

The consideration above implies that we need to understand the d_2-differentials only for $X = \mathbb{R}P^n$ with small values of n. In particular it suffices to determine

- $d: \quad E_2^{0,-4}(*) \quad \to \quad E_2^{3,-6}(*)$ to compute $d(x)$,
- $d: \quad E_2^{1,-2}(\mathbb{R}P^2) \quad \to \quad E_2^{4,-4}(\mathbb{R}P^2)$ to compute $d(y_1)$,
- $d: \quad E_2^{2,-2}(\mathbb{R}P^3) \quad \to \quad E_2^{5,-4}(\mathbb{R}P^3)$ to compute $d(y_2)$,
- $d: \quad E_2^{2,0}(\mathbb{R}P^4) \quad \to \quad E_2^{5,-2}(\mathbb{R}P^4)$ to compute $d(y_3)$,

for all n (cf. Equation (175)). Then all other differentials are determined by the multiplicative structure as mentioned above.

Proposition 19.3. *The differential*

$$d : \mathbb{Z}/2 \cong E_2^{2,0}(\mathbb{R}P^4) \to E_2^{5,-2}(\mathbb{R}P^4) \cong \mathbb{Z}/2 \tag{214}$$

is an isomorphism.

Proof. By Equation (175) and Remark 18.6, the differential is given by the map

$$\begin{aligned} E^{2,0}(\mathbb{R}P^4) \cong \pi_{5,3}(P_4^{\mathbb{R}P^4}, P_6^{\mathbb{R}P^4}) &\to \pi_{4,3}(P_6^{\mathbb{R}P^4}) \\ &\to \pi_{4,3}(P_6^{\mathbb{R}P^4}, P_8^{\mathbb{R}P^4}) \cong E^{5,-2}(\mathbb{R}P^4) \end{aligned} \tag{215}$$

Consider the diagram, where the rows and columns are exact and d is given by the dashed map.

$$\begin{tikzcd}
 & & & \vdots \arrow[d] & \\
 & & & \pi_{5,3}(P_2^{\mathbb{R}P^4}, P_4^{\mathbb{R}P^4}) \arrow[d] & \\
\cdots \arrow[r] & \pi_{5,3}(P_4^{\mathbb{R}P^4}, P_6^{\mathbb{R}P^4}) \arrow[r] \arrow[dr, dotted] & \pi_{4,3}(P_6^{\mathbb{R}P^4}) \arrow[r] \arrow[d] & \pi_{4,3}(P_4^{\mathbb{R}P^4}) \arrow[r] \arrow[d] & \cdots \\
 & & \pi_{4,3}(P_6^{\mathbb{R}P^4}, P_8^{\mathbb{R}P^4}) \arrow[d] & \pi_{4,3}(P_2^{\mathbb{R}P^4}) \arrow[d] & \\
 & & \pi_{3,3}(P_8^{\mathbb{R}P^4}) \arrow[d] & \vdots & \\
 & & \vdots & &
\end{tikzcd} \tag{216}$$

Several groups are zero:

- $\pi_{5,3}(P_2^{\mathbb{R}P^4}, P_4^{\mathbb{R}P^4}) \cong E^{0,2}(\mathbb{R}P^4) = 0$ (cf. equation (175)).
- $\pi_{4,3}(P_2^{\mathbb{R}P^4}) \cong [S^{4,3} \wedge \mathbb{R}P^4, P_2] = 0$, since all cells of $S^{4,3} \wedge \mathbb{R}P^4$ are of the form $S^{4+i,3}$ for $0 \le i \le 4$ and P_2 is $\mathcal{A}_{4,2}$-null.
- Hence, $\pi_{4,3}(P_4^{\mathbb{R}P^4}) = 0$ by exactness.

- It is a lucky coincidence, that we already know that $\pi_{3,3}(P_8^{\mathbb{R}P^4}) = 0$: By part (2) of Proposition 9.5 we obtain

$$\pi_{3,3}(P_8^{\mathbb{R}P^4}) \cong \pi_{3,3}((\mathbb{Z} \times BU)^{\mathbb{R}P^4}) \cong KR^{-3,-3}(\mathbb{R}P^4) \cong KR^{3,0}(\mathbb{R}P^4). \tag{217}$$

  Although we are just about to compute the K-theory of $\mathbb{R}P^4$, we do know that $KR^{3,0}(\mathbb{R}P^4)$ is trivial in this very special case. All groups contributing to $KR^{3,0}(\mathbb{R}P^4)$ are located in the vertical line at $a = -5$ of the E_∞-term of the spectral sequence. But Picture 19 indicates, that the groups in this line of the E_2-term are trivial. Inevitably $KR^{3,0}(\mathbb{R}P^4) = 0$.

Consequently, $d : \mathbb{Z}/2 \to \mathbb{Z}/2$ is the composition of two surjective maps, and thus surjective itself. Clearly, d is even bijective. $\square$

We conclude that $d(y_4) = 0$ (by degree reasons), $d(x) = y_1^3$ (cf. Section 18) and $d(y_3) = y_3^2 y_1$. The computation of $d(y_1)$ and $d(y_2)$ is more subtle:

Proposition 19.4. *The differentials*

$$d : E_2^{1,-2}(\mathbb{R}P^2) \to E_2^{4,-4}(\mathbb{R}P^2) \tag{218}$$

and

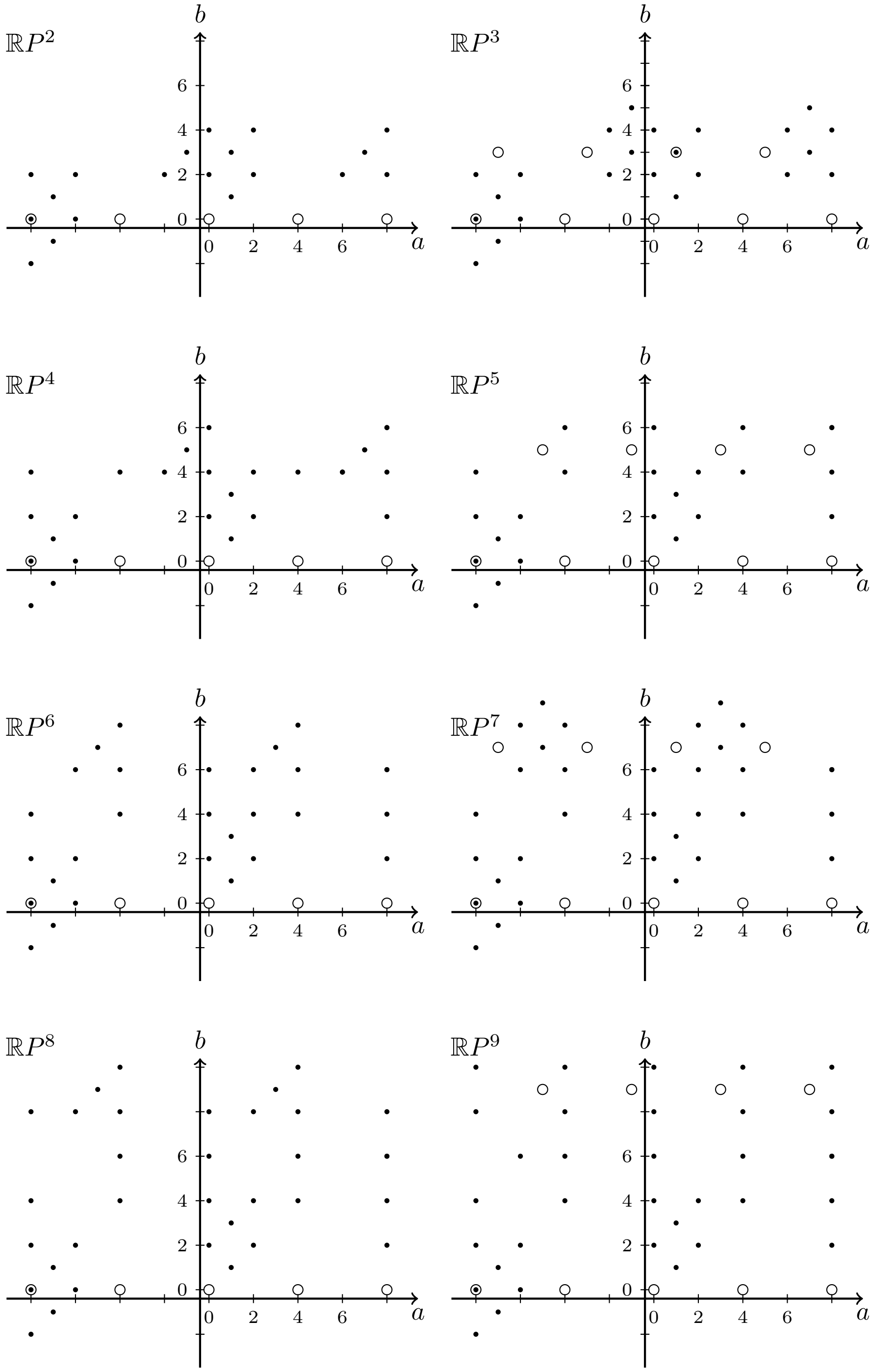
$$d : \mathbb{Z}/2 \cong E^{2,-2}(\mathbb{R}P^3) \to E^{5,-4}(\mathbb{R}P^3) \cong \mathbb{Z}/2 \tag{219}$$

are trivial.

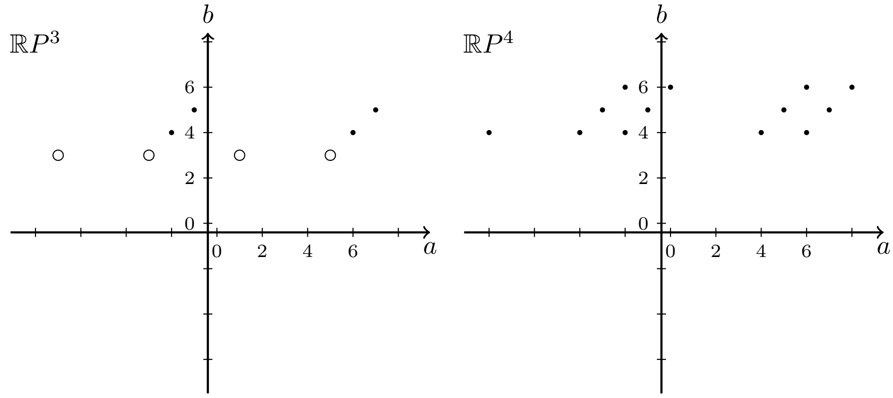
We postpone the proof of this proposition and first calculate the E_3-term, assuming the differentials were trivial. However, we can give some idea why this is true. We know that the spectral sequence converges strongly. The indices are designed in a way such that all elements contributing to $KO^{-i}(\mathbb{R}P^n)$ are located in the vertical line at $a = i$. Hence, in every sheet of the spectral sequence the entrances in the line at $a = 0$ gives an upper bound for the number of elements in $KO^0(\mathbb{R}P^n)$. From arguments independent of the spectral sequence, we will see that the number of elements of $\widetilde{KO}^0(\mathbb{R}P^n)$ is bounded below. If $d(y_1)$ or $d(y_2)$, on the other hand, were non-trivial they would decrease the upper bound below the lower bound, which would be a contradiction.

Remark 19.5. *The calculation of the differentials has the unexpected consequence that $y_2^2 = 0$. A naive guess might have been that $y_2^2 = y_3 y_1^2$, since the non-equivariant cohomology of $\mathbb{R}P^n$ is a truncated polynomial ring. Obviously $d(y_2^2) = 0$, but $d(y_3 y_1^2) = y_3^2 y_1^3 \neq 0$. So these elements can not be equal.*

We now turn to the computation of the E_3-term. The results for $2 \leq n \leq 9$ are indicated in the following pictures (which extend periodically to both sides):


$\mathbb{R}P^2$
$\mathbb{R}P^3$
$\mathbb{R}P^4$
$\mathbb{R}P^5$
$\mathbb{R}P^6$
$\mathbb{R}P^7$
$\mathbb{R}P^8$
$\mathbb{R}P^9$

We can first rule out a large number of elements in the E_2-term, which can't survive to the next page. When we pass from $\mathbb{R}P^{2n}$ to $\mathbb{R}P^{2n+1}$ or $\mathbb{R}P^{2n}$, respectively, the elements $y_4 x^k y_1^l$ resp. $y_3^n x^k y_1^l$ and $y_3^{n-1} y_2 x^k y_1^l$ are added to the E_2-term of the spectral sequence (cf. Picture (213) for the case $n = 1$).
By degree reasons, we have $d(y_4 y_1^l) = d(y_3^n y_1^l) = d(y_3^{n-1} y_2 y_1^l) = 0$. Indeed, these elements are located in the top two non-trivial diagonals of the spectral sequences and are sent to zero by the differentials, which go al least three dimensions up. Moreover, since $d(x^{2k}) = 0$ and $d(x^{2k+1}) = x^{2k} y_1^3$, only the following relevant groups can survive to the E_3-term:



(220)

The pictures for bigger n look just the same, only that they are shifted to the upper left. Note that we did *not* say that all of these groups actually survive to the E_3-term. Instead, we said that all *others* do *not*. We hence only have to examine a handful of remaining groups. The argument also implies that the E_3-term is 8-periodic in the first quadrant.
The elements we consider in the following cases are those we have not ruled out yet. Each case considers the generators of the groups located in the line at $a = i$. In the following we repeatedly use the index α, with values in $\mathbb{N}_+$ to examine all elements of one type at the same time. If we consider a fixed $\mathbb{R}P^n$, the generators will clearly become zero eventually, when α becomes bigger.

- $i = 0$
  The possibly nontrivial elements in the line at $a = 0$ of the E_3-term are $y_3^{4\alpha+1} x^{2\alpha} y_1^2$, $y_3^{4\alpha+1} x^{2\alpha} y_1^2 y_2$, $y_3^{4\alpha+4} x^{2\alpha+2}$ and $y_3^{4\alpha} x^{2\alpha} y_2$. While the first two kinds of elements are sent to $y_3^{4\alpha+2} x^{2\alpha} y_1^3$ and $y_3^{4\alpha+2} x^{2\alpha} y_1^3 y_2$, respectively, the latter two are mapped to zero. None of them is in the image of the differential.

- $i = 1$
  The interesting elements in the E_1-term are $y_3^{4\alpha} x^{2\alpha} y_1$ and $y_3^{4\alpha} x^{2\alpha} y_1 y_2$. All of them are mapped to zero, but for $\alpha \geq 1$ they are hit by $y_3^{4\alpha-1} x^{2\alpha}$ and $y_3^{4\alpha-1} x^{2\alpha} y_2$, respectively. For $n \equiv 3 \bmod 4$, there is an extra $\mathbb{Z}$-summand in the spectral sequence.

- $i = 2$
  Here, we have to consider four types of generators, namely $y_3^{4\alpha} x^{2\alpha} y_1^2$,

$y_3^{4\alpha}x^{2\alpha}y_1^2y_2$, $y_3^{4\alpha+3}x^{2\alpha+2}$ and $y_3^{4\alpha+3}x^{2\alpha+3}y_2$. The first two are mapped to zero, but are the images of $y_3^{4\alpha-1}x^{2\alpha}y_1$ and $y_3^{4\alpha-1}x^{2\alpha}y_1y_2$ if $\alpha \neq 0$. The other two are mapped to $y_3^{4\alpha+4}x^{2\alpha+2}y_1$ and $y_3^{4\alpha+4}x^{2\alpha+2}y_1y_2$, respectively.

- $i = 3$
  The important elements are $y_3^{4\alpha+3}x^{2\alpha+2}y_1$ and $y_3^{4\alpha+3}x^{2\alpha+2}y_1y_2$, which are mapped to $y_3^{4\alpha+4}x^{2\alpha+2}y_1^2$ and $y_3^{4\alpha+4}x^{2\alpha+2}y_1^2y_2$, respectively. So they don't contribute to the E_3-term, unless the images are zero by degree reasons. They are not in the image of d, so this criterion suffices. Hence we have one $\mathbb{Z}/2$-term for $n = 8r$ ($r \neq 0$) or $n = 8r + 6$ and two $\mathbb{Z}/2$-terms for $n = 8r + 7$. Again, there is an extra $\mathbb{Z}$-term for $n = 8r + 1$ or $n = 8r + 5$.

- $i = 4$
  We have to consider the elements $y_3^4\alpha + 3x^{2\alpha+2}y_1^2$, $y_3^4\alpha + 3x^{2\alpha+2}y_1^2y_2$, $y_3^{4\alpha+2}x^{2\alpha+2}$ and $y_3^{4\alpha+2}x^{2\alpha+2}y_2$. The first two are mapped to $y_3^{4\alpha+4}x^{2\alpha+2}y_1^3$ and $y_3^{4\alpha+4}x^{2\alpha+2}y_1^3y_2$, respectively. None of these elements are in the image of d.

- $i = 5$
  The elements to be considered are $y_3^{4\alpha+2}x^{2\alpha+2}y_1$ and $y_3^{4\alpha+2}x^{2\alpha+2}y_1y_2$. All of them are mapped to zero, but they are hit by the elements $y_3^{4\alpha+2}x^{2\alpha+2}y_1$ and $y_3^{4\alpha+2}x^{2\alpha+2}y_1y_2$, respectively. However, there is an extra $\mathbb{Z}$-term for $n = 4r + 3$.

- $i = 6$
  Consider $y_3^{4\alpha+1}x^{2\alpha+2}$, $y_3^{4\alpha+1}x^{2\alpha+2}y_2$, $y_3^{4\alpha+2}x^{2\alpha+2}y_1^2$ and $y_3^{4\alpha+2}x^{2\alpha+2}y_1^2y_2$. The last two are mapped to zero, but are in the image of $y_3^{4\alpha+1}x^{2\alpha+2}y_1$ and $y_3^{4\alpha+1}x^{2\alpha+2}y_1y_2$. The first two are mapped to $y_3^{4\alpha+2}x^{2\alpha+2}y_1$ and $y_3^{4\alpha+2}x^{2\alpha+2}y_1y_2$, respectively. These images can be zero by degree reasons. Hence, we have one $\mathbb{Z}/2$-term for $n = 8r + 2$ or $8r + 4$ and two $\mathbb{Z}/2$-terms for $n = 8r + 3$.

- $i = 7$
  This case is analogous to the case $i = 3$. Here, the important elements are $y_3^{4\alpha+1}x^{2\alpha+2}y_1$ and $y_3^{4\alpha+1}x^{2\alpha+2}y_1y_2$.

This fully determines the E_3-page of the spectral sequence. Note that apart from the lines at $a = 0$ and $a = 4$ there are only very few non-trivial entries. In fact we will see in Propostion 19.9 that the spectral sequence collapses after the E_3-term and we can look at the table of Theorem 19.1 to keep track of the number of non-trivial groups.
We are now able to prove Proposition 19.4.

Lemma 19.6. *The group $\widetilde{KO}^0(\mathbb{R}P^n)$ has a cyclic summand of order at least $\left[\frac{n}{2}\right]$.*

Proof. $\widetilde{KU}^0(S^{2n}) \cong \mathbb{Z}$ is generated by $(H-1)^n := \nu^n$, where H is the canoncial line bundle over S^2. Hence we can use the cofiber sequences $\mathbb{R}P^k \to \mathbb{R}P^{k+1} \to S^{k+1}$, to deduce inductively that the ν^i with $i \leq \left[\frac{n}{2}\right]$ generate $\widetilde{KU}^0(\mathbb{R}P^n)$. Indeed, the cofiber sequence induces a long exact sequence of the form

$$\cdots \to \widetilde{KU}^0(S^{k+1}) \to \widetilde{KU}^0(\mathbb{R}P^{k+1}) \to \widetilde{KU}^0(\mathbb{R}P^k) \to \cdots . \tag{221}$$

The induction basis is trivial, since $\widetilde{KU}^0(\mathbb{R}P^1) \cong \widetilde{KU}^0(S^1) \cong 0$. For even values of k, we have $\widetilde{KU}^0(S^{k+1}) = 0$ and thus, the generators of $\widetilde{KU}^0(\mathbb{R}P^k)$ also generate $\widetilde{KU}^0(\mathbb{R}P^{k+1})$. For odd values of k, on the other hand, $\widetilde{KU}^0(\mathbb{R}P^{k+1})$ is generated by both, the generators of $\widetilde{KU}^0(\mathbb{R}P^k)$ and the single generator $\nu^{\frac{k+1}{2}}$ of $\widetilde{KU}^0(S^{k+1})$.
If $\lambda = \xi - 1$ with ξ denoting the canonical real line bundle, and if $\epsilon : \widetilde{KO}(\mathbb{R}P^n) \to \widetilde{KU}(\mathbb{R}P^n)$ is complexification, we have $\epsilon\lambda = \nu$. Hence, the λ^i are non-trivial generators of $\widetilde{KO}(\mathbb{R}P^n)$. The real line bundle ξ is equivalent to a bundle with structure group $O_1 = \{1, -1\}$, and it is obvious that $\xi \otimes \xi$ is trivial. In other words, $\lambda^2 = -2\lambda$. This shows that $\widetilde{KO}(\mathbb{R}P^n)$ has a cyclic summand of order at least $\left[\frac{n}{2}\right]$. □

Lemma 19.7. *The group $\widetilde{KO}^0(\mathbb{R}P^n)$ is as indicated in Theorem 19.1.*

Proof. Lemma 19.6 gives a lower bound for the number of elements in the group $\widetilde{KO}^0(\mathbb{R}P^n)$. On the other hand, the line at $i = 0$ of the spectral sequence gives an upper bound for the number of elements in $\widetilde{KO}^0(\mathbb{R}P^n)$. For $n \equiv 6, 7$ or 8 mod 8 this suffices to conclude that the group is as indicated in the theorem. From the long exact sequences for $\mathbb{R}P^k \to \mathbb{R}P^{k+1} \to S^{k+1}$ we finally obtain that each $\widetilde{KO}(\mathbb{R}P^n)$ must be cyclic. Indeed, to compute the $\widetilde{KO}(\mathbb{R}P^n)$ for arbitrary n, we just have to choose an $m > n$ with $m \equiv 6, 7$ or 8 mod 8 and inductively use the long exact sequence to show that the groups for $m, m-1, \ldots, n$ are cyclic of the desired order. □

Proof of Proposition 19.4. We have seen that the lower boundary and the upper boundary for the number of elements in $\widetilde{KO}^0(\mathbb{R}P^n)$ coincide, if $d(y_1) = d(y_2) = 0$. If these differentials were non-trivial, the number of entries in the line at $a = 0$ of the E_3-term would decrease, which would be a contradiction. □

Lemma 19.8. *The group $\widetilde{KO^{-4}}(\mathbb{R}P^n)$ is as indicated in Theorem 19.1.*

Proof. This case is analogous to the case $i = 0$. Instead of the sole complexification map ϵ we consider $I^{-2}\epsilon : \widetilde{KO}^{-4}(\mathbb{R}P^n) \to \widetilde{KU}^0(\mathbb{R}P^n)$, where I is the Bott-isomorphism. This allows us to deduce that $\widetilde{KO}^{-4}(\mathbb{R}P^n)$ is cyclic for $n \equiv 2, 3$ or 4 mod 8 and has the indicated number of elements. Inductively this follows for all n. □

Proposition 19.9. *The slice spectral sequence for $\mathbb{R}P^n$ collapses after the E_3-term.*

Proof. The argument is not very different from the one in the last proofs. The differentials $d : E_3^{p,q} \to E_3^{p+5,q-4}$ are trivial for degree reasons, apart from those entering or exiting the lines at $a = 0$ and $a = 4$. Indeed, as indicated before, there are very few non-trivial entries left and we can check this directly for each of them. At the lines $a = 0$ and $a = 4$, on the other hand, every non-trivial differential - both, entering and exiting - would decrease the number of elements in $\widetilde{KO}^0(\mathbb{R}P^n)$ or $\widetilde{KO^{-4}}(\mathbb{R}P^n)$, respectively, which would be a contradiction. □

It remains to solve the residual extension problems.

- $i = 1$

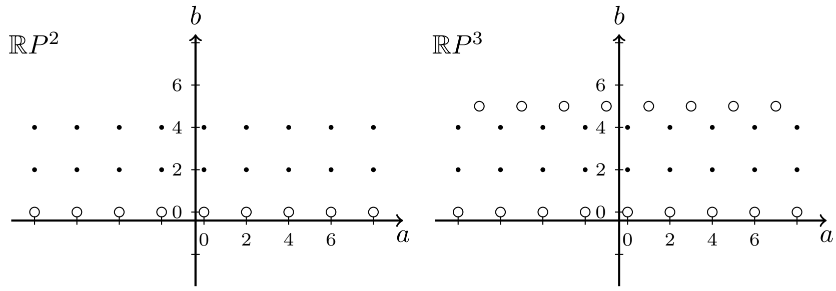
  We have

$$\widetilde{KO}^{-1}(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}/2 & \text{if } n \neq 4r+3 \\ \mathbb{Z}/2 + \mathbb{Z} \text{ or } \mathbb{Z}/2 & \text{if } n = 4r+3 \end{cases} \tag{222}$$

To determine what happens for $n = 4r+3$, we look at the diagram

$$\begin{tikzcd} \widetilde{KO}^{-1}(S^{4r+3}) \arrow[r] \arrow[d, "\epsilon"] & \widetilde{KO}^{-1}(\mathbb{R}P^{4r+3}) \arrow[r] \arrow[d, "\epsilon"] & \widetilde{KO}^{-1}(\mathbb{R}P^{4r+2}) \arrow[d, "\epsilon"] \\ \widetilde{KU}^{-1}(S^{4r+3}) \arrow[r] & \widetilde{KU}^{-1}(\mathbb{R}P^{4r+3}) \arrow[r] & \widetilde{KU}^{-1}(\mathbb{R}P^{4r+2}) \end{tikzcd} \tag{223}$$

where ϵ is the complexification. The spectral sequence for $\mathbb{Z}/2 \wedge \mathbb{R}P^n$ converges to $KU(\mathbb{R}P^n)$. Since $H^{p,q}(\mathbb{Z}/2 \wedge \mathbb{R}P^n) \cong H^p_{sing}(\mathbb{R}P^n)$, the sequence collapses immediately. We draw again the pictures for the cases $n = 2, 3$.



We get analogous results for bigger n. Hence $\widetilde{KU}^k(\mathbb{R}P^{4r+3}) \cong \mathbb{Z}$ and $\widetilde{KU}^k(\mathbb{R}P^{4r+2}) = 0$ for odd values of k. For further purpose we also compute $\widetilde{KU}^k(\mathbb{R}P^n)$ for even values of k: If f is the integral part of $\frac{n}{2}$, $\widetilde{KU}^k(\mathbb{R}P^n)$ has 2^f elements. Since $I^k\epsilon : \widetilde{KO}^0(\mathbb{R}P^n) \to \widetilde{KU}^{2k}(\mathbb{R}P^n)$ is surjective, it has to be cyclic of order 2^f. It remains to determine the vertical maps. Recall from Toda [Tod63], p. 314, that there is an exact sequence

$$\cdots \to \widetilde{KO}^n(X) \xrightarrow{\epsilon} \widetilde{KU}^n(X) \to \widetilde{KO}^{n+2}(X) \to \widetilde{KO}^{n+1}(X) \to \cdots \tag{224}$$

which shows that the image of $\epsilon : \widetilde{KO}^0(S^{2q}) \to \widetilde{KU}^0(S^{2q})$ is $\mathbb{Z}$ if $q \equiv 0$ mod 4 and is $2\mathbb{Z}$ if $q \equiv 2$ mod 4. From the diagram

$$\begin{tikzcd} \widetilde{KO}^{-1}(\mathbb{R}P^{4r+3}) \arrow[r] \arrow[d, "\epsilon"] & \widetilde{KO}^0(S^{4r+4}) \arrow[r] \arrow[d, "\epsilon"] & \widetilde{KO}^0(\mathbb{R}P^{4r+4}) \arrow[r, "i^*"] \arrow[d, "\epsilon"] & \widetilde{KO}^0(\mathbb{R}P^{4r+3}) \arrow[d, "\epsilon"] \\ \widetilde{KU}^{-1}(\mathbb{R}P^{4r+3}) \arrow[r] & \widetilde{KU}^0(S^{4r+4}) \arrow[r] & \widetilde{KU}^0(\mathbb{R}P^{4r+4}) \arrow[r, "i^*"] & \widetilde{KU}^0(\mathbb{R}P^{4r+3}) \end{tikzcd} \tag{225}$$

and the fact that $Ker(i^*) = \mathbb{Z}/2$, it follows that the image of

$$\epsilon : \widetilde{KO}^{-1}(\mathbb{R}P^{4r+3}) \to \widetilde{KU}^{-1}(\mathbb{R}P^{4r+3}) \tag{226}$$

is $\mathbb{Z}$ if r is odd and $2\mathbb{Z}$ if r is even. Using these information, we conclude from diagram (223) that $\widetilde{KO}^{-1}(\mathbb{R}P^{4r+3}) = \mathbb{Z} + \mathbb{Z}/2$.

- $i = 5$

  It is immediate that

$$\widetilde{KO^{-5}}(\mathbb{R}P^{8r+n}) = \begin{cases} \mathbb{Z} & \text{if } n = 3 \text{ or } n = 7 \\ 0 & \text{else} \end{cases} \tag{227}$$

- $i = 3$

  $\widetilde{KO}^{-3}(\mathbb{R}P^n)$ is either $\mathbb{Z}/2 + \mathbb{Z}/2$ or $\mathbb{Z}/4$ for $n = 8r + 7$ and as indicated in the table of Theorem 19.1 for other n.
  Since $\mathbb{R}P^{8r+7}/\mathbb{R}P^{8r+5} \approx S^{8r+6} \vee S^{8r+7}$ we have

$$\underbrace{\widetilde{KO}^{-3}(\mathbb{R}P^{8r+7}/\mathbb{R}P^{8r+5})}_{\cong \mathbb{Z}/2+\mathbb{Z}/2} \to \widetilde{KO}^{-3}(\mathbb{R}P^{8r+7}) \to \underbrace{\widetilde{KO}^{-3}(\mathbb{R}P^{8r+5})}_{\cong \mathbb{Z}} \tag{228}$$

  and hence $\widetilde{KO}^{-3}(\mathbb{R}P^{8r+7}) = \mathbb{Z}/2 + \mathbb{Z}/2$.

- $i = 7$

  Analogous to the case $i = 3$.

- $i = 2$

  We have the following situation

$$\widetilde{KO}^{-2}(\mathbb{R}P^n) = \begin{cases} \begin{matrix}\mathbb{Z}/2 + \mathbb{Z}/2 + \mathbb{Z}/2, \\ \mathbb{Z}/4 + \mathbb{Z}/2, \text{ or } \mathbb{Z}/8\end{matrix} & \text{if } n = 8r + 7, \\ \\ \mathbb{Z}/2 + \mathbb{Z}/2 \text{ or } \mathbb{Z}/4 & \text{if } n = 8r + 6 \text{ or } 8r + 8, \\ \\ \mathbb{Z}/2 & \text{else .} \end{cases} \tag{229}$$

  We will show that $2\widetilde{KO}^{-2}(\mathbb{R}P^n) = 0$. The first step is to verify $4\widetilde{KO}^{-2}(\mathbb{R}P^n) = 0$ or, equivalently, $\widetilde{KO}^{-2}(\mathbb{R}P^{8r+7}) \neq \mathbb{Z}/8$. This follows from the exact sequence of the pair $(\mathbb{R}P^{8r+7}, \mathbb{R}P^{8r+5})$:

$$\begin{aligned} \cdots \to \underbrace{\widetilde{KO}^{-3}(\mathbb{R}P^{8r+7})}_{\cong \mathbb{Z}/2} \to \underbrace{\widetilde{KO}^{-3}(\mathbb{R}P^{8r+5})}_{\cong \mathbb{Z}} \to \underbrace{\widetilde{KO}^{-2}(\mathbb{R}P^{8r+7}/\mathbb{R}P^{8r+5})}_{\cong \mathbb{Z}/2+\mathbb{Z}} \\ \to \widetilde{KO}^{-2}(\mathbb{R}P^{8r+7}) \to \underbrace{\widetilde{KO}^{-2}(\mathbb{R}P^{8r+5})}_{\cong \mathbb{Z}/2} \to \underbrace{\widetilde{KO}^{-1}(\mathbb{R}P^{8r+7}/\mathbb{R}P^{8r+5})}_{\cong \mathbb{Z}} \to \cdots \end{aligned} \tag{230}$$

As in the case $i = 3$ we consider the exact sequence

$$\begin{aligned} \cdots \to \widetilde{KO}^{-2}(\mathbb{R}P^n) \xrightarrow{\epsilon} \widetilde{KU}^{-2}(\mathbb{R}P^n) \xrightarrow{p_*} \widetilde{KO}^{0}(\mathbb{R}P^n) \\ \xrightarrow{\partial} \widetilde{KO}^{-1}(\mathbb{R}P^n) \to \widetilde{KU}^{-1}(\mathbb{R}P^n) \to \cdots \end{aligned} \tag{231}$$

We are only interested in the cases $n = 8r+6, 8r+7$ and $8r+8$. Here we have $\widetilde{KO}^{0}(\mathbb{R}P^n) \cong \mathbb{Z}/2^f$, $\widetilde{KU}^{-2}(\mathbb{R}P^n) \cong \mathbb{Z}/2^f$, where f is the integral part of $\frac{n}{2}$, $\widetilde{KO}^{-1}(\mathbb{R}P^n) \cong \mathbb{Z}/2$ or $\mathbb{Z}/2+\mathbb{Z}$, and $\widetilde{KU}^{-1}(\mathbb{R}P^n) \cong \mathbb{Z}$ or 0.
Therefore $Im(\partial) \cong \mathbb{Z}/2$ and hence $Im(p_*) = Ker(\partial) \cong \mathbb{Z}/2^{f-1}$. Finally we have $Im(\epsilon) = Ker(p_*) \cong \mathbb{Z}/2$, so that $2Im(\epsilon) = 0$ and $Im(\epsilon) \subset 2^{f-1}\widetilde{KU}^{-2}(\mathbb{R}P^n)$.
Now we know that the composition

$$\widetilde{KO}^{-2}(\mathbb{R}P^n) \xrightarrow{\epsilon} \widetilde{KU}^{-2}(\mathbb{R}P^n) \xrightarrow{\rho} \widetilde{KO}^{-2}(\mathbb{R}P^n) \tag{232}$$

is multiplication by 2, where ρ is realization. This yields

$$2\widetilde{KO}^{-2}(\mathbb{R}P^n) = Im(\rho\epsilon) \subset 2^{f-1}\widetilde{KO}^{-2}(\mathbb{R}P^n) = 2^{f-3}\cdot 4\widetilde{KO}^{-2}(\mathbb{R}P^n) = 0 \tag{233}$$

and concludes the proof of the case $i = 2$.

- $i = 6$

  We have one $\mathbb{Z}/2$-term for $n = 8r+2$ or $8r+4$ and two $\mathbb{Z}/2$-terms for $n = 8r+3$. From the exact sequence

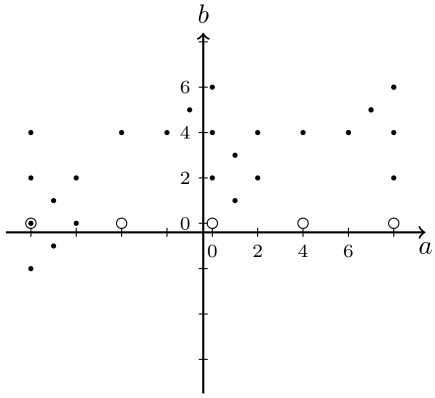
$$\cdots \to \underbrace{\widetilde{KO}^{-6}(\mathbb{R}P^{8r+5})}_{=0} \to \widetilde{KO}^{-6}(\mathbb{R}P^{8r+3}) \to \underbrace{\widetilde{KO}^{-5}(S^{8r+5} \vee S^{8r+4})}_{\cong \mathbb{Z}/2+\mathbb{Z}/2} \to \cdots \tag{234}$$

  we obtain $\widetilde{KO}^{-6}(\mathbb{R}P^{8r+5}) \cong \mathbb{Z}/2+\mathbb{Z}/2$.

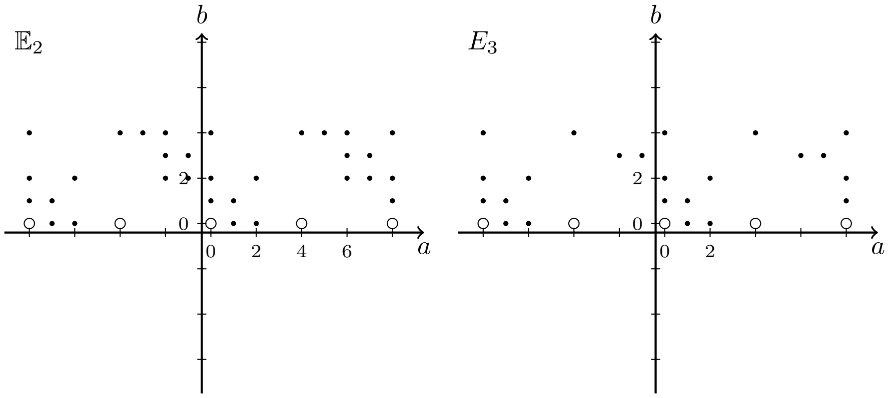
This concludes the proof of Theorem 19.1.

19.1 Comparison to the classical Atiyah-Hirzebruch spectral sequence for KO

Beside the Slice spectral sequence specialized to a space with trivial action there is the classical Atiyah-Hirzebruch spectral sequence converging to KO. One could ask, whether these sequences are isomorphic (at least for $r > r_0$). Our calculations show that this is not the case. An easy example to see this is the space $\mathbb{R}P^4$. We calculated the E_3-term (and thus the E_∞-term) of the slice spectral sequence.



The classical Atiyah-Hirzebruch spectral sequence takes on the form $E_2^{p,q} = H^p(X, KO^q(*))$ with differentials $d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$. If we again take the axes $a = -p - q$ and $b = p$ and $X = \mathbb{R}P^4$, this is



(cf. e.g [Fuj67]).
If we examine the vertical lines $a = 0$ in these pictures, we observe that these sequences induce different filtrations for the group $KO^0(\mathbb{R}P^4)$. The three dots in the picture for the slice spectral sequence are distributed equidistantly along the b-axis. In contrast, there is a "gap" between the dots in the picture for the classical sequence. Even if we would reindex one of these sequences, we would not get rid of this gap. Hence, the filtrations are different, and thus the sequences are not isomorphic.

20 The spectral sequence for $\mathbb{C}P^n$ as a Real space with complex conjugation

Consider the Real space $\mathbb{C}P^n$ where the involution is given by complex conjugation. We already know from the work of Atiyah [Ati66] that

$$KR^{-i}(\mathbb{C}P^n) \cong KR^{-i}(*)[t]/(t^{n+1} - 1). \tag{235}$$

To confirm this result, it is not hard to compute the Atiyah-Hirzebruch slice spectral sequence.

Proposition 20.1. *The cohomology of* $\mathbb{C}P^n$ *is given by*

$$H^{p,q}(\mathbb{C}P^n) \cong \bigoplus_{i=0}^{n} H^{p-2i,q-i}(*). \tag{236}$$

Proof. The cofiber sequence $\mathbb{C}P^n \to \mathbb{C}P^{n+1} \to \mathbb{S}^{2n+2,n+1}$ induces a long exact sequence on cohomology:

$$\cdots \to \tilde{H}^{p,q}(S^{2n+2,n+1}) \to \tilde{H}^{p,q}(\mathbb{C}P^{n+1}) \to \tilde{H}^{p,q}(\mathbb{C}P^n) \to \cdots \tag{237}$$

We will show that this long exact sequence decomposes into split short exact sequences

$$0 \to \tilde{H}^{p,q}(S^{2n+2,n+1}) \to \tilde{H}^{p,q}(\mathbb{C}P^{n+1}) \to \tilde{H}^{p,q}(\mathbb{C}P^n) \to 0. \tag{238}$$

Thus, we obtain

$$\tilde{H}^{p,q}(\mathbb{C}P^{n+1}) \cong \tilde{H}^{p,q}(\mathbb{C}P^n) \oplus \tilde{H}^{p,q}(S^{2n+2,n+1}), \tag{239}$$

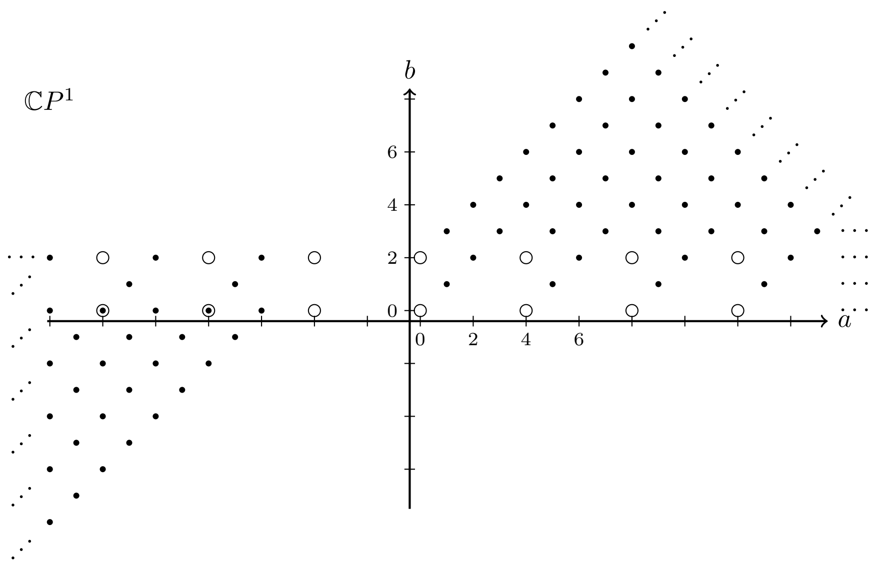
which finishes the proof by induction.
We obtain $\mathbb{C}P^{n+1}$ from $\mathbb{C}P^n$ by attaching a cell $S^{2n+2,n+1}$ via an attaching map $f : S^{2n+2,n} \to \mathbb{C}P^n$. A map $g : \mathbb{C}P^n \to K(\mathbb{Z}(q), p)$ representing an element of $\tilde{H}^{p,q}(\mathbb{C}P^n)$, extends to a map $\mathbb{C}P^{n+1} \to K(\mathbb{Z}(q), p)$, if the composition $g \circ f$ is null-homotopic. Hence, we have to show that f induces a trivial map $\tilde{H}^{p,q}(S^{2n+2,n}) \to \tilde{H}^{p,q}(\mathbb{C}P^n)$.
Since both $H^{p,q}(S^{2n+2,n})$ and $H^{p,q}(\mathbb{C}P^n)$ are modules over $H^{p,q}(*)$, it suffices to realize that

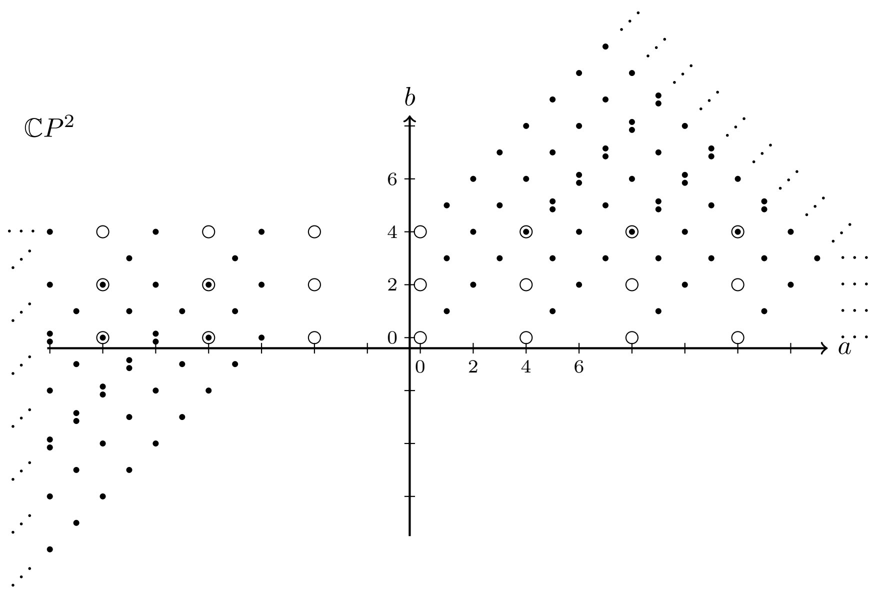
$$\mathbb{Z} \cong \tilde{H}^{2n+2,n}(S^{2n+2,n}) \to \tilde{H}^{2n+2,n}(\mathbb{C}P^n) \cong 0 \tag{240}$$

is trivial. This implies that there exists an extension and thus a splitting.

$\square$

The spectral sequence is given as indicated in the pictures below:





The pictures can easily be extended to general $\mathbb{C}P^n$ by adding circles at the places $(4k, 2j)$ and rays of dots going diagonally up and right. These rays correspond to the cohomology groups of $S^{2n,n}$, and the differentials are the same as the differentials for the point shifted to the right place. In other words, the E_∞-term will consist of copies of $\mathbb{Z}$ in the places $(8k, 2j)$ and $(8k+4, 2j)$ for $0 \leq j \leq 2n$ and copies of $\mathbb{Z}/2$ in the places $(8k+1, 2j+1)$ and $(8k+2, 2j+1)$. This and the long exact sequence

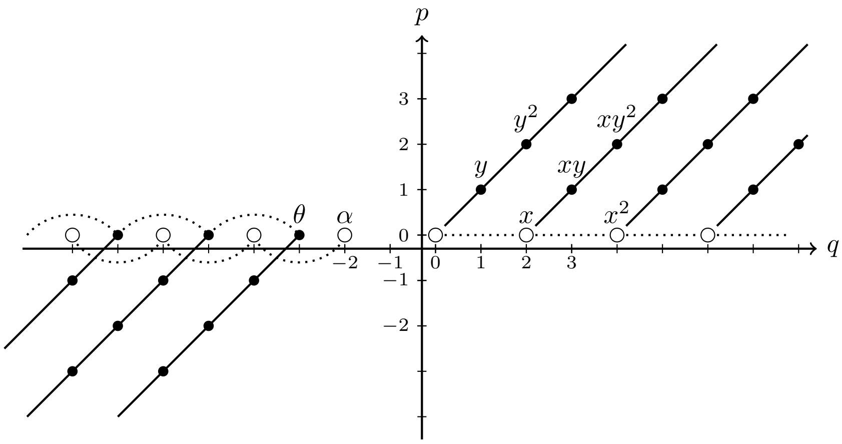
$$\cdots \to K^{-1-q}(X) \to KR^{1-q}(X) \to KR^{-q}(X) \to K^{-q}(X) \to \cdots \qquad (241)$$

suffice to determine the structure of $KR^{-i}(\mathbb{C}P^n)$.

Part VII
Appendix

A $H\mathbb{Z}$ computations

In the unpublished version of [Dug05] he calculates the coefficient ring of $H\mathbb{Z}$. We will repeat this including a few more details. We begin with a picture of the result:



The hollow circles denote $\mathbb{Z}$, while the solid dots represent $\mathbb{Z}/2$. The multiplicative structure is determined by the properties that

(1) the structure is commutative,

(2) the solid lines represent multiplication by $y \in H^{1,1}$,

(3) the dotted lines represent multiplication by $x \in H^{0,2}$ (only a representative set has been drawn),

(4) $\alpha x = 2$.

For any pointed $\mathbb{Z}/2$-spaces X and Y there is an isomorphism

$$[\mathbb{Z}/2_+ \wedge X, Y]_* \to [X, Y]_*^e \tag{242}$$

where $[-,-]_*^e$ denotes homotopy classes of non-equivariant pointed maps. The isomorphism is obtained by the inclusion $\{0\} \hookrightarrow \mathbb{Z}/2$. So for any equivariant spectrum E there are isomorphisms $E^{p,q}(\mathbb{Z}/2) \to E_e^p(*)$, where E_e is the nonequivariant spectrum obtained by forgetting the group action. If E has a product, these isomorphisms give ring maps.
Hence for $H^{*,*}(\mathbb{Z}/2)$ we get

$$H^{p,q}(\mathbb{Z}/2) \cong [\mathbb{Z}/2, K(\mathbb{Z}(q), p)] \cong [*, K(\mathbb{Z}, p)] \cong H^p_{sing}(*) \tag{243}$$

The multiplication is defined by

$$H^{0,q}(\mathbb{Z}/2) \otimes H^{0,t}(\mathbb{Z}/2) \to H^{0,p+t}(\mathbb{Z}/2) \tag{244}$$
$$1 \otimes 1 \mapsto 1 \tag{245}$$

and hence

$$H^{*,*}(\mathbb{Z}/2) \cong \mathbb{Z}[u, u^{-1}] \tag{246}$$

with $deg(u) = (0,1)$ and $1 \in H^{p,q}(\mathbb{Z}/2) \mapsto u^q$.
The computation of $H^{p,q}(*)$ is more involved, and we have to consider different cases and use different arguments each time.
From the construction of $RO(G)$-graded cohomology we know that the dimension axiom holds for representations $V = (p, 0)$. In other words,

$$H^{p,0}(*) = \begin{cases} \mathbb{Z} & p = 0 \\ 0 & \text{else} \end{cases} \tag{247}$$

Now we turn our attention to $H^{p-q,-q}(*)$ for $q > 0$.

$$H^{p-q,-q}(*) \cong \tilde{H}^{p,0}(S^{q,q}) \cong \tilde{H}^p_{sing}(S^{q,q}/(\mathbb{Z}/2); \mathbb{Z}) \cong \tilde{H}^p_{sing}(\Sigma \mathbb{R}P^{q-1}) \tag{248}$$

The first isomorphism is the suspension isomorphism, while the second one is due to the general fact that we have $H^{p,0}(X) = H^p_{sing}(X/(\mathbb{Z}/2); \mathbb{Z})$. This is not hard to see as

$$H^{p,0}(X) \cong [X, K(\mathbb{Z}, p)] \cong [X/(\mathbb{Z}/2), K(\mathbb{Z}, p)]^e \cong H^p_{sing}(X/(\mathbb{Z}/2); \mathbb{Z}) \tag{249}$$

The last isomorphism holds, since $S^{q,q}$ is the suspension of the sphere inside $\mathbb{R}^{q,q}$, which is a $(q-1)$-sphere with antipodal action.
Unfortunately, the analogue for homology is not quite true, which makes the computation of $H^{p+q,q}(*)$ for $q > 0$ harder. However we have

Proposition A.1. *If $\mathbb{Z}/2$ acts freely on X, there is an isomorphism*

$$\tilde{H}_{p,0}(X) \cong \tilde{H}^{sing}_p(X/(\mathbb{Z}/2)) \tag{250}$$

Proof. This is immediate from the definition and true for every topological group instead of $\mathbb{Z}/2$. In Definition 11.3 we defined the homology groups to be

$$H^G_n(X; \underline{\mathbb{Z}}) = H_n(\underline{C}_*(X) \otimes_{\mathscr{B}_G} \underline{\mathbb{Z}}), \tag{251}$$

where $\underline{C}_*(X)$ is the Mackey functor with $\underline{C}_*(X)(G/H) = H_n((X^n)^H, (X^{n-1})^H)$. Since the action of G is free, $\underline{C}_*(X)(G/H)$ is trivial for $H \neq e$, and we have $|G|$ copies of $\mathbb{Z}$ for every equivariant n-cell in $\underline{C}_*(X)(G/e)$. These copies are identified by the equivalence relation, since they are permutated by the morphisms associated with multiplication by elements of G. Hence the chain complex coincides with the cellular complex for X/G.

□

Another tool we need is the Spanier-Whitehead duality.

Proposition A.2. *If X is a wedge summand of a finite G-CW spectrum, and if E is any G-spectrum, then*

$$\nu : DX \wedge E \to F(X, E) \tag{252}$$

is an isomorphism in $\bar{h}G\mathscr{S}$. Therefore, for any representation V,

$$E_V^G(DX) \cong E_G^{-V}(X). \tag{253}$$

Here DX denotes the mapping space $F(X,S)$ also referred to as the dual spectrum. ν is the natural map

$$\nu : F(X,Y) \wedge Z \to F(X, Y \wedge Z) \tag{254}$$

specialized to

$$\nu : F(X,S) \wedge E \to F(X, S \wedge E) \cong F(X,E) \tag{255}$$

Proof. [M$^+$96] Corollary XVI.7.5 □

We can now write (for $p > 0$)

$$H^{p+q,q}(*) \cong \tilde{H}^{p+q,q}(S^{0,0}) \cong \tilde{H}_{-p-q,-q}(S^{0,0}) \cong \tilde{H}_{-p,0}(S^{q,q}). \tag{256}$$

The group action on $S^{q,q}$ is not free, but it fits into a cofiber sequence $S(\mathbb{R}^{q,q}) \hookrightarrow D(\mathbb{R}^{q,q}) \to S^{q,q}$, with one space being contractible and one having a free action. Hence the induced long exact sequence for $H_{*,0}$ should be revealing:

$$\cdots \to H_{n+1,0}(D(\mathbb{R}^{q,q})) \to \tilde{H}_{n+1,0}(S^{q,q}) \to H_{n,0}(S(\mathbb{R}^{q,q})) \to H_{n,0}(D(\mathbb{R}^{q,q})) \to \cdots \tag{257}$$

This sequence shows that for $n \neq 0, 1$

$$\tilde{H}_{n,0}(S^{q,q}) \cong \tilde{H}_{n-1,0}(S(\mathbb{R}^{q,q})) \tag{258}$$

and

$$0 \to \tilde{H}_{1,0}(S^{q,q}) \to H_{0,0}(S(\mathbb{R}^{q,q})) \to \mathbb{Z} \to \tilde{H}_{0,0}(S^{q,q}) \to 0 \tag{259}$$

is exact.
Since the zero-skeleton of $S(\mathbb{R}^{q,q})$ is $\mathbb{Z}/2$, the center map in (259) coincides with the map $H_{0,0}(\mathbb{Z}/2) \to H_{0,0}(*)$ induced by the projection. This map is the same as the transfer map i_* in the Mackey functor $\underline{\mathbb{Z}}$, which is multiplication by 2. Thus, $H^{q-1,q}(pt) = 0$ and $H^{q,q}(pt) = \mathbb{Z}/2$.
Knowing the additive structure we proceed to determine the ring structure. The map $i : \mathbb{Z}/2 \to *$ induces a map $i^* : H^{*,*}(*) \to H^{*,*}(\mathbb{Z}/2)$. Since we already know the target, examining the map will give us information on the source.
Taking the mapping cone of i we get a cofiber sequence $\mathbb{Z}/2_+ \to S^{0,0} \to S^{1,1}$. In fact, we first turn i into a map of pointed spaces, by adding a base point on both sides. In this way we can have a look at the induced long exact sequence for $H^{*,2n}$.

$$\cdots \to H^{-1,2n-1}(*) \to H^{0,2n}(*) \xrightarrow{i^*} H^{0,2n}(\mathbb{Z}/2) \to H^{0,2n-1}(*) \to \cdots \tag{260}$$

We already applied the suspension isomorphism on the first group in this sequence. From our preceding calculations we know that

$$H^{0,2n-1}(*) = 0 = H^{-1,2n-1}(*) \text{ for } n \geq 0, \tag{261}$$

so that i^* is an isomorphism. For $n < 0$ we observe that the sequence reduces to

$$0 \to H^{0,2n}(*) \xrightarrow{i^*} H^{0,2n}(\mathbb{Z}/2) \to \mathbb{Z}/2 \to 0. \tag{262}$$

Hence, i^* is multiplication by 2.
The class y is represented by a map $S^{0,0} \to K(\mathbb{Z}(1), 1)$. Since we explicitly know the homotopy groups of the target, we can see that $K(\mathbb{Z}(1), 1)$ is weakly equivalent to $S^{1,1}$. Again taking the mapping cone we get the cofiber sequence $S^{0,0} \to S^{1,1} \to \mathbb{Z}/2_+ \wedge S^{1,0}$. The induced long exact sequence will give the desired information on multiplication by y.

$$\begin{tikzcd}
\tilde{H}^{p,q}(\mathbb{Z}/2_+ \wedge S^{1,0}) \arrow[r] \arrow[d, "\cong"] & \tilde{H}^{p,q}(S^{1,1}) \arrow[r, "y"] \arrow[d, "\cong"] & \tilde{H}^{p,q}(S^{0,0}) \arrow[r] \arrow[d, "\cong"] & H^{p+1,q}(\mathbb{Z}/2_+ \wedge S^{1,0}) \arrow[d, "\cong"] \\
\tilde{H}^{p}_{sing}(S^1) & H^{p-1,q-1}(*) & H^{p,q}(*) & \tilde{H}^{p+1}_{sing}(S^1)
\end{tikzcd} \tag{263}$$

In all cases of interest, the last group is zero and the middle map is surjective. Since all groups are either $\mathbb{Z}$ or $\mathbb{Z}/2$, this determines the multiplication by y.
The last step is to see what happens with $\theta_n \in H^{0,-2n-1}(*) \cong \mathbb{Z}/2$, when multiplied by x. It will turn out that $x \cdot \theta_{n+1} = \theta_n$. The argument is somewhat different to the ones before.
Let E be the spectrum defined by the cofiber sequence $\Sigma^{0,-2}H\mathbb{Z} \xrightarrow{\cdot x} H\mathbb{Z} \to E$. If we evaluate this sequence at the space $X = *$ once again, we obtain

$$\cdots \to H^{n,-2}(*) \to H^{n,0}(*) \to E^{n,0}(*) \to H^{n+1,-2}(*) \to H^{n+1,0}(*) \to \cdots \tag{264}$$

We already know four of these groups and the maps between them. Hence, for $n \neq 0, 1$, we conclude immediately that $E^{n,0}(*) = 0$. For $n = 0, 1$, we have

$$\begin{aligned}
\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to &E^{0,0}(*) \to 0 \\
0 \to &E^{1,0}(*) \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}
\end{aligned} \tag{265}$$

Hence

$$E^{n,0}(*) = \begin{cases} \mathbb{Z}/2 & n = 0 \\ 0 & \text{else.} \end{cases} \tag{266}$$

For $X = \mathbb{Z}/2$, the sequence is simply

$$\mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \to E^{n,0}(\mathbb{Z}/2) \to \mathbb{Z} \xrightarrow{\cong} \mathbb{Z}, \tag{267}$$

and $E^{n,0}(\mathbb{Z}/2) = 0$. By the axiomatic characterisation of cohomology theories represented by Mackey functors we can conclude that E is the Eilenberg-MacLane cohomology for the Mackey functor $\underline{E}^{0,0}$ and that there is an isomorphism

$$E^{n,0}(X) \cong H^n(X^{\mathbb{Z}/2}; \mathbb{Z}/2). \tag{268}$$

So for $n > 0$ we have

$$E^{0,-n}(*) \cong \tilde{E}^{n,0}(S^{n,n}) \cong \tilde{H}^n_{sing}(S^0) = 0. \tag{269}$$

Hence, from

$$E^{-1,-n}(*) \to H^{0,-n-2}(*) \to H^{0,-n}(*) \to E^{0,-n}(*), \tag{270}$$

we see that multiplication by x induces an isomorphism

$$H^{0,-n-2}(*) \cong H^{0,-n}(*) \text{ for } n \geq 2. \tag{271}$$

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Glossary

$[X,Y]$	the set of homotopy classes of G-maps between two G-spaces X and Y, 24
$[X,Y]_*$	the set of based homotopy classes of based G-maps between two G-spaces X and Y, 24
$[X,Y]^H$	the set of H-equivariant homotopy classes of H-maps between G-spaces X and Y and for a subgroup $H \subset G$, 24
$\underline{\{X,Y\}^G}$	the Mackey functor defined by $\underline{\{X,Y\}^G}(G/H) = \{G/H_+ \wedge X, Y\}^G$ for G-spaces X and Y, 32
$\{X,Y\}^G$	stable G-maps between G-spaces X and Y, 29
BG	classifying space for fiber bundles with structure group G, 19
$\mathscr{B}_G$	the Burnside category, 32
$D(V)$	the unit disk in a G-representation V, 31
$S(V)$	the unit sphere in a G-representation V, 31
$\mathscr{G}$	the orbit category of G, 25
X/H	the orbit space of a G-space X for a subgroup $H \subset G$, 24
$G\mathscr{U}$	the category of G-equivariant compactly generated, weak Hausdorff spaces, 24
$\mathscr{G}\mathscr{U}$	the category of $\mathscr{G}$-spaces, 25
$Herm^{\mathbb{C}}(X)$	the set of isomorphism classes of complex vector bundles provided with a non-degenerate hermitian form over the compact space X, 10
$\mathscr{H}erm^{\mathbb{C}}(X)$	the categories of isomorphism classes of complex vector bundles provided with a non-degenerate hermitian form and maps compatible with the forms, 13
$Herm_n^{\mathbb{C}}(X)$	the set of isomorphism classes of n-dimensional complex vector bundles provided with a non-degenerate hermitian form over the compact space X, 10
$hG\mathscr{U}$	the homotopy category of G-spaces, 25
$\bar{h}G\mathscr{U}$	the category constructed from $hG\mathscr{U}$ by formally inverting the weak equivalences, 25
$H_G^V(X;M)$	ordinary $RO(G)$-graded cohomology with coefficients in the Mackey functor M, 39
$\tilde{H}_G^V(X;M)$	ordinary reduced $RO(G)$-graded cohomology with coefficients in the Mackey functor M, 39
HM	the spectrum representing ordinary $RO(G)$-graded cohomology with coefficients in the Mackey functor M, 40

$hofib(f)$	the homotopy fiber of the map $f : A \to B$, 51
$h\mathscr{R}O(G;U)$	the homotopy category of $\mathscr{R}O(G;U)$, 30
$\bar{h}\mathscr{R}O(G;U)$	the category obtained from $h\mathscr{R}O(G;U)$ by formally inverting the weak equivalences, 30
$K(X)$	$\mathbb{K}$-group (Grothendieck group) of a monoid of vextor bundles over the compact space X, 22
$\tilde{K}(X)$	the reduced $\mathbb{K}$-group of the compact space X, 22
$K(M, V)$	Eilenberg-MacLane space for the Mackey functor M and the representation V, 40
$\Omega^V X$	the loop space of a G-space X with respect to the G-representation V, 29
$O_n(\mathbb{C})$	the subgroup of $GL(n, \mathbb{C})$, which consists of isometries of $\mathbb{C}^n$ provided with the quadratic form $\sum_{i=1}^n (x_i)^2$, 17
$O_{n,p}(\mathbb{K})$	the subgroup of $GL_{n+p}(\mathbb{K})$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$), which consists of isometries of $\mathbb{K}^{n+p}$ provided with the form $\sum_{i=1}^n x_i\bar{y}_i - \sum_{i=n+1}^{n+p} x_i\bar{y}_i$, 17
$\Phi^{\mathbb{K}}(X)$	the set of isomorphism classes of $\mathbb{K}$-vector bundles over the compact space X, 10
$\bar{\Phi}^{\mathbb{R}}(X)$	the set of isomorphism classes of Real vector bundles over the compact Real space X, 10
$\Phi_n^{\mathbb{K}}(X)$	the set of isomorphism classes of n-dimensional $\mathbb{K}$-vector bundles over the compact space X, 9
$\pi_p(B, A)$	p-th relative homotopy group of the pair (B, A), 51
$\Psi^{\mathbb{K}}(X)$	the category of $\mathbb{K}$-vector bundles equipped with a metric and morphisms, which respect this metric, 14
P_V	the Postnikov functor $P_{\mathcal{A}_V}$ for $\mathcal{A}_V = \{S^W \wedge G/H_+ : W \supset V,\ H \le G\}$, 36
$\mathbb{P}_V$	the Postnikov functor $P_{\tilde{\mathcal{A}}_V}$ for $\tilde{\mathcal{A}}_V = \{S^W \wedge G/H_+ : W \supseteq V + 1,\ H \le G\}$, 36
RA^∞	right derived limit of the sequence A^s, 53
$\mathscr{R}O(G;U)$	the category whose objects are the representations embeddable in U and whose morphisms $V \to W$ are G-linear isometries, 30
$\Sigma^V X$	the suspension of a G-space X with respect to the G-representation V, 29
$S(M)$	the Grothendieck group associated with the monoid M, 22
$Sp_{2n}(\mathbb{K})$	the subgroup of $GL_{n2n}(\mathbb{K})$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$), which consists of isometries of $\mathbb{K}^{2n}$ provided with the form $\sum_{i=1}^n x_i y_{i+n} - \sum_{i=1}^n x_{i+n} y_i$, 17

$Sym^{\mathbb{K}}_{+}(X)$ the set of isomorphism classes of $\mathbb{K}$-vector bundles provided with a non-degenerate bilinear symmetric form over the compact space X, 10

$Sym^{\mathbb{K}}_{-}(X)$ the set of isomorphism classes of $\mathbb{K}$-vector bundles provided with a non-degenerate bilinear skew-symmetric form over the compact space X, 10

$\mathscr{S}ym^{\mathbb{K}}_{+}(X)$ the categories of isomorphism classes of $\mathbb{K}$-vector bundles provided with a non-degenerate bilinear symmetric form and maps compatible with the forms, 13

$\mathscr{S}ym^{\mathbb{K}}_{-}(X)$ the categories of isomorphism classes of $\mathbb{K}$-vector bundles provided with a non-degenerate bilinear skew-symmetric form and maps compatible with the forms, 13

$Sym^{\mathbb{K}}_{+,n}(X)$ the set of isomorphism classes of n-dimensional $\mathbb{K}$-vector bundles provided with a non-degenerate bilinear symmetric form over the compact space X, 9

$Sym^{\mathbb{K}}_{-,n}(X)$ the set of isomorphism classes of $2n$-dimensional $\mathbb{K}$-vector bundles provided with a non-degenerate bilinear skew-symmetric form over the compact space X, 9

θ_n the n-dimensional trivial bundle, or the $2n$-dimensional trivial bundle provided with a non-degenerate bilinear skew-symmetric form, 22

$\theta_{n,p}$ the $n + p$-dimensional trivial bundle provided with a non-degenerate bilinear or hermitian form with an n-dimensional subbundle carrying a positiv definite form, and a p-dimensional subbundle carrying a negative definite form, 22

$\mathscr{U}$ the category of compactly generated, weak Hausdorff spaces, 24

$\mathscr{U}^J$ the category of continuous functors $J^{op} \to \mathscr{U}$, 26

$V(H)$ the orthogonal complement of V^H in a G-representation V and for a subgroup $H \subset G$, 31

X^H the set of points of a G-space X,which are fixed under the action of H for a subgroup $H \subset G$, 24

$\underline{\mathbb{Z}}$ the constant Mackey functor with values in $\mathbb{Z}$, 35

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