Stable motivic homotopy groups and periodic self maps at odd primes

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Dissertation

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Contents

Chapter 1

Preliminaries

1.1 Introduction

Outline

In this work we study two topics in the stable motivic homotopy category over the base fields $\mathbb{C}$ and $\mathbb{R}$:

The first topic is the computation of the stable motivic homotopy groups of the spheres at odd primes. In algebraic topology, a wide variety of computational tools have been developed to study the stable homotopy groups of the spheres. Many of them are also available in the motivic context. We use the motivic equivalent of two classical computational devices, the *motivic Adams spectral sequence* and the *motivic Adams-Novikov spectral sequence*, to study the motivic homotopy groups of the spheres. In Proposition 2.4.8 and Proposition 2.4.9 we show that for odd primes over our choice of base fields the classical Adams-Novikov spectral sequence determines the motivic one and its differentials. Apart from torsion groups associated to the existence of nontrivial differentials in the topological Adams-Novikov spectral sequence, the motivic and the classical Adams-Novikov spectral sequence look very much alike.

The second topic of this dissertation is the study of *thick subcategories* and of *periodic self maps* of (finite) motivic spectra. In algebraic topology, this two topics are closely related, and the thick subcategories of finite spectra are all characterized by the property of admitting a self map of a certain type. The fact that the motivic Hopf map η is not nilpotent in the motivic setting and the work of Ruth Joachimi about motivic thick ideals in [Joa] suggest that the picture looks very different in the motivic context. For technical reasons we have to restrict to odd primes.

In Theorem 3.3.11, we prove that periodic motivic self maps defined by algebraic Morava K-theory define a thick subcategory, but we need to make use of a conjectural weakened version of a motivic nilpotence lemma. In Theorem 3.4.13 we lift a construction by Hopkins and Smith in [HS] to the motivic world to show that examples of these self maps exist. Finally, in the last two sections, we use some of our computations in the preceding sections to settle one of the conjectures in [Joa] and to correct an assertion made there about the relation of certain thick subcategories.

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1.2 The motivic stable homotopy category

Most of our computations will be done in the motivic stable homotopy category $\mathcal{SH}_k$, whose objects are $\mathbb{P}^1$-spectra of motivic spaces over the base field k. The construction of this category is due to Voevodsky and Morel (see [Voe] and [MV]) and mimicks the construction of the topological stable homotopy category, where smooth schemes take the place of topological spaces. For our computations we will always restrict the base field k over which these schemes are defined to either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. In contrast to classical topology there are two kinds of spheres in the motivic world, a simplicial and a geometric one; therefore suspensions, homotopy, homology and cohomology are all not singly graded but bigraded; and there are two common conventions for how to grade them. We index them according to the following convention:

Definition 1.2.1. *Define $S^{1,0}$ as the $\mathbb{P}^1$-suspension spectrum of the simplicial sphere $(S^1, 1)$ and $S^{1,1}$ as the $\mathbb{P}^1$-suspension spectrum of $(\mathbb{A}^1 - 0, 1)$. The suspension spectrum of $\mathbb{P}^1$ is then equivalent to $S^{2,1}$. Define*

$$S^{p,q} := (S^{1,0})^{\wedge(p-q)} \wedge (S^{1,1})^q.$$

This relates to the other common notation of $S^\alpha = S^{1,1}$ by $S^{p,q} = S^{p-q+q\alpha}$. The motivic homotopy groups of a motivic spectrum $X \in \mathcal{SH}_k$ are then defined as:

$$\pi_{p,q}(X) := [S^{p,q}, X]_{\mathcal{SH}_k}$$

In contrast to classical algebraic topology, not all motivic spectra are cellular in the following sense:

Definition 1.2.2. *1. The category of cellular spectra $\mathcal{SH}_k^{cell}$ in $\mathcal{SH}_k$ is defined (c.f. [DI2, Definition 2.1]) as the smallest full subcategory that satisfies*

- *The spheres $S^{p,q}$ are contained in the subcategory $\mathcal{SH}_k^{cell}$.*
- *If a spectrum X is contained in the subcategory $\mathcal{SH}_k^{cell}$, then so are all spectra which are weakly equivalent to X.*
- *If $X \to Y \to Z$ is a cofiber sequence and two of the three spectra are contained in the subcategory $\mathcal{SH}_k^{cell}$, then so is the third.*
- *The subcategory $\mathcal{SH}_k^{cell}$ is closed under arbitrary colimits.*

2. The subcategory of finite cellular spectra *$\mathcal{SH}_k^{fin}$ in $\mathcal{SH}_k$ is defined similarly as the smallest full subcategory that satisfies the first three conditions (see [DI2, Definition 8.1]).*

3. Finally, we define the category of quasifinite cellular spectra *$\mathcal{SH}_k^{qfin}$ as the smallest full triangulated subcategory of $\mathcal{SH}_k$ that contains $\mathcal{SH}_k^{fin}$ and* *is closed under retracts. Note that the cofiber of two retracts of finite cell spectra is a retract of a finite cell spectrum by the octahedral axiom. Therefore the spectra in* $\mathcal{SH}_k^{qfin}$ *are exactly finite cell spectra and their retracts.*

4. *By [Roen, Lemma 2.2] a motivic spectrum is cellular if and only if it admits a cell presentation, i.e. it can be built by successively attaching cells* $S^{s,t}$*. A motivic cell spectrum* X *is called* of finite type *if it admits a cell presentation with the following property: there exists a* $k \in \mathbb{N}$ *such that there are no cells in dimensions satisfying* $s - t < k$ *and such that there exist only finitely many cells in dimensions* $(s + t, t)$ *for each* s*.*

Over the base fields $\mathbb{R}$ and $\mathbb{C}$ we have topological realization functors mapping into the stable homotopy category and the stable C_2-equivariant homotopy category:

$$R : \mathcal{SH}_{\mathbb{C}} \to \mathcal{SH}_{top}$$

and

$$R' : \mathcal{SH}_{\mathbb{R}} \to \mathcal{SH}_{C_2}$$

There are many reviews of the construction and basic properties of these functors; one such explicit review of these functors is given in [Joa, 4.3], which we will use as our main reference. Both functors are strict symmetric monoidal left Quillen functors.
The complex realization functor is often called Betti realization and maps the suspension spectrum of a smooth scheme over $\mathbb{C}$ to the suspension spectrum of the topological space of its complex points, endowed with the analytic topology. Similarly, the real realization functor maps the suspension spectrum of a smooth scheme over $\mathbb{R}$ to the space of its complex points, endowed with the analytic topology, and the C_2-action is provided by complex conjugation. Consequently the behaviour of these functors on the sphere spectra is given by the assignments $S^{p,q} \overset{R}{\mapsto} S^p$ and $S^{p,q} \overset{R'}{\mapsto} S^{p-q+q\sigma}$, where σ denotes the sign representation of C_2. Because Betti realization maps the motivic spheres to the topological ones, it induces maps on homotopy groups

$$R : \pi_{pq}(X) \to \pi_p(R(X))$$

for every motivic spectrum $X \in \mathcal{SH}_{\mathbb{C}}$. Therefore for every motivic spectrum $E \in \mathcal{SH}_{\mathbb{C}}$ it also induces maps

$$R : E_{pq}(X) \to R(E)_p(R(X))$$

and

$$R : E^{pq}(X) \to R(E)^p(R(X))$$

on homology and cohomology associated to that spectrum.

The functors R and R' each have a right inverse

$$c : \mathcal{SH}_{top} \to \mathcal{SH}_{\mathbb{C}}$$

and

$$c' : \mathcal{SH}_{C_2} \to \mathcal{SH}_{\mathbb{R}}$$

which are called the constant simplicial presheaf functors. Both are strict symmetric monoidal. It is a result of Levine(see [Lev, Theorem 1]) that c is not only faithful but also full. For c' this only holds after a suitable completion(see [HO, Theorem 1]).

The map of schemes

$$p : \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$$

induces a functor

$$p^* : \mathcal{SH}_{\mathbb{R}} \to \mathcal{SH}_{\mathbb{C}}$$

that relates the different realization functors in the following way(c.f. [HO, Proposition 4.13]):

$$\begin{tikzcd}
\mathcal{SH}_{\mathbb{R}} \arrow[r, "p^*"] \arrow[d, "R'"] & \mathcal{SH}_{\mathbb{C}} \arrow[d, "R"] \\
\mathcal{SH}_{C_2} \arrow[r, "res_e^{\mathbb{Z}/2}"] & \mathcal{SH}_{top}
\end{tikzcd}$$

Here $res_e^{\mathbb{Z}/2}$ denotes the functor which restricts from C_2 to the trivial group.

1.3 Completions

The classic tools for computing homotopy groups of spheres, the Adams spectral sequence and the Adams-Novikov spectral sequence, compute actually not the homotopy groups of the spheres themselves, but their completions and localizations respectively (at a specific prime). For the sake of studying periodic self maps it is also useful to consider one prime at a time, because these maps are detected by a collection of cohomology theories called Morava K-theory, which are defined with regard to a specific prime. In our case this prime will usually be odd, i.e. different from two. Topologically one can implement this by studying the localized or completed homotopy category via the tool of Bousfield localization at an appropiate Moore spectrum. Motivically this works as well(a discussion of this in the motivic setting can be found in [RO, Section 3]): We define the l-completed motivic homotopy category $\mathcal{SH}^{\wedge}_{k,l}$ as the Bousfield localization of the category $\mathcal{SH}_k$ at the mod-l Moore spectrum S/l.

Definition 1.3.1. *Let l be any prime number, and let X be a motivic spectrum in $\mathcal{SH}_k$. The l-completion $X^{\wedge}_l$ of X is the Bousfield localization of X at the mod-l Moore spectrum S/l. One can also describe this completion as:*

$$X^{\wedge}_l := L_{S/l}X \simeq \underleftarrow{holim}\, X/l^n$$

The homotopy groups of X and its l-completion are related by the following short exact sequence([RO, End of section 3]):

$$0 \to Ext^1(\mathbb{Z}/l^\infty, \pi_{**}X) \to \pi_{**}X_l^\wedge \to Hom(\mathbb{Z}/l^\infty, \pi_{*-1,*}X) \to 0$$

Definition 1.3.2. *We define the subcategory $\mathcal{SH}_{k,l}^{\wedge,cell}$ of* l-complete cellular spectra *in $\mathcal{SH}_{k,l}^\wedge$ as the full subcategory of l-completions of cellular spectra. Similarly, we define the subcategories $\mathcal{SH}_{k,l}^{\wedge,fin}$ of l-complete finite cellular spectra and $\mathcal{SH}_{k,l}^{\wedge,qfin}$ of l-complete quasifinite cellular spectra as the full subcategories of l-completions of spectra in $\mathcal{SH}_k^{fin}$ and $\mathcal{SH}_k^{qfin}$.*

Another more exotic completion that we will need to use is the completion at the motivic Hopf map $\eta : \Sigma^{1,1}S \to S$, which is unstably defined as the obvious map $\mathbb{A}^2 - \{0\} \longrightarrow \mathbb{P}^1$. This completion will appear when discussing the convergence of the motivic Adams spectral sequence.

Definition 1.3.3. *Let X be any spectrum in $\mathcal{SH}_k$. Define the* η-completion $X_\eta^\wedge$ of X *as:*

$$X_\eta^\wedge = \underset{\leftarrow}{holim}\ X/\eta^n$$

1.4 Motivic Spanier-Whitehead duality

We are going to make use of Spanier-Whitehead duality when we study periodic self maps. The sources we want to quote use different, but equivalent definitions of dualizability, so we collect a number of basic definitions and facts about Spanier-Whitehead duality that we are going to use in one place. Our primary source is [LMS, III.1] where categorical duality is explained with great detail. Consider a spectrum X in $\mathcal{SH}_k$ or $\mathcal{SH}_{k,l}^\wedge$. Both categories are closed symmetric monoidal categories (see [Jar]), and therefore for an arbitrary motivic spectrum Y there exists a function spectrum $F(X,Y)$. The unit and counit of the canonical tensor-hom adjunction are given by maps

$$\eta_{X,Y} : X \to F(Y, X \wedge Y)$$

and by the evaluation

$$\epsilon_{X,Y} : F(X,Y) \wedge X \to Y$$

and furthermore there is a natural pairing

$$F(X,Y) \wedge F(X',Y') \to F(X \wedge X', Y \wedge Y')$$

which provides a natural map

$$\nu_{X,Y} : F(X,S) \wedge Y \to F(X,Y)$$

by specializing to the case $X' = Y = S$ and using the fact $F(S,Y') \cong Y'$.

Proposition 1.4.1. *Let X be a spectrum in $\mathcal{SH}_k$ or $\mathcal{SH}^{\wedge}_{k,l}$. Then the following three conditions are equivalent:*

1. *The canonical map*

$$\nu_{X,Y} : F(X,S) \wedge Y \to F(X,Y)$$

is an isomorphism for all spectra Y.

2. *The canonical map*

$$\nu_{X,X} : F(X,S) \wedge X \to F(X,X)$$

is an isomorphism.

3. *There is a coevaluation map $coev : S \to X \wedge F(X,S)$ such that the diagram*

$$\begin{tikzcd} S \arrow[r, "coev"] \arrow[d, "\eta_{S,X}"] & X \wedge F(X,S) \arrow[d, "T"] \\ F(X,X) & F(X,S) \wedge X \arrow[l, "\nu_{X,X}"'] \end{tikzcd}$$

commutes, where T denotes the transposition map.

Proof. Clearly the first point implies the second. The second point implies the third, because one can define *coev* as the composite $T \circ \nu^{-1}_{X,X} \circ \eta_{S,X}$. Finally, the third point implies the first (c.f. [LMS, Proposition III.1.3(ii)]) because one can define an inverse to

$$\nu_{X,Y} : F(X,S) \wedge Y \to F(X,Y)$$

as the following composite:

$$\nu^{-1}_{X,Y} : F(X,Y) \cong F(X,Y) \wedge S \overset{id \wedge coev}{\longrightarrow} F(X,Y) \wedge X \wedge F(X,S) \overset{\epsilon_{X,Y} \wedge id}{\longrightarrow}$$
$$\longrightarrow Y \wedge F(X,S) \overset{T}{\longrightarrow} F(X,S) \wedge Y$$

□

Definition 1.4.2. *If X satisfies any of the preceding conditions, it is called* strongly dualizable.
The spectrum $DX = F(X,S)$ is called the (motivic) Spanier-Whitehead dual *of X. By definition, $D := F(-,S)$ is a contravariant functor*

$$D : \mathcal{SH}_k \to \mathcal{SH}_k$$

and similarly $D := F(-,S^{\wedge}_l)$ is a contravariant functor

$$D : \mathcal{SH}^{\wedge}_{k,l} \to \mathcal{SH}^{\wedge}_{k,l}$$

on the category of l-complete spectra. In fact, the obvious map

$$F(-,S) \to F(-,S_l^\wedge)$$

is a completion at l, but we will neither need nor prove it.

The strong dual is in particular reflexive:

Lemma 1.4.3. *If X is strongly dualizable, then $DDX \cong X$.*

Proof. The composite

$$X \wedge DX \xrightarrow{T} DX \wedge X \xrightarrow{\epsilon_{X,S}} S$$

defines an element in

$$\mathrm{Hom}_{\mathcal{SH}_k}(X \wedge DX, S) \cong \mathrm{Hom}_{\mathcal{SH}_k}(X, F(DX,S)) = \mathrm{Hom}_{\mathcal{SH}_k}(X, DDX)$$

and hence a map $\rho : X \to DDX$.
One can define an inverse $\rho^{-1} : DDX \to X$ as follows:(c.f.[LMS, Proposition III.1.3(i)]):

$$\begin{aligned} DDX \cong S \wedge DDX &\xrightarrow{coev \wedge id} X \wedge DX \wedge DDX \xrightarrow{id \wedge T} X \wedge DDX \wedge DX \xrightarrow{id \wedge \epsilon_{DX,S}} \\ &\longrightarrow X \wedge S \cong X \end{aligned}$$

□

Just as in classical topology([Rav2, Proof of Corollary 5.1.5]) the spectrum $DX \wedge X$ has the structure of a homotopy ring spectrum:

Remark 1.4.4. *If X is a strongly dualizable motivic spectrum, then the unit map*

$$e : S \xrightarrow{\eta_{S,X}} F(X,X) \cong F(X,S) \wedge X = DX \wedge X$$

and the multiplication map

$$\mu : DX \wedge X \wedge DX \wedge X \xrightarrow{D(e)} DX \wedge S \wedge X \cong DX \wedge X$$

endow $DX \wedge X$ with the structure of motivic homotopy ring spectrum (in fact an A_∞-structure, but we are not going to use or prove it), where we use

$$X \wedge DX \cong DDX \wedge DX = D(DX \wedge X)$$

in the definition of $D(e)$.

Lemma 1.4.5. *The functor D maps cofiber sequences to cofiber sequences, and the full subcategory of strongly dualizable spectra in $\mathcal{SH}_k$ is thick.*

Proof. For the first statement, let $X \to Y \to Z$ be a cofiber sequence. Because $\mathcal{SH}_k$ is the homotopy category of a pointed monoidal model category, the functor $F(-, A)$ maps cofiber sequences to fiber sequences for any A in $\mathcal{SH}_k$ (c.f. [Hov, 6.6]). In particular this is true for $D(-) = F(-, S)$. Because $\mathcal{SH}_k$ is stable, fiber and cofiber sequences agree, and $DZ \to DY \to DX$ is a cofiber sequence again.
For the second statement we only need to show that a retract of a strongly dualizable spectrum is again strongly dualizable, so let A be a retract of a strongly dualizable spectrum X. Note that by the first point of 1.4.1 we have to show that the canonical map

$$F(A, S) \wedge Y \to F(A, Y)$$

is an isomorphism for all motivic spectra Y, and we already now this statement is true if we replace A with X. But this follows immediately from the following diagram:

$$\begin{tikzcd}
F(X,S)\wedge Y \arrow[r, "\cong"] \arrow[loop above, "id"] \arrow[d, shift left] & F(X,Y) \arrow[loop above, "id"] \arrow[d, shift right] \\
F(A,S)\wedge Y \arrow[r] \arrow[u, bend left] & F(A,Y) \arrow[u, bend right]
\end{tikzcd}$$

□

Lemma 1.4.6. *All spectra in $\mathcal{SH}_{\mathbb{C}}^{qfin}$ are strongly dualizable in $\mathcal{SH}_{\mathbb{C}}$, and $\mathcal{SH}_{\mathbb{C}}^{qfin}$ is closed under taking duals.*
As a consequence, all spectra in $\mathcal{SH}_{k,l}^{\wedge,qfin}$ are strongly dualizable in $\mathcal{SH}_{\mathbb{C}(l)}$, and $\mathcal{SH}_{k,l}^{\wedge,qfin}$ is closed under taking duals.

Proof. Finite cell spectra are contained in the thick subcategory of compact spectra, and compact spectra are dualizable(See [NSO, Remark 4.1] or [Joa, 5.2.7]). Therefore the thick subcategory generated by finite cell spectra is dualizable.
It remains to show that $\mathcal{SH}_{\mathbb{C}}^{qfin}$ is closed under taking duals. The dual of suspensions of the sphere spectrum are suspensions of the sphere spectrum. Therefore, if X is a finite cell spectrum, its Spanier-Whitehead dual DX is also a finite cell spectrum by cellular induction and hence in $\mathcal{SH}_{\mathbb{C}}^{qfin}$.
If X is a retract of a spectrum $F \in \mathcal{SH}_{\mathbb{C}}^{qfin}$ such that $DF \in \mathcal{SH}_{\mathbb{C}}^{qfin}$, with maps $r : F \to X$ and $s : X \to F$ such that $r \circ s = id_X$, then DX is a retract of $DF \in \mathcal{SH}_{\mathbb{C}}^{qfin}$ with maps $Ds : DF \to DX$ and $Dr : DX \to DF$ because $Ds \circ Dr = id_{DX}$.
If $X \to Y \to Z$ is a cofiber sequence and DX and DY are both in $\mathcal{SH}_{\mathbb{C}}^{qfin}$, then $DZ \to DY \to DX$ is a cofiber sequence by 1.4.5, which implies that DZ is in $\mathcal{SH}_{\mathbb{C}}^{qfin}$.

□

1.5 Coefficients of motivic cohomology and the dual motivic Steenrod algebra

One key ingredient for the Adams spectral sequence is knowlegde of the Steenrod algebra or of the dual Steenrod algebra. Motivically, the Steenrod Algebra was described by Voevodsky for fields of characteristic zero and later by Hoyois, Kelly and Østvær in positive characteristic. While some interesting phenomenas happen at the prime two, the motivic Steenrod algebra is more closely related to the classical topological Steenrod algebra at odd primes. To describe the motivic Steenrod algebra it is sufficient to know the coefficients of motivic coholomogy with $\mathbb{Z}/l\mathbb{Z}$-coefficients, so we will compute these coefficients over the base fields $\mathbb{R}$ and $\mathbb{C}$:

Proposition 1.5.1. *Let $l \neq 2$. be a prime*

1. *The coefficients $H\mathbb{Z}/l^{**}$ of motivic cohomology are given as a ring by*
$$H\mathbb{Z}/l^{**} \cong \mathbb{Z}/l[\tau]$$
with $|\tau| = (0,1)$ over $k = \mathbb{C}$.
2. *The coefficients $H\mathbb{Z}/l^{**}$ of motivic cohomology are given as a ring by*
$$H\mathbb{Z}/l^{**} \cong \mathbb{Z}/l[\theta]$$
with $|\theta| = (0,2)$ over $k = \mathbb{R}$.
3. *The map $H\mathbb{Z}/l^{**}_{\mathbb{R}} \to H\mathbb{Z}/l^{**}_{\mathbb{C}}$ is given by $\theta \mapsto \tau^2$.*
4. *For a fixed bidegree p, q with $p \leq q$ there is a commutative square*
$$\begin{tikzcd} H^{p,q}_{\mathbb{R}}(pt, \mathbb{Z}/l) \arrow[r] \arrow[d, "\cong"', "R'"] & H^{p,q}_{\mathbb{C}}(pt, \mathbb{Z}/l) \arrow[d, "\cong"', "R"] \\ H^{p-q+q\sigma}(pt, \underline{\mathbb{Z}/l}) \arrow[r, "res_e^{C_2}"] & H^{p}_{sing}(pt, \mathbb{Z}/l) \end{tikzcd}$$
Here H_k denotes motivic cohomology over the base field k, $H^{p-q+q\sigma}(pt, \underline{\mathbb{Z}/l})$ denotes Bredon cohomology (graded by the trivial representation and the sign representation σ) and H_{sing} denotes singular cohomology. The top map is the one induced by $Spec(\mathbb{C}) \to Spec(\mathbb{R})$, the bottom map is the restriction functor to the trivial group.
In particular, both τ and θ do not vanish under topological realization.

Proof. We know that $H\mathbb{Z}/l^{**} = 0$ for $q < p$ ((cf. [MVW, Theorem 3.6])).
Let $q \geq p$. Then there is an isomorphism from motivic to étale cohomology:
$$H^{p,q}(Spec(k), \mathbb{Z}/l) \cong H^p_{ét}(k, \mu_l^{\otimes q})$$

This isomorphism respects the product structure([GL, 1.2,4.7]).
The étale cohomology groups $H^p_{\acute{e}t}(k,\mu_l^{\otimes q})$ can be computed as the Galois cohomology of the separable closure of the base field (in both cases the complex numbers) with coefficients in the l-th roots of unity. The action of the absolute Galois group G is given by the trivial action if $k=\mathbb{C}$ and by complex conjugation if $k=\mathbb{R}$:

$$H^p_{\acute{e}t}(k,\mu_l^{\otimes q}) \cong H(G,\mu_l^{\otimes q}(\mathbb{C}))$$

1. For $k=\mathbb{C}$, these groups all vanish for $p\neq 0$ by triviality of the Galois action, and they are $\mathbb{Z}/l$ in the degree $p=0$ for all $q\geq 0$. The multiplicative structure is given by the tensor product of the modules.

2. For $k=\mathbb{R}$, we have the following isomorphism of G-modules:

$$\begin{tikzcd} \mu_l\otimes\mu_l \arrow[r] \arrow[d, "\cong"] & \mu_l \arrow[d, "\cong"'] \\ \mathbb{Z}/l\otimes\mathbb{Z}/l \arrow[r, "\cong", "a\otimes b\mapsto ab"'] & Z/l \end{tikzcd}$$

   Here G acts on the top left hand side by complex conjugation on each factor, on the lower left hand side by the assignment $x\mapsto -x$ on each factor, and trivial on the right hand side. Hence $\mu_l^{\otimes q}$ is isomorphic as a G-module to μ_l equipped with the trivial action in degrees with q even, and with the Galois action in degrees with q odd. Since the latter has no nontrivial fixed points, the description above follows additively. The multiplicative statement follows from the same reasoning as for $k=\mathbb{C}$.

3. On the level of Galois cohomology, the map induced by $\mathbb{R}\to\mathbb{C}$ corresponds to the one induced by the map of groups which embeds the trivial group (the absolute Galois group of $\mathbb{C}$) into the Galois group of $\mathbb{R}$. Since all the Galois cohomology groups are concentrated in degree 0, the map is just the inclusion of the fixed points in $\mu_l^{\otimes q}$ under the action of the Galois group of $\mathbb{R}$ into the fixed points of $\mu_l^{\otimes q}$ under the action of the trivial group, and the third statement follows.

4. The statement is a special case of [HO, Thm. 4.18].

□

Remark 1.5.2. *We also have $H\mathbb{Z}/l_{**}=H\mathbb{Z}/l^{-*,-*}$. In an abuse of notation, we denote the elements in $H\mathbb{Z}/l_{**}$ corresponding to τ and θ by the same name, where the bidegree is the same as above multiplied by -1.*

With the knowledge of the coefficients $H\mathbb{Z}/l_{**}$ and the fact that for odd primes, they are concentrated in simplicial degree 0, we can now give a description of the dual motivic Steenrod algebra.

The computation of the motivic mod-l Steenrod algebra over basefields of characteristic 0 is due to Voevodsky in [Voe2]. The implications for the dual motivic Steenrod algebra are for example written down in the introduction of [HKO2].

Proposition 1.5.3. *Let k be a base field of characteristic 0, and let l be an odd prime. The dual motivic Steenrod algebra A_{**} and its Hopf algebroid structure over $k \in \{\mathbb{C}, \mathbb{R}\}$ for l an odd prime can be described as follows:*

$$A_{**} = H\mathbb{Z}/l_{**}[\tau_0, \tau_1, \tau_2, ..., \xi_1, \xi_2, ...]/(\tau_i^2 = 0)$$

Here $|\tau_i| = (2l^i - 1, l^i - 1)$ and $|\xi_i| = (2l^i - 2, l^i - 1)$.
The comultiplication is given by

$$\Delta(\xi_n) = \sum_{i=0}^{n} \xi_{n-i}^{l^i} \otimes \xi_i$$

where $\xi_0 := 1$, and

$$\Delta(\tau_n) = \tau_n \otimes 1 + \sum_{i=0}^{n} \xi_{n-i}^{l^i} \otimes \tau_i$$

Remark 1.5.4. *The τ_i are not related to the element τ defined above. Neither the product nor the coproduct increase the number of τ_i's involved in any given expression in A_{**}. Hence A_{**} can be graded as an A_{**} -comodule by this number. This is similar to the classical situation at odd primes.*

Chapter 2

The motivic Adams and Adams-Novikov spectral sequences at odd primes

2.1 Generalized motivic Adams spectral sequences

The motivic Adams spectral sequence (from now on abbreviated as MASS) was already used by Morel in his computation of the zeroth motivic stable stem(c.f. [Mor]). Dugger and Isaksen have used the MASS for extensive computations over $\mathbb{C}$ at the prime 2(c.f. [DI]), and in fact used the additional information available in the MASS to deduce new information about the classical Adams spectral sequence. In other work Isaksen has extended these computations to the base field $\mathbb{R}$. The construction of the generalized motivic Adams spectral sequence is well known, and mirrors the classical construction. For the sake of completeness, we give an account of its construction and its interaction with complex Betti realization.

Let E be an arbitrary motivic ring spectrum and X a motivic spectrum. Assume furthermore that $E_{**}E$ is flat as a (left) module over the coefficients E_{**}. This is not necessary for the construction, but allows the identification of the E_2-term via homological algebra. In this case one can associate a flat Hopf algebroid to E (See [NSO, Lemma 5.1] for the statement and [Rav, Appendix 1] for the definition and basic properties of Hopf algebroids), and the category of comodules over this Hopf Algebroid is abelian and thus permits homological algebra.

Define $\bar{E}$ as the cofiber of the unit map $S \to E$. Smashing the cofiber sequence $\bar{E} \to S \to E$ with $\bar{E}^s \wedge X$ yields cofiber sequences

$$\bar{E}^{\wedge(s+1)} \wedge X \to \bar{E}^{\wedge s} \wedge X \to E \wedge \bar{E}^{\wedge s} \wedge X$$

giving rise to the following tower, called the canonical E_{**}-Adams resolution:

$$\begin{tikzcd}
\dots \arrow[r] \arrow[d] & \bar{E}^{\wedge(s+1)} \wedge X \arrow[r] \arrow[d] & \bar{E}^{\wedge s} \wedge X \arrow[r] \arrow[d] & \dots \arrow[r] \arrow[d] & \bar{E} \wedge X \arrow[r] \arrow[d] & X \arrow[d] \\
\dots & E \wedge \bar{E}^{\wedge(s+1)} \wedge X & E \wedge \bar{E}^{\wedge s} \wedge X & \dots & E \wedge \bar{E} \wedge X & E \wedge X
\end{tikzcd}$$

The long exact sequences of homotopy groups associated to these cofiber sequences look as follows:

$$\dots \to \pi_{p+1,q}(E \wedge \bar{E}^s \wedge X) \to \pi_{p,q}(\bar{E}^{s+1} \wedge X) \to \pi_{p,q}(\bar{E}^s \wedge X) \to \pi_{p,q}(E \wedge \bar{E}^s \wedge X) \to \dots$$

Together they form a trigraded exact couple

$$\begin{tikzcd}
\pi_{**}(\bar{E}^{\wedge *} \wedge X) \arrow[rr] & & \pi_{**}(\bar{E}^{\wedge *} \wedge X) \arrow[dl] \\
& \pi_{**}(E \wedge \bar{E}^{\wedge *} \wedge X) \arrow[ul] &
\end{tikzcd}$$

and thus give rise to a trigraded spectral sequence $E_r^{s,t,u}(E, X)$ with differentials $d_r : E_r^{s,t,u} \longrightarrow E_r^{s+r,t+r-1,u}$.

Because $E_{**}E$ is flat over E_{**} there is an isomorphism (see [NSO, Lemma 5.1(i)])

$$\pi_{**}(E \wedge E \wedge X) \cong E_{**}(E) \underset{E_{**}}{\otimes} E_{**}(X)$$

allowing us to identify the long exact sequences of homotopy groups of the canonical E_{**}-Adams resolution with the (reduced) cobar complex $C^*(E_{**}(X))$. For this reason the resolution is also referred to as the geometric cobar complex.

The E_2- page of the E-Adams spectral sequence can then be described as:

$$E_2^{s,t,u}(E, X) = \mathrm{Cotor}^{s,t,u}_{E_{**}(E)}(E_{**}, E_{**}(X))$$

Here Cotor denotes the derived functors of the cotensor product in the category of $E_{**}(E)$-comodules and can be computed as the homology of the cobar complex $C^*(E_{**}(X))$.

Remark 2.1.1. *Assume now that $k = \mathbb{C}$. Then Betti realization induces a map of spectral sequences $R_{E,X} : E_r^{s,t,u}(E, X) \to E_r^{s,t}(R(E), R(X))$*

This can be checked by going through the definitions: Because Betti realization preserves cofiber sequences and smash products, we have $R(\bar{E}) = \overline{R(E)}$, and the realization of the canonical E_{**}-Adams resolution for X is the canonical $R(E)_*$-Adams resolution for the topological spectrum $R(X)$. If we consider the induced maps on the long exact sequences of homotopy groups defining the exact couple, we get the following commutative diagram:

$$\begin{tikzcd}
\dots \arrow[r] \arrow[d, "R"] & \pi_{p,*}(\bar{E}^{\wedge(s+1)} \wedge X) \arrow[r] \arrow[d, "R"] & \pi_{p,*}(\bar{E}^{\wedge s} \wedge X) \arrow[r] \arrow[d, "R"] & \pi_{p,*}(E \wedge \bar{E}^{\wedge s} \wedge X) \arrow[r] \arrow[d, "R"] & \dots \arrow[d, "R"] \\
\dots \arrow[r] & \pi_p(\overline{R(E)}^{\wedge(s+1)} \wedge R(X)) \arrow[r] & \pi_p(\overline{R(E)}^{\wedge s} \wedge R(X)) \arrow[r] & \pi_p(R(E) \wedge \overline{R(E)}^{\wedge s} \wedge R(X)) \arrow[r] & \dots
\end{tikzcd}$$

In particular, Betti realization induces a map of exact couples and hence a map of spectral sequences.

By very much the same reasoning we get

Remark 2.1.2. *Assume $k = \mathbb{R}$. The functor $\mathcal{SH}_{\mathbb{R}} \to \mathcal{SH}_{\mathbb{C}}$ associated to $Spec(\mathbb{C}) \to Spec(\mathbb{R})$ gives rise to a map of spectral sequences*

$$E_r^{s,t,u}(E, X) \to E_r^{s,t,u}(E_{\mathbb{C}}, X_{\mathbb{C}}).$$

Convergence in this general situation is discussed by Bousfield in [Bous]; under certain assumptions, the spectral sequence converges completely(c.f.[Bous, Chapter 6]) to a filtration of the homotopy groups of the so called E-nilpotent completion of X, which we will define later in 2.1.3. There are three cases of particular interest for us: The homological motivic Adams spectral sequence, where $E = H\mathbb{Z}/l, X = S$; the motivic Adams spectral sequence for the computation of the coefficients of the l-completed motivic Brown-Peterson spectrum $ABP_l^\wedge$, where $E = H\mathbb{Z}/l, X = ABP_l^\wedge$; and the l-complete motivic Adams-Novikov spectral sequence, where $E = ABP_l^\wedge, X = S$. In all these cases the spectral sequence actually converges strongly either because of a vanishing line or because the spectral sequence degenerates.

One way to define the E-nilpotent completion of X is

Definition 2.1.3. *Let X be a motivic spectrum and E be a motivic ring spectrum. Define the E-nilpotent completion $X_E^\wedge$ of X as the homotopy limit over the semicosimplicial object (i.e. a cosimplicial object without degeneracy maps)*

$$(X_E^\wedge)_n := (X \wedge E^{\wedge n})$$

Here the coface maps are given by the unit map of E.

Remark 2.1.4.
1. *This definition of the E-nilpotent completion is slightly different to the one given in [Bous], which makes use of an alternative Adams tower. There is however an equivalence between the two by the remarks in [DI, 6.7], where the work in [Bous] is adapted to the motivic situation.*
2. *If X has the structure of a motivic ring spectrum, then so has its E-nilpotent completion $X_E^\wedge$, also by [DI, 6.7].*
3. *If X is a module over E, then X is equivalent to its E-nilpotent completion via the canonical map $X \to X_E^\wedge$. (See [DI, 6.9].)*

4. *As explained in [DI, 7.3], the preceding statement implies that the bicompletion $S_{H\mathbb{Z}/l,ABP}$, towards whose homotopy groups the motivic Adams-Novikov spectral sequence converges, can be described as:*

$$S^{\wedge}_{H\mathbb{Z}/l,ABP} \cong S^{\wedge}_{H\mathbb{Z}/l}$$

The arguments given by Dugger and Isaksen are independent of the chosen base field $\mathbb{C}$ and their restriction to the prime 2. They work equally well if l is an odd prime and for the base field $\mathbb{R}$.

The E-nilpotent completion of X is the appropiate spectrum to discuss when one considers the convergence of generalized Adams spectral sequences. It is however helpful to have a more explicit description of it to relate its homotopy groups to the homotopy groups of X that one actually wants to compute.
In the case $E = H\mathbb{Z}/l$, the $H\mathbb{Z}/l$-nilpotent completion has been described more explicitely by Hu, Kriz and Ormsby in [HKO2, Theorem 1], which we restate here for convenience:

Theorem 2.1.5. *Let the base field k be of characteristic 0 and let X be a motivic cell spectrum of finite type(See 1.2.2).*
Then the map $X \longrightarrow X^{\wedge}_{H\mathbb{Z}/l}$ is a completion at l and η, i.e. $X^{\wedge}_{H\mathbb{Z}/l} \simeq X^{\wedge}_{l,\eta}$.
Furthermore, if either $l > 2$, -1 is a sum of squares in k and $cd_l(k) < \infty$ or $l = 2$ and $cd_2(k[i]) < \infty$, then $X \to X^{\wedge}_{H\mathbb{Z}/l}$ is a completion at l. Here $cd_l(k)$ denotes the cohomological dimension of the base field at the prime l.

2.2 The motivic Adams-Novikov spectral sequence

Just as we needed the coefficients of motivic cohomology and the structure of the dual motivic Steenrod algebra for the Adams spectral sequence, we need the coeffients of the (l-completed) algebraic Brown-Peterson spectrum and the algebraic structure of the Hopf algebroid generated by it as input for the motivic Adams-Novikov spectral sequence.

Let $k \subset \mathbb{C}$ and l be an arbitrary prime. Let MGL denote the algebraic cobordism spectrum of Voevodsky (see [Voe, 6.3]) and MU the topological cobordism spectrum. The motivic Brown-Peterson spectrum ABP was first defined by Vezzosi using the Quillen idempotent in [Vez]; and as proven by Hoyois in [Hoy, Remark 6.20], it can equivalently be defined by either the motivic Landweber exact functor theorem or as the quotient of $MGL_{(l)}$ by any regular sequence of elements generating the vanishing ideal of the l-typical formal group law.
We use the same definition as [Joa, 6.3.1]):

Definition 2.2.1. *Choose elements $a_i^{top} \in MU_{2i}$ satisfying the conditions in [Hoy, 6.1, last section] and write a_i for their image in $MGL_{2i,i}$ under the canonical map $L \cong MU_{2*} \to MGL_{2*,*}$. Then $R(a_i) = a_i^{top}$, and we can define the motivic Brown-Peterson spectrum:*

$$ABP := MGL_{(l)}/(a_i | i \neq l^j - 1)$$

Additionally, define $v_n^{top} := a^{top}_{l^n-1}$ *and* $v_n := a_{l^n-1}$.

We summarise a few properties of ABP that we will use later:

Remark 2.2.2. *1. The sphere spectrum* S *and the Brown-Peterson spectrum* ABP *(regardless of the prime* l*) are cell spectra of finite type over an arbitrary basefield.*

2. *If* $k = \mathbb{C}$*, we have* $R(ABP) = BP$ *and* $R(v_n) = v_n^{top}$*.(Mentioned in [Joa, 6.3.2] and clear by construction)*

3. *By [Hoy, Thm. 6.11] we have an isomomorphism* $H\mathbb{Z}/l_{**}(ABP) \cong P_{**}$ *as* A_{**}*-comodules, where* P_{**} *is the polynomial subalgebra*

$$P_{**} = H\mathbb{Z}/l_{**}[\xi_1, \xi_2, ...]$$

of the dual motivic Steenrod algebra.

Both the Adams spectral sequence and the slice spectral sequence converge to the nilpotent completion of the homotopy groups of ABP, so a priori the completion at the motivic Hopf map η might be involved. Because the motivic Hopf is not detected by ABP however, this does not matter:

Proposition 2.2.3. *Let* k *be any field. Then* $ABP^{\wedge}_{\eta,l} \simeq ABP^{\wedge}_l$.

Proof. Since $MGL \wedge \eta = 0$ (for a proof, see e.g. [Joa, Lemma 7.1.1]), we also have $ABP \wedge \eta = 0$ and $ABP^{\wedge}_l \wedge \eta = 0$. Hence all maps in the homotopy limit

$$\underset{\leftarrow}{\mathrm{holim}} ABP^{\wedge}_l/\eta^n$$

are equivalences, and this homotopy limit modells the completion at l and η. □

Proposition 2.2.4. *If* $k = \mathbb{C}$ *and* l *an odd prime, then*

$$\pi_{**}(ABP^{\wedge}_l) = \mathbb{Z}_l[\tau, v_1, v_2, ...]$$

If $k = \mathbb{R}$ *and* l *an odd prime, then*

$$\pi_{**}(ABP^{\wedge}_l) = \mathbb{Z}_l[\theta, v_1, v_2, ...]$$

Here the elements v_i *have bidegree* $(2l^i - 2, l^i - 1)$.

Proof. Let $k = \mathbb{C}$. Consider the motivic Adams spectral sequence for ABP. Since ABP is cellular it converges to $\pi_{**}(ABP^{\wedge}_l)$. The E_2-page has the form $E_2^{s,t,u} = \mathbb{Z}/l[\tau, Q_0, Q_1, ..]$, where Q_n lives in degree $(1, 2l^n - 2, l^n - 1)$. Apart from the grading by weight this agrees with the classical ASS for BP, and the spectral sequence collapses at the E_2-page. The extension problem is solved by considering topological realization over $\mathbb{C}$.

□

Remark 2.2.5. *More generally, the coefficients of the* l*-completed motivic cobordism spectrum* $MGL^{\wedge}_l$ *have been computed over a* l*-low dimensional base field (a base field with exponential characteristic* $p \neq l$ *s.t.* $cd_l(k) \leq 2$*) by Ormsby and Østvær using the slice spectral sequence, cf. [OO, Corollary 2.6]. In particular, these computations cover the case of* $k = \mathbb{R}$ *if* l *is an odd prime* $l \neq 2$.

2.3 The relation of the motivic Adams spectral sequence and the motivic Adams Novikov spectral sequence at odd primes

Both in the classical topological setting (e.g. [Rav, Theorem 4.4.47]), but also motivically over $\mathbb{C}$ (see [DI, Chapter 8]) the Adams spectral sequence and the Adams-Novikov spectral sequence at the prime 2 both provide relevant information about the homotopy groups of the spheres; in fact, one can use knowledge gained in one spectral sequence to deduce differentials in the other and vice versa. In contrast to this, the classical Adams spectral sequence at odd primes yields strictly less information than the classical Adams Novikov spectral sequence. This can be stated precisely via two auxiliary spectral sequences: The first one is the Cartan-Eilenberg spectral sequence, associated to an extension of Hopf algebroids $P_* \to A_* \to E_*$. This extension is split at odd primes, and the Cartan-Eilenberg spectral sequence collapses. The second one is the algebraic Adams Novikov spectral sequence, constructed by filtering the Hopf algebroid of BP by its coaugmentation ideal. These spectral sequences start from the same, albeit differently indexed E_2-term and converge to the E_2-terms of the Adams spectral sequence and the Adams Novikov spectral sequence respectively. An account of the topological situation is given in Ravenels green book ([Rav, 4.4]), and the relevant spectral sequences form the following square:

$$\begin{tikzcd}
\mathrm{Cotor}_{P_*}(H\mathbb{Z}/l_*, \mathrm{Cotor}^{s,t,u}_{E_*}(H\mathbb{Z}/l_*, H\mathbb{Z}/l_*)) \arrow[r, Rightarrow, "AANSS"] \arrow[d, Rightarrow, "CESS"] & Cotor_{BP^\wedge_{l,**}BP^\wedge_l}(BP^\wedge_l, BP^\wedge_l) \arrow[d, Rightarrow, "ANSS"] \\
Cotor_{A_{**}}(H\mathbb{Z}/l_{**}, H\mathbb{Z}/l_{**}) \arrow[r, Rightarrow, "ASS"] & \pi_{**}(S^\wedge_l)
\end{tikzcd}$$

Motivically the situation over $\mathbb{C}$ at the prime 2 is described in [AM, 8.1]. It is even closer to the classical picture at odd primes for both $k = \mathbb{C}$ and $k = \mathbb{R}$, because the relevant Hopf algebroids are just the regraded classical Hopf algebroids base changed to the coefficients of motivic homology. As a consequence, for the purpose of the computation of the homotopy groups of the spheres, we can focus entirely on the motivic Adams-Novikov spectral sequence.

2.3.1 The motivic Cartan Eilenberg spectral sequence

Definition 2.3.1. *Define P_{**} as the polynomial sub-Hopfalgebra of the dual motivic Steenrod Algebra and E_{**} as the exterior part, i.e.*

$$P_{**} = H\mathbb{Z}/l_{**}[\xi_1, \xi_2, ...]$$

$$E_{**} = H\mathbb{Z}/l_{**}[\tau_0, \tau_1, \tau_2, ...]/(\tau_i^2 = 0)$$

Consider the classical extension of Hopf algebras $P_* \to A_* \to E_*$. We can regard all objects here as a bigraded, concentrated in degree 0 with respect to the *second bidegree. By our computation above $H\mathbb{Z}/l_{**}$ is just a polynomial ring in one generator and hence flat over $\mathbb{Z}/l$. Hence by application of $- \underset{\mathbb{Z}/l}{\otimes} H\mathbb{Z}/l_{**}$ to the above extension, we obtain an extension of Hopf algebras $P_{**} \to A_{**} \to E_{**}$ in the sense of [Rav, A1.1.15], where the middle term is precisely the motivic Steenrod algebra over odd primes.*

Associated to such an extension we immediately get a motivic counterpart to [Rav, Theorem 4.4.3]:

Proposition 2.3.2. *1. $Cotor_{E_{**}}(H\mathbb{Z}/l_{**}, H\mathbb{Z}/l_{**}) \cong H\mathbb{Z}/l_{**}[a_0, a_1, ..]$, where $a_i \in Cotor^{1,2l^i-1,l^i-1}_{E_{**}}$ is represented by the cobar cycle $[\tau_i]$.*

2. *There is a motivic Cartan Eilenberg spectral sequence converging to*

$$Cotor_{A_{**}}(H\mathbb{Z}/l_{**}, H\mathbb{Z}/l_{**})$$

(the reindexed E_2-term of the motivic Adams spectral sequence) with the following E_2-page:

$$E_2^{s_1,s_2,t,u} = Cotor^{s_1,t,u}_{P_{**}}(H\mathbb{Z}/l_{**}, Cotor^{s_2,*,*}_{E_{**}}(H\mathbb{Z}/l_{**}, H\mathbb{Z}/l_{**}))$$

and differential $d_r : E_r^{s_1,s_2,t,u} \to E_r^{s_1+r,s_2-r+1,t,u}$.

3. *The P_{**}-coaction on $Cotor_{E_{**}}(H_{**}, H_{**})$ is given by*

$$\psi(a_n) = \sum_i \xi_{n-i}^{l^i} \otimes a_i$$

4. *The motivic Cartan Eilenberg SS collapses at the E_2 page with no non-trivial extensions.*

2.3.2 The motivic algebraic Novikov spectral sequence

Let $k = \mathbb{C}$ and l be an arbitrary prime or let $k = \mathbb{R}$ and let l be an odd prime. Remember that $ABP^{\wedge}_{l,**} = \begin{cases} \mathbb{Z}_l[\tau, v_1, v_2, ...] & k = \mathbb{C} \\ \mathbb{Z}_l[\theta, v_1, v_2, ...] & k = \mathbb{R}, l \neq 2 \end{cases}$.

Remark 2.3.3. *The powers of the ideal $I = (l, v_1, v_2, ..) \subset ABP^{\wedge}_{l,**}$ define a decreasing filtration $I^{r+1} \subset I^r$ of $ABP^{\wedge}_{l,**} = \mathbb{Z}_l[\tau, v_1, v_2, ...]$. The associated graded of $ABP^{\wedge}_{l,**}$ is $E_0 ABP^{\wedge}_{l,**} \cong \mathbb{Z}/l[\tau, q_0, q_1, q_2, ...]$. Here q_0 is the class of l and q_i is the class of v_i for $i \geq 1$, so the grading of q_i is $(1, 2(l^i - 1))$.*
One can check by the same arguments as for $I = (l, v_1, v_2, ..) \subset BP_*$ that I is an invariant ideal with respect to the structure maps of the Hopf Algebroid $ABP^{\wedge}_{l,**}(ABP^{\wedge}_{l,**})$. Since I is invariant, this filtration of the Hopf algebroid induces a Hopf algebroid structure on the associated graded, and the induced filtration on comodules induces a comodule structure on the associated graded of the comodules. In particular we obtain a filtration of the cobar complex as a differential graded comodule.

Proposition 2.3.4. *There is a spectral sequence called the motivic algebraic Adams-Novikov spectral sequence converging to* $Cotor_{ABP^{\wedge}_{l,**}ABP^{\wedge}_l}(ABP^{\wedge}_l, ABP^{\wedge}_l)$. *The* E_1*-page is given by:*

$$E_1 = Cotor_{P_{**}}(H\mathbb{Z}/l_{**}, E_0ABP^{\wedge}_{l,**}) = Cotor_{P_{**}}(H\mathbb{Z}/l_{**}, Cotor^{s,t,u}_{E_{**}}(H\mathbb{Z}/l_{**}, H\mathbb{Z}/l_{**}))$$

Proof. This works as the classical proof(See [Rav, 4.4.4]). □

Therefore we end up with essentially the same square of spectral sequences, regraded in regard to the weight degree and tensored with the coefficients of motivic homoloogy:

$$\begin{tikzcd}
\mathrm{Cotor}_{P_{**}}(H\mathbb{Z}/l_{**}, \mathrm{Cotor}^{s,t}_{E_{**}}(H\mathbb{Z}/l_{**}, H\mathbb{Z}/l_{**})) \arrow[r, Rightarrow, "MAANSS"] \arrow[d, Rightarrow, "MCESS"] & Cotor_{BP^{\wedge}_{l,**}BP^{\wedge}_l}(BP^{\wedge}_l, BP^{\wedge}_l) \arrow[d, Rightarrow, "MANSS"] \\
Cotor_{A_{**}}(H\mathbb{Z}/l_{**}, H\mathbb{Z}/l_{**}) \arrow[r, Rightarrow, "MASS"] & \pi_{**}(S^{\wedge}_l)
\end{tikzcd}$$

Furthermore, just as in the classical topological situation the CESS degenerates at the E_1-page.

2.4 The structure of the motivic Adams-Novikov spectral sequence at odd primes

If $E = ABP^{\wedge}_l$ and $X = S$, the E-Adams spectral sequence is called the l-completed motivic Adams Novikov spectral sequence(from now on abbreviated to MANSS) and was studied for example in [OO, 3],[HKO]. For l-low dimensional fields Ormsby and Østvær have described the E_2-page of the l-primary MANSS in terms of the E_2-page of the classical ANSS and of the coefficients $H\mathbb{Z}_{l**}$ in [OO], in order to calculate $\pi_{1,*}$ over these fields. We want to use their results. First we need to compute $ABP^{\wedge}_{l,**}(ABP^{\wedge}_{l,**})$, which is known by a result of Naumann, Spitzweck and Østvær ([NSO, Lemma 9.1]), as well as the Hopf algebroid structure induced by $ABP^{\wedge}_l$:

$$ABP^{\wedge}_{l,**}(ABP^{\wedge}_l) \cong ABP^{\wedge}_{l,**} \underset{\pi_*(BP)}{\otimes} BP_*(BP)$$

Over $\mathbb{C}$ and $\mathbb{R}$ the Hopf algebroid structure is given by tensoring the Hopf algebroid of the Brown Peterson spectrum BP with $\mathbb{Z}_l[\tau]$ resp. $\mathbb{Z}_l[\theta]$ over $\mathbb{Z}_l$.

Remark 2.4.1.
1. *The Hopf algebroid induced by* $ABP^{\wedge}_l$ *is flat for* $k = \mathbb{R}$ *and* $k = \mathbb{C}$.
2. *Let* $k = \mathbb{C}$ *and* l *be any prime.*
   By [HKO2, Theorem 1] we have $S^{\wedge}_{H\mathbb{Z}/l} \cong S^{\wedge}_l$ *and* $ABP^{\wedge}_{H\mathbb{Z}/l} \cong ABP^{\wedge}_l$, *because the condition regarding cohomological dimension obviously holds and* -1 *is a sum of squares.*

3. *Let $k = \mathbb{R}$ and l be an odd prime $l \neq 2$.*
   By comment 1 on [HKO2, Theorem 1] , the $H\mathbb{Z}/l$-nilpotent completion of the sphere spectrum S is a completion at l and η, and this completion is not equivalent to the completion at l alone.

By the second point of the above remark, this spectral sequence agrees with the $H\mathbb{Z}/l$-complete MANSS considered in [DI].

We recall some facts about the topological ANSS built from the spectrum BP, in particular sparseness, which we are going to need in order to compare the classical the ANSS to the MANSS. Because the structure of the classical and the motivic Adams-Novikov spectral sequence are very similar, we can then use these facts to prove similar statements for the MANSS.

Remark 2.4.2.

1. *There is a vanishing line $E_2^{s,t} = Cotor^{s,t}_{BP_*BP}(BP_*, BP_*) = 0$ if $t < 2s$. If we translate this to the Adams grading $(s, s') := (s, t - s)$, the condition reads as $s' < s$. (cf. [Rav, 5.1.23 (a)])*
2. *$E_2^{0,0} = Z_{(l)}$ and $E_2^{0,t} = 0$ for $t \neq 0$. (cf. [Rav, 5.2.1 (b)])*
3. *If $(s, t) \neq (0, 0)$, then $Cotor^{s,t}_{BP_*BP}(BP_*, BP_*)$ is a finite l-group. (cf. [Rav, 5.2.1 (a)] together with the previous statement). In particular, if we construct the generalized Adams spectral sequence for $BP_l^\wedge$, then this spectral sequence agrees with the ANSS in all bidegrees away from $(0, 0)$, and in this bidegree there is a single copy of $\mathbb{Z}_l$.*
4. *("Sparseness"): $E_2^{s,t} \neq 0$ only if t is divisible by $q = 2l - 2$. This means that a differential d_r can be nontrivial only if $r \equiv 1(\mod q)$. Hence $E_{nq+2} \cong E_{nq+3} \cong ... \cong E_{(n+1)q+1}$.[Rav, 4.4.2]*

The structure of the Hopf algebroid associated to $ABP_l^\wedge$ described above implies a similar decomposition for the cobar complex by its explicite definition:

$$C^*(ABP_l^\wedge) \cong C^*(BP) \underset{\pi_*(BP)}{\otimes} \pi_{**}(ABP_l^\wedge)$$

If $k \in \{\mathbb{C}, \mathbb{R}\}$ and l an odd prime, we can now use the computations of $\pi_{**}(ABP_l^\wedge)$:

$$C^*(ABP_l^\wedge) \cong C^*(BP) \underset{\mathbb{Z}_l}{\otimes} \mathbb{Z}_l[\tau]$$

over $\mathbb{C}$ and

$$C^*(ABP_l^\wedge) \cong C^*(BP) \underset{\mathbb{Z}_l}{\otimes} \mathbb{Z}_l[\theta]$$

over $\mathbb{R}$.

The map induced by $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$ is given by $\theta \mapsto \tau^2$ and by the identity on the other generators.
The universal coefficient theorem of homological algebra, applied to the cobar complex, then yields the following short exact sequence for the E_2-term of the l-complete MANSS, where $E_{2,top}$ denotes the E_2-term of the classical topological ANSS:

Lemma 2.4.3. *The E_2-term of the l-complete MANSS is related to the E_2-term of the classical Adams-Novikov spectral sequence via the short exact sequences*

$$0 \to E_{2,top}^{s,t} \underset{\mathbb{Z}_l}{\otimes} \mathbb{Z}_l \tau^n \to E_2^{s,t,\frac{t}{2}-n} \to Tor_1^{\mathbb{Z}_l}(E_{2,top}^{s-1,t}, \mathbb{Z}_l \tau^n) \to 0$$

over $\mathbb{C}$, and

$$0 \to E_{2,top}^{s,t} \underset{\mathbb{Z}_l}{\otimes} \mathbb{Z}_l \theta^n \to E_2^{s,t,\frac{t}{2}-n} \to Tor_1^{\mathbb{Z}_l}(E_{2,top}^{s-1,t}, \mathbb{Z}_l \theta^n) \to 0$$

over $\mathbb{R}$. Note that $E_{2,top}^{s,t} = 0$ if t is odd because of sparseness.

The following results take the same form as in the case $l = 2$, discussed in [Isa] and [HKO]:

Proposition 2.4.4. *Assume that we know the E_2-page of the classical ANSS (associated to either BP or $BP_l^\wedge$) in a certain range. The E_2 page of the odd-primary MANSS over $\mathbb{C}$ can then be constructed from this information as follows:*

1. *$E_2^{0,0,u} = 0$ if $u > 0$ and $E_2^{0,0,u} \cong \mathbb{Z}_l \tau^u$ if $u \leq 0$. By the multiplicative structure of the MANSS, each $E_2^{s,t,*}$ is a $\mathbb{Z}_l[\tau]$-module, so we can speak of τ-torsion.*
2. *Let $(s,t) \neq (0,0)$: For each group $\mathbb{Z}/l^n$ in $E_2^{s,t}$ of the ANSS, there is a group $\mathbb{Z}/l^n[\tau]$ in $E_2^{s,t}$ of the MANSS, and its generator as a $\mathbb{Z}_l[\tau]$-module has weight $\frac{t}{2}$. There are no other groups in $E_2^{s,t,u}$ of the MANSS.*
3. *The vanishing line in the classical ANSS carries over to the MANSS, so the MANSS converges strongly.*
4. *$E_2^{s,t,u} \neq 0$ only if t is divisible by $q = 2l-2$. This means that a differential d_r can be nontrivial only if $r \equiv 1 (\mod q)$. Hence $E_{nq+2} \cong E_{nq+3} \cong \ldots \cong E_{(n+1)q+1}$.*

Proof. In 2.4.3, the torsion term $\mathrm{Tor}_1^{\mathbb{Z}_l}$ vanishes(the second argument is free over the ground ring). This proves the first two statements. The last two statements can either be proven directly by examining the cobar complex or they can be seen as a corollary of the first two. ☐

Remark 2.4.5. *Let $l \neq 2$. Since $A_{**} \cong A_*^{top} \underset{\mathbb{Z}/l}{\otimes} \mathbb{Z}/l[\tau]$ over $\mathbb{C}$ and $A_{**} \cong A_*^{top} \underset{\mathbb{Z}/l}{\otimes} \mathbb{Z}/l[\theta]$ over $\mathbb{R}$, we can also use these arguments to compute the E_2-page of the MASS in these cases. Because the Tor-term vanishes, it follows that the E_2-term of the MASS is the classical ASS E_2-term (with weights dictated by the cobar representatives of the generators) tensored with the appropiate coefficient ring.*

Now let k=$\mathbb{C}$.
The topological realization functor $R : \mathcal{SH}_{\mathbb{C}} \to \mathcal{SH}_{top}$ induces a map of towers and hence a map of spectral sequences Ψ from the l-completed MANSS to the l-completed topological ANSS, which is just the classical ANSS away from the bidegree $(0,0)$. Since Ψ is a map of spectral sequences and commutes with differentials, it follows that

Lemma 2.4.6. $d_{r-1}(x) = 0 \implies d_{r-1}(\Psi(x)) = 0.$

We now want to show that similar to the case $l = 2$ ([HKO, Lemma 16]) the converse is also true. As a consequence, the τ-primary torsion parts of the MANSS cannot support nontrivial differentials, because they map to zero under the map Ψ.
Before we do this, we need the fact that all τ-primary torsion vanishes in weight 0:

Lemma 2.4.7. *The restriction of the map Ψ to weight 0 is an isomorphism of spectral sequences (where the ANSS is considered a trigraded spectral sequence concentrated in degree 0 with respect to the third grading).*
In particular, if an element $x \in E_r^{s,t,u}$ maps to 0 under Ψ it must be τ-torsion.

Proof. In weight 0, topological realization induces an isomorphism already on the level of exact couples, so Ψ is an isomorphism of spectral sequences at weight 0 for all r.
Now assume $\Psi(x) = 0$. Because $\Psi(\tau^u \cdot x) = \Psi(\tau^u) \cdot \Psi(x) = 1 \cdot \Psi(x) = 0$ and Ψ is an isomorphism in weight 0, it follows that $\tau^u \cdot x = 0$. ☐

We can now prove that the motivic Adams-Novikov spectral sequence can be completely described in terms of the topological Adams-Novikov spectral sequence (compare [HKO, Lemma 16] for the corresponding statement if $l = 2$):

Proposition 2.4.8. *Let $q = 2l - 2$ as above and $k = \mathbb{C}$.*

1. *In each bidegree (s,t) the entry $E_r^{s,t,*}$ of the E_r-page of the l-completed MANSS can be described as follows:*
   The subgroup of elements which are not τ-primary torsion is given by $E_{r,top}^{s,t} \underset{\mathbb{Z}_l}{\otimes} \mathbb{Z}_l[\tau]$. Here the elements with the heighest weight have weight $\frac{t}{2}$.
   In addition, for every nontrivial classical differential
   $$d'_r : E_{r',top}^{s,t} \to E_{r',top}^{s+r',t+r'-1}$$
   with $r' < r$ there is a τ-primary torsion subgroup of order $\frac{r'-1}{2}$ generated by an element in $E_r^{s+r',t+r'-1,\frac{t+r'-1}{2}}$, which neither supports nor receives any nontrival differential. (Note that $r' = nq + 1$ because of sparseness, so r' must be odd). Apart from this there are no further entries.
   In particular, the τ-primary torsion subgroups at $r = nq + 1$ have order of at most $\frac{r-1}{2}$.

2. *For any $x \in E_r$, we have $d_r(\Psi(x)) = 0 \implies d_r(x) = 0$. As noted before, this implies that any τ-primary torsion element cannot support a nontrivial differential.*

3. *For each $E_\infty^{s,t}$ of the classical ANSS, there is a group $E_\infty^{s,t,\frac{t}{2}}[\tau]$ in the E_∞-page of the MANSS. The τ-primary torsion subgroup of the E_∞-page is generated by the groups above.*

Proof. We prove the first two claims by induction over r. The third claim then follows from the other two.
By the previous computations the claims are true for $r = 2$, and in particular there is no τ-primary torsion in this case.
For the induction step, we can restrict to the case $r = nq+1$ by sparsity. Assume all the claims are true for $r = nq + 1$, and we wish to show that they also hold for $r + 1$.
Every nontrivial differential $d_r : E_{r,top}^{s,t} \to E_{r,top}^{s+r,t+r-1}$ in the ANSS lifts to a nontrivial differential $d_r : E^{s,t,\frac{t}{2}} \to E^{s+r,t+r-1,\frac{t}{2}}$ in the MANSS by 2.4.6. Generators in the target have weight $\frac{t+r-1}{2}$, where $t+r-1 = t+nq$ is divisible by two since both t and q are. Hence a τ torsion group of order $\frac{r-1}{2}$ appears at the E_{r+1} page of the MANSS.
Consider a nontrivial differential $d_r : E^{s,t,\frac{t}{2}} \to E^{s+r,t+r-1,\frac{t}{2}}$ in the MANSS. By the induction assumption there can be no τ-primary torsion in the target at weight $\frac{t}{2}$ anymore. Therefore multiplication by $\tau^{\frac{t}{2}}$ is injective. Because all elements in weight 0 are τ-multiples it is in fact an isomorphism between weight $\frac{t}{2}$ and weight 0. In weight 0, Ψ is an isomorphism of spectral sequences by the previous lemma, so $d_r(\Psi(x)) = 0 \implies d_r(x) = 0$ follows. □

Proposition 2.4.9. *Consider k=$\mathbb{R}$. The results of the previous proposition apply if we replace τ with θ.*

Proof. The map of MANSSs induced by $\mathbb{R} \to \mathbb{C}$ is the inclusion of the terms in even weight on each page. □

Let now again be $k = \mathbb{C}$. It is known by a result of Morel ([Mor2]) that $\pi_{s',u}(S) = 0$ if $u > s'$, but it can also be deduced from the vanishing line of the ANSS:

Proposition 2.4.10. $\pi_{s',u}(S_l^\wedge) = 0$ *if $u > s'$.*

Proof. If we index the MANSS according to the usual Adams grading $(s, s') := (s, t - s)$, the column s' converges to $\pi_{s',*}(S_l^\wedge)$. By 2.4.4 the E_2-page of the MANSS has the following vanishing line:
The entry $E_2^{s,t,u}$ is zero if $t < 2s$ or equivalently $s' < s$. Because the weight of a generator in $E_2^{s,s',*}$ is $\frac{s'+s}{2}$ (and the weight of arbitrary elements is lesser or equal to this) and because the weight grows with s, a nonzero element in the column s' can at most have weight s', if the element lies in the maximal nonzero filtration degree $s = s'$. □

Proposition 2.4.11. *$\tau \in E_2^{0,0,-1}$ is a permanent cycle in the MANSS and defines an element $\tau \in \pi_{0,-1}(S_l^\wedge)$. On $\pi_{s',u}(S_l^\wedge)$ multiplication by τ is an isomorphism if $u \leq \frac{s'}{2}+1$ and $s' > 0$, or if $s' = 0$ and $u \leq 0$. Since the groups $\pi_{s',0}(S_l^\wedge)$ in weight 0 are known to be the (completed) classical groups by [Lev] this determines the motivic groups in the given range.*

Proof. It is clear by its position in the MANSS that τ is a permanent cycle.
The multiplication map $E_r^{s,s',u} \xrightarrow{\cdot\tau} E_r^{s,s',u-1}$ is an isomorphism at the E_2-page of the MANSS because there is no τ-primary torsion there. A priori it can fail to be an isomorphism at a given bidegree (s,t) of the E_r-page for one of two reasons: Because an incoming differential $d_{r'}$ $(r' < r)$ hits a τ-multiple of an element or because there is an outgoing differential on the τ-multiple of an element, but not on the element itself.
However, the latter case cannot actually occur: Differentials are τ-linear and do not map into τ-torsion, so we know $d_r(\tau x) = \tau y \implies d_r(x) = y$. In particular outgoing differentials lift to the highest weight $u = \frac{t}{2}$ at each fixed bidegree (s,t).
Therefore multiplication by τ can only fail to be an isomorphism if there is a nontrivial incoming differential $d_{r'} : E_{r'}^{s-r',s'+1,u-1} \to E_{r'}^{s,s',u-1}$ such that a τ-multiple $\tau y \in E_r^{s,s',u-1}$ is in the image but y is not in the image of $d_{r'}$: $E_{r'}^{s-r',s'+1,u} \to E_r^{s,s',u}$.
The statement follows for $s' = 0$ because there are no incoming differentials, so let $s' > 0$. If there is a nontrivial differential of this form, there is an $x \in E_r^{s-r,s'+1',u-1}$ such that $d_r(x) \neq 0$ and $\tau \nmid x$. Then x has the maximal weight $w = \frac{t_x}{2} = \frac{s'+1+s-r}{2}$. Then the lowest possible weight x can have is $w = \frac{s'+2}{2}$ if the differential maps out of the 1-line $(s-r=1)$, because there are no elements in the 0-line for $s > 0$. ☐

Example 2.4.12. 1. *The first nontrivial differential in the classical ANSS is*

$$d_{2l-1}(\beta_{l/l}) = a\alpha_1\beta_1^l$$

(cf. [Rav, Thm.4.4.22]) where a is an unknown constant. If we index the MANSS in the classical Adams grading $s, t-s, u$, these elements live in the following degrees:

Name	s	*t-s*	u
α_1	1	$2l-3$	$l-1$
β_1	2	$2l^2-2l-2$	l^2-l
$\beta_{l/l}$	2	$2l^3-2l^2-2$	l^3-l^2

This provides an example of a τ-torsion group of order $l-1$ in the stem $t-s = 2l^3-2l^2-3$. If we insert for instance $l=3$ this yields a permanent cycle in the MANSS in degree $s=7, t-s=33, u=20$ that is annihilated by τ^2.

2. *Consider the classical ANSS and the elements representing the Hopf maps. The motivic counterpart of the Hopf maps have been defined in [DI3]. For*

$l = 3$, the element α_1 lives in $s = 1, t - s = 3$ and is not involved in any differentials. It represents the Hopf map ν. Its motivic counterpart has the same coordinates and weight 2. Similarly, the element α_2 has coordinates $s = 1, t - s = 7$. It is a permanent cycle and represents the Hopf map σ. Its motivic counterpart has weight 4.
For $l = 5$ the element α_1 in the degree $s = 1, t - s = 7$ similarly is a permanent cycle, representing the Hopf map σ. Its motivic counterpart also has weight 4. Since at odd primes, products of the α-elements all vanish in the classic ANSS, this is true motivically as well. Hence all three elements are nilpotent in the MANSS.

3. *Any classical differential of the form $\mathbb{Z}/l^2 \xrightarrow{\cdot l} \mathbb{Z}/l^2$ in the ANSS would imply an entry of the form $Z/l[\tau]/(l\tau^n)$ in the MANSS. However, I do not know an example of such a differential.*

The η and l-completed homotopy groups of spheres differ from the l-completed ones. However it follows from work of Röndigs, Ormsby and Østvær that they agree in certain bidegrees(c.f. [ROO, 4.3]):

Proposition 2.4.13. *Let l be an odd prime. There is an isomorphism*

$$\pi_{m,n}(S^\wedge_{l,\eta}) \cong \pi_{m,n}(S^\wedge_l)$$

whenever the topological stable stem $\pi_{m-n}(S^\wedge_l) \cong (\pi_{m-n}S)^\wedge_l$ vanishes.

Proof. Röndigs, Spitzweck and Østvær showed in [RSO, Lemma 3.9] that there is a homotopy pullback square for arbitrary elements $\alpha \in \pi_{**}(S)$ and arbitrary motivic spectra E that relates $E^\wedge_\alpha$, $E[\frac{1}{\alpha}]$ and $E^\wedge_\alpha[\frac{1}{\alpha}]$. In our situation of $E = S$ and $\alpha = \eta$ we get the square

$$\begin{tikzcd} S^\wedge_l \arrow[r] \arrow[d] & S^\wedge_l[\frac{1}{\eta}] \arrow[d] \\ S^\wedge_{l,\eta} \arrow[r] & S^\wedge_{l,\eta}[\frac{1}{\eta}] \end{tikzcd}$$

This yields the following long exact sequence of homotopy groups:

$$... \to \pi_{m+1,n}S^\wedge_{l,\eta}[\tfrac{1}{\eta}] \to \pi_{m,n}S^\wedge_l \to \pi_{m,n}S^\wedge_{l,\eta} \oplus \pi_{m,n}S^\wedge_l[\tfrac{1}{\eta}] \to \pi_{m,n}S^\wedge_{l,\eta}[\tfrac{1}{\eta}] \to ...$$

By work of Bachmann([Bac, Chapter 11]) the real realization functor induces an equivalence of categories

$$R : \mathcal{SH}(\mathbb{R})[\tfrac{1}{2}][\tfrac{1}{\eta}] \to \mathcal{SH}_{top}[\tfrac{1}{2}]$$

Note that real realization maps $S^{m,n}$ to S^{m-n} and consequently induces a natural transformation of functors $\pi_{m,n}(-) \implies \pi_{m-n}(-)$. Furthermore the image of the stable map $\eta : S^{1,1} \to S^{0,0}$ under real realization is the map $\cdot 2 : S^0 \to S^0$.

Since both completions are modelled by homotopy inverse limits over the obvious diagrams, this means that the image of the η-completion of a motivic spectrum in $\mathcal{SH}(\mathbb{R})[\frac{1}{2}][\frac{1}{\eta}]$ is the 2-completion in $\mathcal{SH}_{top}[\frac{1}{2}]$ of the image of that spectrum. In particular, this applies to $S^{\wedge}_{l,\eta}[\frac{1}{\eta}]$, which hence must map to $S^{\wedge}_{2,l}[\frac{1}{2}] = 0$. Note that this argument does not apply to $S^{\wedge}_{l,\eta}$, since this spectrum is an object of $\mathcal{SH}(\mathbb{R})[\frac{1}{2}]$ but not of $\mathcal{SH}(\mathbb{R})[\frac{1}{2}][\frac{1}{\eta}]$.
We can therefore split and simplify the long exact sequence associated to the arithmetic square to the isomorphism

$$\pi_{m,n}(S^{\wedge}_l) \cong \pi_{m,n}(S^{\wedge}_{l,\eta}) \oplus \pi_{m,n}(S^{\wedge}_l[\tfrac{1}{\eta}])$$

A corollary of Bachmanns equivalence[Bac, Corollary 42] is the isomorphism

$$\pi_{m,n}(S[\tfrac{1}{2}][\tfrac{1}{\eta}]) \cong \pi_{m-n}S[\tfrac{1}{2}]$$

It is well known that the classical stable homotopy groups of the sphere are finitely generated in each stem and consequently, so are $\pi_{m,n}(S[\frac{1}{2}][\frac{1}{\eta}])$. Therefore the homotopy groups of the l-completion of both spectra are given by the l-completion of their homotopy groups. Applying l-completion to both sides of the above equation then implies the result. □

Example 2.4.14. *Consider the element α_1 of the MANSS at any odd prime. It is an infinite cycle. It's classical counterpart is the element generating the "first" l-torsion in the classical stable stems(in the sense that it is the lowest stem in which l-torsion occurs, namely $2l-3$). Since this element lives in positive weight, the above proposition applies, and α_1 actually represents an element in the l-completed stable motivic stems, not just in the l- and η-completed ones, and generates*

$$\pi_{2l-3,2}(S^{\wedge}_l) \cong \mathbb{Z}/l$$

If we multiply by powers of θ we still get a permanent cycle. Since α_1 has weight 2 and θ decreases weight by 2, its product has weight 0 and does not satisfy the conditions of the proposition anymore, since the l-completed stable stems in the degree in question do not vanish anymore. They are instead being generated by its classical counterpart.
Bachmanns result tells us that $\pi_{2l-3,0}(S^{\wedge}_l[\frac{1}{\eta}]) \cong \mathbb{Z}/l$. The isomorphism

$$\pi_{m,n}(S^{\wedge}_l) \cong \pi_{m,n}(S^{\wedge}_{l,\eta}) \oplus \pi_{m,n}(S^{\wedge}_l[\tfrac{1}{\eta}])$$

therefore implies:

$$\pi_{2l-3,0}(S^{\wedge}_l) \cong \mathbb{Z}/l \oplus \mathbb{Z}/l$$

Chapter 3

Periodic self maps detected by $AK(n)$

3.1 Introduction

There are two famous results by Hopkins and Smith in [HS] that provide a complete description of the thick subcategories in the stable homotopy category of finite topological spectra.

Definition 3.1.1. *A thick subcategory of a tensor triangulated category is a nonempty, full, triangulated subcategory that is closed under retracts. A thick ideal is a thick subcategory that is closed under tensoring with arbitrary objects.*

The thick subcategory theorem states that if we localize at a prime l the thick subcategories (in fact thick ideals) of the category $\mathcal{SH}_{(l)}^{fin}$ are given by a chain

$$\mathcal{SH}_{(l)}^{fin} = \mathcal{C}_0 \supsetneq \mathcal{C}_1 \supsetneq \mathcal{C}_2 \supsetneq ... \supsetneq \mathcal{C}_\infty = \{0\}$$

and each thick ideal $\mathcal{C}_{i+1}, 0 \leq i < \infty$, is characterized by the vanishing of the i-th Morava K-theory $K(i)$, where $K(0) = H\mathbb{Q}$ by convention. The periodicity theorem states that these thick subcategories can also be described by the property of admitting a special kind of periodic self map; a so called v_n-self map that induces an isomorphism in $K(n)$ and nilpotent maps in $K(m),\ m \neq n$. Using the older Nilpotence theorem of Devinatz, Hopkins and Smith in [DHS], Hopkins and Smith showed that the full subcategory $\mathcal{C}_{v_n}$ of finite spectra admitting such self maps is in fact thick, and thus equal to one of the categories $\mathcal{C}_i$. For algebraic reasons (see [Rav2, 3.3.11]) the category $\mathcal{C}_{v_n}$ must be nested in the following way:

$$\mathcal{C}_{n+1} \subset \mathcal{C}_{v_n} \subset \mathcal{C}_n$$

Therefore, by the thick subcategory theorem, the existence of at least one spectrum X_n in $\mathcal{C}_n$ admitting such a self map proves the equality $\mathcal{C}_{v_n} = \mathcal{C}_n$. Using an earlier construction of Smith, they prove that there is indeed such a spectrum

X_n that admits a v_n-self map.

In the motivic setting, a motivic counterpart to the classical Morava K-theory, called algebraic Morava K-theory, was defined by Borghesi in [Bor]. Joachimi studied the thick ideals $\mathcal{C}_{AK(n)}$ described by these algebraic Morava K-theories and proved in [Joa, 9.6.4] that for odd primes over the base field $\mathbb{C}$, the motivic thick ideals $\mathcal{C}_{AK(n)}$ form a similar descending chain as in the classical case. There are however additional thick ideals which are not of this form ([Joa, Chapter 7]). In addition, she relates the thick ideals $\mathcal{C}_{AK(n)}$ to the thick ideals thickid$(c\mathcal{C}_n)$ and $R^{-1}(\mathcal{C}_n)$ provided by the classical thick ideals via the constant simplicial presheaf and Betti realization functors, respectively.

In this chapter we will work over the complex numbers, and the prime l will be odd in this chapter. In particular this prime is implicit in the definition of the motivic Brown-Peterson-spectrum ABP and of the algebraic Morava-K-theory spectrum $AK(n)$.
The aim of this chapter is to study periodic self maps described by $AK(n)$. In the first section we show that the spectra $AK(n)$ admit the structure of a commutative homotopy ring spectrum similar to their classical equivalents.
In the second part we define a motivic analogon to the classical topological v_n^{top}-self maps and show that the existence of such self maps characterizes a thick subcategory.
In the third part we construct an example of such a self map on a suitable motivic spectrum $\mathbb{X}_n$ to show that this thick subcategory is actually nonempty.
In the fourth part we settle conjecture of Joachimi in [Joa] and in the last part we provide a correction, namely a counterexample, to the asserted inclusion

$$\text{thickid}(c\mathcal{C}_2) \subset \mathcal{C}_{AK(1)}$$

in [Joa, Chapter 9, last section] and we identify an error in [Joa, Proposition 8.7.3], on which the assertion is based. The counterexample also proves that the inclusion $\mathcal{C}_{AK(1)} \subset R^{-1}(\mathcal{C}_2)$ is actually proper. In the last part we answer one of the conjectures in Joachimis work([Joa, 7.1.7.3]).

3.2 The algebraic Morava-K-theories $AK(n)$

In [Joa, 6.3.2] Joachimi defines a motivic model of $K(n)$ as a motivic spectrum in $\mathcal{SH}_{\mathbb{C}}$ that maps to $K(n)$ under Betti realization. One obvious example is the image $cK(n)$ of topological Morava K-theory under the constant simplicial presheaf functor. Another example is the spectrum $AK(n)$ originally defined by Borghesi in [Bor]. We will rely on the description provided in [Joa, Def. 6.3.1]:

Definition 3.2.1. *The connective n-th motivic Morava K-theory is defined as*

$$Ak(n) = ABP/(v_0, ..., v_{n-1}, v_{n+1}, v_{n+2}, ...)$$

and the n-th motivic Morava K-theory spectrum $AK(n)$ is defined as:

$$AK(n) = v_n^{-1} ABP/(v_0, ..., v_{n-1}, v_{n+1}, v_{n+2}, ...)$$

In particular, both spectra are $MGL_{(l)}$-modules.

$AK(n)$ and $Ak(n)$ are genuinely motivic in the sense that they are derived from the spectrum representing algebraic cobordism. We will need some of the properties of $AK(n)$ proven in [Joa], namely:

Remark 3.2.2.

1. *Let k=$\mathbb{C}$. The Betti realization of the (connective) motivic Morava K-theory is the classical (connective) Morava K-theory ([Joa, Lemma 6.3.2]):*
$$R_{\mathbb{C}}(AK(n)) = K(n)$$
and
$$R_{\mathbb{C}}(Ak(n)) = k(n)$$

2. *Let k=$\mathbb{C}$. Then by [Joa, Lemma 6.3.7] the coefficients of algebraic Morava K-theory are given by:*
$$AK(n)_{**} = H\mathbb{Z}/(l)_{**} \underset{\mathbb{Z}/(l)}{\otimes} K_*$$

3. *Let $k = \mathbb{C}$. If X is a finite motivic cell spectrum such that $H\mathbb{Z}/(l)^{**}(X)$ is free over the coefficients, the MASS for $Y = Ak(n) \wedge X$ will converge strongly to $Ak(n)_{**}(X)$. (See [Joa, 8.3.3])*

Proposition 3.2.3. *Let $k = \mathbb{R}$. The last statement of the preceding remark remains true in this setting.*

Proof. As noted in [Joa, 8.3.3], it is sufficient to check that the hypotheses of [HKO2, Theorem 1] apply. If l is even, this is clearly the case. If l is odd, we only know that the MASS will converge to the completion at l and η. But the map $Ak(n) \wedge \eta$ induces the zero map in motivic homotopy groups for degree reasons. Since $Ak(n)$ is the l-localization of a cellular spectrum, we can conclude that the map itself is the zero map by [DI2][Remark 7.1]. Hence the η-completion of $Y = Ak(n) \wedge X$ will be Y itself. Hence in either case the proof of [Joa, 8.3.3] applies over $\mathbb{R}$. □

At least for odd primes, the topological spectra $K(n)$ can be shown to be homotopy ring spectra. As remarked in [Joa, Remark 6.3.3(6)], it is not known in general if the motivic Morava K-theory spectrum $AK(n)$ can be endowed with the structure of a motivic homotopy ring spectrum. In the special case $k = \mathbb{C}$, $l \neq 2$ however Joachimi proved that the spectrum

$$AP(n) := ABP/(v_0 = l, v_1, ..., v_{n-1}),$$

another quotient of MGL, admits a unital homotopy associative product [Joa, 9.3], and with the work done by her it is no longer difficult to do the same for

$AK(n)$.

We want to use and extend the results in [Joa, 9.3] and follow the notation used there to make comparison easier. In particular η will not denote the motivic Hopf map in this chapter, but a different map to be defined later. The only exception is the name of our prime l, which is referred to as p in [Joa].
Let $R \in \mathcal{SH}_k$ be a strictly commutative ring spectrum with multiplication map $m : R \wedge R \to R$ and unit map $i : S \to R$. The example that we have in mind is $MGL_{(l)}$, which is a strictly commutative motivic ring spectrum by the reasoning given in the beginning of [Joa, 9.3].

Classically one can study the products on R-modules of the form R/x and use them to gain information about products on quotients of the form R/X where X is a countable regular sequence of homogeneous elements. In contrast to the classical situation, the coefficients of $MGL_{(l)}$ are not known, but the coefficients $MGL_{(l)}/l$ are. Therefore motivically one has to consider R-modules of the form $R/(x, y)$.

In the section immediately preceding [Joa, 9.3.7] and in the proof of [Joa, Lemma 9.3.8] Joachimi constructs a product on quotients of this form and proves the following statement:

Lemma 3.2.4. *Let $y \in \pi_{k',l'}(R)$ and let $x \in \pi_{k,l}(R)$. Define the R-modules $M := R/y$ and $N := M/x$ and denote the structure map of M as $\nu_M : R \wedge M \to M$. Write η' for the canonical map*

$$\eta' : R \to M$$

and η for the canonical map

$$\eta : M \to N$$

.

If $\pi_{2k'+1,2l'}(M) = 0$ and $\pi_{2k+1,2l}(N) = 0$, there are maps of R-modules

$$\mu_M : M \wedge M \to M$$

$$\nu_{M,N} : M \wedge N \to M$$

$$\mu_N : N \wedge N \to N$$

making the following diagrams commute up to homotopy:

$$\begin{tikzcd}
R \wedge R \arrow[rr, "\eta' \wedge \eta'"] \arrow[dr, "1 \wedge \eta'"] \arrow[dd, "m"'] & & M \wedge M \arrow[dd, "\mu_M"] \\
& R \wedge M \arrow[ur, "\eta' \wedge 1"] \arrow[dr, "\nu_M"] & \\
R \arrow[rr, "\eta'"] & & M
\end{tikzcd} \tag{3.1}$$

$$\begin{tikzcd}
M \wedge M \arrow[rr, "\eta\wedge\eta"] \arrow[dr, "1\wedge\eta"] \arrow[dd, "\mu_M"'] & & N \wedge N \arrow[dd, "\mu_N"] \\
& M \wedge N \arrow[ur, "\eta\wedge 1"] \arrow[dr, "\nu_{M,N}"] & \\
M \arrow[rr, "\eta"] & & M
\end{tikzcd} \tag{3.2}$$

In particular, if we choose the maps $\eta' \circ i$ and $\eta \circ \eta' \circ i$ as unit maps, μ_M and μ_N are unital products on M and N respectively.

Furthermore the following result of Joachimi [Joa, Lemma 9.3.8] proves associativity, and we wish to extend it to include commutativity:

Lemma 3.2.5. *If $\pi_{k'+1,l'}(M) = \pi_{2k'+2,2l'}(M) = \pi_{3k'+3,3l'}(M) = 0$, then μ_M is homotopy associative.*
If furthermore $\pi_{k+1,l}(N) = \pi_{2k+2,2l}(N) = \pi_{3k+3,3l}(N) = 0$, then μ_N is also homotopy associative.

We need the following lemma of Joachimi [Joa, Lemma 9.3.3] in the proof of commutativity:

Lemma 3.2.6. *Let R' be a (homotopy) ring spectrum, M' a left R'-module, and $\pi_{k,l}(M') = 0$. Then any R'-module map $\psi : S^{k,l} \wedge R' \to M'$ is homotopically trivial.*

Furthermore we need the following fact, relating the cofibre of a smash products of maps with the smash product of the individual cofibers:

Lemma 3.2.7. *Let*

$$R \xrightarrow{\eta} M \to C_\eta$$

and

$$R \wedge R \xrightarrow{\eta\wedge\eta} M \wedge M \to C_{\eta\wedge\eta}$$

be cofiber sequences.
Then there is a cofiber sequence

$$(R \wedge C_\eta) \vee (C_\eta \wedge R) \to C_{\eta\wedge\eta} \to C_\eta \wedge C_\eta$$

Proof. This cofiber sequence is listed as one of the consequences of the existence of a strong triangulation in [May, Immediately after Definition 4.11]. □

With these two lemmas in place, we are ready to prove:

Proposition 3.2.8. *If the homotopy groups of M satisfy the same assumptions as in 3.2.5, commutativity of the multiplication of R implies that μ_M is homotopy commutative.*
If furthermore the homotopy groups of N satisfy the same assumptions as in 3.2.5, commutativity of the multiplication of R implies that μ_N is also homotopy commutative.

Proof. Let $y \in \pi_{k',l'}(R)$ and $x \in \pi_{k,l}(R)$. Write $M = R/y$ as well as $N = R/(x,y)$, and ϕ for the multiplication-by-y-map. The cofiber sequence defining M is

$$\Sigma^{k',l'} R \xrightarrow{\phi} R \xrightarrow{\eta'} M$$

and if we shift this sequence, we obtain the cofiber sequence

$$R \xrightarrow{\eta'} M \to C_\eta$$

with $C_{\eta'} = \Sigma^{k'+1,l'} R$.
Remember that $m : R \wedge R \to R$ is the product on the ring spectrum R. We want to show that the product $\mu_M : M \wedge M \to M$ is commutative, i.e. that the diagram

$$\begin{tikzcd}
M \wedge M \arrow[r, "\mu_M"] \arrow[d, "T"] & M \\
M \wedge M \arrow[ur, "\mu_M"'] &
\end{tikzcd}$$

commutes up to homotopy, where T is the transposition map. Equivalently, we have to show that

$$\delta := \mu_M \circ (1 - T) : M \wedge M \to M$$

is homotopic to the zero map.
The product m is commutative by assumption and by the commutativity of diagram 3.1 in 3.2.4 we know that $\mu_M \circ (\eta' \wedge \eta') = \eta' \circ m$, so we can compute:

$$\delta \circ (\eta' \wedge \eta') = \mu_M \circ (1-T) \circ (\eta' \wedge \eta') = \mu_M \circ (\eta' \wedge \eta') \circ (1-T) = \eta' \circ m \circ (1-T) = 0$$

We write $C_{\eta' \wedge \eta'}$ for the cofiber of $\eta' \wedge \eta' : R \wedge R \to M \wedge M$. The preceding equation implies that δ factors over a map $\delta' : C_{\eta' \wedge \eta'} \to M$. Consider the following diagram:

$$\begin{tikzcd}
 & R \wedge R \arrow[d, "\eta' \wedge \eta'"] \arrow[dr, "0"] & \\
 & M \wedge M \arrow[r, "\delta"] \arrow[d] & M \\
(R \wedge C_{\eta'}) \vee (C_{\eta'} \wedge R) \arrow[r, "f"] & C_{\eta' \wedge \eta'} \arrow[r] \arrow[ur, "\delta'"] & C_{\eta'} \wedge C_{\eta'} \arrow[u, dashed, "\delta''"']
\end{tikzcd}$$

The bottom row is a cofiber sequence by 3.2.7 and the column is the cofiber sequence we just defined. We wish to show that δ' factors over a map

$$\delta'' : C_{\eta'} \wedge C_{\eta'} \to M$$

as indicated by the dashed line. For this it suffices to show that the composite $\delta' \circ f$ is trivial up to homotopy. But $C_{\eta'}$ was just $\Sigma^{k'+1,l'} R$, and both wedge summands of the source are equivalent to $S^{k'+1,l'} \wedge R \wedge R$. Because we assumed

$\pi_{k'+1,l'}(M) = 0$ 3.2.6 tells us that it is indeed homotopically trivial. Here we choose $R' = R \wedge R$. We can give both source and target of the composite $\delta' \circ f$ a canonical R'-module structure by using the left unit of $R \wedge R$, and the composite is then a map of R'-modules.
But then, using the assumption that $\pi_{2k'+2,2l'}(M) = 0$, the map

$$\delta'' : C_{\eta'} \wedge C_{\eta'} \simeq S^{2k'+2,2l'} \wedge R \wedge R \to M$$

is homotopically trivial by the same reasoning. Because δ' factors over it, it must also be homotopically trivial. This in turn implies that δ is homotopically trivial as desired.
Because we used only the fact that R is a homotopy ring spectrum and not strictly commutativity, and because diagram 3.2 in 3.2.4 commutes, we can then repeat the same proof with M replacing R and N replacing M. Note that this would not have been possible if we worked over R-modules, because it is not clear that R/x is a strictly commutative ring spectrum again. □

Lemma 3.2.9. *Let $k = \mathbb{C}$ and $l \neq 2$. The spectrum $AP(n)$ admits a unital, homotopy associative and homotopy commutative product*

$$\mu_{AP(n)} : AP(n) \wedge AP(n) \to AP(n)$$

and so do the spectra $A_i = AP(n)/(v_{n+1}, ..., v_{n+i})$.

Proof. Except for the statement about commutativity, the first part of this lemma is the content of [Joa, 9.3.9]. The essential argument in the proof of the cited lemma is as follows: if one has a sequence of elements $J \subset R_{**}$ and one knows that $A := R/(J - \{x, y\})$ is a homotopy associative and commutative ring spectrum, then one can describe the product on $R/J \cong A/(x, y) \cong A \underset{R}{\wedge} R/(x, y)$ by

$$(N \underset{R}{\wedge} A) \wedge (N \underset{R}{\wedge} A) \xrightarrow{\tau} (N \wedge N) \underset{R}{\wedge} (A \wedge A) \xrightarrow{id_N \wedge id_N \wedge \mu_A} N \wedge N \underset{R}{\wedge} A \xrightarrow{\mu_N \wedge id_A} N \wedge A$$

and thus has to prove the vanishing of the obstruction groups to associativity only after application of $(-) \underset{R}{\wedge} A$ to the associativity diagram.
Now choose $R = MGL_{(l)}$ and $A = ABP$ and J such that $MGL_{(l)}/J = AP(n)$. Then the relevant obstruction groups are trivial because for odd primes $l \neq 2$, ABP_{**} is concentrated in bidegrees where the first degree is divisible by 4. We can then show that there is a homotopy associative product on $AP(n)$ by induction; because $AP(n)/(v_0, ..., v_n)$, we only have to do finitely many steps, and we can use the fact (see [Joa, Lemma 9.3.7]) that for any sequence $(l) \subset J'$:

$$ABP/(J' \cup \{y\}) \cong MGL_{(l)}/(l, y) \underset{MGL_{(l)}}{\wedge} ABP/J'$$

We can use the same argument to prove commutativity: if we apply $(-) \underset{R}{\wedge} A$ to all the relevant diagrams in 3.2.8, we see that the obstructions to commutativity

lie in groups $\pi_{i,j}(M \underset{R}{\wedge} A)$ and $\pi_{i,j}(N \underset{R}{\wedge} A)$ which are trivial because 4 does not divide i in the relevant bidegrees. Therefore the product on $AP(n)$ is in fact homotopy commutative.
Now consider the spectra A_i. To define them, we add finitely many elements, namely $v_{n+1}, ..., v_{n+i}$, to the sequence J. The proof of [Joa, Lemma 9.3.7] carries through verbatim and we can conclude that there is a product map $A_i \wedge A_i \to A_i$. Similarly, because we had to add only finitely many elements to J, we can repeat the induction argument above for the spectra A_i. This shows that the multiplication on A_i is in fact homotopy associative and homotopy commutative. □

By essentially classical arguments, this allows us to conclude that $Ak(n)$ has the desired ring structure:

Proposition 3.2.10. *Let $k = \mathbb{C}$ and let l be an odd prime. Then the connective algebraic Morava K-theory spectrum*

$$Ak(n) = \operatorname{hocolim}_{\longrightarrow} A_i = ABP/(v_0, v_1, ...v_{n-1}, v_{n+1}, v_{n+2}, ...)$$

admits the structure of a homotopy associative and homotopy commutative motivic ring spectrum.

Proof. By [Joa, Corollary 9.3.5] the elements $v_i, i \neq n$ act trivially on $Ak(n)$. This is in particular the case for $v_0 = l$. Therefore [Str, Lemma 6.7] holds for $A = M = Ak(n)$ (although Strickland considers rings in R-modules, the only necessary modification is replacing the map ρ^* by the map $\rho^* : [R/(l, x_i) \underset{R}{\wedge} B, M] \to [R/x_i \underset{R}{\wedge} B, M] \to [B, M]$), and we can use the arguments of [Str, Proposition 6.8] to conclude that the constructed products on A_i induce a unital, homotopy associative product on $Ak(n)$. As noted in the proof of Stricklands proposition, this product is commutative if and only if the maps $A_i \to Ak(n)$ commute with themself (see [Str, Definition 6.1] for a definition of this notion). Because the product on A_i is commutative, this is the case for every map out of A_i. □

Corollary 3.2.11. *Let $k = \mathbb{C}$ and let l be an odd prime. The algebraic Morava K-theory spectrum $AK(n) = v_n^{-1}Ak(n)$ admits the structure of a commutative and associative motivic homotopy ring spectrum.*

Proof. We have an isomorphism $AK(n) \cong v_n^{-1}MGL_{(l)} \underset{MGL_{(l)}}{\wedge} Ak(n)$ and both smash factors admit a homotopy commutative and associative product([Str, Proposition 6.6]). Therefore we can endow $AK(n)$ with the desired structure as in the proof of 3.2.9. □

It remains to show that this product induces the same product structure on $AK(n)_{**}$ as one would expect from the computation of these coefficients:

Lemma 3.2.12. *The multiplication map*

$$\mu_{AK(n)} : AK(n) \wedge AK(n) \to AK(n)$$

induces the multiplication on $AK(n)_{**}$ given by the multiplication on $K(n)_*$ and the isomorphism $AK(n)_{**} \cong HZ/l_{**} \underset{\mathbb{Z}/l}{\otimes} K(n)_*$ of [Joa, Lemma 6.3.7].

Proof. The proof is similar to the proof of [Joa, Lemma 9.3.10] $\square$

3.3 Thick subcategories characterized by motivic v_n-self maps

In classical topology a v_n-self map is defined (c.f. [HS, Definition 8]) as a map that induces an isomorphism in the n-th Morava K-theory and a nilpotent map in the m-th Morava K-theory for $m \neq n$. Any such map (or more precisely, some power of it) induces multiplication by a power of v_n in Morava K-theory. This property is called asymptotic uniqueness. However, the proof of this relies on the Nilpotence lemma, which does not hold in the motivic setting.
We intent to construct an example v_n-self map $v : \mathbb{X}_n \to \mathbb{X}_n$ on the motivic equivalent of the space X_n used by Hopkins and Smith. However, this space is constructed as a retract of a finite cell spectrum. In classical topology, a retract of a finite cell spectrum is a finite cell spectrum again, but this does not necessarily need to be the case motivically. Therefore, we want to consider the slightly larger thick envelope $\mathcal{SH}_k^{qfin}$ (defined in 1.2.2) of the subcategory of finite spectra in $\mathcal{SH}_{\mathbb{C}}$ in the definition of motivic v_n-self maps and for the study of thick subcategories characterized by v_n-self maps.

Definition 3.3.1. *Let X be a motivic spectrum in $\mathcal{SH}_k^{qfin}$ or $\mathcal{SH}_{k,l}^{\wedge,qfin}$. A map $f : \Sigma^{t,u} X \to X$ is a motivic v_n self-map if it satisfies the following conditions:*

1. *$AK(m)_{**}f$ is nilpotent if $m \neq n$*
2. *$AK(m)_{**}f$ is given by multiplication with an invertible element of $H\mathbb{Q}_{**}$ if $m = n = 0$.*
3. *$AK(m)_{**}f$ is an isomorphism if $m = n$.*

However, the spectrum $AK(n)$ does not exhibit all the pleasant properties of the topological Morava K-theory spectrum, such as the existence of a Künneth isomorphism. We will need to prove the existence of a Künneth-isomorphism in homology for the spectrum $\mathbb{X}_n$ by computing $AK(n)_{**}(\mathbb{X}_n)$.

Let l be an odd prime and let $k = \mathbb{C}$. The aim of this section is to show that the existence of v_n-self maps characterizes thick subcategories in $\mathcal{SH}_{\mathbb{C}}^{qfin}$ and hence also in the motivic homotopy category. We consider only the case $n > 0$.

As mentioned before, the topological nilpotence theorem is a key ingredient in the proof that topological finite cell spectra spectra admitting a v_n-self map form a thick subcategory: A map of finite spectra is nilpotent if and only if it induces zero in all Morava K-theories. The motivic equivalent of this theorem does not hold: For example, the motivic Hopf map η is a non-nilpotent map in $\mathcal{SH}_k$, but induces the zero map in motivic Morava K-theory for degree reasons. It seems likely however that a weaker version of the theorem applies, where we only consider maps of a certain bidegree. For the remainder of this subsection we assume that the following motivic nilpotence conjecture holds:

Conjecture 3.3.2. *Let $k = \mathbb{C}$, let l be an odd prime and $n > 0$ be an integer. If X is a motivic spectrum in $\mathcal{SH}_k^{qfin}$ or $\mathcal{SH}_{k,l}^{\wedge,qfin}$ and $f : \Sigma^{p,q}X \to X$ is a motivic map such that (p,q) is a multiple of $(2l^n - 2, l^n - 1)$, then:*

$$\forall m \in \mathbb{N} : AK(m)_{**}(f) = 0 \implies \exists k \in \mathbb{N} : f^k \simeq 0$$

The known examples of non-nilpotent motivic self maps that induce the zero map in motivic Morava-K-theory (A variety of examples can be found in [Hor], and Boghdan George has constructed a whole family of such maps detected by exotic motivic Morava K-theories) do not contradict this conjecture.

To prove the motivic equivalent of asymptotic uniqueness, we want to use Betti realization to compare the motivic situation to the classical one. To do this, we need to study the effect of Betti Realization on homology groups of $AK(n)$. We will show that the kernel of the map induced by Betti realization is precisely the τ-primary torsion elements. To do this, we need to compare $K(n)_*$ and $AK(n)_{**}$-modules, which is only possible after inverting τ. We also need the fact that the $AK(n)$-homology of a (quasi)-finite motivic cell spectrum is finitely generated over the coefficients:

Lemma 3.3.3. *Let X be a motivic spectrum in $\mathcal{SH}_k^{qfin}$ or $\mathcal{SH}_{k,l}^{\wedge,qfin}$. Then*

1. *$AK(n)_{**}(X)$ is finitely generated as an $AK(n)_{**}$-module.*
2. *$Hom_{AK(n)_{**}}(AK(n)_{**}(X), M)$ is finitely generated as an $AK(n)_{**}$-module for every finitely generated $AK(n)_{**}$-module M. In particular,*

$$End_{AK(n)_{**}}(AK(n)_{**}(X))$$

is finitely generated.

Proof. 1. We will show the statement for finite cell spectra by cellular induction and then show that it also holds for retracts of finite cell spectra. The claim is trivially true for the sphere spectrum since $AK(n)_{**}$ is finitely generated over itself. $AK(n)_{**}$ is a quotient of a polynomial ring in the three variables v_n, v_n^{-1}, τ over the field $\mathbb{F}_l$ and hence Noetherian. Therefore submodules of finitely generated AK_{**}-modules are again finitely generated. In particular, retracts of spectra for which the statement holds

satisfy the statement again.
It remains to show that the statement is true if for a spectrum Z in a cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ if the spectra X and Y already satisfy the statement, that is have finitely generated $AK(n)$-homology. In that case, consider the long exact sequence in $AK(n)$-homology

$$\ldots \to AK(n)_{**}(Y) \xrightarrow{g} AK(n)_{**}(Z) \xrightarrow{\delta} AK(n)_{*-1,*}(X) \xrightarrow{f} AK(n)_{*-1,*}(Y) \to \ldots$$

associated to this cofiber sequence. We can break it up into short exact sequences in the canonical way:

$$0 \to \operatorname{coker}(f) \xrightarrow{\bar{g}} AK(n)_{**}(Z) \xrightarrow{\bar{\delta}_p} \ker(f)[-1] \to 0$$

The two outer terms in the short exact sequence are finitely generated: $\ker(f)[-1]$ as a submodule of a finitely generated module over a Noetherian ring, and $\operatorname{coker}(f)$ as a quotient of a finitely generated module. It is well known that in this case the middle term is also finitely generated.

2. By the first part of this lemma, $AK(n)_{**}(X)$ is finitely generated as an $AK(n)_{**}$-module. Therefore there is a surjection $R^k \to AK(n)_{**}$ from a free and finitely generated $AK(n)_{**}$-module R^k onto $AK(n)_{**}(X)$. Then

$$\operatorname{Hom}_{AK(n)_{**}}(R^k, M) \cong M^k$$

is a free and finitely generated $AK(n)_{**}$-module. Because $AK(n)_{**}$ is a Noetherian ring,

$$\operatorname{Hom}_{AK(n)_{**}}(AK(n)_{**}(X), M)$$

is finitely generated as a submodule of this finitely generated module.

□

Remark 3.3.4. *One can regard $K(n)_*$ and its modules as a bigraded ring and bigraded modules concentrated in degree 0 in regard to the second bidegree. Then every $AK(n)_{**}[\tau^{-1}]$-module has the structure of a bigraded $K(n)_*$-module where v_n^{top} acts via $\tau^{l^n-1}v_n$. (This of course implies that $(v_n^{top})^{-1}$ acts via $\tau^{-l^n+1}v_n^{-1}$, so it only makes sense after inverting τ.) With this module structure, $AK(n)_{**}[\tau^{-1}]$ is free (with basis $\tau^k, k \in \mathbb{Z}, -l^n+1 < k < l^n-1$) and in particular flat as a $K(n)_*$-module. Likewise it is flat as an $AK(n)_{**}$-module, because it is a localization. We will implicitly use this in the following statements and sometimes write $-[\tau^{-1}]$ for $- \underset{AK(n)_{**}}{\otimes} AK(n)_{**}[\tau^{-1}]$, and $-[\tau,\tau^{-1}]$ for $- \underset{K(n)_*}{\otimes} AK(n)_{**}[\tau^{-1}]$ as an abbreviation.*

Lemma 3.3.5. *(Compare [DI, 2.7 + 2.8])*

1. *Let X be a motivic spectrum in $\mathcal{SH}_k^{qfin}$ or $\mathcal{SH}_{k,l}^{\wedge,qfin}$.*
   We can define a map of bigraded $AK(n)_{**}[\tau^{-1}]$-modules (even a map of bigraded algebras if X is a ring spectrum) natural in X
   $$R : AK(n)_{**}(X) \underset{AK(n)_{**}}{\otimes} AK(n)_{**}[\tau^{-1}] \to K(n)_*(R_{\mathbb{C}}(X)) \underset{K(n)_*}{\otimes} AK(n)_{**}[\tau^{-1}]$$
   via the assignment
   $$x \otimes \tau^k \mapsto R_{\mathbb{C}}(x) \otimes \tau^{-q+k}$$
   where q is the motivic weight of $x \in AK(n)_{p,q}(X)$.
   This map is an isomorphism.

2. *The induced map*
   $$\bar{R}_{End,X} : End_{AK(n)_{**}}(AK(n)_{**}(X))[\tau^{-1}] \to End_{K(n)_*}(K(n)_*(R_{\mathbb{C}}(X)))[\tau, \tau^{-1}]$$
   is an isomorphism of bigraded $AK(n)_{**}[\tau^{-1}]$-algebras.

3. *A homogeneous element $f \in End_{AK(n)_{**}}(AK(n)_{**}(X))$ maps to zero under the map*
   $$R_{End,X} : End_{AK(n)_{**}}(AK(n)_{**}(X)) \to End_{K(n)_*}(K(n)_*(R_{\mathbb{C}}(X)))$$
   induced by motivic realization if and only if it is τ-primary torsion.

Proof. 1. The statement about naturality and the module/algebra structure follow from the properties of motivic realization. It remains to show that the map is an isomorphism for spectra X in $\mathcal{SH}_k^{qfin}$ or $\mathcal{SH}_{k,l}^{\wedge,qfin}$. We will prove this using cellular induction, and then show that it remains an isomorphism under taking retracts.

Consider the case of the sphere spectrum $X = S$: The map
$$R : AK(n)_{**}[\tau^{-1}] \to K(n)_* \underset{K(n)_*}{\otimes} AK(n)_{**}[\tau^{-1}]$$
sends τ to τ and $v_n \in AK(n)_{2(l^n-1),l(n-1)}$ to $v_n^{top} \otimes \tau^{-l^n+1} = 1 \otimes \tau^{-l^n+1}\tau^{l^n-1}v_n = v_n$, so it is an isomorphism.

If X is a retract of a spectrum F for which the statement holds, then $AK(n)_{**}(X)$ is a direct summand of $AK(n)_{**}(F)$ and all squares in the following diagram commute:

$$\begin{tikzcd}
AK(n)_{p,q}(X)[\tau^{-1}] \arrow[r, "AK(n)_{**}(s)"] \arrow[rr, bend left, "id"] \arrow[d, "R_X"] & AK(n)_{p,q}(F)[\tau^{-1}] \arrow[r, "AK(n)_{**}(r)"] \arrow[d, "R_F"', "\cong"] & AK(n)_{p,q}(X)[\tau^{-1}] \arrow[d, "R_X"] \\
K(n)_p(R_{\mathbb{C}}(X))[\tau, \tau^{-1}] \arrow[r, "K(n)_*(R_{\mathbb{C}}(s))"'] \arrow[rr, bend right, "id"'] & K(n)_p(R_{\mathbb{C}}(F))[\tau, \tau^{-1}] \arrow[r, "K(n)_*(R_{\mathbb{C}}(r))"'] & K(n)_p(R_{\mathbb{C}}(X))[\tau, \tau^{-1}]
\end{tikzcd}$$

Therefore R_X is surjective and injective via a simple diagram chase.

Finally, suppose $X \to Y \to Z$ is a cofiber sequence and the statement holds for X and Y. Then the long exact sequence for $AK(n)$-homology maps to the long exact sequence for $K(n)$-homology associated to the cofiber sequence $R_{\mathbb{C}}(X) \to R_{\mathbb{C}}(Y) \to R_{\mathbb{C}}(Z)$, and the five lemma tells us that the statement also holds for Z:

$$\begin{tikzcd}
\dots \arrow[r] \arrow[d, "\cong"', "R_X"] & AK(n)_{pq}(Y)[\tau^{-1}] \arrow[r] \arrow[d, "\cong"', "R_Y"] & AK(n)_{pq}(Z)[\tau^{-1}] \arrow[r] \arrow[d, "R_Z"] & AK(n)_{p-1,q}(X)[\tau^{-1}] \arrow[r] \arrow[d, "\cong"', "R_X"] & \dots \arrow[d, "\cong"', "R_Y"] \\
\dots \arrow[r] & K(n)_p(R_{\mathbb{C}}(Y))[\tau][\tau^{-1}] \arrow[r] & K(n)_p(R_{\mathbb{C}}(Z))[\tau][\tau^{-1}] \arrow[r] & K(n)_{p-1}(R_{\mathbb{C}}(X))[\tau][\tau^{-1}] \arrow[r] & \dots
\end{tikzcd}$$

2. Let M be a finitely generated $K(n)_*$-module and N be an arbitrary $K(n)_*$-module. As noted in 3.3.4, $AK(n)_{**}[\tau^-1]$ is a flat $K(n)_*$-module. By [Bour, §2.10, Proposition 11] there is a canonical isomorphism:

$$\mathrm{Hom}_{K(n)_*}(M,N)[\tau,\tau^{-1}] \xrightarrow{\cong} \mathrm{Hom}_{K(n)_*[\tau,\tau^{-1}]}(M[\tau,\tau^{-1}], N[\tau,\tau^{-1}])$$

Likewise, let M be a finitely generated $AK(n)_{**}$-module and N be an arbitrary $AK(n)_{**}$-module. Because $AK(n)_{**}[\tau^{-1}]$ is a flat $AK(n)_{**}$-module, there is also a canonical isomorphism:

$$\mathrm{Hom}_{AK(n)_{**}}(M,N)[\tau^{-1}] \xrightarrow{\cong} \mathrm{Hom}_{AK(n)_{**}[\tau^{-1}]}(M[\tau^{-1}], N[\tau^{-1}])$$

Specializing to the case $M = N = AK(n)_{**}(X)$, these two isomorphisms fit in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{End}_{AK(n)_{**}}(AK(n)_{**}(X))[\tau^{-1}] \arrow[r, "\cong"] \arrow[d] & \mathrm{End}_{AK(n)_{**}[\tau^{-1}]}(AK(n)_{**}(X)[\tau^{-1}]) \arrow[d] \\
\mathrm{End}_{K(n)_*}(K(n)_*(R_{\mathbb{C}}(X)))[\tau,\tau^{-1}] \arrow[r, "\cong"] & \mathrm{End}_{K(n)_*[\tau,\tau^{-1}]}(K(n)_*(R_{\mathbb{C}}(X)))[\tau,\tau^{-1}])
\end{tikzcd}$$

The first statement of the lemma tells us that $K(n)_*[\tau,\tau^{-1}] \cong AK(n)_{**}[\tau^{-1}]$ and $K(n)_*(R_{\mathbb{C}}(X)))[\tau,\tau^{-1}] \cong AK(n)_{**}(X)[\tau^{-1}]$, so the right vertical map is an isomorphism. It follows that the left vertical map is also an isomorphism.

3. Let

$$P : \mathrm{End}_{K(n)_*}(K(n)_*(R_{\mathbb{C}}(X)))[\tau,\tau^{-1}] \to \mathrm{End}_{K(n)_*}(K(n)_*(R_{\mathbb{C}}(X)))$$

be the map of $K(n)_{**}$-algebras defined by sending τ to 1 and elements of $\mathrm{End}_{K(n)_*}(K(n)_*(R_{\mathbb{C}}(X)))$ to themselves. Then we have a commutative diagram of $K(n)_*$-algebras:

$$\begin{tikzcd}
\mathrm{End}_{AK(n)_{**}}(AK(n)_{**}(X))[\tau^{-1}] \arrow[d, "\bar{R}_{End,X}"] & \mathrm{End}_{AK(n)_{**}}(AK(n)_{**}(X)) \arrow[l] \arrow[d, "R_{End,X}"] \\
\mathrm{End}_{K(n)_*}(K(n)_*(R_{\mathbb{C}}(X)))[\tau,\tau^{-1}] \arrow[r, "P"] & \mathrm{End}_{K(n)_*}(K(n)_*(R_{\mathbb{C}}(X)))
\end{tikzcd}$$

A homogeneous element maps to zero under the top horizontal map if and only if it is τ-primary torsion. By the second statement of this lemma, the left vertical map is an isomorphism, and there are no homogeneous elements in the kernel of P. All this together implies the desired result. □

Remark 3.3.6. *If X is dualizable, the map $DX \wedge X = F(X,S) \wedge X \to F(X,X)$ is a weak equivalence, and we have a corresponding isomorphism on homotopy groups $\pi_{pq}(X \wedge DX) \cong \mathrm{End}(X)_{pq}$. With regard to motivic Morava K-theory the situation is more complicated. Using Spanier-Whitehead duality we have:*

$$\begin{aligned} AK(n)_{pq}(X \wedge DX) &= [S, AK(n) \wedge X \wedge DX]_{pq} \\ &= [X, AK \wedge X]_{pq} \\ &= [AK \wedge X, AK \wedge X]_{AK,pq} \end{aligned}$$

The last term is related to $\mathrm{End}_{AK(n)_{**}}(AK(n)_{**}(X))_{pq}$ via the Universal coefficient spectral sequence(c.f [DI2, Prop. 7.7]). The E_2-term of this spectral sequence is given by

$$\mathrm{Ext}_{AK(n)_{**}}(AK(n)_{**}(X), AK(n)_{**}(X))$$

and it converges conditionally to $[AK \wedge X, AK \wedge X]_{AK,pq}$. In particular, if $AK(n)_{**}(X)$ is free or just projective as an $AK(n)_{**}$-module, this spectral sequence collapses at the E_2-page because it is concentrated in the 0-line, and we get an isomorphism:

$$AK(n)_{pq}(X \wedge DX) \cong \mathrm{End}_{AK(n)_{**}}(AK(n)_{**}(X))_{pq}$$

However, there is no general reason why $AK(n)_{**}(X)$ should be free or projective. In contrast to this, all graded modules over the graded field $K(n)_*$ are free, and therefore we always have an isomorphism

$$K(n)_{**}(X \wedge DX) \cong \mathrm{End}_{K(n)_*}(K(n)_*(X))$$

for all finite topological cell spectra X. As a consequence, instead of working with $AK(n)_{**}(X \wedge DX)$, we will work directly with $\mathrm{End}_{AK(n)_{**}}(AK(n)_{**}(X))$ motivically.

Every element in $AK(n)_{**}$ induces a map in $\mathrm{End}_{AK(n)_{**}}(AK(n)_{**}(X))$ given by multiplication with that element. We will denote this map by the same name as the element. We can now prove the motivic equivalent of asymptotic uniqueness:

Lemma 3.3.7. *Let X be a motivic spectrum in $\mathcal{SH}_k^{qfin}$ or $\mathcal{SH}_{k,l}^{\wedge,qfin}$ and*

$$f : X \to X$$

a motivic v_n-self map. Then there exist integers i and j such that:

$$AK(n)_{**}(f^i) = v_n^j$$

Proof. We will use the classical statement for v_n^{top}-self maps in the topological stable homotopy category. In addition, it is known that for any unit u in a $K(n)_*$-algebra that is finitely generated as a $K(n)_*$-module (c.f. [HS, Lemma 3.2] or [Rav2, Proof of Lemma 6.1.1]) there is a power of that element such that $u^i = (v_n^{top})^j$. We will deduce the motivic statement by applying these classical lemmas twice. On the one hand, one can divide out the ideal generated by τ, which yields a finitely generated $K(n)_*$-algebra; on the other hand, one can apply Betti realization.

Our first step is to show that the map

$$\tau : AK(n)_{**}(X) \to AK(n)_{**}(X)$$

can not be a unit in $\mathrm{End}_{AK(n)_{**}}(AK(n)_{**}(X))$:
The element $\tau \in AK(n)_{**}$ is not a unit; if we fix the first degree p in $AK(n)_{pq}$ and vary the height q, then there is a maximum height such that $AK(n)_{pq} = 0$ for all larger heights q. If τ were a unit, all its powers $\tau^k \in AK(n)_{0,-k}$ would need to have an inverse $\tau^{-k} \in AK(n)_{0,k}$ in arbitrarily high weights, which is a contradiction to the previous statement. By the same argument the image of τ cannot be a unit in any finitely generated $AK(n)_{**}$-module. But $\mathrm{End}_{AK(n)_{**}}(AK(n)_{**}(X))$ was finitely generated by 3.3.3, so the multiplication-by-τ-map cannot be a unit.

In the second step, we show that the statement is true modulo τ:
The motivic v_n-self map f induces an isomorphism in $AK(n)_{**}$-homology, i.e. a unit in $\mathrm{End}_{AK(n)_{**}}(AK(n)_{**}(X))$. In the previous step we showed that τ cannot be a unit; this implies that it cannot divide $AK(n)_{**}(f)$, for if it did, τ would also be a unit.
Therefore $AK(n)_{**}(f)$ does not map to zero under the quotient map

$$\mathrm{End}_{AK(n)_{**}}(AK(n)_{**}(X)) \to \mathrm{End}_{AK(n)_{**}/(\tau)}(AK(n)_{**}(X)/(\tau)))$$

and its image $\overline{AK(n)_{**}(f)}$ is thus a unit in the second ring.
If we forget the second bidegree, $AK(n)_{**}/(\tau)$ is isomorphic to $K(n)_*$, and $\mathrm{End}_{AK(n)_{**}/(\tau)}(AK(n)_{**}(X)/(\tau))$ is a finitely generated $K(n)_*$-algebra. In this case we know that there are integers i and j such that $\overline{AK(n)_{**}(f)}^i = (v_n^{top})^j$. Hence

$$AK(n)_{**}(f)^i = v_n^j + \tau\tilde{x}$$

for some element $\tilde{x} \in AK(n)_{**}(X)$.

For the last step, suppose now that $\tilde{x}$ is τ-primary torsion. For the fixed prime l and any $k \in \mathbb{N}$ we can consider powers $AK(n)_{**}(f)^{ikl} = v_n^{jkl} + (\tau\tilde{x})^{kl}$. If k is sufficiently large, the second term vanishes and we are done.
Suppose then that $\tilde{x}$ is not τ-primary torsion. Motivic realization induces a map

$$\mathrm{End}_{AK(n)_{**}}(AK(n)_{**}(X)) \to \mathrm{End}_{K(n)_*}(K(n)_*(X))$$

By the classical statement we know that there are integers i' and j' such that $R_{End,X}(f)^{i'} = (v_n^{top})^{j'}$. Replace i, i' and j, j' with their products $i \cdot i'$ and $j \cdot j'$ and call the result i and j again. Then $AK(n)_{**}(f)^i = v_n^j + \tau\tilde{x}$ realizes to v_n^j, so $\tau\tilde{x}$ realizes to 0. Because τ realizes to 1, $\tilde{x}$ realizes to 0 and by 3.3.5 is therefore 0 itself. □

Lemma 3.3.8. *Asssume that the motivic nilpotence conjecture holds. Let*

$$f : \Sigma^{p,q} X \to X$$

be a motivic v_n-self map and

$$x \in \pi_{p,q}(DX \wedge X)$$

the element corresponding to f under motivic Spanier Whitehead duality. Then there exists an integer $i \in \mathbb{N}$ such that x^i is in the center of $\pi_{p,q}(DX \wedge X)$, considered as a graded commutative ring in the first bidegree.

Proof. The proof is essentially similar to [HS, Lemma 3.5] and [Rav2, Lemma 6.1.2], but we will have to use the motivic Nilpotence conjecture at one point. For all $a \in \pi_{**}(DX \wedge X)$ there is an abstract map of rings

$$ad(a) : \pi_{**}(DX \wedge X) \to \pi_{**}(DX \wedge X)$$

defined by $ad(a)(b) = ab - ba$, and the element a is central if and only if $ad(a)$ is the zero map. This map is realized in homotopy by the composite (here we write R for $DX \wedge X$ and T for the transposition map):

$$S^{p,q} \wedge R \overset{a \wedge id_R}{\to} R \wedge R \overset{1-T}{\to} R \wedge R \overset{\mu}{\to} R$$

We also denote this composite by $ad(a)$.
It now suffices to show that $ad(x)$ is nilpotent because of the following classical formula (proved in [Rav2, Lemma 6.1.2]):

$$ad(x^i)(b) = \sum_{j=1}^{i} \binom{i}{j} ad^j(x)(b)x^{i-j}$$

If we choose $i = l^N$ for a sufficiently large N, all summands in this formula vanish either because of the nilpotence of $ad(x)$ or because the binomial coefficient annihilates $ad(x)$.
Note that $AK(n)_{**}(DX \wedge X)$ is a finitely generated $AK(n)_{**}$-algebra that maps to $K(n)_*(DR(X) \wedge R(X))$ under Betti realization. It follows by the same reasoning as in the proof of Lemma 3.3.7 that a suitable power of $AK(n)_{**}(x)$ is given by v_n^i for some $i \in \mathbb{N}$, which is in the image of $AK(n)_{**}$ in $AK(n)_{**}(DX \wedge X)$ and hence central. Replace x with that power and name it x again. Then $AK(n)_{**}(ad(x))$ is zero, so $ad(x)$ is nilpotent by the nilpotence conjecture. □

Lemma 3.3.9. *Let X be a motivic spectrum in $\mathcal{SH}^{qfin}_{k,(l)}$ or $\mathcal{SH}^{\wedge,qfin}_{k,l}$. Assume that the motivic nilpotence conjecture 3.3.2 holds. If $f, g : X \to X$ are two motivic v_n-self maps, then there exist integers $i, j \in \mathbb{N}$ such that $f^i = g^j$.*

Proof. This lemma corresponds to [HS, Lemma 3.6] and [Rav2, Lemma 6.1.3]. By the previous two lemmas, we can assume that f and g, after replacing them with appropiate powers of themselves, commute with each other in regard to composition, and furthermore that

$$AK(n)_{**}(f^{i'} - g^{j'}) = 0.$$

Using the nilpotence conjecture, we can conclude that $f^{i'} - g^{j'}$ is nilpotent. Then [HS, Lemma 3.4] gives us the desired statement. □

Lemma 3.3.10. *Assume that the motivic nilpotence conjecture 3.3.2 holds. If $f : X \to X$ and $g : Y \to Y$ are two v_n self maps of X and Y and $h : X \to Y$ is any map, then there exist integers $i, j \in \mathbb{N}$ such that $h \circ f^i = g^{l^m} \circ h$.*

Proof. The proof is entirely similar to [Rav2, 6.1.4] □

Theorem 3.3.11. *Let $k = \mathbb{C}$ and l be an odd prime. Assume that the motivic nilpotence conjecture 3.3.2 holds. Then the full subcategories of $\mathcal{SH}^{qfin}_{k,(l)}$ and $\mathcal{SH}^{\wedge,qfin}_{k,l}$ consisting of spectra admitting motivic v_n-self maps are thick.*

Proof. First we prove that the category of spectra admitting motivic v_n-self maps is closed under retracts:
Let $e : X \to Y$ be a retract with right inverse $s : Y \to X$ and assume that there is a v_n-self map $f : X \to X$. By 3.3.8 a power of f commutes with $s \circ e$, so $e \circ f \circ s$ is a v_n-self map.
Furthermore the category of spectra admitting motivic v_n-self maps is closed under cofiber sequences:
Let X and Y be two spectra with motivic v_n-self maps $f : \Sigma^{a,b} X \to X$ and $g : \Sigma^{c,d} Y \to Y$ and let $h : X \to Y$ be any map. By 3.3.10 we can, after replacing the self maps with suitable powers, assume that $(a, b) = (c, d)$ and $h \circ f = g \circ h$. Therefore there exists a map $k : \Sigma^{a,b} C_h \to C_h$ making the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "h"] & Y \arrow[r] & C_h \\
\Sigma a, bX \arrow[u, "f"] \arrow[r, "h"] & \Sigma^{a,b} Y \arrow[u, "g"] \arrow[r] & \Sigma^{a,b} C_h \arrow[u, dotted, "k"]
\end{tikzcd}$$

It follows by the five lemma and basic facts about triangulated categories that k^2 is a v_n-self map on C_h as desired. □

3.4 Existence of a self map on $\mathbb{X}_n$

In [HS] Hopkins and Smith used the Adams spectral sequence to prove the existence of a self map on a spectrum X_n constructed by Smith. In this section we use their proof together with a suitable motivic spectrum $\mathbb{X}_n$ constructed by Joachimi to show that at least one spectrum in $\mathcal{SH}_k^{qfin}$ or $\mathcal{SH}_{k,l}^{\wedge,qfin}$ actually has a motivic v_n-self map. The classical proof relies on computing

$$K(n)_p(X_n \wedge DX_n) \cong \mathrm{End}_{K(n)_*}(K(n)_{**}(X_n))_p$$

via the Adams spectral sequence, so we run into the same kind of problem as in the previous chapter: Because motivically not all graded modules over $AK(n)_{**}$ are free, we first have to show that $AK(n)_{**}(\mathbb{X}_n)$ is in fact free. This also provides us with a Künneth isomorphism for products involving $AK(n)_{**}(\mathbb{X}_n)$.

The proof of the existence of a v_n-self map also relies on the approximation lemma, which relates the cohomology of the Steenrod algebra in certain degrees to the cohomology of certain subalgebras. We need the motivic analogue of this lemma. To this end we need to make two definitions:

Definition 3.4.1. 1. *Let X be a motivic spectrum. Call X k-*bounded below *if $\pi_{m,n} = 0$ for $m \leq k$. Similarly, call a bigraded module $M_{m,n}$ over the motivic Steenrod algebra k-*bounded below *if $M_{m,n} = 0$ for $m \leq k$.*

2. *A module over the motivic Steenrod algebra has a vanishing line of slope m and intercept b if $Ext_A^{s,t,u}(M, H\mathbb{Z}/l^{**}) = 0$ for $s > m(t-s)+b$.*

Note that the preceding definition is exactly like the classic one and the weight is not involved.

Definition 3.4.2. 1. *Let β denote the motivic Bockstein homomorphism, and Sq^i resp. P^i denote the motivic Square- and Power operations as constructed by Voevodsky in [Voe2]. If $l = 2$, define A_n as the subalgebra of the motivic Steenrod algebra generated by $Sq^1, Sq^2, ..., Sq^{2^n}$.*
If $l \neq 2$, define A_n as the subalgebra of the motivic Steenrod algebra generated by $\beta, P^1, ...P^{n-1}$ for $n \neq 0$ and by β for $n = 0$.

2. *Fix the monomial $\mathbb{Z}/l$-basis for the dual motivic Steenrod algebra defined by the elements τ, ξ_i and τ_i (if $l \neq 2$). The elements P_t^s in the motivic Steenrod algebra are defined as the dual elements to $\xi_t^{p^s}$, and the elements Q_i are defined as the dual elements to τ_i if $l \neq 2$ and as $Q_i = P_{i+1}^0$ in the case $l = 2$.*

3. *Write $\Lambda(Q_n)$ for the exterior algebra over the ground ring $H\mathbb{Z}/l^{**}$ in the generator Q_n. This is a subalgebra of the motivic Steenrod algebra.*

We can now prove the motivic analogon to the approximation lemma (c.f. cite[6.3.2]Rav2):

Proposition 3.4.3. *Let M be a bounded below module over the motivic Steenrod algebra such that $Ext_A^{s,t}(M, H\mathbb{Z}/l_{**})$ has a vanishing line of slope m and intercept*

b.
For sufficiently large N the restriction map

$$Ext^{s,t}_{A}(M, H\mathbb{Z}/l_{**}) \to Ext^{s,t}_{A_N}(M, H\mathbb{Z}/l_{**})$$

is an isomorphism in degrees $s \geq m(t-s)+b'$, where b' can be chosen arbitrarily low for sufficiently large N.

Proof. Define C as the kernel of the surjective map of A-modules $A \underset{A_N}{\otimes} M \to M$. As an A_N-module C is given by $M \otimes \overline{A//A_N}$, where $A//A_N = A \underset{A_N}{\otimes} \mathbb{Z}/(l)$ and the bar denotes the augmentation ideal. The motivic Steenrod squares Sq^i live in bidegrees $(2i, i)$ if i is even and $(2i+1, i)$ if it is odd and the motivic Power operations P^i live in bidegrees $(2i(l-1), i(l-1))$. Hence $\overline{A//A_N}$ will be k-bounded below, and k can be chosen arbitrarily high if N is sufficiently large. Therefore C has a vanishing line of the same slope as M and arbitrarily low intercept for sufficiently large N , cf. [HS][4.4]. The short exact sequence defining C and the change-of-rings isomorphism for A_N and A provide the following diagram:

$$\begin{tikzcd}
\mathrm{Ext}^{s-1}_{A_{**}}(C, H\mathbb{Z}/l_{**}) \arrow[d] & \\
\mathrm{Ext}^{s}_{A_{**}}(M, H\mathbb{Z}/l_{**}) \arrow[d] \arrow[dr, "\phi"] & \\
\mathrm{Ext}^{s}_{A_{**}}(A \underset{A_N}{\otimes} M, H\mathbb{Z}/l_{**}) \arrow[r, "\cong"] \arrow[d] & \mathrm{Ext}^{s}_{A_{N,**}}(M, H\mathbb{Z}/l_{**}) \\
\mathrm{Ext}^{s}_{A_{**}}(C, H\mathbb{Z}/l_{**}) &
\end{tikzcd}$$

If the upper and lower term in the diagram vanish - which is the case above the vanishing line of C - the map ϕ is the composite of two isomorphisms and hence an isomorphism itself. $\square$

In [Joa, Theorem 8.5.12] a cell spectrum $\mathbb{X}_n$ has been defined in analogy to the Smith-construction spectrum X_n in [HS](see also [Rav2]). This spectrum is constructed by splitting off a wedge summand of a finite cell spectrum via an idempotent. We need some of the details of the construction of $\mathbb{X}_n$and its properties for the construction of the v_n-self map, so we recall and collect all those that are relevant in one place:

Definition 3.4.4. *The spectrum $\mathbb{X}_n$ is defined as*

$$\mathbb{X}_n = e_V(\mathbb{B}_{(l)}^{\wedge k_V}) = \underset{\to}{hocolim}\ \mathbb{B}_{(l)}^{\wedge k_V} \underset{e_V}{\to} \mathbb{B}_{(l)}^{\wedge k_V} \underset{e_V}{\to} ...$$

where

- *$\mathbb{B}_{(l)}$ is a motivic l-local finite cellular spectrum defined in [Joa, 8.5], implicitly depending on n.*
- *$V = H\mathbb{Z}/l^{**}(\mathbb{B}_{(l)}) = H\mathbb{Z}/l^{**}(a,b)/(a^2, b^{l^n})$, where $|a| = (1,1)$ and $|b| = (2,1)$ ([Joa, 8.5.10])*
- *k_V is an integer dependent on V.*
- *e_V is an idempotent of the groupring $\mathbb{Z}_{(l)}[\Sigma_{k_V}]$, which acts on $\mathbb{B}_{(l)}^{\wedge k_V}$ by permuting the smashfactors and adding maps.*
- *On the level of cohomology, the effect of this idempotent is to split of a free, nonzero $H\mathbb{Z}/l^{**}$-submodule of $V^{\otimes k_V}$. In particular, the motivic cohomology of $\mathbb{X}_n$ is bounded below as a module over the Steenrod algebra.*

Furthermore Joachimi proves the following statements about $\mathbb{X}_n$:

Theorem 3.4.5. *1. $AK(s)_{**}(\mathbb{X}_n) = 0$ for $s < n$ and $AK(n)(\mathbb{X}_n) \neq 0$ ([Joa, Theorem 8.5.12])*

2. Because the operation Q_n acts trivially on $H\mathbb{Z}/l^{**}(\mathbb{B}_{(l)})$ and hence on $H\mathbb{Z}/l^{**}(\mathbb{B}_{(l)}^{\wedge k_V})$, and $H\mathbb{Z}/l^{**}(\mathbb{X}_n)$ is a $H\mathbb{Z}/l^{**}$-submodule of this module, Q_n acts trivially on $H\mathbb{Z}/l^{**}(\mathbb{X}_n)$ (see [Joa, 8.5.2]).

3. $R(\mathbb{X}_n) = X_n$. ([Joa, 8.6])

By 1.4.6 $\mathbb{X}_n$ is dualizable, and its dual is the retract of a finite cell spectrum. Because the spectrum $\mathbb{X}_n$ is dualizable, it satisfies the expected relation between homology and cohomology once we show that its cohomology is free:

Lemma 3.4.6. *1. Let E be a cellular motivic ring spectrum and X be a dualizable cellular motivic spectrum. If $E^{**}(X)$ is a free module over the coefficients E^{**}, then $Hom_{E^{**}}(E^{**}(X), E^{**}) \cong E_{**}(X)$.*

2. Let X be a dualizable cellular motivic spectrum such that

- *$H\mathbb{Z}/l^{**}(X)$ is free over $H\mathbb{Z}/l^{**}$*
- *Q_n acts trivially on $H\mathbb{Z}/l^{**}(X)$.*

Then we have an additive bigraded isomorphism

$$Ext^{s,t,u}_{\Lambda(Q_n)}(H\mathbb{Z}/l^{**}(X), H\mathbb{Z}/l^{**}) \cong H\mathbb{Z}/l_{**}(X)[v_n]$$

where $|v_n| = (1, 2(l^n-1), l^n-1)$. Here s is the homological degree and t,u correspond to the internal bidegree. (The result also holds multiplicatively, but we are not going to need this.)

Proof. 1. This is the content of [Joa, 8.1.2], using the universal coefficient spectral sequence of [DI2, 7.7] and the fact that this spectral sequence collapses if $E^{**}(X)$ is free over E^{**}. Note that the cited corollary is stated only for finite cell spectra and the case $E = H\mathbb{Z}/l$, but the only properties of X actually used are cellularity and dualizability, and that the proof also works for any cellular motivic ring spectrum E.

2. This is a classical result that can be proven in the following way:
Consider the following resolution of free $\Lambda(Q_n)$-modules

$$\ldots \overset{\cdot Q_n}{\to} \Lambda(Q_n) \overset{\cdot Q_n}{\to} \Lambda(Q_n) \overset{\cdot Q_n}{\to} \Lambda(Q_n) \overset{\epsilon}{\to} H\mathbb{Z}/l^{**}$$

where the last map is the projection $\epsilon : \Lambda(Q_n) \to H\mathbb{Z}/l^{**}$ and apply $(-) \underset{H\mathbb{Z}/l^{**}}{\otimes} H\mathbb{Z}/l^{**}(X)$.
The resulting long exact sequence

$$\ldots \overset{\cdot Q_n}{\to} \Lambda(Q_n) \underset{H\mathbb{Z}/l^{**}}{\otimes} H\mathbb{Z}/l^{**}(X) \overset{\cdot Q_n}{\to} \Lambda(Q_n) \underset{H\mathbb{Z}/l^{**}}{\otimes} H\mathbb{Z}/l^{**}(X) \overset{\epsilon}{\to} H\mathbb{Z}/l^{**}(X)$$

is a resolution of the $\Lambda(Q_n)$-module $H\mathbb{Z}/l^{**}(X)$. Here we use the assumption that Q_n acts trivially on this module in the claim that the last map is a map of $\Lambda(Q_n)$-modules.
Now apply $\mathrm{Hom}_{\Lambda(Q_n)}((-), H\mathbb{Z}/l^{**})$ and take cohomology. All maps are zero because the target has the trivial $\Lambda(Q_n)$-module structure. Using the isomorphism from the previous part, we can rewrite degreewise:

$$\begin{aligned} &\mathrm{Hom}_{\Lambda(Q_n)}(\Lambda(Q_n) \underset{H\mathbb{Z}/l^{**}}{\otimes} H\mathbb{Z}/l^{**}(X), H\mathbb{Z}/l^{**}) \\ &\cong \mathrm{Hom}_{H\mathbb{Z}/l^{**}}(H\mathbb{Z}/l^{**}(X), H\mathbb{Z}/l^{**}) \\ &\cong H\mathbb{Z}/l_{**}(X) \end{aligned}$$

□

Recall that the coefficient rings of the classical Morava K-theories are graded fields in the sense that all graded modules over it are free. This is not true of the motivic Morava K-theories in general. The algebraic Morava K-theory of the spectrum $\mathbb{X}_n$ however is free and finitely generated. To see this, we need to go through the steps of its construction.

Proposition 3.4.7. *Let $k \in \{\mathbb{R}, \mathbb{C}\}$ and l be an odd prime. Then $AK(n)_{**}(\mathbb{X}_n)$ is a free, finitely generated $AK(n)_{**}$-module.*

Proof. To prove the statement it suffices to show that

- $AK(n)_{**}(\mathbb{X}_n)$ is a finitely generated $AK(n)_{**}$-module
- $AK(n)_{**}(\mathbb{X}_n)$ has no τ-torsion (over $\mathbb{C}$) or θ-torsion (over $\mathbb{R}$)

We are going to show both claims in three steps: First we compute $Ak(n)(\mathbb{B}_{(l)})$ using the motivic Adams spectral sequence. We show that it is finitely generated and does not have τ/θ-torsion, which implies that $AK(n)(\mathbb{B}_{(l)})$ is finitely generated and torsionfree. Then we use the Künneth theorem to show the same statement for $AK(n)(\mathbb{B}_{(l)}^{\wedge k_V})$. Finally we use the definition of the idempotent defining $\mathbb{X}_n$ to show that $AK(n)_{**}(\mathbb{X}_n)$ satisfies both claims.
We begin with the first step: The motivic Adams spectral sequence for $Ak(n) \wedge \mathbb{B}_{(l)}$ converges strongly to $Ak_{**}(\mathbb{B}_{(l)})$ ([Joa, 8.3.3]). We claim that there are no nontrivial differentials in this spectral sequence. Note that we can restrict to the case $k = \mathbb{C}$, because we have a map of motivic Adams spectral sequences, induced by $\mathbb{R} \to \mathbb{C}$ and given by $\theta \mapsto \tau^2$. Hence if there are no nontrivial differentials over $\mathbb{C}$ there are also none over $\mathbb{R}$.
The E_2-term of this motivic Adams spectral sequence can be written as

$$\mathrm{Ext}_{\Lambda(Q_n)}(H\mathbb{Z}/l^{**}(\mathbb{B}_{(l)}), H\mathbb{Z}/l^{**})$$

by change of rings ([Joa, 8.2.3]). Recall that $H\mathbb{Z}/l^{**}(\mathbb{B}_{(l)}) = H\mathbb{Z}/l^{**}(a,b)/(a^2, b^{l^n})$. The element Q_n acts trivially on this free and finitely generated $H\mathbb{Z}/l^{**}$-module, which implies by the previous lemma

$$\mathrm{Ext}^{***}_{\Lambda(Q_n)}(H\mathbb{Z}/(l)^{**}(\mathbb{B}_{(l)})), H\mathbb{Z}/l^{**}) \cong H\mathbb{Z}/l_{**}[v_n] \underset{H\mathbb{Z}/(l)_{**}}{\otimes} H\mathbb{Z}/(l)_{**}(\mathbb{B}_{(l)})$$

The right hand side is a tensor product of polynomial algebras, and the position of the polynomial generators and of v_n in the spectral sequence imply that they cannot support a nontrivial differential at any stage. In the following sketch of the spectral sequence in an abuse of notation a and b denote the dual of the cohomology classes with the same name. Note that the spectral sequence to the right of the depicted area looks very similar to the displayed area - the same elements appear in the same configuration, just multiplied by some power of v_n. In the standard Adams grading the differential d_r maps one entry to the left and r entries up. Thus it is clear that no potentially nontrivial differential can have a target different from zero.

$$\begin{array}{cc|cccccc}
 & \cdots & & & & & & \\
 & & & & & & & av_n \\
\mathrm{s} & 1 & & & & & & v_n \\
 & & & ab & ab^2 & & ab^{l^n-1} & \\
 & 0 & a & b & b^2 & \cdots & b^{l^n-1} & \\
\hline
 & & 0 & 1 & 2 & \cdots & l^n-1 & 2(l^n-1) \\
 & & & & & \mathrm{t-s} & &
\end{array}$$

Therefore $Ak(n)(\mathbb{B}_{(l)})$ is finitely generated over $Ak(n)_{**}$ and does not have τ-primary torsion. For all cellular spectra X we have $AK(n)_{**}(X) \cong v_n^{-1}Ak(n)_{**}(X)$.

Therefore $AK(n)(\mathbb{B}_{(l)})$ is free and finitely generated over $Ak(n)_{**}$.
The second step is now easy: Since AK is a cellular spectrum and since we just proved that the cellular spectrum $\mathbb{B}_{(l)}$ has free AK-homology over the coefficients, we can apply the Künneth theorem (([DI2][Remark 8.7])) and obtain

$$AK(n)_{**}(\mathbb{B}_{(l)}^{\wedge k_V}) \cong AK(n)_{**}(\mathbb{B}_{(l)})^{\otimes k_V}$$

Therefore also $AK(n)_{**}(\mathbb{B}_{(l)}^{\wedge k_V})$ is free and finitely generated over the coefficients.
For the last step, note that $AK(n)_{**}(\mathbb{X}_n)$ is a finitely generated $AK(n)_{**}$-module as well, since it is a submodule of the finitely generated module $AK(n)_{**}(\mathbb{B}_{(l)}^{\wedge k_V})$ over the noetherian ring $AK(n)_{**}$.
It remains to show that no torsion occurs. The idempotent $e_V \in Z_{(l)}[\Sigma_{k_V}]$ acts on $AK(n)_{**}(\mathbb{B}_{(l)}^{\wedge k_V})$ by permutation of the tensor factors and multiplication by integers. No τ-Torsion can occur in $e_V(AK(n)_{**}(\mathbb{B}_{(l)}^{\wedge k_V})) = AK(n)_{**}(\mathbb{X}_n)$ because the order of an element in a fixed bidegree in $AK(n)_{**}(\mathbb{B}_{(l)}^{\wedge k_V})$ is the same as that of the τ-multiples of that element. Consequently, $AK(n)_{**}(\mathbb{X}_n)$ is a free $AK(n)_{**}$-module. □

Definition 3.4.8. *Define $R = D\mathbb{X}_n \wedge \mathbb{X}_n$. It is a quasifinite cell spectrum by definition and by 1.4.4 it can be endowed with the structure of a motivic homotopy ring spectrum, with unit map $e : S \to D\mathbb{X}_n \wedge X_n$ and multiplication map $\mu : R \wedge R \to R$.*

As a corollary of the preceding proposition we get the following:

Corollary 3.4.9. *Let $R = D\mathbb{X}_n \wedge \mathbb{X}_n$. There are Künneth-isomorphisms*

1.
$$AK(n)^{**}(R) \overset{\cong}{\to} AK(n)^{**}(D\mathbb{X}_n) \underset{AK(n)^{**}}{\otimes} AK(n)^{**}(\mathbb{X}_n)$$

2.
$$AK(n)_{**}(R) \overset{\cong}{\to} AK(n)_{**}(D\mathbb{X}_n) \underset{AK(n)_{**}}{\otimes} AK(n)_{**}(\mathbb{X}_n)$$

Proof. The Milnor short exact sequence for $\mathbb{X}_n$ and $AK(n)$-cohomology is

$$0 \to \underset{\leftarrow}{\lim}^1 AK(n)^{*-1,*}(\mathbb{B}_{(l)}^{\wedge k_V}) \to AK(n)^{**}(\mathbb{X}_n) \to \underset{\leftarrow}{\lim}\, AK(n)^{**}(\mathbb{B}_{(l)}^{\wedge k_V}) \to 0$$

Because the map e_V over which the homotopy colimit defining $\mathbb{X}_n$ is taken is an idempotent, the system $AK(n)^{**}(\mathbb{B}_{(l)}^{\wedge k_V})$ is Mittag-Leffler, which implies that the $\underset{\leftarrow}{\lim}^1$-term vanishes. By the same argument, we have:

$$\underset{\leftarrow}{\lim}\, AK(n)^{**}(D\mathbb{X}_n \wedge \mathbb{B}_{(l)}^{\wedge k_V}) \cong AK(n)^{**}(D\mathbb{X}_n \wedge \mathbb{X}_n)$$

Here the limit is taken over the maps $id \wedge e_V$.
Since $\mathbb{B}_{(l)}^{\wedge k_V}$ is the l-localization of a finite cell spectrum and $AK(n)^{**}(\mathbb{B}_{(l)}^{\wedge k_V})$ is

a free module over $AK(n)^{**}$, we can use the Künneth-isomorphism of Dugger and Isaksen [DI2, Remark 8.7] to see that

$$AK(n)^{**}(D\mathbb{X}_n \wedge \mathbb{B}_{(l)}^{\wedge k_V}) \xrightarrow{\cong} AK(n)^{**}(D\mathbb{X}_n) \underset{AK(n)^{**}}{\otimes} AK(n)^{**}(\mathbb{B}_{(l)}^{\wedge k_V})$$

It remains to rewrite the inverse limit over the right hand side: $AK(n)^{**}(D\mathbb{X}_n)$ is a free $AK(n)^{**}$-module because $AK(n)_{**}(\mathbb{X}_n)$ is a free $AK(n)_{**}$-module, so using the earlier isomorphism we get:

$$\lim_{\leftarrow} \left(AK(n)^{**}(D\mathbb{X}_n) \underset{AK(n)^{**}}{\otimes} AK(n)^{**}(\mathbb{B}_{(l)}^{\wedge k_V})\right) \cong AK(n)^{**}(D\mathbb{X}_n) \underset{AK(n)^{**}}{\otimes} AK(n)^{**}(\mathbb{X}_n)$$

The Künneth-isomorphism in $AK(n)$-homology can either be derived from the one in cohomology or from the Künneth-isomorphism of the l-local finite cell spectra $\mathbb{B}_{(l)}^{\wedge k_V}$ and the fact that homology commutes with direct limits. □

We also need the following vanishing line:

Lemma 3.4.10. *Let l be odd. The A^{**}-module*

$$\mathrm{Ext}_{A^{**}}(H\mathbb{Z}/l^{**}(R), H\mathbb{Z}/l^{**})$$

has a vanishing line of slope $1/2(l^n - 1)$.

Proof. Over odd primes, the motivic Steenrod-algebra is just the classical Steenrod algebra (where the generators are understood to live in the appropiate motivic bidegrees) base changed to $H\mathbb{Z}/l^{**}$. Similarly, $H\mathbb{Z}/l^{**}(D\mathbb{X}_n \wedge \mathbb{X}_n)$ corresponds to $H\mathbb{Z}/l_*(DX_n \wedge X_n)$ basechanged to $H\mathbb{Z}/l^{**}$, where the generators are once again understood to live in the appropiate bidegree.
Consequently $\mathrm{Ext}_{A^{**}}(H\mathbb{Z}/l^{**}(D\mathbb{X}_n \wedge \mathbb{X}_n), H\mathbb{Z}/l^{**})$, which maps to the classical Ext-term $\mathrm{Ext}_{A^*_{\mathrm{top}}}(H\mathbb{Z}/l^*(DX_n \wedge X_n), H\mathbb{Z}/l^*)$, is just that classical Ext-term base changed to $H\mathbb{Z}/l^{**}$ and in particular does not contain τ-torsion (over $\mathbb{C}$) or θ-torsion (over $\mathbb{R}$). The existence of the vanishing line then follows from the existence of a vanishing line with the same slope in the classical case for the spectrum X_n(see [Rav2, 6.3.1]). □

Furthermore, we need the following duality isomorphisms:

Proposition 3.4.11.

1. $\mathrm{Hom}_{H\mathbb{Z}/l^{**}}(H\mathbb{Z}/l^{**}(R), H\mathbb{Z}/l^{**}) \cong H\mathbb{Z}/l_{**}(R)$
2. $AK(n)_{**}(D\mathbb{X}_n) \cong AK(n)^{**}(\mathbb{X}_n) \cong \mathrm{Hom}_{AK(n)_{**}}(AK(n)_{**}(\mathbb{X}_n), AK(n)_{**})$
3. $AK(n)^{**}(D\mathbb{X}_n) \cong AK(n)_{**}(\mathbb{X}_n) \cong \mathrm{Hom}_{AK(n)^{**}}(AK(n)^{**}(\mathbb{X}_n), AK(n)^{**})$

Proof. 1. $R = D\mathbb{X}_n \wedge \mathbb{X}_n$ is a dualizable cell spectrum since $\mathbb{X}_n$ and $D\mathbb{X}_n$ are. Therefore we can consider the universal coefficient spectral sequence of [DI2, 7.7]. As explained in [Joa, 8.1.2], this spectral sequence collapses if $H\mathbb{Z}/l^{**}(R)$ is free over $H\mathbb{Z}/l^{**}$. (Note that the cited corollary is stated for finite cell spectra, but the only properties actually used are cellularity and dualizability.) To show the freeness of $H\mathbb{Z}/l^{**}(R)$ as a $H\mathbb{Z}/l^{**}$-module, observe that $H\mathbb{Z}/l^{**}(\mathbb{X}_n)$ is free by construction ([Joa, 8.5.3]). This implies the existence of a Künneth isomorphism for $\mathbb{X}_n$, and thus

$$H\mathbb{Z}/l^{**}(R) = H\mathbb{Z}/l^{**}(D\mathbb{X}_n) \underset{H\mathbb{Z}/l^{**}}{\otimes} H\mathbb{Z}/l^{**}(\mathbb{X}_n)$$

is free.

2. The first isomorphism follows directly from the canonical bijection. The second isomorphism is proven by the same argument as in the proof of part 1, using the universal coefficient spectral sequence [DI2, 7.7] together with the facts that AK is a cellular spectrum and that $AK(n)_{**}(\mathbb{X}_n)$ is free over the coefficients.

3. This is proven just as in part 1 or part 2.

□

Corollary 3.4.12. *1. There exists a well defined coevaluation map*

$$coev : AK_{**} \to AK_{**}(\mathbb{X}_n)^{\vee} \underset{AK_{**}}{\otimes} AK_{**}(\mathbb{X}_n)$$

Here $(-)^{\vee}$ denotes the linear dual $\mathrm{Hom}_{AK_{**}}(-, AK_{**})$. It is induced by the map $T \circ e : S \to \mathbb{X}_n \wedge D\mathbb{X}_n$, where $e : S \to D\mathbb{X}_n \wedge \mathbb{X}_n$ is the unit map of $R = D\mathbb{X}_n \wedge \mathbb{X}_n$ and T is the map that transposes the two factors.

2. Under the composition

$$AK_{**} \to AK_{**}(R) \to \mathrm{Hom}_{AK_{**}}(AK_{**}(\mathbb{X}_n), AK_{**}(\mathbb{X}_n))$$

an element $v \in AK_{**}$ maps to multiplication by that element.

Proof. 1. The coevalution map of 1.4.1, which is the same as $T \circ e$, induces the claimed map in $AK(n)$-homology, together with the identification

$$AK(n)_{**}(D\mathbb{X}_n) \cong AK^{**}(\mathbb{X}_n) \cong \mathrm{Hom}_{AK_{**}}(AK_{**}(\mathbb{X}_n), AK_{**})$$

of the preceding proposition. Because $AK(n)_{**}(\mathbb{X}_n)$ is a free and finitely generated $AK(n)_{**}$-module, there is also an algebraic coevalution defined via choosing a basis as for a vector space, and the two maps coincide since they both satisfy the equivalent of the condition of the first point of 1.4.1 for projective and finitely generated modules.

2. The element $1 \in AK_{**}$ maps to the coevaluation of $AK_{**}(\mathbb{X}_n)$ under the first map, using the identification $AK_{**}(R) \cong AK_{**}(\mathbb{X}_n)^\vee \underset{AK_{**}}{\otimes} AK_{**}(\mathbb{X}_n)$ implied by the Künneth and duality isomorphisms. Hence an element of AK_{**} maps to that element times the coevaluation. The coevaluation maps to the identity under the second map. Consequently an element in AK_{**} times the coevaluation maps to multiplication by that element.

□

We now have all the ingredients to use the classical proof in the motivic setting ([HS, Theorem 4.12], see also [Rav2, 6.3]):

Theorem 3.4.13. *Let $k = \mathbb{C}$ and l be an odd prime. The spectrum $\mathbb{X}_n$ has a motivic v_n self-map f satisfying*

$$AK(m)_* f = \delta_{mn} v_n^{p^{N_m}}$$

for a sufficiently large integer N_m.

Proof. The aim is to construct a permanent cycle

$$v \in \mathrm{Ext}_{A_{**}}(H\mathbb{Z}/l_*(R), H\mathbb{Z}/l_*)$$

that maps to a power of v_n in $Ak(n)_{**}(R)$ and to a nilpotent element in $Ak(m)_{**}(R)$ if $m \neq n$. The diagram below will specify the meaning of "maps". Under motivic Spanier Whitehead duality such a class corresponds to a self-map of the described form on $\mathbb{X}_n$.
The cohomology of the point, $H\mathbb{Z}/l^{**}$, is concentrated in simplicial degree 0. Therefore the operations Q_n act trivially on this module over the motivic Steenrod algebra. They act trivially on $H\mathbb{Z}/l^{**}(R)$ since they act trivially on $H^{**}(\mathbb{X}_n)$. If we write $P(v_n)$ for the polynomial algebra in one generator with respect to the base ring $H\mathbb{Z}/(l)_{**}$, this provides us with the following isomorphisms of trigraded algebras:

$$\mathrm{Ext}^{***}_{\Lambda(Q_n)}(H\mathbb{Z}/l^{**}, H\mathbb{Z}/l^{**}) \xrightarrow{\cong} P(v_n) \otimes H\mathbb{Z}/l_{**} \tag{3.3}$$

$$\mathrm{Ext}^{***}_{\Lambda(Q_n)}(H\mathbb{Z}/l^{**}(R), H\mathbb{Z}/l^{**}) \xrightarrow{\cong} P(v_n) \otimes H\mathbb{Z}/l_{**}(R) \tag{3.4}$$

Here v_n has homological degree 1 and internal bidegree $(2(l^n - 1, l^n - 1)$.

Together with the change-of-rings morphisms related to the subalgebras $\Lambda(Q_n)$

and A_N these fit into the following diagram:

$$\begin{tikzcd}
\mathrm{Ext}_{A^{**}}(H\mathbb{Z}/l^{**}, H\mathbb{Z}/l^{**}) \arrow[r, "i"] \arrow[d, "\phi"] & \mathrm{Ext}_{A^{**}}(H\mathbb{Z}/l^{**}(R), H\mathbb{Z}/l^{**}) \arrow[d, "\phi"] \\
\mathrm{Ext}_{A_N^{**}}(H\mathbb{Z}/l^{**}, H\mathbb{Z}/l^{**}) \arrow[r, "i"] \arrow[d, "\lambda"] & \mathrm{Ext}_{A_N^{**}}(H\mathbb{Z}/l^{**}(R), H\mathbb{Z}/l^{**}) \arrow[d, "\lambda"] \\
\mathrm{Ext}_{\Lambda(Q_n)}(H\mathbb{Z}/l^{**}, H\mathbb{Z}/l^{**}) \arrow[r, "i"] \arrow[d, "\cong(1)"] & \mathrm{Ext}_{\Lambda(Q_n)}(H\mathbb{Z}/l^{**}(R), H\mathbb{Z}/l^{**}) \arrow[d, "\cong(2)"] \\
P(v_n) \underset{H\mathbb{Z}/(l)_{**}}{\otimes} H\mathbb{Z}/l_{**} \arrow[r, "i"] \arrow[d] & P(v_n) \underset{H\mathbb{Z}/(l)_{**}}{\otimes} H\mathbb{Z}/l_{**}(R) \arrow[d] \\
Ak(n)_{**} \arrow[r, "i"] & Ak(n)_{**}(R)
\end{tikzcd}$$

Step 1: Consider the element $\widetilde{v_n} \in \mathrm{Ext}_{\Lambda(Q_n)}(H\mathbb{Z}/l_*(R), H\mathbb{Z}/l_*)$ that corresponds to $v_n \otimes 1 \in P(v_n) \otimes H_{**}(R)$ under the isomorphism (2).

Proposition 3.4.14. *$\forall N \geq n$ there is an integer $t > 0$ and an element $x \in Ext_{A_{N,**}}(H\mathbb{Z}/l_*, H\mathbb{Z}/l_*)$ such that $\lambda(x) = v_n^t$. The image of x under i is central in $Ext_{A_{N,**}}(H\mathbb{Z}/l_*(R), H\mathbb{Z}/l_*)$, where central is meant in respect to graded commutativity in the first, but not in the second bidegree.*

Proof. This statement is a corollary of [HS, Theorem 4.12]. Since the motivic cohomology of the point $H\mathbb{Z}/(l)_{**} = \mathbb{Z}/(l)[\tau]$ is concentrated in simplicial degree 0, the action of the motivic Steenrod algebra is trivial on this module. Hence we can basechange the statement of the cited theorem to $\mathbb{Z}/(l)[\tau]$. $\square$

Step 2: The module $\mathrm{Ext}_{A^{**}}(H\mathbb{Z}/l^{**}(R), H\mathbb{Z}/l^{**})$ has a vanishing line of slope $1/(2^l - 2)$ and a fixed intercept b. By the motivic approximation lemma, the morphism

$$\phi : \mathrm{Ext}_{A^{**}}(H\mathbb{Z}/l^*(R), H\mathbb{Z}/l^{**}) \to \mathrm{Ext}_{A_N^{**}}(H\mathbb{Z}/l^{**}(R), H\mathbb{Z}/l^{**})$$

is an isomorphism above a line with slope $1/2(l^n - 1)$ and arbitrarily low intercept for sufficiently large N. Since the element x(and therefore also $i(x)$) has tridegree $(t, 2(l^n - 1), (l^n - 1))$, it lies above that line for a sufficiently large choice of N. Define $y \in \mathrm{Ext}_{A^{**}}(H\mathbb{Z}/l^{**}(R), H\mathbb{Z}/l^{**})$ as the preimage of $i(x)$ under ϕ. Since $i(x)$ is central (in the graded sense with respect to the first bidegree but not with respect to the second) in $\mathrm{Ext}_{A_N^{**}}(H\mathbb{Z}/l^{**}(R), H\mathbb{Z}/l^{**})$, it commutes with all elements in the image of ϕ, in particular with all elements above the line defined by the approximation lemma.

Step 3: The element y and its powers, as well as the images of y and its powers under the differentials of the motivic Adams spectral sequence all satisfy the requirement of the last statement, so they commute with each other. By induction we can assume that a power $\tilde{y}$ of y survives up to the rth page. We wish to show that $\tilde{y}^l$ is a r-cycle, i.e. $d_r(\tilde{y}^l) = 0$. This is true since $d_r(\tilde{y}^l) = l \cdot \tilde{y}^{l-1} d_r(\tilde{y}) = 0$. After a finite number of pages, the differential will point in the area of the spectral sequence above the vanishing line, and we can stop the process. We end with a power $\tilde{y}$ of y that is a permanent cycle in the motivic Adams spectral sequence and hence represents an element of $Ak(n)_{**}(R)$.

Step 4: The permanent cycle $\tilde{y}$ represents an element $\bar{y} \in \pi_{**}(R)$. Choose m such that v_n^m has the same degree as $\bar{y}$. By the exact same arguments as in [HS] we can choose a power of $\bar{y}$ such that $Ak(n)_{**}(\bar{y}^g) = v_n^{gm}$ and define f as the map corresponding to that power of $\bar{y}$ under motivic Spanier Whitehead duality.

Step 5: For $m \neq n$ it follows just as in the topological case that the image of v in $AK(m)_{**}$ is nilpotent either for trivial reasons $(m < n)$ or because of a vanishing line with tighter slope in the Adams spectral sequence computing $Ak(m)_{**}(R)$ $(m > n)$. ☐

3.5 The relation of $\mathcal{C}_\eta$ and $\mathcal{C}_{AK(n)}$

As a corollary of the Künneth isomorphism, we can settle one of the open conjectures in Ruth Joachimis dissertation [Joa, Conjecture 7.1.7.3] which concerns the relation of the thick ideal thickid(C_η) generated by the cone of the motivic Hopf map C_η and the thick ideals $\mathcal{C}_{AK(n)}$ characterized by the vanishing of motivic Morava K-theory.

Lemma 3.5.1. *Let $m \in \mathbb{N}$ be any integer. Then the coefficients of the cone C_η of $\eta : \Sigma^{1,1} S \to S$ in $AK(m)_{**}$-homology are given by:*

$$AK(m)_{**}(C_\eta) \cong AK(m)_{**} \oplus AK(m)_{*-2,*-1}$$

In particular, they are free over $AK(m)_{**}$.

Proof. The long exact sequence induced by the cofiber sequence

$$S^{1,1} \to S^{0,0} \to C_\eta \to S^{2,1}$$

defining C_η splits into short exact sequences

$$0 \to AK(m)_{**} \to AK(m)_{**}(C_\eta) \to AK(m)_{*-2,*-1} \to 0$$

because η induces the zero map in $AK(m)_{**}$-homology. The sequence splits because the outer terms are free $AK(m)_{**}$-modules, yielding the result. ☐

Corollary 3.5.2. *Let $m \in \mathbb{N}$. In the case $m < n$ we have*

$$AK(m)_{**}(C_\eta \wedge \mathbb{X}_n) \cong 0$$

and in the case $m = n$ we have:

$$AK(n)_{**}(C_\eta \wedge \mathbb{X}_n) \cong AK(n)_{**}(C_\eta) \underset{AK(n)_{**}}{\otimes} AK(n)_{**}(\mathbb{X}_n) \neq 0$$

Proof. By the preceding lemma the finite cell spectrum C_η has free $AK(m)$-homology and thus satisfies the requirements of the Künneth formula [DI2, Remark 8.7].
Application of the Künneth formula yields:

$$AK(m)_{**}(C_\eta \wedge \mathbb{X}_n) \cong AK(m)_{**}(C_\eta) \underset{AK(m)_{**}}{\otimes} AK(m)_{**}(\mathbb{X}_n)$$

If $m < n$ the factor $AK(m)_{**}(X) = 0$ vanishes by 3.4.5. This implies the first part of the statement. If $m = n$ the result contains

$$AK(n)_{**}(\mathbb{X}_n) \underset{AK(n)_{**}}{\otimes} AK(n)_{**} = AK(n)_{**}(\mathbb{X}_n) \neq 0$$

as a direct summand, so $AK(n)_{**}(C_\eta \wedge \mathbb{X}_n)$ cannot vanish. □

Proposition 3.5.3. *The spectrum $\mathbb{X}_{n+1}$ is contained in the intersection of thick ideals thickid$(C_\eta) \cap \mathcal{C}_{AK(n)}$, but not in thickid$(C_\eta) \cap \mathcal{C}_{AK(n+1)}$. In particular, these intersections are nonzero and distinct for all $n \in \mathbb{N}$.*

Proof. Clearly $C_\eta \wedge \mathbb{X}_{n+1}$ is in the thick ideal generated by C_η. The preceding corollary tells us on the one hand that $C_\eta \wedge \mathbb{X}_{n+1} \in \mathcal{C}_{AK(n)}$, and on the other hand that $C_\eta \wedge \mathbb{X}_{n+1} \notin \mathcal{C}_{AK(n+1)}$. □

3.6 A counterexample to a statement about thick subcategories in [Joa]

In this section we construct a counterexample to the inclusion

$$\text{thickid}(c\mathcal{C}_2) \subset \mathcal{C}_{AK(1)}$$

claimed in [Joa, Chapter 9, last section], based on an error in [Joa, Proposition 8.7.3].
Let l be an odd prime, and consider the topological mod-l Moore spectrum $S/l \in \mathcal{SH}$. We can easily compute its $K(1)$-homology:

Lemma 3.6.1. $K(1)_*(S/l) \cong K(1)_* \oplus K(1)_{*-1}$

Proof. The Moore spectrum is defined via the cofiber sequence $S \xrightarrow{\cdot l} S \to S/l$ and the map induced by l is trivial in $K(1)$-homology. Therefore the long exact sequence in $K(1)$-homology induced by this cofiber sequence splits up into short exact sequences, and these short exact sequences split because all graded $K(1)$-modules are free. □

In [Ada] Adams proved the existence of a non-nilpotent self map

$$v : \Sigma^{2l-2} S/l \to S/l$$

on the Moore spectrum which induces an isomorphism in $K(1)$-homology; namely multiplication by the invertible element v_1^{top}. Consequently, the $K(1)$-homology of the cone C_v vanishes: $K(1)_*(C_v) = 0$, or equivalently $C_v \in \mathcal{C}_2$.

Applying the constant simplicial presheaf functor c to the construction gives us the cofiber sequence

$$\Sigma^{2l-2,0} S/l \xrightarrow{cv} S/l \to C_{cv}$$

in $\mathcal{SH}_{\mathbb{C}}$. The cone C_{cv} of cv is equivalent to $c(C_v)$ because c is a triangulated functor, and the Moore spectrum is mapped to the Moore spectrum ($cS/l = S/l$ because $cl = l$.) We can compute the $AK(1)$-homology of the mod-l-Moore spectrum using the same argument as in the topological case:

$$AK(1)_{**}(S/l) \cong AK(1)_{**} \oplus AK(1)_{*-1,*}$$

However, the algebraic Morava K-theory of C_{cv} does not vanish:

Lemma 3.6.2. $AK(1)(C_{cv})) \cong AK(1)_{**}(S/l)/(\tau^{l-1}) \neq 0$

Proof. The cofiber sequence $S/l \overset{cv}{\to} S/l \to C_{cv}$ induces a long exact sequence in $AK(1)$-homology:

$$... \to AK(1)_{p+(2l-2),q}(S/l) \overset{AK(1)_{**}(cv)}{\longrightarrow} AK(1)_{pq}(S/l) \to AK(1)_{pq}(C_{cv}) \to ...$$

The map $AK(1)_{**}(cv)$ must be given by multiplication with $\tau^{l-1}v_1$, because Betti realization maps $AK(1)_{**}(cv)$ to multiplication with v_1^{top} and there is only one map realizing to this in the appropiate bidegree. This map is injective but, unlike the topological case, no longer an isomorphism. Hence the long exact sequence splits into short exact sequences

$$0 \to AK(1)_{pq}(cS/l) \overset{\cdot\tau^{l-1}v_1}{\longrightarrow} AK(1)_{pq}(cS/l) \to AK(1)_{pq}(C_{cv}) \to 0$$

and because v_1 is invertible, the last term is isomorphic to $AK(1)_{**}/(\tau^{l-1})$. □

Because $\mathcal{C}_{AK(1)}$ was defined by the vanishing of $AK(1)$-homology and $AK(1)_{**}(C_{cv}) \neq 0$ does not vanish, we have $C_{cv} \notin \mathcal{C}_{AK(1)}$. On the other hand, we have shown that $C_v \in \mathcal{C}_2$. Because $R(C_{cv}) = C_v$, this implies $C_{cv} \in R^{-1}(\mathcal{C}_2)$. Therefore we can conclude the following corollary from the preceding lemma:

Corollary 3.6.3. *The inclusion*

$$\mathcal{C}_{AK(1)} \subsetneq R^{-1}(\mathcal{C}_2)$$

is proper.

Furthermore we have $C_{cv} = cC_v \in \text{thickid}(c\mathcal{C}_2)$. Therefore cC_v is our desired counterexample and proves:

Proposition 3.6.4. *thickid*$(c\mathcal{C}_2) \not\subset \mathcal{C}_{AK(1)}$

Remark 3.6.5. *The mistake on which the incorrect assertion is based occurs in [Joa, Proposition 8.7.3]. This proposition states that for a finite topological CW spectrum Y, $AK(n)^{**}(cY) = 0$ if and only if $K(n)_*(Y) = 0$. In the proof of this proposition Joachimi shows that the differentials in the motivic Atiyah-Hirzebruch spectral sequence are determined by the differentials of the topological Atiyah-Hirzebruch spectral sequence, and that the E_2-page of the motivic spectral sequence is given by adjoining a generator τ to each entry in the topological spectral sequence, where all entries are generated in motivic weight 0. The problem that now occurs is that the differentials in the motivic spectral sequence do not preserve the weight, but lower it. Hence a nontrivial differential can generate τ-primary torsion in the spectral sequence. The above example shows that this in fact happens.*

This argument can in fact be made for any topological spectrum $X \in \mathcal{C}_{n+1} \backslash \mathcal{C}_{n+2}$. Any such spectrum has nontrivial $K(n)$-homology and a self map $v : \Sigma^m X \to X$ that induces multiplication by some power of v_n^{top}. We know by 3.3.5 that the map $AK(n)_{**}(cX) \to K(n)_*(X)$ induced by Betti realization is surjective and its kernel is exactly the τ-primary torsion elements. In particular we know that $AK(n)_{**}(cX) \neq 0$, and the self map provides us with a motivic map cv. This map induces multiplication by the same power of $\tau^{l-1}v_n$ in $AK(n)$-homology - up to a possible error term, which has to be τ-primary torsion. We can eliminate this error term by taking sufficiently large l-fold powers of this map. We end up with a v_n^{top}-self v' of X whose image cv' under the constant simplicial presheaf funtor c induces multiplication by some power of $\tau^{l-1}v_n$ in $AK(n)$-homology. In particular, its cone has nonvanishing $AK(n)$-homology by the same argument as for our earlier counterexample and thus proves:

Proposition 3.6.6. *thickid*$(c\mathcal{C}_{n+1}) \not\subset \mathcal{C}_{AK(n)}$

Just as before, this also proves:

Corollary 3.6.7. *The inclusion*

$$\mathcal{C}_{AK(n)} \subsetneq R^{-1}(\mathcal{C}_{n+1})$$

is proper.

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