On Atiyah-Segal completion for Hermitian K-theory

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Herman Rohrbach

Abstract

The Atiyah-Segal completion theorem states that the completion of equivariant complex K-theory may be computed using the Borel construction, which is a generalization of the construction of the classifying space. This thesis investigates whether there is an analogue of the Atiyah-Segal completion theorem for Hermitian K-theory, also known as Grothendieck-Witt theory. We compute the higher Grothendieck-Witt groups of the motivic classifying space of a split torus over an arbitrary field of characteristic not two. We also compute the higher Grothendieck-Witt groups of projective bundles over a divisorial base scheme X such that $\frac{1}{2} \in \mathcal{O}_X(X)$, as well as the higher Grothendieck-Witt groups of even Grassmannians over a field of zero characteristic. These computations rely on semi-orthogonal decompositions that behave well with respect to duality. Using these results, we prove the Atiyah-Segal completion theorem for Grothendieck-Witt theory in the special case of the split torus. This completion theorem is an important first step in finding a more general Atiyah-Segal completion theorem, moreover it is likely to be a key ingredient in the proof of such a generalization.

Contents

1 Introduction

This thesis stands on the shoulders of some of the greatest mathematicians of the 20th century: those of Michael Atiyah and Graeme Segal, who laid the foundations for equivariant K-theory; of Alexander Grothendieck, the architect of modern algebraic geometry; and of Ernst Witt, the founder of the theory of quadratic forms over arbitrary fields. It stands to reason that their most influential ideas form the heart of the mathematical objects and results that were named after them. The story presented here begins with the legacy of Segal's equivariant K-theory as presented in [75] and the Atiyah-Segal completion theorem [2, theorem 2.1], and, through numerous twists and turns, ties it to the marriage of Grothendieck's concept of group completion and Witt's ideas about quadratic forms known as both Hermitian K-theory and Grothendieck-Witt theory.

The development of equivariant cohomology began in the 1950's with Cartan and Borel. Given a topological space X acted on by a topological group G, one can ask for a kind of singular cohomology of X that reflects the group action. Since ordinary singular cohomology only depends on X and not on the action, a modification is necessary. The idea, then, is essentially to construct a homotopy quotient $X//G$ using a technique now known as the *Borel construction*, and to define the *G-equivariant singular cohomology* of X as the singular cohomology of $X//G$. When X is a point, the Borel construction yields the *classifying space* $\mathrm{B}G$ of the group G, which shows that the cohomology of classifying spaces lies at the core of equivariant cohomology.

The *Atiyah-Segal completion theorem* is a famous result in equivariant topological complex K-theory, which builds on the ideas of equivariant singular cohomology. For a compact Lie group G, the representation ring $R(G)$ is the G-equivariant K-theory of a point. In the case of singular cohomology, the G-equivariant singular cohomology of the point is defined as the singular cohomology of the classifying space $\mathrm{B}G$, and that's the end of the story. By contrast, $R(G)$ is not isomorphic to the ordinary K-theory of $\mathrm{B}G$, but the Atiyah-Segal completion theorem shows that the representation ring becomes isomorphic to it after taking the *completion* $R(G)^\wedge$ of $R(G)$ with respect to the *augmentation ideal.* The Atiyah-Segal completion theorem contains this result as an important special case.

One of the main features of topological cohomology theories is that they are representable objects in the *stable homotopy category*, which is obtained from the category of (topological) spectra by inverting the weak homotopy equivalences. The framework of *motivic homotopy theory* developed in [66] is the analogue of this situation in algebraic geometry. In it, the affine line $\mathbb{A}^1$ takes the place of the contractible unit interval in topology, which is the reason it is also called *$\mathbb{A}^1$-homotopy theory* sometimes. Topological spectra are replaced by *motivic spectra* or *$\mathbb{P}^1$-spectra*, and the homotopy classes of these represent $\mathbb{A}^1$-invariant cohomology theories in algebraic geometry. Among the cohomology theories that have been proven to be representable by motivic spectra are orientable theories such as motivic cohomology, algebraic K-theory and algebraic cobordism [86, section 6], as well as non-orientable theories such as Witt theory and Grothendieck-Witt theory (also known as Hermitian K-theory) [38, section 5].

It is natural to ask whether one can extend Atiyah-Segal completion to cohomology theories in the framework of motivic homotopy theory; the formulation of equivariant Chow groups in [83] and [28] paved the way for such a project, and the completion theorem was extended to algebraic K-theory in [57].

This thesis provides a foray into the extension of Atiyah-Segal completion to *Hermitian K-theory*, which is also known as algebraic *Grothendieck-Witt theory.* It is an extension of algebraic K-theory incorporating the concept of *symmetric forms.* Grothendieck-Witt theory has evolved from *triangular Witt theory*, which was developed by Balmer in [3] and [4] at the turn of the century, using the notion of triangulated category introduced by Verdier in [84]. Grothendieck-Witt theory is an active area of research, with many analogues of results for algebraic K-theory still unproven. For example, the ring structure of the Grothendieck-Witt theory of projective spaces and Grassmannians has not yet been described.

Extensions of Atiyah-Segal completion are useful, because they provide a bridge between equivariant and non-equivariant theory, and it is often possible to leverage results from representation theory to compute the equivariant theory. Furthermore, Atiyah-Segal completion is a good measure for the "niceness" of an equivariant theory.

In line with recent developments in the field, the results of this thesis can very likely be rephrased in the language of Poincaré ∞-categories as in [15–17], but it was written using the language of dg categories before these preprints became available.

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2 Preliminaries

Before exploring the Atiyah-Segal theorem for Grothendieck-Witt theory, we will give an (incomplete) overview of the preliminaries to this work. These preliminaries include the classical Atiyah-Segal completion theorem, a concise guide to motivic homotopy theory, and some details regarding perfect complexes. Other topics, such as classifying spaces, semi-orthogonal decompositions and Grothendieck-Witt theory have their own introductory chapters because more details and intermediate results are required.

2.1 The Atiyah-Segal completion theorem

A *complex vector bundle* over a topological space X can be thought of as a continuous family of complex vector spaces parametrized by X. One might wonder how many different (up to isomorphism) complex vector bundles exist over a given X, whether these isomorphism classes can be organized into some algebraic object and what kind of information about X such an algebraic object contains; the study of these questions is known as *topological* K*-theory*. More concretely, the K-theory $\mathrm{K}(X)$ of a topological space is the *group completion* of the abelian monoid $(\mathrm{Vect}(X), \oplus)$ of isomorphism classes of vector bundles over X. Thus $\mathrm{K}(X)$ is an abelian group, and tensor product of vector bundles gives it the structure of a commutative ring. This extends to a $\mathbb{Z}$-graded cohomology theory, denoted by $\mathrm{K}^*(X)$.

Although some information contained in $\mathrm{Vect}(X)$ is lost when passing to $\mathrm{K}(X)$ since K-theory considers vector bundles up to *stable equivalence* instead of up isomorphism, the theory has proven powerful and flexible. Two famous applications of K-theory are the solution of the *Hopf invariant one* problem by Adams and Atiyah and the description of *vector fields on spheres* by Adams. For a detailed introduction to topological K-theory, see [50].

Recall that a compact Lie group is a compact smooth differentiable manifold G that has a group structure compatible with the smooth manifold structure. Let X be a topological space, which is acted on by a compact Lie group G. The purpose of *equivariant* K*-theory* is to study complex vector bundles E with a G-action over X, such that the structure map $E \to X$ is compatible with the G-action on X. The foundations of equivariant K-theory can be found in [75].

Now let X_G be the homotopy type of $(X \times \mathrm{E}G)/G$, where $\mathrm{E}G$ is the universal G-torsor over the classifying space $\mathrm{B}G$. Given a G-vector bundle V over X, the space $(V \times \mathrm{E}G)/G$ is a vector bundle over X_G, which defines a map

$$\alpha : \mathrm{K}^*_G(X) \longrightarrow \mathrm{K}^*(X_G).$$

When X is a point, $\mathrm{K}^*_G(X) \cong R(G)$. For any other compact G-space X, the unique map $\pi : X \to *$ induces a pullback map $\pi^* : R(G) \to \mathrm{K}^*_G(X)$, which gives $\mathrm{K}^*_G(X)$ the structure of an $R(G)$-module. Furthermore, there is a natural map $\mathrm{K}^*_G(X) \to \mathbb{Z}$ defined by taking the rank of a vector bundle, which is known as the *augmentation map*. In the case of $X = *$, its kernel $I_G \subset R(G)$ is called the *augmentation ideal*.

The theorem of central interest is the following.

Theorem 2.1.1 (Atiyah-Segal completion theorem)**.** *Let X be a compact G-space such that $\mathrm{K}^*_G(X)$ is finite over the representation ring $R(G)$. Then there is an isomorphism*

$$\alpha : \mathrm{K}^*_G(X)^\wedge \longrightarrow \mathrm{K}^*(X_G),$$

where $\mathrm{K}^*_G(X)^\wedge$ is the I_G-adic completion of $\mathrm{K}^*_G(X)$.

Remark 2.1.2. The cited [2, theorem 2.1] contains the slightly stronger result that there is an isomorphism of pro-rings whose limit is the isomorphism in the theorem above.

Atiyah and Segal address *real topological* K*-theory, or* KO*-theory*, the topological analogue of Hermitian K-theory, in [2, section 7], using KR*-theory* as a more general invariant. In short, the KR-theory of a compact space with involution X is the K-theory of the category of complex vector bundles with an anti-linear involution that is compatible with the involution on X. If the involution on X is trivial, then its KR-theory is isomorphic to its KO-theory via the functor that sends a real vector bundle V over X to its complexification $V \otimes \mathbb{C}$ with the obvious anti-linear involution $v \otimes z \mapsto v \otimes \bar{z}$. The inverse of that functor sends a complex vector bundle V over X with anti-linear involution to the fixed points of this involution.

2.2 Motivic homotopy theory

In the twentieth century, the study of topological spaces up to homotopy became a prolific area of mathematical research, as the realization grew that homotopy groups contain profound information about the spaces studied.

At the turn of the century, as algebraic geometry and categorical methods matured, Morel and Voevodsky were led to consider the possibility of a homotopy theory for algebraic varieties. In such a theory, the unit interval, which is fundamental to the definition of a homotopy between continuous maps, would have to be replaced by a suitably contractible, one-dimensional variety. Morel and Voevodsky chose the affine line $\mathbb{A}^1$, developed the concept of a *site with a unit interval* and demonstrated that an appropriate category of schemes could be equipped with such a structure in their landmark paper [66] and the related texts [86] and [65], spawning what they called $\mathbb{A}^1$-*homotopy theory*, and what is now also known as *motivic homotopy theory*.

One of the great advantages of motivic homotopy theory is that many cohomology theories in algebraic geometry become representable by *motivic spectra*, yielding a general framework for the study of cohomology theories and their operations. Conversely, if a cohomology theory satisfies a certain collection of axioms, such as being $\mathbb{A}^1$-invariant, then it is necessarily representable [86, section 6]. Furthermore, motivic spectra can be organized into a triangulated category called the *stable motivic homotopy category*, enabling the use of triangulated methods. The proof of the Bloch-Kato conjecture [87], for example, relies crucially on this framework.

Motivic homotopy theory is a broad and active area of modern research, and we cannot hope to explain all of its aspects here. In this section, we will focus mainly on the part of the theory needed for the construction of classifying spaces (see chapter 3), as well as a working definition of motivic spectra.

In order to do homotopy theory on a category, it requires the notion of a homotopy between morphisms. The concept of homotopy originated in topology, but a curious analogue appeared in the study of chain complexes: the chain homotopy. In the 1960's, Quillen distilled a categorical framework for homotopy theory from these examples, by showing that the category of topological spaces as well as the category of chain complexes of abelian groups can be equipped with the structure of a *model category*, thereby generalizing the context of homotopy theory. For a basic and readable introduction to the theory of model categories, see [27].

Definition 2.2.1. A *model category* is a category $\mathcal{C}$ together with three classes of morphisms $\mathbf{w}$, $\mathbf{c}$ and $\mathbf{f}$ called *weak equivalences*, *cofibrations* and *fibrations*, respectively, such that

(M1) the category $\mathcal{C}$ admits small limits and small colimits;
(M2) the class of weak equivalences contains all isomorphisms;
(M3) if f and g are morphisms of $\mathcal{C}$ such that the composition gf exists, and if two of f, g and gf are weak equivalences, then so is the third; and
(M4) the pairs $(\mathbf{c}, \mathbf{f} \cap \mathbf{w})$ and $(\mathbf{c} \cap \mathbf{w}, \mathbf{f})$ are weak factorization systems on $\mathcal{C}$.

If the category $\mathcal{C}$ satisfies (M1), then a triple $(\mathbf{w}, \mathbf{c}, \mathbf{f})$ of classes of morphisms satisfying (M2-M4) is called a *model structure* on $\mathcal{C}$.

Sometimes, condition (M1) is weakened to the existence of finite limits and colimits. It is possible to define the notion of (left and right) homotopy in a model category. Subsequently, one obtains the notion of the *homotopy category* $\mathrm{Ho}(\mathcal{C})$ of a model category $\mathcal{C}$, the study of which can be thought of as the homotopy theory of $\mathcal{C}$.

The category of schemes **Sch**, while admitting finite limits, does not admit pushouts except in some special cases such as the gluing construction, thus does not admit a model structure. The category $\mathbf{Sm}_k$ of smooth schemes of finite type over a field k suffers the same fate, but it turns out that restricting to this category allows the use of *homotopy purity*, also known as the *Thom isomorphism*, which is often useful. We can easily adjoin all limits and colimits by passing to the category $\mathbf{PSh}(\mathbf{Sm}_k)$ of presheaves of sets on $\mathbf{Sm}_k$, but this category still does not admit a suitable model structure. This is fixed by replacing the category of sets by the category $\mathbf{sSet}_*$ of *pointed simplicial sets*.

Simplicial sets are combinatorial models for topological spaces. The singular complex of a topological space is a fundamental example of a simplicial set. In fact, taking the singular complex of a space is a functor which is right adjoint to another functor called *geometric realization*. We will give a quick overview of simplicial sets for the uninitiated, for a wonderful and complete introduction see [68].

Definition 2.2.2. The *simplex category* Δ is the category whose objects are non-empty totally ordered finite sets and whose maps are *order-preserving morphisms*. A morphism $f : X \to Y$ of totally ordered sets is order-preserving if, for $x_1 \leq x_2$ in X, $f(x_1) \leq f(x_2)$ in Y. A small model of Δ is given by the sets $[n] = \{0 < 1 < \cdots < n\}$ with $n \in \mathbb{N}$.

Any order-preserving morphism between totally ordered finite sets can be written as a composition of elementary order-preserving morphisms, the *face* and *degeneracy* maps.

Definition 2.2.3. For $i \leq n+1$, the *face map* $\delta_i^n : [n] \hookrightarrow [n+1]$ is the unique order-preserving injection that does not have i in its image. For $i < n$, the *degeneracy map* $\sigma_i^n : [n] \twoheadrightarrow [n-1]$ is the unique order-preserving surjection such that $\sigma_i^n(i+1) = \sigma_i^n(i) = i$.

Definition 2.2.4. A *simplicial set* is a functor $S : \Delta^{\mathrm{op}} \to \mathbf{Set}$, in other words, a presheaf of sets on the simplex category. The *category of simplicial sets* $\mathbf{sSet}$ is the presheaf category $\mathbf{PSh}(\Delta)$. Let S be a simplicial set. We write S_n for $S([n])$. The elements of S_0 and S_1 are called the *vertices* and *edges* of S, respectively. A *pointed simplicial set* is a functor $S : \Delta^{\mathrm{op}} \to \mathbf{Set}_*$, where $\mathbf{Set}_*$ is the category of pointed sets. We denote by $\mathbf{sSet}_*$ the corresponding category of pointed simplicial sets.

Definition 2.2.5. For $n \in \mathbb{N}$, the *standard n-simplex* Δ^n is the representable presheaf $\Delta(-,[n])$. The *face map* $d_i^n : \Delta^{n+1} \to \Delta^n$ is the map $\Delta(-,[n+1]) \to \Delta(-,[n])$ induced by composition with the face map δ_i^n from definition 2.2.3. Similarly, the *degeneracy map* $s_i^n : \Delta^{n-1} \to \Delta^n$ is the map induced by composition with the degeneracy map σ_i^n.

Lemma 2.2.6. *For a simplicial set S and $n \in \mathbb{N}$, there is a bijection $S_n \to \mathbf{sSet}(\Delta^n, S)$.*

Proof. This follows from the Yoneda lemma. □

Definition 2.2.7. For $S \in \mathbf{sSet}$ and $n \in \mathbb{N}$, we call an element of S_n an *n-simplex of S*. An n-simplex σ is called *degenerate* if there exists an $(n-1)$-simplex $\tau \in S_{n-1}$ such that $s_i^n(\tau) = \sigma$ for some i. Note that for the standard n-simplex Δ^n, every m-simplex with $m > n$ is degenerate, since any order-preserving morphism $[m] \to [n]$ factors through a degeneracy map.

Returning to motivic homotopy theory, one obtains the category $\mathbf{Spc}_*(k)$ of *pointed simplicial presheaves on* $\mathbf{Sm}_k$, which are functors $\mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{sSet}_*$. Any category of pointed simplicial presheaves inherits a pointwise smash product from $\mathbf{sSet}_*$.

Here is a sketch of the construction of the model category $\mathbf{Spc}_*(k)$, carried out in [24, section 8] as an example of a more general procedure. There is a general model structure on $\mathbf{Spc}_*(k)$ called the *projective structure* or the *Bousfield-Kan model structure*, which exists on any category of pointed simplicial presheaves. Its fibrations and weak equivalences are maps which are pointwise fibrations and weak equivalences in simplicial sets, respectively. However, this model structure does not take into account any topology on $\mathbf{Sm}_k$, so one *localizes* with respect to the *Nisnevich topology* on $\mathbf{Sm}_k$. This is called the *universal model category* for $\mathbf{Sm}_k$, considered as a site with the Nisnevich topology, in [24, definition 7.2]. Finally, one last localization is needed to make the affine line contractible. When we write $\mathbf{Spc}_*(k)$, we shall understand its model structure to be the localization in which the affine line is contractible. The objects of $\mathbf{Spc}_*(k)$ are called *(pointed) motivic spaces*. This construction is different from the original construction in [66], but equivalent by [24, proposition 8.1]. The homotopy category $\mathbf{H}(k)$ of $\mathbf{Spc}_*(k)$ is called the *unstable motivic homotopy category*, analogously to the unstable homotopy category of topological spaces.

Remark 2.2.8. It is possible to construct an ∞-category $\mathcal{H}(k)_*$ whose homotopy category coincides with $\mathbf{H}(k)$ by [69, section 2.4.1]. Again, we start with the category $\mathbf{Sm}_k$ of smooth schemes of finite type over k. Then, we take the *nerve* $N(\mathbf{Sm}_k)$, which is an ∞-category. We equip $N(\mathbf{Sm}_k)$ with the usual Nisnevich topology using [61, remark 6.2.2.3]. Next, we take the ∞-category $\mathcal{P}(N(\mathbf{Sm}_k))$ of presheaves on $N(\mathbf{Sm}_k)$ and sheafify with respect to the Nisnevich topology to obtain the ∞-topos $\mathbf{Sh}_{\mathrm{Nis}}(N(\mathbf{Sm}_k))$. Note that the sheafification functor is left adjoint to the inclusion $\mathbf{Sh}_{\mathrm{Nis}}(N(\mathbf{Sm}_k)) \subset \mathcal{P}(N(\mathbf{Sm}_k))$ by [61, proposition 5.5.4.15]. Now we would have to take the hypercompletion of $\mathbf{Sh}_{\mathrm{Nis}}(N(\mathbf{Sm}_k))$ for technical reasons, but it turns out that $\mathbf{Sh}_{\mathrm{Nis}}(N(\mathbf{Sm}_k))$ is already hypercomplete by [66, proposition 3.1.16]. Finally, we take the localization of $\mathbf{Sh}_{\mathrm{Nis}}(N(\mathbf{Sm}_k))$ with respect to the class of all projection maps $X \times \mathbb{A}^1 \to X$ with $X \in \mathbf{Sm}_k$ to obtain $\mathcal{H}(k)$. The ∞-category $\mathcal{H}(k)$ admits a final object, so we may consider the ∞-category $\mathcal{H}(k)_*$ of pointed objects, which is the *underlying ∞-category* of the model category $\mathbf{Spc}_*(k)$.

The unstable motivic homotopy category has a curious property its topological sibling does not share: there are two different circles! One is the simplicial circle S^1 obtained from the constant functor $S^1 : \mathbf{Sm}_k \to \mathbf{sSet}_*$ sending $X \in \mathbf{Sm}_k$ to $\Delta^1/\partial\Delta^1$, the other one is the *Tate circle* $\mathbb{G}_m$ obtained by pointing the punctured affined $\mathbb{A}^1 - 0$ at 1. The smash product of these two circles is the projective line $\mathbb{P}^1$.

Lemma 2.2.9. *The smash product of S^1 and $\mathbb{G}_m$ is $S^1 \wedge \mathbb{G}_m = \mathbb{P}^1$.*

Proof. The projective line admits a Zariski cover $\mathbb{P}^1 = U \cup V$, such that $U \cong \mathbb{A}^1$, $V \cong \mathbb{A}^1$ and $U \cap V \cong \mathbb{G}_m$. Hence there is a pushout diagram

$$\begin{tikzcd} \mathbb{G}_m \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

in $\mathbf{Spc}(k)$. The maps $\mathbb{G}_m \to \mathbb{A}^1$ are both cofibrations, so the above diagram corresponds to the homotopy pushout diagram

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & S^1 \wedge \mathbb{G}_m,
\end{tikzcd}$$

whence $\mathbb{P}^1 = S^1 \wedge \mathbb{G}_m$, as was to be shown. □

In light of this lemma, we regard spheres in $\mathbf{Spc}(k)$ as having two parameters.

Definition 2.2.10. For natural numbers $p \geq q$, let $S^{p,q} = (S^1)^{\wedge p-q} \wedge \mathbb{G}_m^{\wedge q}$. In particular, $S^1 = S^{1,0}$, $\mathbb{G}_m = S^{1,1}$ and $\mathbb{P}^1 = S^{2,1}$.

It is possible to construct a reasonable category of *motivic spectra* $\mathbf{Spt}(k)$, in which the functor $S^{p,q} \wedge -$ becomes invertible. The homotopy category of this category is called the *stable motivic homotopy category* and is denoted by $\mathbf{SH}(k)$, or simply $\mathbf{SH}$ if no confusion can arise. It is a triangulated category, where the shift functor is given by smashing with the simplicial circle $S^{1,0}$. There is an infinite suspension functor

$$\Sigma^\infty_{\mathbb{P}^1} : \mathbf{Spc}_*(k) \longrightarrow \mathbf{Spt}(k),$$

which sends a pointed motivic space X to its $\mathbb{P}^1$-suspension spectrum $\Sigma^\infty_{\mathbb{P}^1}(X)$. If X is an unpointed scheme, we denote by X_+ the pointed motivic space $X \sqcup *$. Of course, the model category $\mathbf{Spt}(k)$ also has an underlying ∞-category, which can be obtained by equipping the ∞-category $\mathcal{H}(k)_*$ of remark 2.2.8 with the symmetric monoidal structure induced by the smash product and then *formally inverting* $(\mathbb{P}^1, \infty)$ as in [69, definition 2.38].

For most of the purposes of this thesis, it will suffice to think of $\mathbf{Spc}_*(k)$ as a closed model category containing $\mathbf{Sm}_k$ as a full subcategory, with an infinite suspension functor $\Sigma^\infty_{\mathbb{P}^1} : \mathbf{Spc}_*(k) \to \mathbf{Spt}(k)$. The functor $\Sigma^\infty_{\mathbb{P}^1}$ and its adjoint $\Omega^\infty_{\mathbb{P}^1}$ become invertible on the stable motivic homotopy category $\mathbf{SH}(k)$. Furthermore, $\mathbf{SH}(k)$ is triangulated with translation the simplicial suspension $\Sigma_s = S^1_s \wedge -$, the inverse of which is denoted by Ω_s. As in the case of the classical stable homotopy category, an object $E \in \mathbf{SH}(k)$ defines a cohomology theory on $\mathbf{Sm}_k$. However, these motivic cohomology theories are *bigraded*, because of the existence of the two different circles.

Definition 2.2.11. Let $E \in \mathbf{SH}(k)$ and $X \in \mathbf{Sm}_k$ with suspension spectrum $\Sigma^\infty_{\mathbb{P}^1} X_+$. For $p, q \in \mathbb{Z}$, the cohomology theory E on X in bidegree (p,q) is defined as the abelian group

$$E^{p,q}(X) = \mathbf{SH}(k)(\Sigma^\infty_{\mathbb{P}^1} X_+, S_s^{p-2q} \wedge (\mathbb{P}^1)^{\wedge q} \wedge E).$$

Note that it is indeed an abelian group since $\mathbf{SH}(k)$ is triangulated and therefore additive.

2.3 Perfect complexes

Thomason was working on the algebraic K-theory of schemes, when one night, his deceased friend Tom Trobaugh appeared to him in a dream, telling him to consider perfect complexes. Thomason, who was sure that his friend's remark in the dream had been wrong, pursued the idea anyway and ended up laying the foundations of the modern algebraic K-theory of schemes [80], crediting Trobaugh as co-author. Since then, categories of perfect complexes have become the standard way of organizing the algebraic K-theory of schemes.

The previous approach to algebraic K-theory was Quillen's *Q-construction* for exact categories [67], but Waldhausen [89] realized that algebraic K-theory could be defined for *Waldhausen categories*, which Waldhausen himself called *categories with weak equivalences and cofibrations.* The category of perfect complexes of sheaves of modules on a scheme admits such a structure, and is explained in [73, section 3]. See also [77, Tag 08CL] and [77, Tag 08C3] for more details on perfect complexes.

In this thesis, we organize perfect complexes into *dg categories*, which will be studied in much more detail in chapter 5. For a scheme X over a suitable base scheme S, the category of chain complexes of $\mathcal{O}_X$-modules can be enriched in the category of $\mathcal{O}_S$-modules, which makes it into an $\mathcal{O}_S$-linear dg category. Most of the dg categories we will encounter in this thesis will be of this form.

Let X be a scheme.

Definition 2.3.1. A *strictly perfect complex of $\mathcal{O}_X$-modules* is a bounded complex of finite locally free $\mathcal{O}_X$-modules. The strictly perfect complexes of $\mathcal{O}_X$-modules form a dg category, which is denoted by $\mathrm{sPerf}(X)$.

Definition 2.3.2. A *perfect complex M of $\mathcal{O}_X$-modules* is a complex of $\mathcal{O}_X$-modules such that there exists an affine open cover $\{U_i\}_i$ of X such that each $M|_{U_i}$ is quasi-isomorphic to a strictly perfect complex. The *category* $\mathrm{Perf}(X)$ *of perfect complexes on* X is the full dg subcategory of the dg category $\mathrm{Ch}(\mathcal{O}_X)$ of chain complexes of $\mathcal{O}_X$-modules consisting of the perfect complexes of $\mathcal{O}_X$-modules.

Let X be a scheme with an action of a group scheme G.

Definition 2.3.3. An *perfect complex M of G-equivariant $\mathcal{O}_X$-modules* is a complex of G-equivariant $\mathcal{O}_X$-modules such that there exists an affine open cover $\{U_i\}_i$ of X such that each $M|_{U_i}$ is quasi-isomorphic to a bounded complex of finite locally free G-equivariant $\mathcal{O}_X$-modules. The *category* $\mathrm{Perf}^G(X)$ *of G-equivariant perfect complexes on* X is the full dg subcategory of the dg category $\mathrm{Ch}^G(X)$ of chain complexes of G-equivariant $\mathcal{O}_X$-modules consisting of the perfect complexes of G-equivariant $\mathcal{O}_X$-modules.

Definition 2.3.4. The scheme X has the *resolution property* if every coherent $\mathcal{O}_X$-module is the quotient of a finite locally free $\mathcal{O}_X$-module.

Sometimes, the resolution property is formulated by requiring that every quasi-coherent $\mathcal{O}_X$-module is a directed colimit of finitely presented $\mathcal{O}_X$-modules

Definition 2.3.5. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and let $f \in \mathcal{L}(X)$ be a global section. Define

$$X_f = \{x \in X \mid \mathcal{L}_x \cong f_x\mathcal{O}_X\},$$

that is, X_f consists of the points $x \in X$ such that $f(x) \neq 0$, or equivalently, $f_x \notin \mathfrak{m}_x\mathcal{L}_x$.

A set $X_f \subset X$ as above, with f a global section of an invertible sheaf, is an open subscheme of X, as explained in [34, (5.5.2)]. The following definition is [44, definition 2.2.4].

Definition 2.3.6. A collection $\{\mathcal{L}_i \mid i \in I\}$ of line bundles on X is said to be an *ample family of line bundles* if the open subschemes X_f, with $f \in \mathcal{L}_i^{\otimes n}$ for some $i \in I$ and $n \in \mathbb{N}$, form a basis for the topology on X.

For equivalent definitions, see [44, proposition 2.2.3].

Proposition 2.3.7. *If X is a quasi-compact quasi-separated scheme admitting an ample family of line bundles, then X has the resolution property.*

Proof. Let $\{\mathcal{L}_i : i \in I\}$ is an ample family of line bundles on X and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. By [44, proposition 2.2.3(ii)], there are families $(m_i)_{i\in I}$ with $m_i \geq 0$ and $(n_i)_{i\in I}$ with $n_i > 0$ such that there is a surjective morphism of sheaves

$$\bigoplus_{i\in I} \left(\mathcal{L}_i^{\otimes -n_i}\right)^{\oplus m_i} \longrightarrow \mathcal{F}.$$

Hence $\mathcal{F}$ is the quotient of a finite locally free $\mathcal{O}_X$-module, and the proof is done. □

Definition 2.3.8. A quasi-compact quasi-separated scheme admitting an ample family of line bundles is called *divisorial*.

The following proposition is [73, proposition 3.4.8].

Proposition 2.3.9. *Let X be a divisorial scheme. Then the fully faithful inclusion* $\mathrm{sPerf}(X) \subset \mathrm{Perf}(X)$ *induces an equivalence* $D(\mathrm{sPerf}(X)) \simeq D(\mathrm{Perf}(X))$ *of derived categories.*

For the rest of this thesis, we will only consider divisorial schemes, and we shall use $\mathrm{sPerf}(X)$ and $\mathrm{Perf}(X)$ interchangeably.

3 Classifying spaces

The ultimate goal of this thesis is to prove Atiyah-Segal completion for Grothendieck-Witt theory, which means *equivariant* Grothendieck-Witt theory and therefore classifying spaces are involved, as discussed in the introduction. This warrants a rigorous account of the existing theory of classifying spaces in motivic homotopy theory.

The idea of *classifying spaces* is that they *classify* torsors over an arbitrary base. Agnostically speaking, the classifying space $\mathrm{B}G$ of a group G ought to come equipped with a *universal G-torsor* $\mathrm{E}G \to \mathrm{B}G$ such that any G-torsor $T \to X$ corresponds to a unique map $X \to \mathrm{B}G$ with the property that

$$\begin{array}{ccc} T & \longrightarrow & \mathrm{E}G \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathrm{B}G \end{array}$$

is a pullback diagram, the slogan being that *every torsor is the pullback of the universal torsor along its classifying map.*

Classifying spaces were originally constructed for topological groups, within the category of topological spaces. Unfortunately, the classifying space of a group scheme does not exist in the category of schemes. One of the advantages of motivic homotopy theory is that this situation is remedied, but in multiple ways: classifying spaces can be constructed naively, as Nisnevich sheaves, or as the sheafification of these Nisnevich sheaves in some finer topology such as the étale topology.

The classifying spaces of interest will be those of linear algebraic groups, so an overview of linear algebraic groups is given in the first subsection 3.1. A brief introduction to torsors in the category of schemes is given in subsection 3.3. Two different flavors of classifying spaces in the motivic setting are defined and examined in subsection 3.5. Subsection 3.6 gives the construction of geometric classifying spaces, which will be used throughout this thesis. The fact that geometric classifying spaces model étale classifying spaces is an essential ingredient in the proof of Atiyah-Segal completion.

3.1 Linear algebraic groups

Matrices with coefficients in a field k model the morphisms in k-linear algebra. The group of linear automorphisms of a vector space V is usually called the *general linear group* of V, denoted by $\mathrm{GL}(V)$, and consists of *invertible* $(n\times n)$-matrices once a basis is chosen. If $V = k^n$, then $\mathrm{GL}(V)$ is also written as $\mathrm{GL}_n(k)$. The subgroups of $\mathrm{GL}(V)$ are known as *matrix groups* or *linear groups*. Examples of linear groups include the general linear group $\mathrm{GL}_n(k)$ itself, the special linear group $\mathrm{SL}_n(k)$ of matrices with determinant 1, the orthogonal group $\mathrm{O}_n(k)$ of matrices whose transpose and inverse coincide, and the special orthogonal group $\mathrm{SO}_n(k) = \mathrm{O}_n(k)\cap\mathrm{SL}_n(k)$. It is natural to ask whether these linear groups give rise to functors $\mathbf{Alg}_k \to \mathbf{Grp}$ from the category of k-algebras to the category of groups, and if they do, whether they are representable by *group schemes*. The answer to both these questions is affirmative (even when k is only a commutative ring), and the consequent study of the resulting group schemes is the modern theory of linear algebraic groups. The following definitions are central to the subject.

Definition 3.1.1. Fix a base scheme S. A *group scheme over S* is a group object in the category of schemes $\mathbf{Sch}/S$, which is the same as a representable presheaf of groups on $\mathbf{Sch}/S$ via the Yoneda embedding.

For more on group objects in categories, see [63, section III.6]. Since a general linear group $\mathrm{GL}_n(k)$ is the group of invertible $(n \times n)$-matrices, and a matrix is invertible precisely when its determinant is invertible, a general linear group *scheme* GL_n should reflect this property in a formal way.

Definition 3.1.2. The *general linear group* $\mathrm{GL}_{n,\mathbb{Z}}$ *over* $\mathbb{Z}$ is the scheme

$$\operatorname{Spec}\mathbb{Z}[\{T_{ij} \mid 1 \le i, j \le n\}, \det{}^{-1}],$$

where det is the determinant of the formal $(n \times n)$-matrix (T_{ij}). For an arbitrary scheme S, the *general linear group* $\mathrm{GL}_{n,S}$ *over* S is defined as

$$\mathrm{GL}_{n,S} = \mathrm{GL}_{n,\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} S.$$

More generally still, if $E \to S$ is a vector bundle over a scheme S, then the *general linear group* $\mathrm{GL}(E)$ is the group scheme $\underline{\mathrm{Aut}}(E)$ over S of linear automorphisms of E.

If there is no risk of confusion, $\mathrm{GL}_{n,S}$ will simply be denoted by GL_n. When $S = \operatorname{Spec} A$ is affine, one also writes $\mathrm{GL}_{n,A}$ instead of $\mathrm{GL}_{n,S}$. By construction, for a scheme X, the group $\mathrm{GL}_{n,\mathbb{Z}}(X)$ of X-valued points is the group of invertible $(n \times n)$-matrices with coefficients in the ring of global sections $\mathcal{O}_X(X)$. For a field k in particular, the group $\mathrm{GL}_{n,\mathbb{Z}}(k)$ corresponds to the classical general linear group. Note that $\mathrm{GL}_{n,S}$ is canonically isomorphic to $\mathrm{GL}(\mathbb{A}^n_S)$. Furthermore, the general linear group $\mathrm{GL}(E)$ of a vector bundle $E \to S$ is locally isomorphic to GL_n, where n is the rank of E, as E is locally trivial.

The philosophy is that linear algebraic groups should be subgroups of general linear groups that are given by further algebraic properties of the formal matrix, such as the determinant being 1 or the transpose being invertible.

Definition 3.1.3. A group scheme G over S is called a *linear algebraic group* if there exists a closed immersion $G \to \mathrm{GL}(E)$ for some vector bundle E over S.

When $S = \operatorname{Spec} k$ is a point, $\mathrm{GL}(E) \cong \operatorname{Spec} k[T_{ij}, \det^{-1}]$ and a closed immersion into $\mathrm{GL}(E)$ corresponds uniquely to an ideal $I \subset k[T_{ij}, \det^{-1}]$. Such an ideal is necessarily generated by polynomial relations on the T_{ij}, in other words, algebraic properties of the formal matrix (T_{ij}), as demonstrated in the following examples.

Example 3.1.4. The *special linear group* $\mathrm{SL}_{n,\mathbb{Z}}$ is the linear algebraic group

$$\operatorname{Spec} \mathbb{Z}[T_{ij}]/(\det -1),$$

with notation as in definition 3.1.2. It is the subgroup of $\mathrm{GL}_{n,\mathbb{Z}}$ whose functor of points assigns to a scheme X the group $\mathrm{SL}_n(\mathcal{O}_X(X))$ of $(n \times n)$-matrices with coefficients in $\mathcal{O}_X(X)$ and determinant 1.

Example 3.1.5. A *split torus of rank n* is a linear algebraic group T that is isomorphic to the subgroup $\mathbb{G}^n_{m,\mathbb{Z}} \subset \mathrm{GL}_{n,\mathbb{Z}}$ of invertible diagonal matrices. In more detail, for a scheme X, $\mathbb{G}^n_{m,\mathbb{Z}}(X)$ consists of diagonal $(n \times n)$-matrices whose entries on the diagonal are in the unit group $\mathbb{G}_m(X) \cong \mathcal{O}_X(X)^\times$ of the ring $\mathcal{O}_X(X)$.

3.2 Representation theory of linear algebraic groups

This section contains a few key concepts of the representation theory of linear algebraic groups. Let G be a linear algebraic group over a base scheme S.

Definition 3.2.1. A *finite representation of* G is a homomorphism of group schemes $\rho : G \to \mathrm{GL}_n(\mathcal{E})$ for some finite locally free $\mathcal{O}_S$-module $\mathcal{E}$.

If $S = \operatorname{Spec} k$ for some field k, then a finite G-representation corresponds uniquely to a morphism $G(k) \to \mathrm{GL}_n(V)$, which is a classical representation of the group $G(k)$ of k-rational points of G. These two different but equivalent incarnations of a G-representation are often used interchangably.

Assume $S = \operatorname{Spec} k$. Then $G = \operatorname{Spec} A$ for some Hopf algebra A over k. It is a useful fact that A-comodules correspond to G-representations.

Lemma 3.2.2. *The data of a representation of G is equivalent to that of an A-comodule.*

Another useful fact is that a finite G-representation always admits a composition series.

Lemma 3.2.3. *Let M be a finite G-representation. Then M admits a composition series.*

Proof. Consider M as an A-comodule. Note that M is a finite dimensional over k by assumption. Hence M is finite length as k-comodule, and it follows that M is finite length as A-comodule. By the Jordan-Hölder theorem, M admits a composition series. ☐

3.3 Torsors in sites

Torsors are the objects classified by classifying spaces, so if there is to be any hope of defining and constructing some kind of *universal* torsor, one must understand what a torsor is. The material in this section is based on [85, section 4.4.1].

It is useful to think of a torsor as a group that has forgotten its neutral element, ignoring the subtleties of the theory, which will be presented hereafter. Let $\mathcal{C}$ be a site and G a group object in $\mathcal{C}$.

Definition 3.3.1. Let X and Y be G-objects in $\mathcal{C}$, with action maps $a : G\times X \to X$ and $b : G \times Y \to Y$. A map $f : X \to Y$ is called

(i) *G-equivariant* if the diagram

$$\begin{tikzcd} G \times X \arrow[r, "\mathrm{id}\times f"] \arrow[d, "a"] & G \times Y \arrow[d, "b"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes; and

(ii) *G-invariant* if f is G-equivariant and the action of G on Y is trivial.

Definition 3.3.2. Let Y be an object of $\mathcal{C}$. A *trivial G-torsor X over Y in $\mathcal{C}$* is a G-object X in $\mathcal{C}$, together with a G-invariant map $f : X \to Y$, such that there exists a commutative diagram

$$\begin{tikzcd} G \times Y \arrow[rr, "g"] \arrow[dr, "\mathrm{pr}_2"'] & & X \arrow[dl, "f"] \\ & Y, & \end{tikzcd}$$

where g is a G-equivariant isomorphism.

Intuitively, a trivial G-torsor over Y consists of copies of G, parametrized by Y. A G-torsor over Y is an object over Y that is a trivial G-torsor *locally* on Y, and this is where the topology on $\mathcal{C}$ begins to play a role.

Definition 3.3.3. A *G-torsor X over Y in $\mathcal{C}$* is an object X in $\mathcal{C}$ with an action of G, together with a G-invariant map $f : X \to Y$, such that there exists a covering $\{U_i \to Y\}$ of Y for which the projection map $X \times_Y U_i \to U_i$ is a trivial G-torsor over U_i for each i.

Sometimes, G-torsors are also called *principal G-bundles*, since they are trivial G-bundles locally on the target. The following proposition gives a convenient alternative description of G-torsors.

Proposition 3.3.4. *Let X be a G-object in $\mathcal{C}$ with an invariant map $f : X \to Y$. The following are equivalent.*

(i) The morphism f defines a G-torsor over Y.
(ii) The morphism f is a covering of Y and the arrow $(a, \mathrm{pr}_2) : G \times X \to X \times_Y X$ is an isomorphism.

Proof. To see that (ii) $\Rightarrow$ (i), note that there is a commutative diagram

$$\begin{tikzcd} G \times X \arrow[rr, "(a,\mathrm{pr}_2)"] \arrow[dr, "\mathrm{pr}_2"'] & & X \times_Y X \arrow[dl, "\mathrm{pr}_2"] \\ & X. & \end{tikzcd}$$

For an object T in $\mathcal{C}$, $(a, \mathrm{pr}_2)_T(g(h, x)) = (ghx, gx) = g(a, \mathrm{pr}_2)_T(h, x)$, so (a, pr_2) is a G-equivariant isomorphism and $\mathrm{pr}_2 : X \times_Y X \to X$ is a trivial G-torsor. Therefore f gives X the structure of a G-torsor over Y.

See the proof of [85, proposition 4.43] for the other direction. □

Example 3.3.5. Here is the prototypical example of a nontrivial torsor in the category $\mathbf{Sch}_{\text{ét}}$ of schemes with the étale topology. Let $K \subset L$ be a finite Galois extension with group G', and let G be the discrete group scheme over $\operatorname{Spec} K$ corresponding to G'. Then the map $\operatorname{Spec} L \to \operatorname{Spec} K$ becomes an étale G-torsor if $\operatorname{Spec} L$ is equipped with the Galois action, which can be described as follows. Let $\alpha \in L$ be a primitive element, so that $L = K[\alpha]$. Note that

$$G = \coprod_{g \in G'} \operatorname{Spec} K = \operatorname{Spec}\left(\prod_{g \in G'} K\right).$$

The Galois action $a : G \times \operatorname{Spec} L \to \operatorname{Spec} L$ is given by the map

$$L \to \prod_{g \in G'} L, \quad \alpha \mapsto (\alpha g)_{g \in G'}.$$

Here the action of G' on L is written as a right action so that a is a left action.

There is an alternative and more general definition of torsors in terms of sheaves, for which it suffices that G is a sheaf of groups on a site $\mathcal{C}$. Note that this means that G is not necessarily representable. The interested reader may compare and contrast this definition with [77, Tag 03AH].

Definition 3.3.6. Let $\mathcal{C}$ be a site, X a sheaf of sets on $\mathcal{C}$ and G a sheaf of groups on $\mathcal{C}$. A *G-torsor* is a morphism $\pi : Y \to X$ of sheaves on $\mathcal{C}$, where Y is equipped with a G-action $a : G \times Y \to Y$, that satisfies the following conditions.

(i) The morphism π is G-invariant, that is, $\pi(a(g, y)) = \pi(y)$ for all $g \in G$ and $y \in Y$.

(ii) The morphism $G \times Y \to Y \times Y$ of sheaves given by $(g, y) \mapsto (a(g, y), y)$ is a monomorphism. In this case, we call the action $a : G \times Y \to Y$ *categorically free.*

(iii) Let Y/G be the coequalizer of the diagram

$$G \times Y \underset{\mathrm{pr}_2}{\overset{a}{\rightrightarrows}} Y$$

of sheaves on $\mathcal{C}$. The canonical morphism $Y/G \to X$ induced by π is an isomorphism of sheaves.

A *morphism of G-torsors* $Y \to Y'$ is a commutative diagram

$$\begin{tikzcd} Y \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & Y' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

of G-equivariant morphisms of sheaves on $\mathcal{C}$.

The fact that a G-torsor $\pi : Y \to X$ is *locally* trivial is now hidden in the fact that Y/G is a coequalizer and is therefore the *sheafification* of the naive quotient presheaf $U \mapsto Y(U)/G(U)$.

Generalizing even further, we can replace the category **Set** with the category of simplicial sets **sSet**, in which case G becomes a sheaf of simplicial groups. Since motivic spaces are sheaves of simplicial sets, we will need the definition from [66, section 4.1] in the context of motivic homotopy theory, but note that this definition is essentially the same as definition 3.3.6; we simply replace sheaves with simplicial sheaves.

3.4 Free actions in the category of schemes

Free actions in the category of schemes feature prominently in the construction of quotient schemes. Quotient schemes, in turn, are central to the construction of geometric classifying spaces. This section provides the necessary background to make the latter construction precise.

Intuitively, the action of a group on some object is *free* if it has no fixed points. However, this is not quite accurate in the category of schemes and one can distinguish between actions that are *set-theoretically free*, meaning they have no fixed T-valued points for arbitrary T, and actions that are *scheme-theoretically free* or *strictly free*, meaning that the graph of the action map is a closed immersion. Let G be an S-group scheme acting on an S-scheme X with action morphism $a : G \times_S X \to X$. Let $(a, \mathrm{pr}_2) : G \times X \to X \times X$ be the graph of the action morphism.

Definition 3.4.1. The action of G on X is called

(i) *proper* if (a, pr_2) is proper;
(ii) *set-theoretically free* if (a, pr_2) is a monomorphism; and
(iii) *strictly free* (or *scheme-theoretically free*) if (a, pr_2) is a closed immersion.

Note that an action is set-theoretically free precisely if, for each S-scheme T, all its T-valued points have trivial stabilizers. Proper set-theoretically free actions correspond to strictly free ones by a result of Grothendieck.

Lemma 3.4.2. *If the action of G on X is proper and set-theoretically free, then it is strictly free.*

Proof. A proper monomorphism is a closed immersion by [36, corollaire 18.12.6]. $\square$

An important example of a strictly free action is that of a linear algebraic group on the general linear group, which will turn out to be useful when constructing geometric classifying spaces.

Lemma 3.4.3. *Let $i : G \to \mathrm{GL}_{n,S}$ be a linear algebraic group. Then the translation action $a : G \times \mathrm{GL}_{n,S} \to \mathrm{GL}_{n,S}$ given by $(g, x) \mapsto gx$ is strictly free.*

Proof. The map
$$\mathrm{GL}_{n,S} \times \mathrm{GL}_{n,S} \longrightarrow \mathrm{GL}_{n,S} \times \mathrm{GL}_{n,S}$$
given by $(x, y) \mapsto (xy, y)$ is an isomorphism and therefore a closed immersion, which shows that the translation action of $\mathrm{GL}_{n,S}$ on itself is strictly free. Since closed immersion are stable under pullback, $i \times \mathrm{id} : G \times \mathrm{GL}_{n,S} \to \mathrm{GL}_{n,S} \times \mathrm{GL}_{n,S}$ is a closed immersion. The composition of these two closed immersion is the map
$$(a, \mathrm{pr}_2) : G \times \mathrm{GL}_{n,S} \longrightarrow \mathrm{GL}_{n,S} \times \mathrm{GL}_{n,S},$$
which is therefore also a closed immersion, as was to be shown. $\square$

Fix a quasi-compact separated group scheme G over a base scheme S. Let X and Y be quasi-compact quasi-separated schemes over S with a G-action. The following lemmas are useful for determining when an action is proper or free.

Lemma 3.4.4. *Assume that the action of G on both X and Y is proper. Then the induced action of G on $X \times_S Y$ is proper.*

Proof. As the fiber product of proper maps,
$$(a_X, \mathrm{pr}_2) \times (a_Y, \mathrm{pr}_2) : G \times X \times G \times Y \longrightarrow X \times X \times Y \times Y$$
is proper. Since G is separated, the diagonal map $G \to G \times G$ is a closed immersion. Hence the composition
$$G \times X \times Y \longrightarrow G \times X \times G \times Y \longrightarrow X \times X \times Y \times Y$$
which defines the action of G on $X \times Y$ is proper, as was to be shown. $\square$

For the computation of the Grothendieck-Witt theory of the classifying space of a linear algebraic group G, it will be useful to know whether certain fppf-quotients X/G exist as schemes. The following result, [21, théorème 4.C], provides a partial answer.

Theorem 3.4.5. *Let S be a locally noetherian scheme of dimension ≤ 1 and G a group scheme locally of finite type over S with a closed subgroup H that is flat over S. Then the fppf-quotient G/H exists as a scheme.*

This theorem applies nicely to the case of linear algebraic groups.

Corollary 3.4.6. *Let E be a vector bundle over a locally noetherian scheme S of dimension ≤ 1 and let $G \subset \mathrm{GL}(E)$ be a linear algebraic group that is flat over S. Then the fppf-quotient $\mathrm{GL}(E)/G$ exists as a scheme.*

Proof. Since $\mathrm{GL}(E)$ is of finite type over S and $G \subset \mathrm{GL}(E)$ is closed by definition, the result follows from theorem 3.4.5. $\square$

Note that $S = \operatorname{Spec} \mathbb{Z}$ is noetherian of dimension 1, so as long as one considers linear algebraic groups $G \subset \mathrm{GL}_{n,\mathbb{Z}}$ that are flat over $\operatorname{Spec} \mathbb{Z}$, the fppf-quotient $\mathrm{GL}_{n,\mathbb{Z}}/G$ exists as a scheme; the same goes for $S = \operatorname{Spec} \mathbb{Z}[1/2]$. In particular, it is a universal categorical quotient and therefore stable under pullback.

3.5 Nisnevich and étale classifying spaces

Since the category of motivic spaces admits colimits, classifying spaces of group objects in motivic homotopy theory can be shown to exist. The objects of $\mathbf{Spc}_*(k)$ are simplicial sheaves on the Nisnevich site $\mathbf{Sm}_k$, so it most natural to first construct a classifying space that classifies Nisnevich torsors. One can then consider the étale sheafification of the resulting Nisnevich sheaf, which classifies étale torsors, and regard it as a Nisnevich sheaf. This is possible because the étale topology is finer than the Nisnevich topology, so any étale sheaf is automatically a Nisnevich sheaf. Typically, one considers étale torsors in algebraic geometry, since one would like finite Galois extensions, as in example 3.3.5, to be torsors; any theory of torsors should mesh nicely with the theory of étale fundamental groups. As a result, the étale classifying space is a slightly elusive object in motivic homotopy theory, and we will see how we can construct it geometrically in section 3.6.

A word of warning: in the preliminary section 2.2, we defined motivic spaces as simplicial *presheaves* on the Nisnevich site $\mathbf{Sm}_k$, because that makes their construction easier. In this section, however, we stay close to the source material [66] by working with simplicial *sheaves* instead. As already mentioned, the two theories are equivalent by [24, proposition 8.1]. Furthermore, whenever $\mathbf{Spc}_*(k)$ is mentioned, we consider the simplicial model structure before $\mathbb{A}^1$-localization.

Definition 3.5.1. Let $\mathcal{C}$ be a category. The *nerve of* $\mathcal{C}$ is the simplicial set $\mathrm{N}(\mathcal{C}) : \Delta^{\mathrm{op}} \to \mathbf{Set}$ which sends $[n]$ to the set of composable n-tuples $(f_0, \ldots, f_{n-1})$ of morphisms in $\mathcal{C}$. A degeneracy map inserts an identity map in a given n-tuple, and a face map replaces an adjacent pair of morphisms by the composition.

Recall that a *monoid* can be defined as a category with a single object.

Definition 3.5.2. Let F be a pointed presheaf of monoids on $\mathbf{Sm}_k$. Then the *nerve of* F is the simplicial presheaf $\mathrm{N}(F) : \mathbf{Sm}_k \to \mathbf{sSet}$ sending a scheme X to the nerve of the monoid $F(X)$.

To define classifying spaces of simplicial sheaves of groups, we need the notion of a bisimplicial set.

Definition 3.5.3. A *bisimplicial set* is a functor $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathbf{Set}$, and its *diagonal* is the simplicial set obtained by precomposing with the diagonal functor $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$.

Let G be a group object in $\mathbf{Spc}_*(k)$, that is, a Nisnevich sheaf of simplicial groups on $\mathbf{Sm}_k$; in practice, G will often be a linear algebraic group over some field k. The following definition can be found on [66, section 4.1, p. 123].

Definition 3.5.4. The *Nisnevich classifying space* $\mathrm{B}_{\mathrm{nis}}G$ *of* G is the diagonal simplicial sheaf of the bisimplicial sheaf $([n],[m]) \mapsto \mathrm{N}(G_n)_m$, where $\mathrm{N}(G_n)$ is the nerve of the sheaf of groups G_n, regarded as sheaf of monoids. Note that $\mathrm{B}_{\mathrm{nis}}G_n = \mathrm{N}(G_n)_n$.

As discussed in the introduction to this chapter, classifying spaces of torsors, in the sense of definition 3.3.6, should be equipped with a universal G-torsor. The following definition is the first step in the construction of such a universal torsor in our current context.

Definition 3.5.5. Let X be a Nisnevich sheaf of sets on $\mathbf{Sm}_k$. Define $\mathrm{E}\,X : \mathbf{Sm}_k \to \mathbf{sSet}$ by letting $\mathrm{E}\,X_n : \mathbf{Sm}_k \to \mathbf{Set}$ be the sheaf X^{n+1}. The face maps $X^{n+1} \to X^n$ are given by projections and the degeneracy maps $X^n \to X^{n+1}$ are given by diagonals.

Let G be a Nisnevich sheaf of groups on $\mathbf{Sm}_k$, $T \in \mathbf{Sm}_k$ and $n \in \mathbb{N}$. Then there is a natural map $\mathrm{E}\,G(T)_n \to \mathrm{N}(G)(T)_n$, given by

$$(g_0, \ldots, g_n) \longmapsto (g_0^{-1}g_1, g_1^{-1}g_2, \ldots, g_{n-1}^{-1}g_n),$$

which yields a natural map $\mathrm{E}\,G \to \mathrm{N}(G)$. Note that $\mathrm{E}\,G$ admits a natural left G-action, and that $\mathrm{E}\,G/G \to \mathrm{N}(G)$ is an isomorphism with inverse $(g_1, \ldots, g_n) \mapsto (1, g_1, g_1g_2, \ldots, g_1g_2\cdots g_n)$.

Now if G is a group object in $\mathbf{Spc}_*(k)$, then $\mathrm{E}\,G$ is defined as the diagonal of the bisimplicial sheaf of groups $(n,m) \mapsto \mathrm{E}(G_n)_m$. Again, there are a natural map $\mathrm{E}\,G \to \mathrm{B}_{\mathrm{nis}}G_n$ which factors through the isomorphism $\mathrm{E}\,G/G \to \mathrm{B}_{\mathrm{nis}}G_n$.

We will need the notion of local fibrations from [46, section II.4.2] in the next lemma. A morphism $f : X \to Y$ of simplicial presheaves on a site $\mathcal{C}$ has the *local right lifting property* with respect to a map of simplicial sets $c : K \to L$ if there exists, for every object $U \in \mathcal{C}$ and every commutative square

$$\begin{tikzcd} K \arrow[r] \arrow[d, "c"] & X(U) \arrow[d, "f"] \\ L \arrow[r] & Y(U) \end{tikzcd}$$

a covering $\{U_i \to U\}_{i \in I}$ of U such that the diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d, "c"] & X(U) \arrow[r] & X(U_i) \arrow[d, "f"] \\ L \arrow[r] \arrow[urr, dashed] & Y(U) \arrow[r] & Y(U_i) \end{tikzcd}$$

admits a lift $L \to Y(U_i)$ for every $i \in I$. A *local fibration* is a morphism $f : X \to Y$ of simplicial presheaves that has the local right lifting property with respect to all horn inclusions $\Lambda^n_i \subset \Delta^n$ with $n \geq 1$ and $0 \leq i \leq n$. A *local trivial fibration* is a morphism $f : X \to Y$ of simplicial presheaves that has the local right lifting property with respect to all boundary inclusions $\partial\Delta^n \subset \Delta^n$ with $n \geq 0$. Note that although local trivial fibrations are motivic weak equivalences, the converse is not true.

The following lemma [66, lemma 4.1.12] shows that $\mathrm{B}_{\mathrm{nis}}G$ classifies G-torsors up to local trivial fibrations.

Lemma 3.5.6. *Let $X \in \mathbf{Spc}_*(k)$ and let $\mathcal{E} \to X$ be a G-torsor over X. Then there exist a local trivial fibration $p_{\mathcal{E}} : Y \to X$, and a map $f_{\mathcal{E}} : Y \to \mathrm{B}_{\mathrm{nis}}G$ in $\mathbf{Spc}_*(k)$ such that $f^*_{\mathcal{E}}\,\mathrm{E}\,G \cong p^*_{\mathcal{E}}\mathcal{E}$.*

Proof. Let $Y = (\mathrm{E}\,G \times \mathcal{E})/G$. Then $p_{\mathcal{E}} : Y \to X$ is a local trivial fibration, whose fibers are locally isomorphic to the contractible space $\mathrm{E}\,G$. Furthermore, there is a map $f_{\mathcal{E}} : Y \to \mathrm{B}_{\mathrm{nis}}G$, which is the quotient by G of the projection $\mathrm{E}\,G \times \mathcal{E} \to \mathrm{E}\,G$. Since $f^*_{\mathcal{E}}\,\mathrm{E}\,G \cong p^*_{\mathcal{E}}\mathcal{E}$ by construction, the proof is done. □

If G has simplicial dimension zero, that is, if G is just a sheaf of groups, then we define $H^1(X, G)$ to be the set of isomorphism classes of G-torsors over X, where X is some motivic space. We state the following result [66, lemmas 4.1.13-4.1.16] without proof.

Lemma 3.5.7. *Let G be a group object of $\mathbf{Spc}_*(k)$ of simplicial dimension zero. Let $\mathscr{B}G$ be a fibrant replacement of $\mathrm{B}_{\mathrm{nis}}G$ in the simplicial model structure on $\mathbf{Spc}_*(k)$.*

- *(i) If $f : X \to Y$ is a trivial local fibration in $\mathbf{Spc}_*(k)$, then the induced map $H^1(Y, G) \to H^1(X, G)$ is a bijection.*
- *(ii) For $X \in \mathbf{Spc}_*(k)$, there is a natural bijection*

$$H^1(X, G) \longrightarrow \mathbf{H}(k)(X, \mathrm{B}_{\mathrm{nis}}G),$$

  given by $\mathcal{E} \mapsto f_{\mathcal{E}} \circ p^{-1}_{\mathcal{E}}$, where $f_{\mathcal{E}}$ and $p_{\mathcal{E}}$ are as in lemma 3.5.6.
- *(iii) There exists a G-torsor $\mathscr{E}G \to \mathscr{B}G$ such that for any $X \in \mathbf{Spc}_*(k)$, the map $\mathbf{Spc}_*(k)(X, \mathscr{B}G) \to H^1(X, G)$ given by $f \mapsto f^*\mathscr{E}G$ defines a bijection*

$$\mathbf{H}(k)(X, \mathscr{B}G) \longrightarrow H^1(X, G).$$

Summarizing, $\mathrm{B}_{\mathrm{nis}}G$ is a classifying space for G in the unstable motivic homotopy category $\mathbf{H}(k)$.

Since the étale topology on $\mathbf{Sm}_k$ is finer than the Nisnevich topology, there is a continuous functor of sites

$$\pi : (\mathbf{Sm}_k)_{\text{ét}} \longrightarrow (\mathbf{Sm}_k)_{\mathrm{Nis}},$$

which yields the étale sheafification functor π^* on simplicial Nisnevich sheaves and its right adjoint π_*, as well as the total right derived functor $\mathbf{R}\pi_*$.

Definition 3.5.8. Let G be a group object in $\mathbf{Spc}_*(k)$ of simplicial dimension zero. Then the *étale classifying space* $\mathrm{B}_{\text{ét}}G$ is the object

$$\mathrm{B}_{\text{ét}}G = \mathbf{R}\pi_*\pi^*\mathrm{B}_{\mathrm{nis}}G$$

in the unstable motivic homotopy category $\mathbf{H}(k)$.

If $\mathscr{B}_{\text{ét}}G$ is a fibrant model of the classifying space of π^*G in the category of simplicial étale sheaves, then $\pi_*\mathscr{B}_{\text{ét}}G \cong \mathrm{B}_{\text{ét}}G$. Again, there is a natural bijection

$$H^1(X, G) \longrightarrow \mathbf{H}(k)(X, \mathrm{B}_{\text{ét}}G)$$

as in lemma 3.5.7(iii). There is a natural map $\mathrm{B}_{\text{nis}}G \to \mathrm{B}_{\text{ét}}G$ in $\mathbf{H}(k)$, adjoint to the identity on $\pi^*\mathrm{B}_{\text{nis}}G$. By [66, lemma 4.1.18], this map is an isomorphism if and only if G is a sheaf in the étale topology and if all étale G-torsors are also Nisnevich G-torsors, in other words, if

$$H^1_{\text{Nis}}(X, G) \cong H^1_{\text{ét}}(X, G)$$

for all $X \in \mathbf{Sm}_k$. In the next section, we will see how we can construct a geometric model for the étale classifying space $\mathrm{B}_{\text{ét}}G$.

3.6 Geometric classifying spaces

As noted before, having a geometric model $\mathrm{B}_{\text{gm}}G$ for the étale classifying space $\mathrm{B}_{\text{ét}}G$ provides one with a means of computing the cohomology of $\mathrm{B}_{\text{ét}}G$ in good cases. Thus far, they have been of fundamental importance in the proof of every variant of the Atiyah-Segal completion theorem. The construction presented in this subsection is a special case of the more general construction for quotient stacks found in [40, section 2], namely the case where $\mathfrak{S} = \operatorname{Spec} k$ is a point in the notation of [40]. The original construction comes from [66, section 4.2].

Let L_{fppf} be the fppf sheafification functor on simplicial presheaves on $\mathbf{Sm}_k$. We denote $\operatorname{Spec} k$ by $*$. Let G be an fppf sheaf of groups (of simplicial dimension zero) on $\mathbf{Sm}_k$ acting on a simplicial fppf sheaf of sets U on $\mathbf{Sm}_k$.

Definition 3.6.1. For a simplicial fppf sheaf X and an fppf G-torsor $\pi : T \to X$, the *twist of U by π* is defined as $U_\pi = L_{\text{fppf}}((U \times T)/G)$.

Definition 3.6.2. The *fppf classifying space* $\mathrm{B}_{\text{fppf}}G$ *of G* is defined as the fppf sheaf of groupoids

$$\mathrm{B}_{\text{fppf}}G = L_{\text{fppf}}(*/G).$$

The fppf classifying space $\mathrm{B}_{\text{fppf}}G$

There is analogue of lemma 3.5.7 for $\mathrm{B}_{\text{fppf}}G$. The construction of geometric classifying spaces hinges on the following lemma [40, lemma 2.1], which allows the construction of $\mathrm{B}_{\text{fppf}}G$ as the fppf quotient of the auxiliary sheaf U.

Lemma 3.6.3. *If $U_\pi \to X$ is a motivic equivalence for all schemes X and all fppf G-torsors $\pi : T \to X$, then the morphism*

$$L_{\text{fppf}}(U/G) \longrightarrow \mathrm{B}_{\text{fppf}}G$$

induced by $U \to *$ is a motivic equivalence.

The following string of definitions and lemmas is used to construct a U satisfying the condition of lemma 3.6.3 as a colimit of fppf quotients. In many cases, for example when G is a linear algebraic group, these fppf quotients are representable by schemes, which justifies calling such a colimit construction a *geometric* classifying space.

Definition 3.6.4. A *system of vector bundles over a scheme S* is a diagram $(V_i)_{i\in I}$ of vector bundles over S, where the maps are vector bundle inclusions and I is a filtered partially ordered set. A system of vector bundles over S is called:

(i) *saturated* if for every $i \in I$ there exists $j \geq i$ such that the map $V_i \to V_j$ is isomorphic under V_i to the inclusion $(\mathrm{id}, 0) : V_i \to V_i \times_S V_i$; and
(ii) *complete* if for every affine $X \in \mathbf{Sch}_S$ and every vector bundle E over X, there exists an $i \in I$ and a vector bundle inclusion $E \to V_i \times_S X$.

The following example gives a convenient complete and saturated system of vector bundles over S.

Example 3.6.5. The system of vector bundles $(\mathbb{A}^n_S)_{n\in\mathbb{N}}$ over S with maps $\mathbb{A}^n_S \to \mathbb{A}^{n+1}_S$ given by $v \mapsto (v, 0)$ is saturated by definition. Now let $X = \operatorname{Spec} A$ be an affine scheme and E a vector bundle over X. Then E corresponds to a finitely generated projective A-module (cf. [33, proposition 11.7]) and is therefore a direct summand of a finite free A-module of rank n. Thus there is a vector bundle inclusion $E \to \mathbb{A}^n_X$, which shows that this system of vector bundles is also complete.

Lemma 3.6.6. *Let $X = \operatorname{Spec} A$ be an affine scheme, V a vector bundle over X and $i : Z \to X$ a closed immersion. Then any global section of $i^*V \to Z$ lifts to a global section of $V \to X$.*

Proof. Let $\mathcal{E}$ be a locally free sheaf on X such that $V \cong \underline{\operatorname{Spec}}(\operatorname{Sym}(\mathcal{E}^\vee))$. Then the canonical map $\mathcal{E} \to i_*i^*\mathcal{E}$ is an epimorphism, since i is a closed immersion. An element $s \in i_*i^*\mathcal{E}(X)$ corresponds to a global section of $i^*V \to Z$. Note that s also corresponds to a map of $\mathcal{O}_X$-modules $\mathcal{O}_X \to i_*i^*\mathcal{E}$. Furthermore, $\mathcal{O}_X$ is projective since X is affine and therefore by Serre's criterion $H^i(X, \mathcal{F}) = 0$ for any quasi-coherent $\mathcal{F}$ and $i > 0$. Thus s lifts to a map $\mathcal{O}_X \to \mathcal{E}$, which defines a global section of $V \to X$, and the proof is finished. $\square$

Lemma 3.6.7. *Let $(V_i)_{i\in I}$ be a saturated system of vector bundles over a scheme S and let $U_i \subset V_i$ be an open subscheme for each $i \in I$. Suppose that*

(i) there exists an $i \in I$ such that $U_i \to S$ admits a section; and
(ii) for each $i \in I$, the isomorphism $V_i \times_S V_i \cong V_{2i}$ induces an inclusion $(U_i \times_S V_i) \cup (V_i \times_S U_i) \subset U_{2i}$.

Then the simplicial presheaf $U_\infty = \operatorname{colim}_i U_i$ on $\mathbf{Sm}_S$ is motivically contractible.

Proof. If $U_\infty(X) \to S(X)$ is a weak equivalence of simplicial sets for all smooth affine S-schemes X, then the canonical morphism of simplicial presheaves $U_\infty \to S$ is a motivic equivalence. Let X be an arbitrary smooth affine scheme over S. Then it suffices to show that $U_\infty(X) = \operatorname{Map}(\mathbb{A}^\bullet_S \times_S X, U_\infty)$ is a trivial Kan complex. Here, $\mathbb{A}^\bullet_S$ is the motivic space whose sheaf of n-simplices is given by $\mathbb{A}^n_S$, with the usual face and degeneracy maps. Consider a lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "f"] \arrow[d] & \operatorname{Map}(\mathbb{A}^\bullet_S \times_S X, U_\infty) \\ \Delta^n, \arrow[ur, dashed] & \end{tikzcd}$$

and replace f by the corresponding map $\partial\mathbb{A}^n_X \to U_\infty$, where $\partial\mathbb{A}^n_X$ is the boundary of the algebraic n-simplex. Since $\partial\mathbb{A}^n_X$ is a compact object of $\mathbf{PSh}(\mathbf{Sm}_S)$, there exists an $i \in I$ such that f factors through U_i. Increasing i if necessary, assumption (i) ensures that $U_i \to S$ admits a section $s : S \to U_i$. Consider the diagram

$$\begin{tikzcd}
\partial\mathbb{A}^n_X \arrow[rr, "f"] \arrow[dr, "f'"] \arrow[ddr, bend right] & & U_i \arrow[d] \\
& V_i \times_S \mathbb{A}^n_X \arrow[r] \arrow[d] & V_i \arrow[d] \\
& \mathbb{A}^n_X \arrow[r] & S,
\end{tikzcd}$$

and note that f' is a section of $V_i \times_S \partial\mathbb{A}^n_X \to \partial\mathbb{A}^n_X$ that factors through $U_i \times_S \mathbb{A}^n_X$. Hence, by lemma 3.6.6, there exists a morphism $g' : \mathbb{A}^n_X \to V_i \times_S \mathbb{A}^n_X$ that lifts f'. Composing g' with the projection $V_i \times_S \partial\mathbb{A}^n_X \to V_i$ yields a lift $g : \mathbb{A}^n_X \to V_i$ of f. Let $Z_i \subset V_i$ be a closed subscheme such that $Z_i = V_i - U_i$ as topological spaces. Then $g^{-1}(Z_i) \cap \partial\mathbb{A}^n_X = \emptyset$. Let $s' : \mathbb{A}^n_X \to U_i \times_S \mathbb{A}^n_X$ be the base change of s to $\mathbb{A}^n_X$ and note that $s'(g^{-1}(Z_i)) \cap f'(\partial\mathbb{A}^n_X) = \emptyset$. Hence,

$$g^{-1}(Z_i) \sqcup \partial\mathbb{A}^n_X \xrightarrow{s' \sqcup f'} V_i \times_S \mathbb{A}^n_X$$

is a section and admits a lift $h' : \mathbb{A}^n_X \to V_i \times_S \mathbb{A}^n_X$ by lemma 3.6.6, which yields a map $h : \mathbb{A}^n_X \to V_i$ by composing h' with the projection $V_i \times_S \mathbb{A}^n_X \to V_i$.

Now g and h are such that if $g(x) \notin U_i$, then $h(x) \in U_i$ for all $x \in \mathbb{A}^n_X$. Thus, by assumption (ii), the map $(g, h) : \mathbb{A}^n_X \to V_i \times_S V_i$ factors through U_{2i} under the isomorphism $V_i \times_S V_i \cong V_{2i}$. Consequently, (g, h) is a solution to the above lifting problem and $U_\infty(X)$ is a trivial Kan complex, as was to be shown. $\square$

Let $i : V \to W$ be an inclusion of vector bundles over $*$. The subpresheaf $U \subset \underline{\mathrm{Hom}}(V, W)$ of vector bundle inclusions, defined on objects $T \in \mathbf{Sm}_k$ by

$$U(T) = \{\text{linear maps } f : V \times T \to W \times T \text{ of maximal rank}\},$$

is representable by an open subscheme by [33, proposition 16.18]. Now let $G \subset \mathrm{GL}(V)$ be a closed subgroup and define an action $a : G \times \underline{\mathrm{Hom}}(V, W) \to \underline{\mathrm{Hom}}(V, W)$ by $(g, f) \mapsto f \circ g^{-1}$. The following fact is of key importance in the construction of geometric classifying spaces.

Lemma 3.6.8. *The action of G on the open subscheme $U \subset \underline{\mathrm{Hom}}(V, W)$ of vector bundle inclusions is strictly free.*

Proof. First assume that $G = \mathrm{GL}(V)$. Since $(a, \mathrm{pr}_2) : G \times U \to U \times U$ being a closed immersion is local on the target and therefore local on S, it may be assumed that $S = \operatorname{Spec} A$ is affine and that V and W are free of rank m and n, respectively. Hence $G \cong \mathrm{GL}_{m,A}$. Consider a lifting problem

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r] \arrow[d] & G \times U \arrow[d] \\
\operatorname{Spec} B \arrow[r] \arrow[ur, dashed] & U \times U
\end{tikzcd}$$

where $B \to A$ is a ring extension. The map $\operatorname{Spec} B \to U \times U$ corresponds to a pair (y, x) of elements of $\mathrm{Mat}_{n\times m}(B)$. Likewise the map $\operatorname{Spec} A \to G \times U$ corresponds to a pair $(g, \tilde{x})$ with $g \in G(A)$ and $\tilde{x} \in \mathrm{Mat}_{n\times m}(A)$. The commutativity of the solid diagram ensures that $\tilde{x} = x$ and that $xg^{-1} = y$. Since x is of maximal rank m, it contains an invertible submatrix $x' \in \mathrm{GL}_m(B)$. Note that $x'g^{-1} = y'$ for some submatrix y' of y, so $g^{-1} = x'^{-1}y'$ has entries in B. Moreover, $xg^{-1} = y$ is also of maximal rank m, which shows that $g^{-1} \in \mathrm{GL}_m(B)$. Thus there exists a unique dotted arrow $(g, x) : \operatorname{Spec} B \to G \times U$ making the diagram commute. It follows that (a, pr_2) is a closed immersion by theorem A.2.1.

For an arbitrary closed subgroup $G \subset \mathrm{GL}(V)$, the map

$$G \times U \longrightarrow \mathrm{GL}(V) \times U \xrightarrow{(a,\mathrm{pr}_2)} U \times U$$

is a composition of closed immersions and therefore a closed immersion, which concludes the proof. $\square$

Finally, here is the theorem that constructs the geometric classifying space, see [40, theorem 2.7] for the more general statement in the appropriate category of stacks.

Theorem 3.6.9. *Let V be a vector bundle over $*$ and $G \subset \mathrm{GL}(V)$ a closed subgroup that is flat and finitely presented. Let $(V_i)_{i\in I}$ be a complete saturated system of vector bundles over $*$. For $i \in I$, let $U_i \subset \underline{\mathrm{Hom}}(V, V_i)$ be the open subscheme of vector bundle inclusions. Let $U_\infty = \operatorname{colim}_{i\in I} U_i$. Then the morphism*

$$L_{\mathrm{fppf}}(U_\infty/G) \longrightarrow \mathrm{B}_{\mathrm{fppf}}G$$

induced by the canonical map $U_\infty \to *$ is a motivic equivalence.

Proof. By lemma 3.6.3, it suffices to prove that for any G-torsor $\pi : T \to X$ in $\mathbf{Sm}_k$, the map $(U_\infty)_\pi \to X$ is a motivic equivalence. By [41, lemma 4.6(1)], one need only consider G-torsors $\pi : T \to X$ where X is affine; let $\pi : T \to X$ be such a G-torsor. Then the schemes $W_i = \underline{\mathrm{Hom}}(V_\pi, V_i \times X)$ define a saturated system of vector bundles over X. Now $(U_i)_\pi \subset W_i$, and it will be shown that the saturated system of vector bundles $(W_i)_{i\in I}$ with the opens $(U_i)_\pi \subset W_i$ satisfy the conditions of lemma 3.6.7, with X taking the place of S in the statement of the lemma.

For condition (i), note that sections of $(U_i)_\pi \to X$ are precisely the vector bundle inclusions $V_\pi \to V_i \times X$. Since X is affine and the system of vector bundles $(V_i)_{i\in I}$ over S is complete, it follows that there is an $i \in I$ for which there exists such a vector bundle inclusion.

Condition (ii) can be seen to hold as follows. Let Y be any scheme over X. Then a pair of morphisms of vector bundles $f, g : V_\pi \times_X Y \to V_i \times Y$, where one of the two is an inclusion, fits in a commutative diagram

$$\begin{tikzcd}
V_\pi \times_X Y \arrow[dr, dashed] \arrow[drr, "f"] \arrow[ddr, "g"'] & & \\
& V_{2i} \times Y \arrow[r] \arrow[d] & V_i \times Y \arrow[d] \\
& V_i \times Y \arrow[r] & Y,
\end{tikzcd}$$

and the dotted arrow exists by the universal property of the pullback. Because either f or g is a vector bundle inclusion, so is the dotted arrow.

Hence lemma 3.6.7 applies and it follows that $(U_\infty)_\pi \to X$ is a motivic equivalence, as was to be shown. $\square$

The fppf quotients $L_{\mathrm{fppf}}(U_i/G)$ in the above theorem are often representable by schemes (cf. [21, théorème 5] and corollary 3.4.6), in which case it follows that $L_{\mathrm{fppf}}(U_\infty/G)$ is a colimit of representables; this is why this construction is called *geometric*.

If G is representable by a smooth group scheme, then it is necessarily an fppf sheaf since the fppf topology is subcanonical. Furthermore,

$$H^1_{\mathrm{fppf}}(X, G) \cong H^1_{\text{ét}}(X, G)$$

for all $X \in \mathbf{Sm}_k$ since any G-torsor $T \to X$ is a smooth morphism and therefore it has étale-local sections by [36, corollaire 17.16.3]. Thus it follows that the natural map $\mathrm{B}_{\text{ét}}G \to \mathrm{B}_{\mathrm{fppf}}G$ is an isomorphism in $\mathbb{H}(k)$ whenever G is representable by a smooth group scheme, which will always be the case in this thesis.

Now we tie the construction of $\mathrm{B}_{\mathrm{fppf}}G$ to the concept of an *admissible gadget with a nice G-action* [66, definitions 4.2.1, 4.2.4]. These gadgets are useful for cohomology computations involving the classifying space $\mathrm{B}_{\text{ét}}G$ since the cohomology of their constituent parts is often well-understood.

Definition 3.6.10. Let $X \in \mathbf{Sm}_k$. An *admissible gadget over X* is a sequence $(V_i, U_i, f_i)_{i\geq 1}$, where the V_i are vector bundles over X, the U_i are open subschemes of V_i and the $f_i : U_i \to U_{i+1}$ are monomorphisms over X such that:

(i) for any field k and closed point $x : \operatorname{Spec} k \to X$, there exists an $i \geq 1$ such that $U_i \times_X \operatorname{Spec} k$ has a k-rational point; and
(ii) for any i there exists $j > i$ such that the morphism $U_i \to U_j$ factors through the morphism $U_i \to V_i^2 - (V_i - U_i)^2$ of the form $v \mapsto (0, v)$.

Let V be a vector bundle over $*$ and $G \subset \mathrm{GL}(V)$ a closed subgroup that is flat and finitely presented.

Definition 3.6.11. Let (V_i, U_i, f_i) be an admissible gadget over S. A *nice action of G on (V_i, U_i, f_i)* is an action of G on V_i for each i, such that:

(i) for each $i \geq 1$, the open subscheme $U_i \subset V_i$ is G-invariant, the map f_i is G-equivariant and the factorization of condition (ii) of definition 3.6.10 can be chosen to consist of G-equivariant maps;
(ii) the action of G on U_i is strictly free; and
(iii) for any étale G-torsor $T \to X$ over a smooth S-scheme X, there exists an i such that the morphism $(U_i \times T)/G \to X$ is Nisnevich locally an epimorphism.

Lemma 3.6.12. *Let V be a vector bundle on $*$. Let $V_i = \underline{\mathrm{Hom}}(V, \mathbb{A}^i_k)$ for $i \in \mathbb{N}$, and $f_i : V_i \to V_{i+1}$ the monomorphism induced by the inclusion $\mathbb{A}^i_k \to \mathbb{A}^{i+1}_k$ on the first i coordinates. Let $U_i \subset V_i$ be the open subscheme of vector bundle inclusions. Let G act on V_i via the standard action on V. Then $(V_i, U_i, f_i)_{i\in\mathbb{N}}$ is an admissible gadget over $*$ with a nice G-action.*

Proof. Given a closed point $x : \operatorname{Spec} k' \to \operatorname{Spec} k$, the pullback of V is a finite dimensional vector space over k', so there exists a $i \in \mathbb{N}$ such that $V \times \operatorname{Spec} k' \cong \mathbb{A}^i_{k'}$. Hence $U_i \times \operatorname{Spec} k'$ has a k'-rational point, which shows that (i) of definition 3.6.10 is satisfied. Point (ii) of definition 3.6.10 is satisfied by construction.

Since G acts on each V_i via V, the maps f_i are G-equivariant, so condition (i) of definition 3.6.11 is satisfied. Condition (ii) of definition 3.6.11 is also satisfied, by lemma 3.6.8. Let $T \to X$ be an étale G-torsor over $X \in \mathbf{Sm}_k$. By the proof of theorem 3.6.9, for $i \in \mathbb{N}$ the sections of $(U_i \times T)/G \to X$ are vector bundle inclusions $(V \times T)/G \to \mathbb{A}^i_X$. Let i be the rank of V. Then $(V \times T)/G$ has rank i over X, so $(V \times T)/G$ is Zariski locally isomorphic to $\mathbb{A}^i_X$. Hence (V_i, U_i, f_i) satisfies condition (iii) of definition 3.6.11, and the proof is finished. □

We can use the above lemma to construct admissible gadgets with nice actions for arbitrary $X \in \mathbf{Sm}_k$ with an action of G.

Definition 3.6.13. Let X be a smooth scheme over k with a G-action and let (V_i, U_i, f_i) be an admissible gadget over $*$ with a nice G-action. The *motivic borel space* X_G is the motivic space

$$X_G = \operatorname*{colim}_i (X \times U_i)/G.$$

The following examples constitute the most important geometric classifying spaces considered in this thesis.

Example 3.6.14. Let $T \cong \mathbb{G}_m^t$ be a split torus over k of rank t. Then $T \subset \mathrm{GL}_t$ is the inclusion of invertible diagonal matrices. Set $V = \mathbb{A}^t_k$ and let $(V_i, U_i, f_i)_{i \in \mathbb{N}}$ be the admissible gadget with nice T-action of lemma 3.6.12. Let $i \in \mathbb{N}_{\geq t}$. Note that for $X \in \mathbf{Sm}_k$, $U_i(X)$ is the set of $(i \times t)$-matrices M of maximal rank t with entries in $\mathcal{O}_X(X)$. Then a computation shows that $U_i/T \cong (\mathbb{P}^i_k)^t$, and it follows $\mathrm{B_{gm}}T \cong \operatorname{colim}_{i \in \mathbb{N}} (\mathbb{P}^i_k)^t$. Note that $\mathrm{B_{gm}}T$ is a product of t copies of the "infinite projective space" $\mathrm{B_{gm}}\mathbb{G}_m$.

More generally, for $X \in \mathbf{Sm}_k$ with trivial T-action, the motivic borel space X_T is $\operatorname{colim}_{i \in \mathbb{N}} (\mathbb{P}^i_X)^t$.

Example 3.6.15. Let $G \cong \mathrm{GL}_t$ be a general linear group over k for some $t \in \mathbb{N}$. Let $i \in \mathbb{N}_{\geq t}$. Then $U_i/G = \mathrm{GL}_i / P = \mathrm{Gr}_k(t, i)$, where $P \subset \mathrm{GL}_i$ is the parabolic subgroup

$$P = \begin{pmatrix} \mathrm{GL}_t & * \\ 0 & \mathrm{GL}_{i-t} \end{pmatrix}.$$

Hence $\mathrm{B_{gm}}G \cong \operatorname{colim}_{i \in \mathbb{N}} \mathrm{Gr}_k(t, i)$ is a colimit of Grassmannians, of which we can think as an infinite Grassmannian.

4 Milnor exact sequence

The Milnor exact sequence is instrumental in computing the cohomology of classifying spaces, since it provides a means of detecting whether a cohomology theory commutes with a given colimit. This section is dedicated to the Milnor exact sequence in motivic homotopy theory, which takes the following form for a motivic spectrum E and an $\mathbb{N}$-filtered colimit $X = \operatorname{colim}_{i\in\mathbb{N}} X_i$ of motivic spaces:

$$0 \to \lim_i{}^1 E^{p-1,q}(X_i) \to E^{p,q}(X) \to \lim_i E^{p,q}(X_i) \to 0.$$

Thus the $\lim^1$-term should be understood as the obstruction to the equality of the cohomology of the colimit X and the limit of the cohomologies of the X_i. In particular, the Milnor exact sequence can be applied to the classifying spaces of linear algebraic groups, since these can be constructed as $\mathbb{N}$-filtered colimits. Any treatment of the Atiyah-Segal completion theorem would be incomplete without considering the appropriate Milnor exact sequence.

4.1 Mapping telescopes

We give a construction of the mapping telescope of a sequence of cofibrations in a simplicial model category that is of independent interest, as well as useful in the proof of the existence of the Milnor exact sequence.

Remark 4.1.1. Here is a remark to avoid potential confusion. When we write about "*the* homotopy (co)limit" of some diagram in a model category $\mathcal{C}$, we mean some explicit construction of the homotopy colimit in $\mathcal{C}$ itself, not in $\mathrm{Ho}(\mathcal{C})$. Thus it makes sense to ask whether the colimit and the homotopy colimit of a given diagram are weakly equivalent; if they are, they are isomorphic in $\mathrm{Ho}(\mathcal{C})$.

Let $\mathcal{C}$ be a simplicial model category. Regard $\mathbb{N}$ as a poset with the usual partial order. Let $X : \mathbb{N} \to \mathcal{C}$ be a sequence of cofibrations in $\mathcal{C}$, with X_0 cofibrant. Denote the cofibration $X(i \le i+1)$ by $f_i : X_i \to X_{i+1}$.

Definition 4.1.2. The *mapping telescope of* X is the pushout T in the pushout diagram

$$\begin{tikzcd} \coprod_{m\in\mathbb{N}} X_m \arrow[r, "\iota_1"] \arrow[d, "\iota_0"] & \coprod_{m\in\mathbb{N}} X_{2m+1}\otimes\Delta[1] \arrow[d] \\ \coprod_{m\in\mathbb{N}} X_{2m}\otimes\Delta[1] \arrow[r] & T. \end{tikzcd} \tag{4.1.3}$$

where the map ι_i for $i = 0, 1$ is given on X_m by $\mathrm{id}_{X_m} \otimes 1$ if the parities of m and i coincide, and $f_m \otimes 0$ otherwise. Note that the ι_i are cofibrations, so that T is in fact a homotopy pushout.

Lemma 4.1.4. *With notation as above, the mapping telescope T of X is weakly equivalent to* $\operatorname{colim} X_i$.

Proof. I will construct T inductively, so that the required weak equivalence will be a colimit of weak equivalences.

(i) Define $T_0 = X_0 \otimes \Delta[1]$. Let $i_0 : X_0 \to T_0$ be the map $\mathrm{id}_{X_0} \otimes 1$ induced by the trivial cofibration of simplicial sets $1 : \Delta[0] \to \Delta[1]$. Since X_0 is cofibrant, the map i_0 is a trivial cofibration by [37, proposition 9.3.9]. In particular, T_0 is cofibrant. Let $p_0 : T_0 \to X_0$ be the projection map induced by $\Delta[1] \to \Delta[0]$. Then p_0 is a weak equivalence by two-out-of-three, since $p_0 i_0 = \mathrm{id}_{X_0}$ and i_0 are weak equivalences.

(ii) Fix $n \in \mathbb{N}$. Assume that T_m, a trivial cofibration $i_m : X_m \to T_m$ and a weak equivalence $p_m : T_m \to X_m$ have been defined for all $m \leq n$. Define T_{n+1} as the following pushout:

$$\begin{array}{ccc} X_n & \xrightarrow{f_n \otimes 0} & X_{n+1} \otimes \Delta[1] \\ \downarrow{\scriptstyle i_n} & & \downarrow \\ T_n & \xrightarrow{t_n} & T_{n+1}. \end{array}$$

The right vertical map, being the pushout of a trivial cofibration, is a trivial cofibration, so the composition

$$X_{n+1} \xrightarrow{\mathrm{id}_{X_{n+1}} \otimes 1} X_{n+1} \otimes \Delta[1] \longrightarrow T_{n+1}$$

is a composition of trivial cofibrations and therefore itself a trivial cofibration; denote it by i_{n+1}. Clearly, the diagram

$$\begin{array}{ccc} X_n & \xrightarrow{f_n \otimes 0} & X_{n+1} \otimes \Delta[1] \\ \downarrow{\scriptstyle i_n} & & \downarrow \\ T_n & \xrightarrow{f_n p_n} & X_{n+1} \end{array}$$

commutes, so there is a unique map $p_{n+1} : T_{n+1} \to X_{n+1}$ such that $p_{n+1} t_n = f_n p_n$ and $p_{n+1} i_{n+1} = \mathrm{id}_{X_{n+1}}$ by the universal property of the pushout T_{n+1}. Hence p_{n+1} is a weak equivalence by two-out-of-three. The map t_n is also a cofibration, since it is the pushout of the cofibration $f_n \otimes 0$.

Combining (i) and (ii) yields a sequence of cofibrations $T' : \mathbb{N} \to \mathcal{C}$, the colimit of which is clearly T. Furthermore, there is a map $p : T' \to X$ which is an objectwise weak equivalence of cofibrant objects, so by [37, theorem 18.5.3] the induced map of homotopy colimits is a weak equivalence. Since the homotopy colimits of T' and X are weakly equivalent to their respective colimits by [37, theorem 19.9.1], T is weakly equivalent to $\operatorname{colim} X_i$, as was to be shown. □

4.2 The motivic Milnor exact sequence

In this section, we construct the motivic Milnor exact sequence, using the fact that the category $\mathbf{Spt}_{\mathbb{P}^1}(k)$ of $\mathbb{P}^1$-spectra is a stable, proper simplicial model category. As noted before, this exact sequence is a useful tool for computing the cohomology of certain colimits of motivic spaces, such as the geometric classifying spaces of theorem 3.6.9. Such a computation will usually consist of (i) an assertion that the $\lim^1$-*term* of the sequence vanishes, and (ii) a computation of the limit of the cohomology groups of each space in the colimit diagram.

First, we prove an auxiliary lemma.

Lemma 4.2.1. *Let $\mathcal{C}$ be a stable model category with suspension functor Σ. Let*

$$X \underset{f_1}{\overset{f_0}{\rightrightarrows}} Y \xrightarrow{g} Z$$

be a coequalizer diagram in $\mathcal{C}$. Assume that the homotopy coequalizer of f_0 and f_1 exists and that Z is weakly equivalent to it. Then there exists a distinguished triangle

$$X \xrightarrow{[f_0]-[f_1]} Y \xrightarrow{[g]} Z \xrightarrow{\alpha} \Sigma X$$

in the stable homotopy category $\mathrm{Ho}(\mathcal{C})$.

Proof. By assumption

$$X \underset{[f_1]}{\overset{[f_0]}{\rightrightarrows}} Y \xrightarrow{[g]} Z,$$

is a coequalizer diagram in $\mathrm{Ho}(\mathcal{C})$. Since $\mathcal{C}$ is stable, $\mathrm{Ho}(\mathcal{C})$ is triangulated and in particular additive, yielding a coequalizer diagram

$$X \underset{0}{\overset{[f_0]-[f_1]}{\rightrightarrows}} Y \xrightarrow{[g]} Z,$$

which by definition of the triangulated structure on $\mathrm{Ho}(\mathcal{C})$ fits into a distinguished triangle

$$X \xrightarrow{[f_0]-[f_1]} Y \xrightarrow{[g]} Z \xrightarrow{\alpha} \Sigma X,$$

as was to be shown. $\square$

The model structure of the category $\mathbf{Spt}_{\mathbb{P}^1}(k)$ of $\mathbb{P}^1$-spectra is stable, proper and simplicial by [47, theorem 2.9]. Let $f : X \to Y$ be a map of $\mathbb{P}^1$-spectra and consider the pushout square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle g} \\ X \wedge \Delta[1] & \longrightarrow & Z. \end{array} \tag{4.2.2}$$

Note that i is a cofibration. Since the model structure on $\mathbf{Spt}_{\mathbb{P}^1}(k)$ is proper, Z is weakly equivalent to the homotopy pushout of f and i by [37, proposition 13.5.4]. Since $X \wedge \Delta[1]$ is contractible, it follows that this construction defines a cofiber sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1], \tag{4.2.3}$$

of $\mathbb{P}^1$-spectra, which becomes a distinguished triangle when passing to $\mathbf{SH}(k)$, since $[g] : Y \to Z$ is the coequalizer of $[f]$ and the zero map in $\mathbf{SH}(k)$.

Remark 4.2.4. Lemma 4.2.1 is particularly useful when f_0 and f_1 are cofibrations in $\mathbf{Spt}_{\mathbb{P}^1}(k)$, in which case the coequalizer Z automatically coincides with the homotopy coequalizer of f_0 and f_1 by [37, proposition 19.9.1]. Note that the map $[f_0] - [f_1]$ is not necessarily represented by a map $f : X \to Y$, but it is represented by a map $f : X' \to Y'$ with X' and Y' weakly equivalent to X and Y, respectively. Thus, when passing to $\mathbf{SH}(k)$, the pushout square 4.2.2 defining the cofiber sequence for f becomes isomorphic to the pushout square

$$\begin{tikzcd} X \arrow[r, "{[f_0]-[f_1]}"] \arrow[d] & Y \arrow[d, "{[g]}"] \\ * \arrow[r] & Z, \end{tikzcd}$$

showing that the maps $[f_0] - [f_1]$ and $[g]$ really fit in a distinguished triangle in $\mathbf{SH}(k)$.

Lemma 4.2.1 also applies to coequalizer diagrams in the category of pointed motivic spaces $\mathbf{Spc}_*(k)$, since the suspension $\mathbb{P}^1$-spectrum functor $\Sigma^\infty_{\mathbb{P}^1} : \mathbf{Spc}_*(k) \to \mathbf{Spt}_{\mathbb{P}^1}(k)$ has a right adjoint and therefore preserves colimits.

Bousfield and Kan prove the existence of a $\lim^1$-exact sequence of homotopy groups of simplicial sets in [13, theorem IX.3.1], which might be adapted to the current setting. However, we choose to present a more direct proof here. Note that the proof works for any stable, proper and simplicial model category.

Theorem 4.2.5 (Milnor sequence)**.** *Let $E \in \mathbf{SH}(k)$ be a $\mathbb{P}^1$-spectrum in the motivic stable homotopy category, with associated cohomology theory $E^{*,*}$. Let $X : \mathbb{N} \to \mathbf{Spc}_*(k)$ be a sequence of cofibrations and denote the cofibration $X(i \le i+1)$ by $f_i : X_i \to X_{i+1}$. Then there is a short exact sequence*

$$0 \to \lim{}^1 E^{p-1,q}(X) \to E^{p,q}(\operatorname*{colim}_i X_i) \to \lim_i E^{p,q}(X_i) \to 0 \tag{4.2.6}$$

for all $p, q \in \mathbb{Z}$.

Proof. See [64] for a proof of the similar statement on the stable homotopy category $\mathbf{SH}$ of $\mathbf{Top}$. This proof will be analogous to the one found there, though slightly more technical in nature.

Since $\mathbf{Spc}_*(k)$ has a simplicial model category structure, X has a mapping telescope T as in definition 4.1.2. Now T fits in a coequalizer diagram

$$\coprod_{i\in\mathbb{N}} X_i \overset{\iota_0}{\underset{\iota_1}{\rightrightarrows}} \coprod_{i\in\mathbb{N}} X_i \otimes \Delta[1] \longrightarrow T,$$

where the ι_i from definition 4.1.2 have been replaced by their compositions with the canonical inclusions

$$\coprod_{i\in\mathbb{N}} X_{2i} \longrightarrow \coprod_{i\in\mathbb{N}} X_i \qquad \text{and} \qquad \coprod_{i\in\mathbb{N}} X_{2i+1} \longrightarrow \coprod_{i\in\mathbb{N}} X_i.$$

By lemma 4.1.4 and remark 4.2.4, T is weakly equivalent to $\operatorname{colim} X_i$ in $\mathbf{Spt}_{\mathbb{P}^1}(k)$. Hence the observation that $\coprod_{i\in\mathbb{N}} X_i$ is weakly equivalent to $\coprod_{i\in\mathbb{N}} X_{2i} \otimes \Delta[1]$ together with an application of lemma 4.2.1 yields a distinguished triangle

$$\coprod_{i\in\mathbb{N}} X_i \xrightarrow{\ g\ } \coprod_{i\in\mathbb{N}} X_i \xrightarrow{\ h\ } \operatorname{colim}_i X_i \longrightarrow \Sigma_s \left(\coprod_{i\in\mathbb{N}} X_i \right)$$

in $\mathbf{SH}(k)$, where $g = \iota_0 - \iota_1$. Let $P = \coprod_{i \in \mathbb{N}} X_i$ and $C = \operatorname{colim}_i X_i$. For $q \in \mathbb{Z}$, the functor $\mathbf{SH}(k)(-, \Sigma_t^q E)$, which we shall also denote by the shorthand $[-, \Sigma_t^q E]$, is cohomological by e.g. [77, Tag 0149]. Thus, we apply $[-, \Sigma_t^q E]$ to the above triangle to obtain a long exact sequence of abelian groups

$$\ldots \longrightarrow [\Sigma_s P, \Sigma_t^q E] \longrightarrow [C, \Sigma_t^q E] \xrightarrow{h^*} [P, \Sigma_t^q E] \xrightarrow{g^*} [P, \Sigma_t^q E] \longrightarrow \ldots.$$

Note that $[\Sigma_s P, \Sigma_t^q E] = [P, \Omega_s \Sigma_t^q E] = E^{q-1,q}(P)$ by definition 2.2.11. By shifting, we can concoct a diagram

$$\begin{tikzcd}
E^{p-1,q}(P) \arrow[r, "g_{p-1}^*"] & E^{p-1,q}(P) \arrow[r] \arrow[d] & E^{p,q}(C) \arrow[r, "h_p^*"] \arrow[dr, dashed] & E^{p,q}(P) \arrow[r, "g_p^*"] & E^{p,q}(P) \\
& \operatorname{coker} g_{p-1}^* \arrow[ur, dashed] & & \ker g_p^* \arrow[u] &
\end{tikzcd}$$

for any $p, q \in \mathbb{Z}$, where the top row is exact and the dotted morphisms fit in a short exact sequence. It remains to show that $\operatorname{coker} g_{p-1}^* = \lim^1{}_i E^{p-1,q}(X_i)$ and $\ker g_p^* = \lim_i E^{p,q}(X_i)$. First note that

$$E^{p,q}(P) = \prod_{i \in \mathbb{N}} E^{p,q}(X_i).$$

The map $g^* : E^{p,q}(P) \to E^{p,q}(P)$ is given by $g_0^* - g_1^*$, that is, for $(a_i)_i \in E^{p,q}(P)$,

$$(a_i)_i \mapsto ((-1)^i (a_i - f_i^*(a_{i+1})))_i.$$

Multiply g^* with $((-1)^i)_i$, and note that this leaves its kernel and cokernel unchanged. Now g^* is given by

$$(a_i)_i \mapsto (a_i - f_i^*(a_{i+1}))_i.$$

The kernel of this map is equal to $\lim_i E^{p,q}(X_i)$, and its cokernel is $\lim^1{}_i E^{p,q}(X_i)$ by definition. See section 3.5 of [92] for more details about the $\lim^1$-term. □

4.3 The example of motivic cohomology

In this section, we will apply the motivic Milnor exact sequence to the example of motivic cohomology to show that Totaro's definition of the the Chow ring of a classifying space from [83] is the right one in the context of motivic homotopy theory. Since Totaro defined the Chow ring of a classifying space using an admissible gadget as in definition 3.6.10, the proof of this result boils down to checking that the $\lim^1$-term in a particular instance of the motivic Milnor exact sequence vanishes. We will generalize Totaro's definition to include Bloch's higher Chow groups [10] along the way, for a smoother comparison to motivic cohomology.

Integral motivic cohomology is given on $\mathbb{P}^1$-spectra by the Eilenberg-MacLane spectrum $\mathbf{H}\mathbb{Z}$ as

$$\mathbf{H}\mathbb{Z}^{p,q}(E) = \mathbf{SH}(k)(E, S_s^{p-2q} \wedge (\mathbb{P}^1)^{\wedge q} \wedge \mathbf{H}\mathbb{Z}),$$

see for example [88, section 3.1]. For $X \in \mathbf{Sm}/k$ and $p, q \in \mathbb{Z}$, there are isomorphisms $\mathbf{H}\mathbb{Z}^{p,q}(X) \cong \mathrm{CH}^q(X, 2q-p)$, where $\mathrm{CH}^q(X, m)$ are the higher Chow groups as defined in [10]. In particular, one retrieves the ordinary Chow groups $\mathbf{H}\mathbb{Z}^{2q,q}(X) = \mathrm{CH}^q(X)$.

Let G be a linear algebraic group over a field k. We will show that the construction from [83] of the Chow ring $\mathrm{CH}^*(\mathrm{B}G)$ of the classifying space of G coincides with the cohomology ring $\mathbf{H}\mathbb{Z}^{2*,*}(\mathrm{B}_{\mathrm{gm}}G)$ of the motivic geometric classifying space $\mathrm{B}_{\mathrm{gm}}G$. We will define the higher Chow groups $\mathrm{CH}^q(\mathrm{B}G, m)$ for all $q, m \in \mathbb{N}$ and prove

$$\mathrm{CH}^q(\mathrm{B}G, m) \cong \mathbf{H}\mathbb{Z}^{2q-m,q}(\mathrm{B}_{\mathrm{gm}}G),$$

which is a slightly stronger result.

We first prove a few auxiliary lemmas, adapting the proof of [83, theorem 1.1] to the current context, in order to show that Totaro's method of defining Chow groups of classifying spaces also works for higher Chow groups, and that these coincide with the motivic integral cohomology of $\mathrm{B}_{\text{ét}}G$. The same technique is used in [28, definition-proposition 1] and in [28, section 2.7], equivariant higher Chow groups are defined in a similar way.

Let $\mathcal{D}$ be the category whose objects are pairs (V, S) where V is a representation of G and $S \subset V$ is a G-invariant closed subset such that G acts freely on $V - S$ and the geometric quotient $(V - S)/G$ exists as a quasi-projective variety. There are sufficiently many such pairs by [83, remark 1.4]. Morphisms $f : (V, S_V) \to (W, S_W)$ in $\mathcal{D}$ are inclusions $f : V \to W$ of G-representations such that $f^{-1}(S_W) \subset S_V$, which implies $\mathrm{codim}_V(S_V) \leq \mathrm{codim}_W(S_W)$. Note that $\mathcal{D}$ is a non-empty category and that we can produce pairs (V, S) with arbitrarily large $\mathrm{codim}_V(S)$. In fact, the admissible gadget of lemma 3.6.12 produces an infinite sequence of such pairs, with closed subsets of increasing codimension.

Remark 4.3.1. The existence of a geometric quotient can be ignored if one works in the category of smooth algebraic spaces over k, rather than schemes. In this case, as long as G acts freely on $V - S$ the quotient $(V - S)/G$ always exists as an algebraic space, c.f. the discussion at the beginning of [28, section 2.2]. Still, the existence of a geometric quotient is often useful in computations. Note that the category of motivic spaces over k also contains all smooth algebraic spaces over k (since algebraic spaces are presheaves on $\mathbf{Sm}_k$).

Lemma 4.3.2. *Let $(V, S), (V, S') \in \mathcal{D}$ such that both S and S' have codimension $\geq d$ in V. Then it holds that*

$$\mathrm{CH}^q((V - S)/G, m) \cong \mathrm{CH}^q((V - S')/G, m)$$

for all $q < d$ and $m \in \mathbb{N}$.

Proof. Since it holds that $S \subset S \cup S'$ and $S \cup S'$ is also a G-invariant closed subset of codimension $\geq d$ on whose complement G acts freely, I may assume without loss of generality that $S \subset S'$. Let $X = (V - S)/G$, $U = (V - S')/G$ and $Y = (S' - S)/G$. There is a long exact localization sequence of higher Chow groups

$$\cdots \to \mathrm{CH}^{q-d}(Y, m) \to \mathrm{CH}^q(X, m) \to \mathrm{CH}^q(U, m) \to \mathrm{CH}^{q-d}(Y, m-1) \to \dots$$

by [11, corollary (0.2)] (c.f. [10, theorem 3.1]; the proof contained a mistake, which was rectified in [11]). As $\mathrm{CH}^{q-d}(Y, m) = 0$ for $q < d$ and $m \in \mathbb{N}$,

$$\mathrm{CH}^q(X, m) \cong \mathrm{CH}^q(U, m)$$

for $q < d$ and $m \in \mathbb{N}$, as was to be shown. □

Lemma 4.3.3. *Let* $(V, S_V), (W, S_W) \in \mathcal{D}$ *such that* S_V *and* S_W *have codimension* $\geq d$. *Then*

$$\mathrm{CH}^q((V - S_V)/G, m) \cong \mathrm{CH}^q((W - S_W)/G, m)$$

for $q < d$ *and* $m \in \mathbb{N}$.

Proof. Consider $((V - S_V) \times W)/G$ and $(V \times (W - S_W))/G$ as open subsets of $V \oplus W$, once again as in the proof of [83, theorem 1.1]. These are both vector bundles; the former over $(V - S_V)/G$, the latter over $(W - S_W)/G$. Note that both $S_V \times W$ and $V \times S_W$ have codimension $\geq d$ in $V \oplus W$. Thus, an application of lemma 4.3.2 yields

$$\mathrm{CH}^q(((V - S_V) \times W)/G, m) \cong \mathrm{CH}^q((V \times (W - S_W))/G, m)$$

for $q < d$ and $m \in \mathbb{N}$. Then it follows from $\mathbb{A}^1$-homotopy invariance for higher Chow groups [10, theorem 2.1] that

$$\mathrm{CH}^q(((V - S_V) \times W)/G, m) \cong \mathrm{CH}^q((V - S_V)/G, m)$$
$$\mathrm{CH}^q((V \times (W - S_W))/G, m) \cong \mathrm{CH}^q((W - S_W)/G, m)$$

for $q, m \in \mathbb{N}$, which yields the desired result. □

Lemmas 4.3.2 and 4.3.3 inspire the following definition.

Definition 4.3.4. The *higher Chow group* $\mathrm{CH}^q(\mathrm{B}G, m)$ *of the classifying space* $\mathrm{B}G$ is the group $\mathrm{CH}^q((V - S)/G, m)$ for any $(V, S) \in \mathcal{D}$ such that $q < \mathrm{codim}_V(S)$.

Note that the symbol $\mathrm{B}G$ in the above definition does not actually represent a geometric object, but is rather a placeholder for the geometric classifying space $\mathrm{B_{gm}}G$, as we will show next by relating the integral motivic cohomology of $\mathrm{B_{gm}}G$ to the higher Chow groups of the classifying space in the sense of definition 4.3.4.

Theorem 4.3.5. *There is a natural isomorphism*

$$\mathbf{H}\mathbb{Z}^{2q-m,q}(\mathrm{B_{gm}}G) \to \mathrm{CH}^q(\mathrm{B}G, m), \tag{4.3.6}$$

where $\mathrm{CH}^q(\mathrm{B}G, m)$ *is as in definition 4.3.4.*

Proof. Let (V_i, U_i, f_i) be an admissible gadget over k with a nice G-action, as in definitions 3.6.10 and 3.6.11, which exists by lemma 3.6.12. Let $U_\infty = \mathrm{colim}_i\, U_i$. By theorem 4.2.5, there is an exact sequence

$$0 \to \lim_i{}^1 \mathrm{CH}^q(U_i/G, m+1) \to \mathbf{H}\mathbb{Z}^{2q-m,q}(U_\infty/G) \to \lim \mathrm{CH}^q(U_i/G, m) \to 0$$

for all $q, m \in \mathbb{N}$. Note that $\mathrm{B_{gm}} = U_\infty/G$, so it remains to be shown is that the $\lim^1$-term vanishes. It follows from lemma 4.3.3 that, for fixed $q, m \in \mathbb{N}$, the inverse system of Chow groups $\mathrm{CH}^q(U_i/G, m+1)$ is constant for large enough i and therefore satisfies the Mittag-Leffler condition (c.f. [77, Tag 0594]). Thus, by [30, corollary 6], the $\lim^1$-term vanishes, and the proof is done. □

Remark 4.3.7. One can also consider algebraic K-theory, which is similarly represented by a motivic spectrum by [86, section 6.2]. In this case, the $\lim^1$-term of the Milnor exact sequence vanishes in many cases by [57, theorem 7.5], yielding an analogue of theorem 4.3.5 for algebraic K-theory.

5 Semi-orthogonal decompositions

The theory of semi-orthogonal decompositions has become an invaluable tool in algebraic K-theory. It provides a way of describing the structure of certain "stable" categories, such as stable ∞-categories, pretriangulated dg categories and triangulated categories. We extend the theory of semi-orthogonal decompositions to include duality, thus making it compatible with Grothendieck-Witt theory. All the calculations of Grothendieck-Witt spectra in this thesis make use of such decompositions, which renders the theory indispensable for our strategy of proof of the Atiyah-Segal completion theorem.

The structure of a ring or scheme is to a large extent reflected in its derived category, so explicit descriptions of the derived category yield deep cohomological information. One tool for giving such explicit descriptions is the *semi-orthogonal decomposition*. A category with a semi-orthogonal decomposition can be thought of as being semi-simple with respect to K-theory. Their existence already loomed in [89] and [80], which proved additivity for algebraic K-theory using auxiliary categories of exact sequences.

Before we define semi-orthogonal decompositions, we review the theory of dg categories, which is the main categorical language of this thesis. The most relevant example of a dg category here is the dg category of (bounded) chain complexes of sheaves of modules on a scheme X. The key idea is to replace the morphism set of chain maps between two complexes by a morphism complex, yielding an enrichment in chain complexes of the classical category of chain complexes. This provides a flexible framework for categorical constructions that are not available in the language of triangulated categories, such as functorial mapping cones and more generally homotopy limits and colimits. It is essentially the idea of *enhanced triangulated categories*, as introduced in [12].

The definitions and results of this section are not new and mainly sourced from Keller's excellent overview article [53], work by Toën [81] and Tabuada [78], and [58, section 2].

5.1 dg Modules over a commutative ring

Let k be a commutative ring (or a sheaf of commutative rings, such as the structure sheaf $\mathcal{O}_X$ of a scheme X). The morphism object between two objects of a k-linear dg category is a differential graded k-module. As such, the dg category of dg k-modules forms the backbone of the theory of k-linear dg categories, and we will describe this category in this section, starting with the objects.

Definition 5.1.1. A *differential graded k-module*, or more briefly a *dg k-module*, is a graded k-module

$$M = \bigoplus_{i \in \mathbb{Z}} M^i$$

together with a morphism of k-modules $d : M \to M$ called the *differential of M* such that $d(M^i) \subset M^{i+1}$ for all $i \in \mathbb{Z}$ and $d^2 = 0$.

Note that a differential graded k-module can also be viewed as a complex of k-modules. This perspective is often useful, and will be leveraged throughout, using the cohomological grading convention; by a complex we shall always mean a *cochain complex*. Furthermore, every differential graded k-module has an underlying graded k-module, which is obtained by forgetting the differential.

Definition 5.1.2. Let M be a dg k-module. The *degree* $|x|$ or $\deg x$ of a homogeneous element $x \in M^i$ is defined as $|x| = i$.

Definition 5.1.3. A *morphism of dg k-modules* $f : M \to N$ is a morphism of graded k-modules such that $d_N f = f d_M$, where d_M and d_N are the differentials of M and N, respectively. Classically, morphisms of dg k-modules are called *chain maps*. We will denote the abelian group of chain maps $M \to N$ by $Z^0(M, N)$.

Definition 5.1.4. The *category of dg k-modules* is the category whose objects are dg k-modules and whose morphisms are chain maps of dg k-modules as in definitions 5.1.1 and 5.1.3.

There are additive functors Z^i, B^i and H^i from the category of dg k-modules to $\mathbf{Mod}_k$, defined for a dg k-module M by the i-th cycles $Z^i M = \ker(d : M^i \to M^{i+1})$, the i-th boundaries $B^i M = \operatorname{im}(d : M^{i-1} \to M^i)$ and the i-th cohomology $H^i M = Z^i M / B^i M$, respectively.

Definition 5.1.5. Let M and N be dg k-modules. Their *tensor product* $M \otimes N$ is the dg k-module whose n-th component is

$$(M \otimes N)^n = \bigoplus_{i+j=n} M^i \otimes N^j,$$

and whose differential is given by

$$d(x \otimes y) = (dx \otimes y) + (-1)^{|x|}(x \otimes dy)$$

on simple tensors $x \otimes y \in M^i \otimes N^j$ with $i, j \in \mathbb{Z}$.

Definition 5.1.6. Let M be a dg k-module and $n \in \mathbb{Z}$. Define Σ to be the dg k-module consisting of a single copy of k in degree -1. Let $\Sigma^n = \Sigma^{\otimes n}$. The *$n$-th shift* $\Sigma^n M$ of M is the tensor product $\Sigma^n \otimes M$. For $i \in \mathbb{Z}$, $(\Sigma^n M)^i = M^{i+n}$ and $d^i : M^{i+n} \to M^{i+n+1}$ is given by $(-1)^n d_M^{i+n}$, by definition of the tensor product. The dg k-module $\Sigma^n M$ will also be denoted by $M[n]$.

Definition 5.1.7. Let M and N be dg k-modules. Let $[M, N]$ be the dg k-module whose n-th component is given by the morphisms of graded k-modules $\Sigma^n M \to N$, and whose differential is given by

$$df = d_N f - (-1)^{|f|} f d_M$$

for $f \in [M, N]^n$. A morphism $f \in [M, N]^n$ is called *homogeneous of degree n*. A homogeneous morphism f is called *closed* if $df = 0$. Note that chain maps are precisely the closed homogeneous morphisms of degree 0.

From now on, when we write $f \in [M, N]$, we will always understand f to be homogeneous, unless explicitly indicated otherwise. The following result is folklore, and its proof amounts to checking the relevant definitions.

Proposition 5.1.8. *The category of dg k-modules has a closed symmetric monoidal structure with tensor product $\otimes$, internal hom-functor $[-,-]$, unit $\mathbb{1} = k$ and symmetry $\tau : M \otimes N \to N \otimes M$ given by $\tau(x \otimes y) = (-1)^{|x||y|}(y \otimes x)$.*

Proposition 5.1.8 allows us to upgrade the category of dg k-modules to a dg category.

Definition 5.1.9. The *dg category of dg k-modules* $\mathbf{dgMod}_k$ is the category whose objects are dg k-modules and whose mapping complex $[M, N]$ between two objects M and N is the dg k-module $[M, N]$ of definition 5.1.7.

The underlying k-linear category $Z^0\,\mathbf{dgMod}_k$ is obtained by taking Z^0 of each mapping complex $[M, N]$. Note that $Z^0[M, N]$ is the set of closed morphisms $M \to N$, so that $Z^0\,\mathbf{dgMod}_k$ is the category of dg k-modules from definition 5.1.4. Limits and colimits exist in $\mathbf{dgMod}_k$ and may be computed degreewise.

We may equip $\mathbf{dgMod}_k$ with the *projective model structure* [39, theorem 2.3.11], in which the weak equivalences are the quasi-isomorphisms and the fibrations the surjections. This model structure is compatible with the monoidal structure, making $\mathbf{dgMod}_k$ a *symmetric monoidal model category* in the sense of [39, definition 4.2.6].

The category $\mathbf{dgMod}_{\mathbb{Z}}$ is closely related to the category $\mathbf{sAb}$ of simplicial abelian groups via the Dold-Kan correspondence [48], a modern account of which can be found in [32, chapter III]. The general version [23, Satz 3.6] due to Dold and Puppe states that for an abelian category $\mathcal{A}$, the *normalized chain complex functor* $N : \mathcal{A}^{\Delta^{\mathrm{op}}} \to \mathrm{Ch}_+(\mathcal{A})$ to the category of non-negatively graded chain complexes is an equivalence of categories, which admits an explicit inverse. Since we use a cohomological grading convention, we should write $\mathrm{Ch}^-(\mathcal{A})$ for the category of non-positively graded cochain complexes instead of $\mathrm{Ch}_+(\mathcal{A})$. In our current case, $\mathcal{A}$ is the abelian category of k-modules $\mathbf{Mod}_k$, and $\mathrm{Ch}^-(\mathbf{Mod}_k)$ is a full subcategory of $Z^0\,\mathbf{dgMod}_k$. The Dold-Kan correspondence makes it possible to construct an n-simplex in $\mathbf{dgMod}_k$ as the normalized chain complex $N(k[\Delta^n])$ of the free k-module on Δ^n. For $n = 0$, this yields the monoidal unit $N(k[\Delta^0]) = \mathbb{1}$, and for $n = 1$, we obtain the *standard interval in chain complexes* $\mathrm{I} = N(k[\Delta^1])$, which is the complex

$$\mathrm{I}: \qquad \dots \longrightarrow 0 \longrightarrow k \xrightarrow{(1,-1)} k \oplus k \longrightarrow 0 \longrightarrow \dots$$

concentrated in degrees $[-1, 0]$. There is a split monomorphism $(0, 1) : \mathbb{1} \to \mathrm{I}$, and its quotient is the *cone object* Γ. Hence the exact sequence

$$\begin{tikzcd}
\mathbb{1}: \arrow[d] & \dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & k \arrow[r] \arrow[d, "{(0,1)}"] & 0 \arrow[r] & \dots \\
\mathrm{I}: \arrow[d] & \dots \arrow[r] & 0 \arrow[r] & k \arrow[r, "{(1,-1)}"] \arrow[d, equal] & k \oplus k \arrow[r] \arrow[d] & 0 \arrow[r] & \dots \\
\Gamma: & \dots \arrow[r] & 0 \arrow[r] & k \arrow[r, equal] & k \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

in $\mathbf{dgMod}_k$ is analogous to the fiber sequence $S^0 \hookrightarrow I_+ \twoheadrightarrow I$ in pointed topological spaces, where $I_+ = I \sqcup *$ is the unit interval with a disjoint base point. Recall that, in pointed topological spaces, S^0 is the unit for the smash product, smashing with I_+ constructs the *reduced cylinder*, and smashing with I constructs the *cone* of a space, making it contractible. Similar properties hold for $\mathbb{1}$, I and Γ in $\mathbf{dgMod}_k$.

Definition 5.1.10. Let $A, B \in \mathbf{dgMod}_k$. The *cone* ΓA of A is the tensor product $\Gamma \otimes A$. Similarly, the *cylinder* $\mathrm{I}A$ of A is the tensor product $\mathrm{I} \otimes A$.

The classical concept of chain homotopy can be reformulated using the cylinder. Note that for $A \in \mathbf{dgMod}_k$, there are two maps $(0,1) : \mathbb{1} \to \mathrm{I}$ and $(1,0) : \mathbb{1} \to \mathrm{I}$, which we can think of as the endpoints of $\mathrm{I}A$.

Definition 5.1.11. Let $f, g : A \to B$ be chain maps in $\mathbf{dgMod}_k$. A *chain homotopy* $f \sim g$ is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "(0,1)"] \arrow[dr, "f"'] & \mathrm{I}A \arrow[d] & A \arrow[l, "(1,0)"'] \arrow[dl, "g"] \\ & B. & \end{tikzcd}$$

We leave it as an exercise for the reader to check that the above definition is equivalent to the classical definition of chain homotopy; that Γ is chain contractible (i.e. $0 \sim \mathrm{id}_\Gamma$); and that for contractible C, $C \otimes A$ is contractible and $[C, A]$ is acyclic for all $A \in \mathbf{dgMod}_k$.

It will be useful to give $Z^0\,\mathbf{dgMod}_k$ the structure of an *exact category* (see definition 5.4.1) by setting conflations to be those sequences $A \to B \to C$ such that for each $i \in \mathbb{Z}$, the sequence $A^i \to B^i \to C^i$ is split exact in $\mathbf{Mod}_k$. Using this structure, Γ fits in an exact sequence $\mathbb{1} \to \Gamma \to \Sigma$ in $Z^0\,\mathbf{dgMod}_k$, given by the commutative diagram

$$\begin{tikzcd} \mathbb{1}: \arrow[d] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & k \arrow[r] \arrow[d, equal] & 0 \\ \Gamma: \arrow[d] & 0 \arrow[r] & k \arrow[r, equal] \arrow[d, equal] & k \arrow[r] \arrow[d] & 0 \\ \Sigma: & 0 \arrow[r] & k \arrow[r] & 0 \arrow[r] & 0, \end{tikzcd}$$

which is clearly degreewise split, a splitting being given by reversing the vertical arrows. Note that neither of these two splittings are chain maps. The resulting commutative square

$$\begin{tikzcd} \mathbb{1} \arrow[r] \arrow[d] & 0 \arrow[d] \\ \Gamma \arrow[r] & \Sigma \end{tikzcd}$$

is cocartesian in $\mathbf{dgMod}_k$, in the same way that S^1 is the pushout of the inclusion $S^0 \to I$ along the map $S^0 \to *$. This demonstrates the analogy between the shift functor on $\mathbf{dgMod}_k$ and the suspension functor on the stable homotopy category. The cone of a chain map in $\mathbf{dgMod}_k$ is a functorial colimit construction, analogous to the construction of cofibers in topology.

Definition 5.1.12. Let $f \in [A, B]$ be a chain map in $\mathbf{dgMod}_k$. The *cone* $\mathrm{cone}(f)$ *of* f is the pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ \Gamma A \arrow[r] & \mathrm{cone}(f) \end{tikzcd}$$

in $\mathbf{dgMod}_k$.

Since ΓA is contractible and $A \to \Gamma A$ is a cofibration in the model structure on $\mathbf{dgMod}_k$, the cone of a chain map f is also a homotopy cokernel of f. The cone construction in $\mathbf{dgMod}_k$ serves as a blueprint for the cone construction in other dg categories. In particular, we will define the *arrow dg category* $\mathbf{Ar}(\mathbf{dgMod}_k)$ and show that the cone construction is a functor $C : \mathbf{Ar}(\mathbf{dgMod}_k) \to \mathbf{dgMod}_k$.

5.2 dg Categories

As in the previous section, let k be a commutative ring or a sheaf of commutative rings.

Definition 5.2.1. A *dg k-category* is a category $\mathcal{A}$ enriched in $\mathbf{dgMod}_k$. It is *small* if the collection of objects is a set, as usual. It is *pointed* if it is a equipped with a zero object, called the *base point* and denoted 0. Thus a *small pointed dg k-category* consists of the following data:

(i) a non-empty set of objects $\mathrm{Ob}(\mathcal{A})$;
(ii) for each pair of objects $A, B \in \mathcal{A}$ a dg k-module $\mathcal{A}(A, B)$ called the *mapping complex* from A to B;
(iii) for each object $A \in \mathcal{A}$ a unit morphism $1_A \in \mathcal{A}(A, A)^0$;
(iv) for any three objects $A, B, C \in \mathcal{A}$ a morphism of dg k-modules

$$\circ : \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \to \mathcal{A}(A, C)$$

called *composition*, satisfying the usual associative and unital conditions; and
(v) a choice of zero object 0.

We shall mean by *dg category* a pointed dg k-category, to ease notation. For an unpointed dg category $\mathcal{A}$, we denote by $\mathcal{A}_+$ the pointed dg category obtained by adding a disjoint basepoint to $\mathcal{A}$. For $n \in \mathbb{Z}$ and $A, B \in \mathcal{A}$, an element $f \in \mathcal{A}(A, B)^n$ is called a *homogeneous morphism of degree n*, and if $df = 0$, the morphism f is said to be *closed*.

Note that $\mathbf{dgMod}_k$ is the prototypical example of a dg category by the above definition. Let $\mathcal{A}$ be a dg category and $A \in \mathcal{A}$. Let $1 = 1_A$ and let $d : \mathcal{A}(A, A) \to \mathcal{A}(A, A)$ be the differential. By definition, $1 = 1 \circ 1 = \circ(1 \otimes 1)$, and since $\circ$ is a morphism of dg k-modules, it holds that

$$d(1) = d(1 \circ 1) = \circ(d(1 \otimes 1)) = (d(1) \circ 1) + (1 \circ d(1)),$$

which yields $d(1) = 0$. Similarly,

$$d(f \circ g) = d(\circ(f \otimes g)) = \circ(d(f \otimes g)) = (df \circ g) + (-1)^{|f|}(f \circ dg),$$

so if both $df = 0$ and $dg = 0$, it holds that $d(f \circ g) = 0$. These observations show that $Z^0\mathcal{A} \subset \mathcal{A}$ is a k-linear category with the same objects as $\mathcal{A}$, which can be thought of as the *underlying category* of $\mathcal{A}$.

More generally, each dg category $\mathcal{A}$ has induced categories $Z^n\mathcal{A}$, $B^n\mathcal{A}$ and $H^n\mathcal{A}$ for $n \in \mathbb{Z}$, whose objects are the objects of $\mathcal{A}$ and whose morphism sets are $Z^n(\mathcal{A}(A, B))$, $B^n(\mathcal{A}(A, B))$ and $H^n(\mathcal{A}(A, B))$, respectively.

Definition 5.2.2. For a dg category $\mathcal{A}$, its *underlying category* is the category $Z^0\mathcal{A}$, which has the same objects as $\mathcal{A}$ and the closed morphisms of degree zero as its morphisms. The *homotopy category* of $\mathcal{A}$ is the category $H^0\mathcal{A}$. We call a morphism $f : A \to B$ of $Z^0\mathcal{A}$ an *equivalence* if its image in $H^0\mathcal{A}$ is an isomorphism.

Definition 5.2.3. Let $\mathcal{A}$ and $\mathcal{B}$ be dg categories. A *dg functor* $F : \mathcal{A} \to \mathcal{B}$ is an enriched functor of categories enriched in $\mathbf{dgMod}_k$. Thus such an F consists of the following data:

(i) a map $F : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$ on objects sending the base point of $\mathcal{A}$ to the base point of $\mathcal{B}$; and
(ii) for each pair of objects $A, B \in \mathcal{A}$ a morphism $\mathcal{A}(A, B) \to \mathcal{B}(FA, FB)$ of dg k-modules, respecting composition and units.

Definition 5.2.4. The *category of dg categories* $\mathbf{dgCat}_k$ has small pointed dg categories as its objects and dg functors as its morphisms.

Let $\mathcal{A}, \mathcal{B} \in \mathbf{dgCat}_k$ be dg categories and let $F, G : \mathcal{A} \to \mathcal{B}$ be dg functors.

Definition 5.2.5. A *natural transformation of dg functors* $\alpha : F \to G$ is a collection $\{\alpha_A \mid A \in \mathcal{A}, \alpha_A \in Z^0\mathcal{B}(FA, GA)\}$ such that

$$\begin{array}{ccc} FA & \xrightarrow{\alpha_A} & GA \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ FB & \xrightarrow{\alpha_B} & GB \end{array}$$

commutes for all $A, B \in \mathcal{A}$ and $f \in \mathcal{A}(A, B)$.

Definition 5.2.6. If there exists a dg functor $F' : \mathcal{B} \to \mathcal{A}$ such that FF' and $F'F$ are naturally isomorphic to the respective identity functors on $\mathcal{A}$ and $\mathcal{B}$, then F is called an *equivalence of dg categories.*

Definition 5.2.7. The *tensor product dg category* $\mathcal{A} \otimes \mathcal{B}$ is the dg category whose objects are pairs $(A, B) \in \mathcal{A} \times \mathcal{B}$, where all objects $(A, 0_\mathcal{B})$ and $(0_\mathcal{A}, B)$ are identified with the base point $0_{\mathcal{A}\otimes\mathcal{B}}$, so that $\mathrm{Ob}(\mathcal{A}\otimes\mathcal{B}) = \mathrm{Ob}(\mathcal{A}) \wedge \mathrm{Ob}(\mathcal{B})$, and whose mapping complexes are

$$(\mathcal{A} \otimes \mathcal{B})((A_1, B_1), (A_2, B_2)) = \mathcal{A}(A_1, A_2) \otimes \mathcal{B}(B_1, B_2),$$

where the composition is given by

$$(f_2 \otimes g_2) \circ (f_1 \otimes g_1) = (-1)^{|g_2||f_1|}((f_2 \circ f_1) \otimes (g_2 \circ g_1))$$

for homogeneous morphisms $f_1 \in \mathcal{A}(A_1, A_2)$, $f_2 \in \mathcal{A}(A_2, A_3)$, $g_1 \in \mathcal{B}(B_1, B_2)$ and $g_2 \in \mathcal{B}(B_2, B_3)$.

Definition 5.2.8. The *homomorphism dg category* $\mathbf{dgFun}(\mathcal{A}, \mathcal{B})$ has dg functors $F : \mathcal{A} \to \mathcal{B}$ as its objects. For $F, G \in \mathbf{dgFun}(\mathcal{A}, \mathcal{B})$, define $[F, G]$ as the graded k-module for which $\alpha \in [F, G]^i$ is given by a collection $\{\alpha_A \mid A \in \mathcal{A}, \alpha_A \in \mathcal{B}(FA, GA)^i\}$ such that

$$\begin{array}{ccc} FA & \xrightarrow{\alpha_A} & GA \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ FB & \xrightarrow[(-1)^{|\alpha||f|}\alpha_B]{} & GB \end{array}$$

commutes for all homogeneous $f \in \mathcal{A}(A, B)$. Then define a differential d on $[F, G]$ by setting $(d\alpha)_A = d_{\mathcal{B}(FA,GA)}\alpha_A$ for $\alpha \in [F, G]$ and $A \in \mathcal{A}$.

The category of dg categories $\mathbf{dgCat}_k$ becomes a closed symmetric monoidal category with the tensor product of definition 5.2.7 and the mapping dg categories of definition 5.2.8.

Definition 5.2.9. Let $\mathcal{C}$ be a small category and $\mathcal{A}$ a dg category. Let $k[\mathcal{C}]$ be the dg category whose objects are those of $\mathcal{C}$ and whose mapping complexes are given by $k[\mathcal{C}](A, B)^0 = k[\mathcal{C}(A, B)]$ and $k[\mathcal{C}](A, B)^i = 0$ for $i \neq 0$, where $k[\mathcal{C}(A, B)]$ is the free k-module on the set $\mathcal{C}(A, B)$. The *functor dg category* $\mathbf{Fun}(\mathcal{C}, \mathcal{A})$ is the dg category $\mathbf{dgFun}(k[\mathcal{C}]_+, \mathcal{A})$.

To describe $\mathbf{Fun}(\mathcal{C}, \mathcal{A})$ more concretely, let us consider $F \in \mathbf{Fun}(\mathcal{C}, \mathcal{A})$ for $\mathcal{C}$ a small category and $\mathcal{A}$ a dg category. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Since $k[\mathcal{C}]_+(X, Y) = k[\mathcal{C}(X, Y)]$ is a complex concentrated in degree zero, f is closed of degree zero. Hence, as $F : k[\mathcal{C}]_+ \to \mathcal{A}$ is a dg functor, $F(f)$ is also closed of degree zero. Since homogeneous morphisms in $k[\mathcal{C}]_+$ are freely generated by morphisms in $\mathcal{C}$, it follows that F is a functor $\mathcal{C} \to Z^0\mathcal{A}$.

Definition 5.2.10. Let $\mathcal{A}$ be a dg category and let $[1]$ be the category consisting of two objects with a single non-identity morphism between them. The *arrow category* $\mathbf{Ar}(\mathcal{A})$ *of* $\mathcal{A}$ is the dg category $\mathbf{Fun}([1], \mathcal{A})$. The objects of $\mathbf{Ar}(\mathcal{A})$ are the morphisms of $Z^0\mathcal{A}$, and for $f, g \in \mathbf{Ar}(\mathcal{A})$, the mapping complex $\mathbf{Ar}(\mathcal{A})(f, g)$ is given in degree n by the collection of those $(\alpha, \beta) \in \mathcal{A}(A, A')^n \oplus \mathcal{B}(B, B')^n$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\ A' \arrow[r, "g"] & B' \end{tikzcd}$$

commutes, and the differential of $(\alpha, \beta) \in \mathbf{Ar}(\mathcal{A})(f, g)^n$ is given by $(d\alpha, d\beta)$.

Example 5.2.11. For a complex A and $i \in \mathbb{Z}$, let $A^{\leq i}$ be the *naive truncation*

$$A : \qquad \ldots \longrightarrow A^{i-1} \longrightarrow A^i \longrightarrow 0 \longrightarrow \ldots$$

of A.

Let $\mathscr{C}_k$ be the full subcategory of $\mathbf{dgMod}_k$ whose objects are (representatives of the isomorphism classes of) the bounded complexes of finitely generated free k-modules. Then $\mathscr{C}_k$ is a full dg subcategory of $\mathbf{dgMod}_k$, since the tensor products and mapping complexes of such complexes are again bounded complexes of finitely generated free k-modules in $\mathscr{C}_k$.

The unit of $\mathscr{C}_k$ is the unit $\mathbb{1}$ of $\mathbf{dgMod}_k$, and the cone object Γ, the interval I and the shift object Σ of section 5.1 are contained in $\mathscr{C}_k$. The category $\mathbf{ffMod}_k$ of finite free k-modules embeds fully faithfully into $\mathscr{C}_k$ via $k^{\oplus m} \mapsto \mathbb{1}^{\oplus m}$ for $m \in \mathbb{N}$. Consider an object

$$M : \qquad \ldots \longrightarrow 0 \longrightarrow M^a \longrightarrow M^{a+1} \longrightarrow \ldots \longrightarrow M^{b-1} \longrightarrow M^b \longrightarrow 0 \longrightarrow \ldots$$

in $\mathscr{C}_k$, concentrated in degrees $[a, b]$. Then M is the cone of the morphism d^{b-1} : $\Sigma^{-1}M^{\leq b-1} \to M^b$, where $\Sigma^{-b}M^b$ is the complex with a single copy of M^b in degree

b. Note that $\Sigma^{-b}M^b = (\Sigma^{-b})^{\oplus m}$ for some $m \in \mathbb{N}$. Continuing iteratively, one sees that M can be constructed from $\mathbf{ffMod}_k$ by taking finitely many shifts and cones. Hence $\mathscr{C}_k$ is the closure of $\mathbf{ffMod}_k$ (and even of $\mathbb{1}$) in $\mathbf{dgMod}_k$ under shifts and cones.

5.3 dg Modules over a dg category

Just as it is often useful to think of a scheme X as a representable functor $\mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$, it can be useful to think of an object of a dg category $\mathcal{A}$ as a representable dg functor $\mathcal{A}^{\mathrm{op}} \to \mathbf{dgMod}_k$.

Definition 5.3.1. Let $\mathcal{A}$ be a dg category. A *dg $\mathcal{A}$-module* is a dg functor $M : \mathcal{A}^{\mathrm{op}} \to \mathbf{dgMod}_k$.

Example 5.3.2. Let $\mathcal{A}$ be a dg category with a single object X and $\mathrm{End}(X) = R$. Then a dg $\mathcal{A}$-module $M : \mathcal{A} \to \mathbf{dgMod}_k$ is the same as a dg R-module. This mirrors the situation for a k-algebra R: an R-module M is the same as a functor $M : \mathcal{A} \to \mathbf{Mod}_k$ where $\mathcal{A}$ is a pre-additive category with a single object X such that $\mathrm{End}(X) = R$.

Definition 5.3.3. Let $\mathcal{A}$ be a dg category. The *dg category of dg $\mathcal{A}$-modules $C_{dg}(\mathcal{A})$* is the dg category $\mathbf{dgFun}(\mathcal{A}^{\mathrm{op}}, \mathbf{dgMod}_k)$.

Lemma 5.3.4 (dg Yoneda embedding)**.** *Let $\mathcal{A}$ be a dg category. Then there is a fully faithful dg functor $h : \mathcal{A} \to C_{dg}(\mathcal{A})$ given by $A \mapsto \mathcal{A}(-, A)$. For $A \in \mathcal{A}$, the image $h(A)$ will be denoted h_A.*

Proof. The proof is analogous to the classical Yoneda lemma. □

Similarly to the classical case, we define a dg module $F : \mathcal{A}^{\mathrm{op}} \to \mathbf{dgMod}_k$ to be *representable* if it is isomorphic to $h_A \in C_{\mathrm{dg}}(\mathcal{A})$ for some $A \in \mathcal{A}$. We can define shifts and cones of dg modules pointwise: the cone ΓF is the dg module $A \mapsto \Gamma \otimes F(A)$, and for $n \in \mathbb{Z}$, the shift $\Sigma^n F$ is the dg module $A \mapsto \Sigma^n \otimes F(A)$. More generally, we can define a dg functor

$$\mathbf{dgMod}_k \otimes C_{\mathrm{dg}}(\mathcal{A}) \longrightarrow C_{\mathrm{dg}}(\mathcal{A})$$

by $M \otimes F \mapsto (A \mapsto M \otimes F(A))$. Of course, we can also take the underlying category of dg modules $C(\mathcal{A}) = Z^0 C_{\mathrm{dg}}(\mathcal{A})$, which turns out to be a *Frobenius category* as in definition 5.4.2(v).

The dg category $C_{\mathrm{dg}}(\mathcal{A})$ is a cofibrantly generated $\mathbf{dgMod}_k$-model category [82, section 3]. If the (co)limit of a diagram of representable objects in $C_{\mathrm{dg}}(\mathcal{A})$ is itself representable, then the (co)limit of the corresponding diagram in $\mathcal{A}$ exists. We will see in section 5.4 that this is true in particular for cones of morphisms in pretriangulated dg categories.

Definition 5.3.5. The *derived category $D(\mathcal{A})$ of a dg category $\mathcal{A}$* is the localization of $H^0 C_{\mathrm{dg}}(\mathcal{A})$ with respect to the quasi-isomorphisms.

In many cases of interest, the derived category $D(\mathcal{A})$ will be too large, and it will be more convenient to construct a related, smaller category, see example 5.5.5(iii).

5.4 Exact and pretriangulated dg categories

As defined in [67, section 2], an *exact category* is an additive category with a distinguished collection of sequences $A \rightarrowtail B \twoheadrightarrow C$, which is to satisfy some properties to mimic the behaviour of abelian categories. By [52, appendix A.1], we can use the following definition of an exact category.

Definition 5.4.1. Let $\mathcal{E}$ be an additive category. A sequence

$$A \xrightarrow{i} B \xrightarrow{d} C$$

is *exact* if i is a kernel of d and d is a cokernel of i. Let ε be a collection of pairs (i, d) that are exact sequences, called *conflations*. The morphism i of a conflation is called an *inflation*, and d is called a *deflation*. Suppose that ε has the following properties:

(i) the identity on the zero object 0 is a deflation;
(ii) the composition of deflations is a deflation;
(iii) for any deflation $d : X \twoheadrightarrow Y$ and any morphism $f : Y' \to Y$, there exists a pullback square

$$\begin{tikzcd} X' \arrow[r, two heads, "d'"] \arrow[d, "f'"] & Y' \arrow[d, "f"] \\ X \arrow[r, two heads, "d"] & Y \end{tikzcd}$$

such that d' is a deflation; and
(iv) for any inflation $i : X \rightarrowtail Y$ and any morphism $f : X \to X'$, there exists a pushout square

$$\begin{tikzcd} X \arrow[r, tail, "i"] \arrow[d, "f"] & Y \arrow[d, "f'"] \\ X' \arrow[r, tail, "i"] & Y' \end{tikzcd}$$

such that i' is an inflation.

Then the pair $(\mathcal{E}, \varepsilon)$ is called an *exact category*. Inflations are usually denoted by feathered arrows $A \rightarrowtail B$, and deflations by two-headed arrows $A \twoheadrightarrow B$. An *exact functor* $F : (\mathcal{E}, \varepsilon) \to (\mathcal{E}', \varepsilon')$ is a functor such that $F(\varepsilon) \subset \varepsilon'$, i.e. F preserves conflations.

A common example is the category of finite locally free sheaves on a scheme, where the conflations are the usual exact sequences of locally free sheaves. This category can be embedded fully faithfully in the abelian category of quasi-coherent sheaves on the scheme.

Like in an abelian category, in an exact category there are the notions of injective and projective objects. In the special case where there are enough injective and projective objects, and these two types objects moreover coincide, we obtain a *Frobenius category* giving rise to a *stable category*, which is triangulated.

Definition 5.4.2. Let $(\mathcal{E}, \varepsilon)$ be an exact category.

(i) An object I of $\mathcal{E}$ is called *injective* if the functor

$$\mathcal{E}(-,I) : \mathcal{E}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$$

is exact, when the abelian category $\mathbf{Ab}$ is endowed with the canonical exact structure in which all short exact sequences are conflations.

(ii) An object P of $\mathcal{E}$ is called *projective* if the functor

$$\mathcal{E}(P,-) : \mathcal{E} \longrightarrow \mathbf{Ab}$$

is exact.

(iii) The exact category $\mathcal{E}$ is said to have *enough injectives* if every object A of $\mathcal{E}$ admits an inflation $A \to I$ into an injective object I.

(iv) The exact category $\mathcal{E}$ is said to have *enough projectives* if every object A of $\mathcal{E}$ admits a deflation $P \to A$ out of a projective object P.

(v) The exact category $\mathcal{E}$ is called a *Frobenius category* if it has enough injectives, enough projectives, and injective and projective objects coincide.

Definition 5.4.3. Let $\mathcal{A}$ be a dg category. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ with $gf = 0$ in the underlying category $Z^0\mathcal{A}$ is called *exact* if there exist $r \in \mathcal{A}(B,A)^0$ and $s \in \mathcal{A}(C,B)^0$ such that $rf = \mathrm{id}_A$ and $gs = \mathrm{id}_C$ and $fr + sg = \mathrm{id}_B$. The dg category $\mathcal{A}$ is called *exact* if the exact sequences of $Z^0\mathcal{A}$ turn $Z^0\mathcal{A}$ into an exact category.

Note that the retractions s and r in the above definition are not morphisms of $Z^0\mathcal{A}$, so not all exact sequences of $Z^0\mathcal{A}$ are necessarily split. In light of the above definition, being exact is a *property* of a dg category, and does not amount to *added structure*, as is the case for additive categories with an exact structure. Note als that any dg functor between exact dg categories preserves exact sequences.

Definition 5.4.4 (dg category of bounded complexes)**.** Let $\mathcal{E}$ be a k-linear exact category. Let $\mathrm{Ch}^b(\mathcal{E})$ be the category of bounded chain complexes in $\mathcal{E}$ with the usual chain maps (respecting the grading and commuting with differentials) as morphisms. For chain complexes M and N, let the mapping complex $[M,N]$ be given for $n \in \mathbb{Z}$ by

$$[M,N]^n = \prod_{i\in\mathbb{Z}} \mathcal{E}(M^i, N^{i+n}),$$

with differential $df = d_N f - (-1)^{|f|} f d_M$ for $f \in [M,N]^n$. This gives $\mathrm{Ch}^b(\mathcal{E})$ the structure of a dg category, the *dg category of bounded complexes in $\mathcal{E}$*.

Definition 5.4.5. For a dg category $\mathcal{A}$, let $\mathscr{C}_k\mathcal{A}$ be the tensor product dg category $\mathscr{C}_k \otimes \mathcal{A}$ of definition 5.2.7. A dg category $\mathcal{A}$ is called *pretriangulated* if it is exact and the functor $\mathcal{A} \to \mathscr{C}_k\mathcal{A}$ given by $A \mapsto \mathbb{1} \otimes A$ is a natural equivalence.

If $\mathcal{A}$ is a pretriangulated dg category, its homotopy category $H^0\mathcal{A}$ is triangulated by [12, proposition 3.2]. In fact, most triangulated categories one encounters in practice are the homotopy category of a pretriangulated dg category. It was precisely this observation that led Bondal and Kapranov to the definition of pretriangulated dg categories [12, definition 3.1] in the first place, and to call a triangulated category $\mathcal{T}$ *enhanced* if there exists a pretriangulated dg category $\mathcal{A}$ such that $\mathcal{T} \simeq H^0\mathcal{A}$.

Note that $\mathscr{C}_k\mathcal{A}$ is a dg category containing all the shifts and cones of objects in $\mathcal{A}$. Its dg modules are isomorphic to dg $\mathcal{A}$-modules, so there is a Yoneda embedding $h : \mathscr{C}_k\mathcal{A} \to C_{\mathrm{dg}}(\mathcal{A})$, and for any $M \in \mathscr{C}_k$ and $A \in \mathcal{A}$,

$$h_{M\otimes A} = M \otimes h_A.$$

Indeed, it is the defining feature of a pretriangulated dg category $\mathcal{A}$ that it is closed under shifts and cones, so that its homotopy category $H^0\mathcal{A}$ is triangulated.

Lemma 5.4.6. *Let $\mathcal{A}$ be a dg category. Then $\mathcal{A}$ is pretriangulated if and only if*

(i) the dg category $\mathcal{A}$ is exact;
(ii) for all objects $A \in \mathcal{A}$ and $n \in \mathbb{Z}$, $\Sigma^n h_A$ is representable; and
(iii) for any morphism $f : A \to B$ in $Z^0\mathcal{A}$, the dg $\mathcal{A}$-module X in the pushout square

$$\begin{tikzcd} h_A \arrow[r, "h_f"] \arrow[d] & h_B \arrow[d] \\ \Gamma h_A \arrow[r] & X \end{tikzcd}$$

in $C_{\mathrm{dg}}(\mathcal{A})$ is representable. This pushout square is called the cone construction *and $X = \mathrm{cone}(f)$ is called the* cone of f.

Proof. First, assume that $\mathcal{A}$ is pretriangulated. Then $\mathcal{A}$ is exact by definition.

To prove (ii), let $f : A \to B$ be a closed morphism of degree 0 in $\mathcal{A}$. For $n \in \mathbb{Z}$ and $A \in \mathcal{A}$, the object $\Sigma^n \otimes A$ in $\mathscr{C}_k\mathcal{A}$ is isomorphic to the image of some $\Sigma^n A \in \mathcal{A}$ under the equivalence $\mathcal{A} \to \mathscr{C}_k\mathcal{A}$, and $\Sigma^n A$ represents the dg $\mathcal{A}$-module $\Sigma^n h_A$.

For the proof of (iii), note that the tensor product $\Gamma \otimes A \in \mathscr{C}_k\mathcal{A}$ is isomorphic to an object $\Gamma A \in \mathcal{A}$. Since $\mathcal{A}$ is exact and $A \hookrightarrow \Gamma A$ is an inflation with cokernel ΣA, the pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ \Gamma A \arrow[r] & \mathrm{cone}(f) \end{tikzcd}$$

exists in $\mathcal{A}$. In the pushout square

$$\begin{tikzcd} h_A \arrow[r, "f^*"] \arrow[d] & h_B \arrow[d] \\ \Gamma h_A \arrow[r] & X \end{tikzcd}$$

in $C_{\mathrm{dg}}(\mathcal{A})$, $\Gamma h_A = h_{\Gamma A}$, so it follows that $X = h_{\mathrm{cone}(f)}$.

For the converse, assume that $\mathcal{A}$ is a dg category satisfying (i)-(iii). As $\mathscr{C}_k$ is the closure of $\mathbf{ffMod}_k$ in $\mathbf{dgMod}_k$ under shifts and cones by example 5.2.11, it follows from our assumptions that the canonical functor $\mathcal{A} \to \mathscr{C}_k\mathcal{A}$ is essentially surjective, and since it is always fully faithful, it is an equivalence and $\mathcal{A}$ is pretriangulated. $\square$

In light of the above lemma, it is useful to think of pretriangulated dg categories as exact dg categories that are "closed under taking shifts and cones". Our main source of pretriangulated dg categories will be categories of bounded complexes in an exact category $\mathcal{E}$.

Lemma 5.4.7. *Let $\mathcal{E}$ be an exact category and $\mathrm{Ch}^b(\mathcal{E})$ the corresponding dg category of bounded complexes in $\mathcal{E}$. Then $\mathrm{Ch}^b(\mathcal{E})$ is pretriangulated. In particular, $H^0\,\mathrm{Ch}^b(\mathcal{E})$ is the triangulated category of chain complexes in $\mathcal{E}$ up to chain homotopy.*

Proof. To ease notation, let $\mathcal{A} = \mathrm{Ch}^b(\mathcal{E})$. By lemma 5.4.6, it suffices to show that $\mathcal{A}$ is exact and closed under shifts and cones. The latter is a fact from homological algebra, so it remains to be shown that $\mathcal{A}$ is exact.

Note that the maps of $Z^0\mathcal{A}$ are chain maps of complexes. Suppose given a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ with $gf = 0$ in $Z^0\mathcal{A}$. Moreover, suppose we have maps

$$A^i \underset{r_i}{\overset{f_i}{\rightleftarrows}} B^i \underset{s_i}{\overset{g_i}{\rightleftarrows}} C^i$$

for each $i \in \mathbb{Z}$ such that $r_i f_i = \mathrm{id}$, $g_i s_i = \mathrm{id}$ and $f_i r_i + s_i g_i = \mathrm{id}$. Then $dr_i f_i = d$ and $r_{i+1} d f_i = r_{i+1} f_{i+1} d = d$. Since f_i is a monomorphism, it follows that $dr_i = r_{i+1}d$, which shows that the collection $(r_i)_{i\in\mathbb{Z}}$ is actually a chain map r. A similar argument shows that $s = (s_i)_{i\in\mathbb{Z}}$ is a chain map. Hence the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is exact in $Z^0\mathcal{A}$ if and only if it is termwise split exact. Since $\mathcal{E}$ is exact and in particular additive, it follows that $\mathcal{A}$ is indeed exact.

The statement about the homotopy category $H^0\,\mathrm{Ch}^b(\mathcal{E})$ follows from the definition of H^0. □

The properties of being exact and pretriangulated transfer to certain functor categories and dg categories as follows.

Lemma 5.4.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be pointed dg categories. The following statements hold.*

(i) If $\mathcal{B}$ is exact then so is $\mathbf{dgFun}(\mathcal{A}, \mathcal{B})$.
(ii) If $\mathcal{B}$ is pretriangulated then so is $\mathbf{dgFun}(\mathcal{A}, \mathcal{B})$.
(iii) If $\mathcal{B}$ is pretriangulated then so is $\mathscr{C}_k\mathcal{B}$.

Proof. This is [72, lemma 1.10]. □

Definition 5.4.9. The *pretriangulated hull* $\mathrm{ptr}(\mathcal{A})$ *of a dg category* $\mathcal{A}$ is a dg functor $\mathcal{A} \to \mathrm{ptr}(\mathcal{A})$ which is universal among dg functors $\mathcal{A} \to \mathcal{B}$ where $\mathcal{B}$ is pretriangulated.

An explicit construction of the pretriangulated hull is given in [72, definition 1.16], and refines an earlier construction from [12]. It admits an action $\otimes : \mathscr{C}_k \otimes \mathrm{ptr}(\mathcal{A}) \to \mathrm{ptr}(\mathcal{A})$ by [72, remark 1.18], which plays an important role in the construction of products on Grothendieck-Witt spectra.

Definition 5.4.10. A dg functor $F : \mathcal{A} \to \mathcal{B}$ is called a *Morita equivalence* if it induces an equivalence $D(\mathcal{A}) \to D(\mathcal{B})$ of derived categories.

Let $\mathcal{A}$ be a dg category and let $C(\mathcal{A}) = Z^0 C_{\mathrm{dg}}(\mathcal{A})$ be the category of dg $\mathcal{A}$-modules, which is Frobenius by [53, lemma 3.3]. Recall that a *Frobenius category* is an exact category which has enough injectives and projectives, and in which injectives and projectives coincide. If $\mathcal{A}$ is pretriangulated, then $Z^0\mathcal{A}$ is a Frobenius subcategory of $C(\mathcal{A})$, and $H^0\mathcal{A}$ is triangulated.

One of the main advantages of dg categories over triangulated categories is that cones in dg categories are functorial, as announced at the end of section 5.1. For us specifically, it provides a framework for higher Hermitian K-theory as developed in [72].

Lemma 5.4.11. *Let $\mathcal{A}$ be a pretriangulated dg category. Then the cone construction that assigns to a map $f : A \mapsto B$ its cone* $\mathrm{cone}(f)$ *defines a dg functor*

$$\mathrm{cone} : \mathbf{Ar}(\mathcal{A}) \longrightarrow \mathcal{A}.$$

Proof. By lemma 5.4.6, each morphism of $Z^0\mathcal{A}$ admits a unique cone. Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\ A' \arrow[r, "g"] & B' \end{tikzcd}$$

be a homogeneous morphism $f \to g$ in $\mathbf{Ar}(\mathcal{A})$. Taking cones, we obtain a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[r, "\beta"] \arrow[d] & B' \arrow[dd] \\ \Gamma A \arrow[r] \arrow[d, "\Gamma\alpha"] & \mathrm{cone}(f) \arrow[dr, dashed] & \\ \Gamma A' \arrow[rr] & & \mathrm{cone}(g) \end{tikzcd}$$

in which a unique dotted arrow exists since the upper left square is a pushout in $\mathcal{A}$. Hence the cone construction $\mathrm{cone} : \mathbf{Ar}(\mathcal{A}) \to \mathcal{A}$ defines a dg functor. □

Let $f : A \to B$ be a morphism in $Z^0\mathcal{A}$, where $\mathcal{A}$ is a pretriangulated dg category. Note that a homogeneous map $\mathrm{cone}(f) \to C$ in $\mathcal{A}$ is completely determined by a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ \Gamma A \arrow[r] & C, \end{tikzcd}$$

and that the image of ΓA under the Yoneda embedding is pointwise contractible. It is therefore useful to think of $\mathrm{cone}(f)$ as a homotopy cokernel of f, since $A \to B \to \mathrm{cone}(f)$ is an exact triangle in the triangulated category $H^0\mathcal{A}$.

5.5 Localization of dg categories

For triangulated categories, there are good notions of localization and quotients, which makes it possible to construct exact sequences of triangulated categories. This behavior turns out to be a shadow of a more general result for dg categories.

In [79], it is shown that the category **dgCat** admits a model structure with the quasi-equivalences as its weak equivalences, and the homotopy category Ho(**dgCat**) with respect to this model structure is studied in more detail in [81], where the mapping space between two dg categories $\mathcal{A}$ and $\mathcal{B}$ is described as the nerve of a certain category of $(\mathcal{A}, \mathcal{B})$-bimodules.

For most applications, including those in this thesis, we study invariants of dg categories that are invariant under quasi-equivalence, so that we may invert the quasi-equivalences. One way to do this is by taking the *Dwyer-Kan localization* $L(\mathbf{dgCat})$ with respect to the quasi-equivalences, which gives a generalization of the homotopy category $\mathrm{Ho}(\mathbf{dgCat})$ with simplicial sets as mapping spaces. In particular, by [81, corollary 8.7], for a set of closed morphisms S in a dg category $\mathcal{A}$, there exists a localization $L_S(\mathcal{A})$ in $L(\mathbf{dgCat})$ with a natural morphism $\mathcal{A} \to L_S(\mathcal{A})$ which is universal among morphisms $\mathcal{A} \to \mathcal{B}$ that invert the morphisms of S, in that they become isomorphisms in $H^0\mathcal{B}$. We will state the precise result in this section. For a detailed treatise, see [81, sections 2, 8.2].

Definition 5.5.1. A dg functor $F : \mathcal{A} \to \mathcal{B}$ is called

(i) *quasi-fully faithful* if for each pair of objects $A, B \in \mathcal{A}$ the morphism of mapping complexes $F : \mathcal{A}(A, B) \to \mathcal{B}(F(A), F(B))$ is a quasi-isomorphism;
(ii) *quasi-essentially surjective* if the induced functor $H^0F : H^0\mathcal{A} \to H^0\mathcal{B}$ is essentially surjective;
(iii) a *quasi-equivalence* if it is both quasi-fully faithful and quasi-essentially surjective; and
(iv) a *fibration* if it satisfies the following conditions:
  (a) for all $A, B \in \mathcal{A}$, the map $\mathcal{A}(A, B) \to \mathcal{B}(F(A), F(B))$ is an epimorphism (i.e. a fibration of dg k-modules); and
  (b) for all $A \in \mathcal{A}$ and for all isomorphisms $g : H^0F(A) \to B'$ in $H^0\mathcal{B}$, there exists an isomorphism $f : A \to B$ in $H^0\mathcal{A}$ such that $H^0F(f) = g$.

The fibrations and quasi-equivalences define a cofibrantly generated model structure on $\mathbf{dgCat}$ [79]. In this model structure, every object $\mathcal{A} \in \mathbf{dgCat}$ is fibrant, and there exists a cofibrant replacement functor $Q : \mathbf{dgCat} \to \mathbf{dgCat}$ such that for $\mathcal{A} \in \mathbf{dgCat}$, the natural map $Q\mathcal{A} \to \mathcal{A}$ is the identity on objects [81, proposition 2.3]. Taking the Dwyer-Kan localization [26, definition 4.1], we obtain the simplicially enriched category $L(\mathbf{dgCat})$, whose underlying category is the homotopy category of $\mathbf{dgCat}$:

$$\pi_0 L(\mathbf{dgCat}) = \mathrm{Ho}(\mathbf{dgCat}).$$

For objects $\mathcal{A}, \mathcal{B} \in L(\mathbf{dgCat})$, the mapping space $L(\mathbf{dgCat})(\mathcal{A}, \mathcal{B})$ can be represented by an explicit simplicial set $\mathrm{Map}(\mathcal{A}, \mathcal{B})$ [37, chapter 17]. By [81, theorem 6.1], $\mathrm{Ho}(\mathbf{dgCat})$ admits a closed symmetric monoidal structure such that for any two objects $\mathcal{A}, \mathcal{B} \in \mathrm{Ho}(\mathbf{dgCat})$, there is an object $\mathrm{RHom}(\mathcal{A}, \mathcal{B}) \in \mathrm{Ho}(\mathbf{dgCat})$ that represents the functor $\mathcal{C} \mapsto \mathrm{Map}(\mathcal{C} \otimes^{\mathbf{L}} \mathcal{A}, \mathcal{B})$, where $\otimes^{\mathbf{L}}$ is the total left derived functor of the tensor product on $\mathbf{dgCat}$ of definition 5.2.7.

Let S be a set of morphisms in $H^0\mathcal{A}$ for some dg category $\mathcal{A}$. For a dg category $\mathcal{B}$, let $\mathrm{Map}_S(\mathcal{A}, \mathcal{B}) \subset \mathrm{Map}(\mathcal{A}, \mathcal{B})$ be the full simplicial subset whose vertices are the morphisms $F : \mathcal{A} \to \mathcal{B}$ in $L(\mathbf{dgCat})$ such that $H^0F : H^0\mathcal{A} \to H^0\mathcal{B}$ sends morphisms in S to isomorphisms. Then we have the following theorem [81, corollary 8.7, corollary 8.8].

Theorem 5.5.2. *The $L(\mathbf{sSet})$-enriched functor*

$$\mathrm{Map}_S(\mathcal{A}, -) : L(\mathbf{dgCat}) \to L(\mathbf{sSet})$$

is corepresented by an object $L_S(\mathcal{A})$ of $L(\mathbf{dgCat})$, unique up to unique isomorphism in $L(\mathbf{dgCat})$. Moreover, for $\mathcal{B} \in \mathbf{dgCat}$, the natural morphism $\mathcal{A} \to L_S(\mathcal{A})$ in $L(\mathbf{dgCat})$ induces a quasi-fully faithful functor

$$\mathrm{RHom}(L_S(\mathcal{A}), \mathcal{B}) \to \mathrm{RHom}(\mathcal{A}, \mathcal{B})$$

whose quasi-essential image is given by the morphisms $\mathcal{A} \to \mathcal{B}$ in $L(\mathbf{dgCat})$ sending morphisms in S to isomorphisms in $H^0\mathcal{B}$.

Definition 5.5.3. Let $\mathcal{A} \in L(\mathbf{dgCat})$ and let $S \subset \mathcal{A}$ be a set of morphisms. The *localization $L_S(\mathcal{A})$ of $\mathcal{A}$ with respect to S* is an object $L_S(\mathcal{A})$ of $L(\mathbf{dgCat})$ corepresenting the $L(\mathbf{sSet})$-enriched functor $\mathrm{Map}_S(\mathcal{A}, -)$, which exists by theorem 5.5.2.

If $\mathcal{A}$ is pretriangulated, then so is its localization $L_S(\mathcal{A})$. As an application, we obtain a natural theory of dg quotients. This will be useful for defining exact sequences of dg categories and semi-orthogonal decompositions.

Definition 5.5.4. Let $\mathcal{B}$ be a dg subcategory of a dg category $\mathcal{A}$. Let S be the set of morphisms $B \in 0$ in $\mathcal{A}$ with $B \in \mathcal{B}$. Then the *quotient dg category $\mathcal{A}/\mathcal{B}$* is the localization $L_S(\mathcal{A})$.

Example 5.5.5. Here are some relevant examples of quotients, c.f. [53, section 4.4].

(i) Let $A \in \mathcal{A}$. The quotient $\mathcal{A}/\langle A \rangle$ is the localization of $\mathcal{A}$ with respect to the morphism $A \to 0$ in $H^0\mathcal{A}$. More generally, if $\{A_i \mid i \in I\}$ is a set of objects in $\mathcal{A}$, then $\mathcal{A}/\langle A_i \mid i \in I \rangle$ is the localization of $\mathcal{A}$ with respect to the set of morphisms $\{A_i \to 0 \mid i \in I\}$ in $H^0\mathcal{A}$.

(ii) Let $\mathcal{A} = \mathrm{Ch}^b(\mathcal{E})$ be the dg category of bounded chain complexes in an exact category $\mathcal{E}$. Let S be the set of chain homotopy equivalences. Then S consists precisely of the isomorphisms in $H^0\mathcal{A}$, and therefore $L_S(\mathcal{A}) = \mathcal{A}$.

(iii) Again, let $\mathcal{A} = \mathrm{Ch}^b(\mathcal{E})$. Let S be the set of quasi-isomorphisms in $H^0\mathcal{A}$. Then there is an equivalence of triangulated categories

$$H^0 L_S(\mathcal{A}) \simeq D^b(\mathcal{E}),$$

where $D^b(\mathcal{E})$ is the bounded derived category of $\mathcal{E}$. Alternatively, let T be the set of morphisms $A \to 0$ where A is *acyclic*, i.e. has vanishing cohomology. Note that

$$L_T(\mathcal{A}) = \mathrm{Ch}^b(\mathcal{E}) / \mathrm{Ac}^b(\mathcal{E}),$$

where $\mathrm{Ac}^b(\mathcal{E}) \subset \mathrm{Ch}^b(\mathcal{E})$ is the dg subcategory of bounded acyclic complexes. The localization $L_T(\mathcal{A})$ is quasi-equivalent to the localization $L_S(\mathcal{A})$. We can show this using the universal property of the localization. There is a natural morphism $L_T(\mathcal{A}) \to L_S(\mathcal{A})$ since $T \subset S$. Now suppose that $f : A \to B$ is a quasi-isomorphism in $\mathcal{A}$. Then its cone $\mathrm{cone}(f)$ is acyclic. Hence the induced functor $H^0\mathcal{A} \to H^0 L_T(\mathcal{A})$ sends the exact triangle $A \to B \to \mathrm{cone}(f) \to A[1]$

in $H^0\mathcal{A}$ to the exact triangle $A \to B \to 0 \to A[1]$ in $H^0 L_T(\mathcal{A})$, which shows that f is mapped to an isomorphism in $H^0 L_T(\mathcal{A})$. The universal property of $L_S(\mathcal{A})$, yields a natural morphism $L_S(\mathcal{A}) \to L_T(\mathcal{A})$ that is an inverse to $L_T(\mathcal{A}) \to L_S(\mathcal{A})$ by uniqueness. In light of this example, it makes sense to define the *bounded derived dg category* $D^b_{\mathrm{dg}}(\mathcal{E}) = L_T(\mathcal{A})$. In case $\mathcal{E} = \mathrm{Vect}(X)$ for some scheme X with an ample family of line bundles, we understand $D^b_{\mathrm{dg}}(X)$ to mean $D^b_{\mathrm{dg}}(\mathcal{E})$. Note that $H^0 D^b_{\mathrm{dg}}(\mathcal{E})$ is the usual bounded derived category $D^b(\mathcal{E})$.

Remark 5.5.6. Here is a word of warning about the "category of perfect complexes on a scheme". Let X be a scheme. The *dualizable objects* in the usual derived category $D(X)$ are precisely the perfect complexes, so the category of dualizable objects in $D(X)$ is often denoted $\mathrm{Perf}(X)$, in which case it is a triangulated category. Another convention is that $\mathrm{Perf}(X)$ denotes a certain sub-∞-category of dualizable objects. We follow yet another convention, in which $\mathrm{Perf}(X)$ is the localization with respect to the quasi-isomorphisms of the full pretriangulated dg subcategory of perfect complexes of the pretriangulated dg category $D_{\mathrm{dg}}(X)$ of complexes of $\mathcal{O}_X$-modules. If X satisfies the resolution property, each perfect complex of $\mathcal{O}_X$-modules is quasi-isomorphic to a strictly perfect complex of $\mathcal{O}_X$-modules and therefore to a bounded complex of finite locally free sheaves on X. In this case

$$\mathrm{Perf}(X) = \mathrm{sPerf}(X) = D^b_{\mathrm{dg}}(\mathrm{Vect}(X)) = \mathrm{Ch}^b(\mathrm{Vect}(X))/\,\mathrm{Ac}^b(\mathrm{Vect}(X)),$$

where the equalities should be understood as equivalences we pretend to be identities, and this is the only case we will encounter in this thesis.

5.6 Semi-orthogonal decompositions of dg categories

We have now developed the technical machinery we need in order to define semi-orthogonal decompositions of dg categories.

Semi-orthogonal decompositions, or rather their precursor in the form of exceptional collections, were among the first applications of pretriangulated dg categories to be considered in [12]. Given a collection of objects in a triangulated category, it is not obvious how to determine the triangulated category generated by this collection, but the triangulated category generated by a collection of objects in a pretriangulated dg category admits a natural description [12, theorem 4.1]. An *exceptional collection* in a pretriangulated dg category is a collection of objects satisfying certain conditions such that they split the dg category into smaller subcategories. More generally, a *semi-orthogonal decomposition* of a pretriangulated dg category splits it into smaller subcategories, without requiring the existence of special objects. They play an important role in modern computations of algebraic K-theory and Grothendieck-Witt theory, see e.g. [54] and [51].

The material in this section may be compared to that of [58, section 2] for the classical case of triangulated categories, and to that of [62, section II.7.2.1] for the case of stable ∞-categories, the theory of dg categories being "somewhere in the middle". Every dg category yields an ∞-category by applying the *differential graded nerve* [60, construction 1.3.1.6]. Furthermore, it is shown in [20] that there is an equivalence between the underlying ∞-category of the Morita model category structure on the category of k-linear dg categories on the one hand, and the ∞-category of idempotent-complete k-linear small stable ∞-categories on the other.

In linear algebra, we may think of orthogonality as a geometric condition, two vectors being orthogonal if their inner product vanishes. Generalizing, we define the orthogonal complement $W^\perp$ of a subspace W of a vector space V equipped with an inner product $\xi : V \times V \to V$ as those vectors $v \in V$ such that $\xi(w, v) = 0$ for all $w \in W$. Crucially, $V = W \oplus W^\perp$, which in effect reduces the study of V to the study of W and $W^\perp$. If we imagine that ξ is another bilinear form which is not symmetric, we end up instead with a "left" and a "right" orthogonal of W via the formulas $\xi(w, v) = 0$ and $\xi(v, w) = 0$, respectively. Abstracting the situation further, we may take instead of a vector space V an additive category $\mathcal{C}$ with a full additive subcategory $\mathcal{B} \subset \mathcal{C}$, and consider the morphism functor $\mathrm{Hom}_\mathcal{C} : \mathcal{C} \times \mathcal{C} \to \mathbf{Ab}$. We define the *left orthogonal* ${}^\perp\mathcal{B}$ *(resp. the right orthogonal* $\mathcal{B}^\perp$*) of* $\mathcal{B}$ to be the full subcategory of $\mathcal{A}$ consisting of the objects $A \in \mathcal{A}$ such that $\mathcal{C}(A, B) = 0$ (resp. $\mathcal{C}(B, A) = 0$) for all $B \in \mathcal{B}$. We can then wonder to what extent $\mathcal{A}$ decomposes as a "direct sum" $\mathcal{B} \oplus \mathcal{B}^\perp$, and semi-orthogonal decompositions are a formal tool for studying this question in the context of dg categories. For dg categories, we also take into account the model structure on $\mathbf{dgMod}_k$, which takes over the role of $\mathbf{Ab}$.

Definition 5.6.1. Let $\mathcal{B} \subset \mathcal{A}$ be a (not necessarily full) dg subcategory of a dg category. The *left orthogonal* ${}^\perp\mathcal{B}$ *(resp. the right orthogonal* $\mathcal{B}^\perp$*) of* $\mathcal{B}$ is the full dg subcategory consisting of the objects $A \in \mathcal{A}$ such that the chain complex $\mathcal{A}(A, B)$ (resp. $\mathcal{A}(B, A)$) is acyclic for all $B \in \mathcal{B}$.

Note that $\mathcal{B} \subset ({}^\perp\mathcal{B})^\perp$ and $\mathcal{B} \subset {}^\perp(\mathcal{B}^\perp)$ by definition.

Lemma 5.6.2. *If $\mathcal{B} \subset \mathcal{A}$ is an inclusion of pretriangulated dg categories, then the left orthogonal ${}^\perp\mathcal{B}$ and the right orthogonal $\mathcal{B}^\perp$ are pretriangulated.*

Proof. Let $\mathcal{B}^\perp$ be the right orthogonal of $\mathcal{B}$ in $\mathcal{A}$. By lemma 5.4.6, we need to show that $\mathcal{B}^\perp$ is exact and contains all shifts and cones.

Let $A_1 \to A_2 \to A_3$ be an exact sequence in $Z^0\mathcal{A}$ with two out of three terms in $\mathcal{B}^\perp$, and call the remaining term A. Note that this also determines an exact triangle in the triangulated category $H^0\mathcal{A}$. For $B \in \mathcal{B}$, applying the hom-functor $H^0\mathcal{A}(B, -)$ to the exact triangle yields a long exact cohomology sequence

$$\ldots \longrightarrow H^{i+1}\mathcal{A}(B, A_3) \longrightarrow H^i\mathcal{A}(B, A_1) \longrightarrow H^i\mathcal{A}(B, A_2) \longrightarrow H^i\mathcal{A}(B, A_3) \longrightarrow \ldots$$

By assumption, two out of each three consecutive terms in this sequence are zero, so it follows that $H^i\mathcal{A}(B, A) = 0$ for each $i \in \mathbb{Z}$. Therefore, $A \in \mathcal{B}^\perp$, yielding that $\mathcal{B}^\perp$ is exact.

Let $A \in \mathcal{B}^\perp$ and $n \in \mathbb{Z}$. As $\mathcal{A}$ is pretriangulated,

$$H^i\mathcal{A}(B, A[n]) = H^i(\mathcal{A}(B, A)[n]) = H^{i+n}\mathcal{A}(B, A) = 0$$

for all $B \in \mathcal{B}$, which shows that $A[n] \in \mathcal{B}^\perp$. Thus, $\mathcal{B}^\perp$ is closed under taking shifts.

Let $f : A \to A'$ be a closed morphism of degree 0 in $\mathcal{B}^\perp$ and $B \in \mathcal{B}$. Let $b : B \to \mathrm{cone}(f)$ be another closed morphism of degree 0. Then there is a diagram in $H^0\mathcal{A}$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & B \arrow[r, equal] \arrow[d, dashed] & B \arrow[r] \arrow[d, "b"] & 0 \arrow[d] \\
A \arrow[r, "f"] & A' \arrow[r, "g"] & \mathrm{cone}(f) \arrow[r, "h"] & A[1],
\end{tikzcd}$$

where the rows are exact triangles. Note that $hb = 0$ by assumption, so the diagram can be completed to a morphism of triangles because $H^0\mathcal{A}$ is triangulated. As there are no nonzero morphisms $B \to A'$ in $H^0\mathcal{A}$, it follows that $b = 0$ and $H^0\mathcal{A}(B, \mathrm{cone}(f)) = 0$. Shifting B shows that $\mathcal{A}(B, \mathrm{cone}(f))$ is acyclic, so $\mathcal{B}^\perp$ has cones. This finishes the proof that $\mathcal{B}^\perp$ is pretriangulated.

The proof that ${}^\perp\mathcal{B}$ is pretriangulated is similar. □

If $\mathcal{A}$ is a pretriangulated dg category and $S \subset \mathcal{A}$ is a collection of objects of $\mathcal{A}$, then $\langle S \rangle$ denotes the full pretriangulated dg subcategory of $\mathcal{A}$ *generated* by S, obtained as follows: consider $S \subset H^0\mathcal{A}$, take the smallest triangulated subcategory $\mathcal{S} \subset H^0\mathcal{A}$ containing S, and define $\langle S \rangle$ to be the full pretriangulated dg subcategory of $\mathcal{A}$ whose objects are those of $\mathcal{S}$. For a dg subcategory $\mathcal{B} \subset \mathcal{A}$, the pretriangulated dg subcategory $\langle \mathcal{B} \rangle \subset \mathcal{A}$ is defined by regarding $\mathcal{B}$ as its collection of objects.

Definition 5.6.3. Let $\mathcal{A}$ be a pretriangulated dg category. A pair of full pretriangulated dg subcategories $\mathcal{A}_-, \mathcal{A}_+$ forms a *semi-orthogonal decomposition* $\langle \mathcal{A}_-, \mathcal{A}_+ \rangle$ *of* $\mathcal{A}$ if

(i) for all objects $A_- \in \mathcal{A}_-$ and $A_+ \in \mathcal{A}_+$, the mapping complex $\mathcal{A}(A_+, A_-)$ is acyclic ("there are no morphisms from right to left"); and
(ii) for every object $A \in \mathcal{A}$, there exists a closed morphism $f : A_+ \to A$ with $A_+ \in \mathcal{A}_+$ whose cone $A_- = \mathrm{cone}(f)$ is in $\mathcal{A}_-$.

The following lemma shows that a semi-orthogonal decomposition of a pretriangulated dg category is the same as a semi-orthogonal decomposition of the underlying triangulated category.

Lemma 5.6.4. *Let $\mathcal{A}$ be a pretriangulated dg category with full pretriangulated dg subcategories $\mathcal{A}_-$ and $\mathcal{A}_+$. The following are equivalent.*

(i) The pair $\langle \mathcal{A}_-, \mathcal{A}_+ \rangle$ is a semi-orthogonal decomposition of $\mathcal{A}$.
(ii) The inclusion $H^0\mathcal{A}_- \subset H^0\mathcal{A}$ admits a left adjoint and ${}^\perp\mathcal{A}_- = \mathcal{A}_+$.
(iii) The inclusion $H^0\mathcal{A}_+ \subset H^0\mathcal{A}$ admits a right adjoint and $\mathcal{A}_+{}^\perp = \mathcal{A}_-$.

Proof. We will prove only that (i) is equivalent to (ii), the proof that (i) is equivalent to (iii) being similar.

First note that $\mathcal{A}(A_+, A_-)$ is acyclic if and only if $H^0\mathcal{A}(A_+, A_-[n]) = 0$ for all $n \in \mathbb{Z}$, so $\mathcal{A}(A_+, A_-)$ being acyclic for all $A_+ \in \mathcal{A}_+$ and $A_- \in \mathcal{A}_-$ is equivalent to $H^0\mathcal{A}(A_+, A_-)$ being zero for all $A_+ \in \mathcal{A}_+$ and $A_- \in \mathcal{A}_-$, since $\mathcal{A}_-$ is pretriangulated.

Let $f : A \to B$ be a morphism in $H^0\mathcal{A}$. Since $\langle \mathcal{A}_-, \mathcal{A}_+ \rangle$ is a semi-orthogonal decomposition of $\mathcal{A}$, we obtain two exact triangles

$$A_+ \xrightarrow{a_+} A \xrightarrow{a_-} A_-$$

$$B_+ \xrightarrow{b_+} B \xrightarrow{b_-} B_-$$

in $H^0\mathcal{A}$. Then there is a morphism of exact triangles

$$\begin{tikzcd} A_+ \arrow[r, "a_+"] \arrow[d] & A \arrow[r, "a_-"] \arrow[d, "b_- f"] & A_- \arrow[d, dashed, "f_-"] \\ 0 \arrow[r] & B_- \arrow[r, equal] & B_- \end{tikzcd}$$

in which the dotted arrow $f_- : A_- \to B_-$ exists and is unique because of [77, Tag 0FWZ]. Thus we can define a functor $F : H^0\mathcal{A} \to H^0\mathcal{A}_-$ by choosing an $A_- \in \mathcal{A}_-$ for each $A \in \mathcal{A}$, which is left adjoint to the inclusion $H^0\mathcal{A}_- \to H^0\mathcal{A}$ by construction.

Conversely, suppose that $F : H^0\mathcal{A} \to H^0\mathcal{A}_-$ is left adjoint to the inclusion $H^0\mathcal{A}_- \to H^0\mathcal{A}$ and that ${}^\perp\mathcal{A}_- = \mathcal{A}_+$. Let $A \in \mathcal{A}$. Then there is a canonical morphism $A \to F(A)$ in $H^0\mathcal{A}$, which we extend to an exact triangle $A \to F(A) \to B$. If $A \in \mathcal{A}_-$, then the canonical morphism $A \to F(A)$ is an isomorphism, as the inclusion $H^0\mathcal{A}_- \to H^0\mathcal{A}$ is fully faithful. Hence, the image of the exact triangle $A \to F(A) \to B$ under F is isomorphic to the exact triangle $F(A) = F(A) \to 0$, so $F(B) = 0$. This yields $H^0\mathcal{A}(B, A_-) = H^0\mathcal{A}_-(F(B), A_-) = 0$ by adjunction for any $A_- \in \mathcal{A}_-$, so it follows that $B \in \mathcal{A}_+$. Rotation of triangles yields an exact triangle $B[1] \to A \to F(A)$, which is induced by a morphism $B[1] \to A$ with $B[1] \in \mathcal{A}_+$, whose cone is in $\mathcal{A}_-$. □

Lemma 5.6.4 gives a feasible way of constructing a semi-orthogonal decomposition from an inclusion $\mathcal{B} \subset \mathcal{A}$ of pretriangulated dg categories – we can try to construct a right or left adjoint on the level of homotopy categories, and if we succeed we define the other component of the decomposition as the appropriate orthogonal, much as we would define the orthogonal of a subspace of a vector space.

It is possible to define semi-orthogonal decompositions using the notion of a *split exact sequence of dg categories*, which is defined as a sequence

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

in **dgCat** that is both a fiber and a cofiber sequence in a localization of the model structure on **dgCat**, see [71, section 2.1.2] and [18, definition 3.1], but we will not need to work in this generality.

Recall that for a triangulated category $\mathcal{T}$ and a thick subcategory $\mathcal{T}'$, the *Verdier quotient* $\mathcal{T}/\mathcal{T}'$ is the unique (up to unique equivalence) triangulated category such that any exact functor $F : \mathcal{T}' \to \mathcal{S}$ with $\mathcal{T} \to \mathcal{T}' \to \mathcal{S}$ trivial factors uniquely through the canonical functor $\mathcal{T}' \to \mathcal{T}'/\mathcal{T}$. Recall furthermore that a sequence $\mathcal{T}_1 \to \mathcal{T}_2 \to \mathcal{T}_3$ of triangulated categories is called *exact* if the composition is trivial, $\mathcal{T}_1 \to \mathcal{T}_2$ turns $\mathcal{T}_1$ into a thick subcategory of $\mathcal{T}_2$, and the induced functor $\mathcal{T}_2/\mathcal{T}_1 \to \mathcal{T}_3$ is an equivalence of triangulated categories.

Definition 5.6.5. A sequence

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

of pretriangulated dg categories is called *quasi-exact* if the associated sequence

$$H^0\mathcal{A} \to H^0\mathcal{B} \to H^0\mathcal{C}$$

of triangulated categories is exact. It is called *split exact* if $H^0\mathcal{A} \to H^0\mathcal{B}$ admits a right adjoint.

Semi-orthogonal decompositions correspond to split exact sequences by lemma 5.6.4 and the following result.

Lemma 5.6.6. *Let $\langle \mathcal{A}_-, \mathcal{A}_+ \rangle$ be a semi-orthogonal decomposition of $\mathcal{A}$. Then $H^0\mathcal{A}_-$ is equivalent as a triangulated category to the Verdier quotient $H^0\mathcal{A}/H^0\mathcal{A}_+$.*

Proof. We must show that $H^0\mathcal{A}_-$ satisfies the universal property of the Verdier quotient. Let $F : H^0\mathcal{A} \to H^0\mathcal{A}_-$ be the left adjoint to the inclusion $H^0\mathcal{A}_- \subset H^0\mathcal{A}$, and for $A \in H^0\mathcal{A}$ let $A_- = F(A)$. Let $G : H^0\mathcal{A} \to H^0\mathcal{B}$ be an exact functor such that each $A_+ \in H^0\mathcal{A}_+$ is mapped to the zero object in $H^0\mathcal{B}$. Then a morphism $f : A \to A'$ in $H^0\mathcal{A}$ can be completed to a morphism of exact triangles

$$\begin{tikzcd} A_+ \arrow[r] \arrow[d, dashed] & A \arrow[r] \arrow[d, "f"] & A_- \arrow[d, "F(f)"] \\ A'_+ \arrow[r] & A' \arrow[r] & A'_-. \end{tikzcd}$$

As $G(A_+) = 0$, the image of this morphism of triangles under G is

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & G(A) \arrow[r] \arrow[d, "G(f)"] & G(A_-) \arrow[d, "G(F(f))"] \\ 0 \arrow[r] & G(A') \arrow[r] & G(A'_-), \end{tikzcd}$$

and since G is exact, this is a morphism of exact triangles. It follows that $G(A) \to G(A_-)$ is an isomorphism for all $A \in H^0\mathcal{A}$. Hence G factors through F, as was to be shown. $\square$

Now we expand the definition of a semi-orthogonal decomposition by allowing more components, which will make the concept more flexible in practice.

Definition 5.6.7. Let $\mathcal{A}$ be a pretriangulated dg category with full pretriangulated dg subcategories $\mathcal{A}_1, \ldots, \mathcal{A}_n$ for some $n \in \mathbb{N}_{\geq 2}$. For $0 \leq i \leq n$, Define $\mathcal{A}_{\leq i} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_i \rangle$ to be the smallest full pretriangulated dg subcategory containing each $\mathcal{A}_j$ for $j \leq i$, and define $\mathcal{A}_{\geq i}$ similarly. Note that $\mathcal{A}_{\leq 0} = 0$. Then $\mathcal{A}_1, \ldots, \mathcal{A}_n$ form a *semi-orthogonal decomposition* $\langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$ *of* $\mathcal{A}$ if $\langle \mathcal{A}_{\leq i}, \mathcal{A}_{\geq i+1} \rangle$ is a semi-orthogonal decomposition (definition 5.6.3) of $\mathcal{A}$ for each $1 \leq i \leq n-1$.

Remark 5.6.8. Unpacking definition 5.6.7, we see that subcategories $\mathcal{A}_1, \ldots, \mathcal{A}_n$ of $\mathcal{A}$ form a semi-orthogonal decomposition if and only if for all $i > j$, $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$, the mapping complex $\mathcal{A}(A_i, A_j)$ is acyclic, and each $A \in \mathcal{A}$ admits a filtration

$$0 = A_{\geq n+1} \longrightarrow A_{\geq n} \longrightarrow \ldots \longrightarrow A_{\geq 2} \longrightarrow A_{\geq 1} = A$$

with $A_{\geq i} \in \mathcal{A}_{\geq i}$ for each $1 \leq i \leq n+1$, where the cone of $A_{\geq i+1} \to A_{\geq i}$ lies in $\mathcal{A}_i$.

The following prototypical example of a semi-orthogonal decomposition was given by Bernstein, Gelfand and Gelfand in [9].

Example 5.6.9. Let S be a scheme and let $X = \mathbb{P}^n_S$ be the projective space over X. Let $\mathcal{A} = \mathrm{Perf}(X)$ be the pretriangulated dg category of perfect complexes of $\mathcal{O}_X$-modules, which is equivalent to the quotient dg category

$$\mathrm{Ch}^b(\mathrm{Vect}(X))/\,\mathrm{Ac}^b(\mathrm{Vect}(X)),$$

where $\mathrm{Ac}^b(\mathrm{Vect}(X)) \subset \mathrm{Ch}^b(\mathrm{Vect}(X))$ denotes the dg subcategory of acyclic objects. For $i \in \mathbb{Z}$, define $\mathcal{A}_i = \langle \mathcal{O}_X(i) \rangle$ to be the full pretriangulated dg subcategory of $\mathcal{A}$ generated by the line bundle $\mathcal{O}_X(i)$. Then $\mathcal{A}(\mathcal{O}_X(i), \mathcal{O}_X(j))$ is acyclic for all $j+n \geq i > j$, which follows from the fact that the sheaf cohomology $H^*(X, \mathcal{O}_X(-r))$ vanishes for $r = 1, \dots n$. Furthermore, by a dg version of [73, lemma 3.5.2], $\mathcal{A}$ is generated by the line bundles $\mathcal{O}_X(i)$ with $i \leq 0$. The Koszul complex of X, which we will see in detail in section 7.2, shows that $\mathcal{O}_X(-n-1)$ is generated by $\mathcal{O}_X(i)$ with $-n \leq i \leq 0$. Hence $\langle \mathcal{A}_{-n}, \dots, \mathcal{A}_0 \rangle$ is a semi-orthogonal decomposition of $\mathcal{A}$. Note that we can always twist a semi-orthogonal decomposition by a line bundle to obtain another semi-orthogonal decomposition. In particular, $\langle \mathcal{A}_i, \dots, \mathcal{A}_{i+n} \rangle$ is a semi-orthogonal decomposition of $\mathcal{A}$ for each $i \in \mathbb{Z}$. This trick will be useful once we start dealing with duality.

Here is another extension of the definition of a semi-orthogonal decomposition following [71, definition 3.1] by allowing infinitely many components, which has applications in equivariant K- and GW-theory.

Definition 5.6.10. Let $\mathcal{A}$ be a pretriangulated dg category. For any pre-ordered set $(P, \leq)$ and a collection of full pretriangulated subcategories $\{\mathcal{A}_i \subset \mathcal{A} \mid i \in P\}$, the subcategories $\mathcal{A}_i$ form a *pre-ordered semi-orthogonal decomposition of* $\mathcal{A}$ if

(i) for all $i \in P$, the inclusion $H^0\mathcal{A}_i \subset H^0\mathcal{A}$ admits both a left and a right adjoint;
(ii) for all $i, j \in P$ such that $i < j$, $\mathcal{A}_i \subset {\mathcal{A}_j}^{\perp}$; and
(iii) $\mathcal{A}$ is the smallest pretriangulated dg subcategory containing $\mathcal{A}_i$ for each $i \in P$.

Example 5.6.11. Let $G = \mathbb{G}_m$ be the multiplicative group over a base field k. The pretriangulated dg category $\mathcal{A} = \mathrm{Perf}^G(k)$ of G-equivariant perfect complexes over k is equivalent to the bounded derived dg category of finite dimensional graded k-vector spaces. For $i \in \mathbb{Z}$, let $k(i)$ be the graded vector space with $k(i)_j = \delta_{ij}k$, where δ_{ij} is the Kronecker delta, and let $\mathcal{A}_i = \langle k(i) \rangle$. Then $\mathcal{A}_i$ is equivalent to $\mathrm{Perf}(k)$. Note that $\mathcal{A}(k(i), k(j)) = 0$ for all $i, j \in \mathbb{Z}$ such that $i \neq j$. The smallest pretriangulated dg category containing each $\mathcal{A}_i$ is $\mathcal{A}$ itself, and since we only consider finite dimensional k-vector spaces, each $V \in \mathcal{A}$ is contained in $\mathcal{A}_{[i,j]}$ for some $i, j \in \mathbb{Z}$. Hence $\langle \mathcal{A}_i \mid i \in \mathbb{Z} \rangle$ is an example of a semi-orthogonal decomposition as in definition 5.6.10. See proposition 8.1.1 for a generalization of this semi-orthogonal decomposition which is crucial in the proof of the Atiyah-Segal completion theorem for split tori.

A useful feature of semi-orthogonal decompositions is that they are stable under equivalences of pretriangulated dg categories.

Proposition 5.6.12. *Let $F : \mathcal{A} \to \mathcal{B}$ be an equivalence of pretriangulated dg categories and let $\langle \mathcal{A}_-, \mathcal{A}_+ \rangle$ be a semi-orthogonal decomposition of $\mathcal{A}$. Then the image $\langle F(\mathcal{A}_-), F(\mathcal{A}_+) \rangle$ is a semi-orthogonal decomposition of $\mathcal{B}$.*

Proof. Since F preserves shifts and cones, $F(\mathcal{A}_-)$ and $F(\mathcal{A}_+)$ are full pretriangulated dg subcategories of $\mathcal{B}$.

For $A_- \in \mathcal{A}_-$ and $A_+ \in \mathcal{A}_+$, as F is fully faithful, $\mathcal{B}(F(A_+), F(A_-)) = \mathcal{A}(A_+, A_-)$, which is acyclic by assumption. Let $B \in \mathcal{B}$ and let $A \in \mathcal{A}$ such that $F(A) \cong B$, which exists since F is essentially surjective. Then A admits a sequence

$$A_+ \xrightarrow{f} A \longrightarrow \operatorname{cone}(f)$$

with $A_+ \in \mathcal{A}_+$ and $\operatorname{cone}(f) \in \mathcal{A}_-$ by assumption. Taking the image of this sequence under F yields a sequence

$$F(A_+) \xrightarrow{F(f)} F(A) \longrightarrow F(\operatorname{cone}(f)),$$

where $F(\operatorname{cone}(f)) \cong \operatorname{cone}(F(f))$. As $F(A) \cong B$, there is a sequence

$$F(A_+) \xrightarrow{g} B \longrightarrow \operatorname{cone}(g)$$

with $\operatorname{cone}(g) \in F(\mathcal{A}_+)$. Therefore, $\langle F(\mathcal{A}_-), F(\mathcal{A}_+)\rangle$ is a semi-orthogonal decomposition of $\mathcal{B}$, as was to be shown. □

Corollary 5.6.13. *Let $\mathcal{A}$ be a pretriangulated dg category with a semi-orthogonal decomposition $\langle \mathcal{A}_i \mid i \in \mathbb{Z}\rangle$. Let $F : \mathcal{A} \to \mathcal{A}$ be an equivalence of dg categories. Then $\langle F(\mathcal{A}_i) \mid i \in \mathbb{Z}\rangle$ is a semi-orthogonal decomposition of $\mathcal{A}$.*

Proof. By proposition 5.6.12, $\langle F(\mathcal{A}_{\leq i}), F(\mathcal{A}_{>i})\rangle$ is a semi-orthogonal decomposition of $\mathcal{A}$ for each $i \in \mathbb{Z}$. By assumption, each $A \in \mathcal{A}$ is contained in $\mathcal{A}_{[i,j]}$ for some $i, j \in \mathbb{Z}$, so each $F(A)$ is contained in $F(\mathcal{A}_{[i,j]}) = (F\mathcal{A})_{[i,j]}$ for some $i, j \in \mathbb{Z}$. Since F is an equivalence, each $A \in \mathcal{A}$ is isomorphic to $F(B)$ for some $B \in \mathcal{A}$, and the result follows. □

The situation where this corollary is the most useful for us is the following. Let X be a scheme with the resolution property of definition 2.3.4 and let $\mathcal{A} = \operatorname{Perf}(X)$. Let $\mathcal{L}$ be a line bundle on X. Then tensoring by $\mathcal{L}$ defines an autoequivalence of $\mathcal{A}$, and therefore sends semi-orthogonal decompositions to semi-orthogonal decompositions.

Remark 5.6.14. Semi-orthogonal decompositions are closely related to the notion of t-structures. A *t-structure* on a pretriangulated dg category $\mathcal{A}$ is given by a pair of full dg subcategories $\mathcal{A}_{\leq 0}, \mathcal{A}_{\geq 0}$ such that

(i) for all $A \in \mathcal{A}_{\geq 0}$, $B \in \mathcal{A}_{\leq 0}$, $\mathcal{A}(A, B[1])$ is acyclic;
(ii) $\mathcal{A}_{\leq 0}[1] \subset \mathcal{A}_{\leq 0}$ and $\mathcal{A}_{\geq 0}[-1] \subset \mathcal{A}_{\geq 0}$; and
(iii) for any $A \in \mathcal{A}$, there exists a closed morphism $f : A_+ \to A$ with $A_+ \in \mathcal{A}_{\geq 0}$ such that the cone $\operatorname{cone}(f)$ lies in $\mathcal{A}_{\leq 0}[1]$.

Note that if $\mathcal{A}_{\leq 0}$ is closed under desuspension, or equivalently if $\mathcal{A}_{\geq 0}$ is closed under suspension, then $\mathcal{A}_{\leq 0}$ and $\mathcal{A}_{\geq 0}$ are pretriangulated, and $\langle \mathcal{A}_{\leq 0}, \mathcal{A}_{\geq 0}\rangle$ is a semi-orthogonal decomposition of $\mathcal{A}$. In this case the *heart* $\mathcal{A}^{\heartsuit} = \mathcal{A}_{\leq 0} \cap \mathcal{A}_{\geq 0}$ of the t-structure consists solely of nullhomotopic objects, c.f. [62, remark 7.2.1.6].

6 Grothendieck-Witt theory

In many areas of mathematics, involutions and duality appear naturally, the prototypical example being the dual of a vector space over a field k. Ignoring these natural structures leads to coarser invariants, so it is desirable to design invariants that incorporate these structures. This line of research was arguably initiated by Ernst Witt in the 1930's, who studied quadratic forms over fields and gave the theory its name. In subsequent years a more general theory developed, allowing the study of quadratic forms over schemes [55, 56]. This culminated in the definition of the triangular Witt groups [3, 4], which provide a general framework for studying triangulated categories with duality. The derived categories of fields, rings and schemes can all be equipped with various dualities, and their Witt groups continue to be studied, e.g. [7, 42, 51].

Grothendieck-Witt theory developed in the wake of the development of Witt theory, and can be thought of as an amalgam of Witt theory and algebraic K-theory, capturing both invariants simultaneously. It is sometimes also called Hermitian K-theory to emphasize the connection to both K-theory and Hermitian phenomena. Complex K-theory in topology has a real counterpart KO. Some of the development of the algebraic theory has been informed by this topological counterpart through comparison maps, as in [96]. Like algebraic K-theory, both Grothendieck-Witt theory and Witt theory are cohomological invariants which are representable in the stable motivic homotopy category [38, section 5].

We claim no originality in this section, instead snaking our way through [5], [74] and [72] for our definitions and results.

6.1 Grothendieck-Witt groups of categories with duality

A finite dimensional vector space V is non-canonically isomorphic to its dual $V^\vee = \mathrm{Hom}(V,k)$. Any isomorphism $\phi : V \to V^\vee$ corresponds to a perfect pairing $\phi' : V \otimes V \to k$ via $\phi'(v,w) = \phi(v)(w)$. Furthermore, if the characteristic of the base field is not two, isomorphisms $\phi : V \to V^\vee$ correspond bijectively to quadratic forms on V. Grothendieck-Witt theory formalizes the study of such isomorphisms. First, we extract a more general concept from the category of finite dimensional vector spaces over a field k, equipped with its duality functor.

Definition 6.1.1. An *exact category with duality* is a triple $(\mathcal{E}, *, \omega)$ consisting of an exact category $\mathcal{E}$ together with an exact involution $* : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ called the *duality on* $\mathcal{E}$ and a natural isomorphism $\omega : \mathrm{Id}_{\mathcal{E}} \to * \circ *^{\mathrm{op}}$ called the *double dual identification* such that $(\omega_V)^* \circ \omega_{V^*} = \mathrm{id}_{V^*}$.

Fix an exact category with duality $(\mathcal{E}, *, \omega)$.

Definition 6.1.2. A *symmetric space* (V, ϕ) in $\mathcal{E}$ is an object V of $\mathcal{E}$ together with an isomorphism $\phi : V \to V^*$ such that $\phi^*\omega_V = \phi$. An *isometry of symmetric spaces* (V,ϕ) *and* (W,ψ) is an isomorphism $f : V \to W$ such that $f^*\psi f = \phi$.

The *orthogonal sum* $(V,\phi) \perp (W,\psi)$ of two symmetric spaces (V,ϕ) and (W,ψ) is the symmetric space

$$\begin{pmatrix} \phi & 0 \\ 0 & \psi \end{pmatrix} : V \oplus W \to V^* \oplus W^*$$

Definition 6.1.3. A symmetric space (V,ϕ) in $\mathcal{E}$ is called *metabolic* if there exists an exact sequence

$$L \overset{\alpha}{\rightarrowtail} V \xrightarrow{\alpha^*\phi} L^*.$$

In this case, the subspace $L \subset V$ is called a *Lagrangian of* (V,ϕ). A symmetric space (V,ϕ) is called *hyperbolic* if it is isometric to a symmetric space $H(L)$, given by

$$\begin{pmatrix} 0 & 1 \\ \omega_L & 0 \end{pmatrix} : L \oplus L^* \longrightarrow L^* \oplus L^{**}.$$

Let $M(\mathcal{E})$ be the monoid of isometry classes in $\mathcal{E}$ whose binary operation is given by the orthogonal sum.

Definition 6.1.4. The *Grothendieck-Witt group* $\mathrm{GW}_0(\mathcal{E})$ *of* $\mathcal{E}$ is the Grothendieck group $K(M(\mathcal{E}))$ modulo the relation $[(V,\phi)] = [H(L)]$, where (V,ϕ) is a metabolic space with Langrangian L. The *Witt group* $\mathrm{W}(\mathcal{E})$ *of* $\mathcal{E}$ is the Grothendieck group $K(M(\mathcal{E}))$ modulo the relation $[(V,\phi)] = 0$ for all metabolic spaces (V,ϕ).

If $\mathcal{E} = \mathrm{Vect}(X)$ is the exact category of finite locally free sheaves on a scheme X with a line bundle $\mathcal{L}$ and duality $\mathcal{F}^* = \mathscr{H}om(\mathcal{F},\mathcal{L})$, then $\mathrm{GW}_0(\mathcal{E})$ is denoted $\mathrm{GW}_0(X,\mathcal{L})$. If $\mathcal{L} = \mathcal{O}_X$, it is often surpressed in the notation.

We can associate to an exact category with duality $\mathcal{E}$ its bounded derived category $\mathcal{A} = D^b(\mathcal{E})$. This is an example of a *triangulated category with duality.* The triangulated category $\mathcal{A}$ comes equipped with a shift functor, which can be used to transform the duality on $\mathcal{E}$ into new dualities. In [72, section 3], gives a detailed exposition of triangulated categories with duality and the Grothendieck-Witt groups of such categories. We will only present the necessary definitions and an example.

Definition 6.1.5. A *triangulated category with duality* $(\mathcal{A},\sharp,\mathrm{can},\lambda)$ is a triangulated category $\mathcal{A}$ together with an additive duality functor $\sharp : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$ and natural isomorphisms $\mathrm{can} : \mathrm{Id}_{\mathcal{A}} \to \sharp\sharp^{\mathrm{op}}$ and $\lambda : \sharp \to \Sigma\sharp\Sigma^{\mathrm{op}}$, where Σ is the shift functor on $\mathcal{A}$, such that

(i) the square

$$\begin{tikzcd} \Sigma \arrow[r, "\mathrm{can}\,\Sigma"] \arrow[d, "\Sigma\,\mathrm{can}"'] & \Sigma\sharp\sharp^{\mathrm{op}} \arrow[d, "\lambda\sharp^{\mathrm{op}}\Sigma"] \\ \sharp\sharp^{\mathrm{op}}\Sigma & \Sigma\sharp\Sigma^{\mathrm{op}}\sharp^{\mathrm{op}}\Sigma \arrow[l, "\Sigma\sharp\lambda^{\mathrm{op}}"] \end{tikzcd}$$

commutes;

(ii) for each $A \in \mathcal{A}$, $\mathrm{can}_A^\sharp\,\mathrm{can}_{A^\sharp} = \mathrm{id}_{A^\sharp}$; and

(iii) for each exact triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

in $\mathcal{A}$, the dual triangle

$$C^\sharp \xrightarrow{g^\sharp} B^\sharp \xrightarrow{f^\sharp} A^\sharp \xrightarrow{\Sigma(h^\sharp)\lambda_A} \Sigma(C^\sharp)$$

is also exact.

Comparing definition 6.1.5 to definition 6.1.1, we see that $\sharp$ plays the role of $*$ and can plays the role of ω. The natural transformation λ ensures that the duality $\sharp$ behaves well with respect to the shift Σ, as it does in the case of complexes, where the dual of a complex is obtained by dualizing each differential and inverting the grading. In this case, shifting a complex, dualizing it and shifting again is the same as simply dualizing it, up to sign, and this sign constitutes λ, whence the term *δ-duality* originates that can be found in [3].

A morphism of triangulated categories with duality

$$(F, \rho, \phi) : (\mathcal{A}_1, \sharp_1, \mathrm{can}_1, \lambda_1) \longrightarrow (\mathcal{A}_2, \sharp_2, \mathrm{can}_2, \lambda_2)$$

is a triangle functor $(F, \rho) : \mathcal{A}_1 \to \mathcal{A}_2$ together with a duality compatibility isomorphism $\phi : F\sharp_1 \to \sharp_2 F$ satisfying some further compatibility conditions, see [72, definition 3.4].

Note that definition 6.1.2 can be used in an additive category with duality $(\mathcal{A}, *, \omega)$, so in particular we can define symmetric spaces in a triangulated category with duality as symmetric spaces in the underlying additive category. Note also that, by definition, the category of exact triangles in a triangulated category with duality is an additive category with duality, so that we may consider symmetric spaces in the category of exact triangles.

Definition 6.1.6. Let $(\mathcal{A}, \sharp, \mathrm{can}, \lambda)$ be a triangulated category with duality. The *Grothendieck-Witt group* $\mathrm{GW}_0(\mathcal{A})$ *of* $\mathcal{A}$ is the abelian group generated by isometry classes of symmetric spaces $[A, \phi]$ in $\mathcal{A}$, subject to the relations:

(i) $[A, \phi] + [B, \psi] = [A \oplus B, \phi \oplus \psi]$; and
(ii) if

$$\begin{tikzcd}
A_0 \arrow[r, "f"] \arrow[d, "\phi_0"] & A_1 \arrow[r, "g"] \arrow[d, "\phi_1"] & A_2 \arrow[r, "h"] \arrow[d, "\phi_2"] & \Sigma A_0 \arrow[d, "\Sigma\phi_0"] \\
A_2^\sharp \arrow[r, "g^\sharp"] & A_1^\sharp \arrow[r, "f^\sharp"] & A_0^\sharp \arrow[r, "\Sigma(h^\sharp)\lambda_A"] & \Sigma(A_2^\sharp)
\end{tikzcd}$$

is a symmetric space in the category of exact triangles of $\mathcal{A}$, then

$$[A_1, \phi_1] = \left[A_0 \oplus A_2, \begin{pmatrix} 0 & \phi_2 \\ \phi_0 & 0 \end{pmatrix} \right].$$

The *Witt group* $\mathrm{W}_0(\mathcal{A})$ *of* $\mathcal{A}$ is the quotient of $\mathrm{GW}_0(\mathcal{A})$ by the relation $[A_1, \phi_1] = 0$ for $[A_1, \phi_1]$ as in (ii) above.

In a category of chain complexes with duality, dualizing a complex inverts its grading. Hence if we shift a complex, take its dual, shift it again, and take its dual again, we expect to end up with the complex we started with, up to signs on the differential and the canonical double dual identification on the terms of the complex. This turns out to be true, and can be formalized with the following definition.

Definition 6.1.7. Let $(\mathcal{A}, \sharp, \mathrm{can}, \lambda)$ be a triangulated category with duality. We define $\mathcal{A}^{[1]}$ as the triangulated category with duality

$$(\mathcal{A}, \Sigma\sharp, -(\lambda\sharp^{\mathrm{op}}) \circ \mathrm{can}, -\Sigma\lambda),$$

and we call $\Sigma\sharp$ the *shifted duality*. Here, $-(\lambda\sharp)\circ \mathrm{can}$ is the negation of the vertical composition of the natural transformations $\mathrm{can} : 1 \to \sharp\sharp^{\mathrm{op}}$ and $\lambda\sharp^{\mathrm{op}} : \sharp\sharp^{\mathrm{op}} \to \Sigma\sharp\Sigma^{\mathrm{op}}\sharp^{\mathrm{op}}$. Similarly, we define $\mathcal{A}^{[-1]}$ as the triangulated category with duality

$$(\mathcal{A}, \sharp\Sigma, (\sharp\lambda^{\mathrm{op}})^{-1}\circ \mathrm{can}, -\lambda\Sigma).$$

Let $\mathcal{A}^{[0]} = \mathcal{A}$. For $n \in \mathbb{Z}_{>0}$, we define $\mathcal{A}^{[n]} = (\mathcal{A}^{[n-1]})^{[1]}$ and $\mathcal{A}^{[-n]} = (\mathcal{A}^{[-n+1]})^{[-1]}$ recursively.

Definition 6.1.8. Let $\mathcal{A}$ be a triangulated category with duality. For $n \in \mathbb{Z}$, the *n-th shifted Grothendieck-Witt group of $\mathcal{A}$* is defined as

$$\mathrm{GW}_0^{[n]}(\mathcal{A}) = \mathrm{GW}_0(\mathcal{A}^{[n]}).$$

Similarly, the *n-th shifted Witt group of $\mathcal{A}$* is defined as $\mathrm{W}_0^{[n]}(\mathcal{A}) = \mathrm{W}_0(\mathcal{A}^{[n]})$.

Example 6.1.9. Let k be a field. The bounded derived category $\mathcal{A} = D^b(k)$ of k is a triangulated category which can be equipped with a duality $\vee : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$ as follows. Let $* : \mathrm{Vect}(k)^{\mathrm{op}} \to \mathrm{Vect}(k)$ be the standard duality on k-vector spaces, with canonical double identification the evaluation map $\mathrm{ev} : \mathrm{Id}_{\mathrm{Vect}(k)} \to **^{\mathrm{op}}$. Let V be a complex of k-vector spaces with differential d. Then, for $i \in \mathbb{Z}$,

$$(V^\vee)^i = (V^{-i})^* \qquad (d^\vee)^i = (-1)^{i+1}(d^{-i-1})^*.$$

The dual of a chain map $f : V \to W$ is the chain map $f^\vee : W^\vee \to V^\vee$ given in each component by $(f^\vee)^i = f^*_{-i}$. The canonical double dual identification $\mathrm{can}_V : V \to V^{\vee\vee}$ is given by $(\mathrm{can}_V)^i = (-1)^i \mathrm{ev}_{V^i}$. The signs are forced by the sign conventions in $\mathbf{dgMod}_k$, see definition (5.1.1). With these definitions, $(\mathcal{A}, \vee, \mathrm{can}, 1)$ becomes a triangulated category with duality. The duality $\vee$ is actually given by the dg functor $[-, k] : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$, and this is how many dualities in chain complexes arise. We define $\mathrm{GW}_0^{[n]}(k) = \mathrm{GW}_0^{[n]}(\mathcal{A})$.

Now suppose that k is quadratically closed of characteristic not two. Under the identification $k \to \mathrm{Hom}(k, k)$ given by $1 \mapsto (1 \mapsto 1)$, the identity map $1 : k \to k$ is a symmetric form. Let $a \in k$. Since the square

$$\begin{tikzcd} k \arrow[r, "\sqrt{a}^{-1}"] \arrow[d, "1"'] & k \arrow[d, "a"] \\ k & k \arrow[l, "\sqrt{a}^{-1}"] \end{tikzcd}$$

commutes, $[k, 1] = [k, a]$ in $\mathrm{GW}_0^{[0]}(k)$. Hence the class in $\mathrm{GW}_0^{[0]}(k)$ of a symmetric form is determined completely by its dimension, yielding $\mathrm{GW}_0^{[0]}(k) = \mathbb{Z}$. To compute the shifted Grothendieck-Witt groups of k, we use the fundamental exact sequence

$$\mathrm{GW}_0^{[n]}(k) \xrightarrow{F_n} \mathrm{K}_0(k) \xrightarrow{H_{n+1}} \mathrm{GW}_0^{[n+1]}(k) \longrightarrow \mathrm{W}_0^{[n+1]}(k) \longrightarrow 0$$

of [91, theorem 2.6], or the more general version [72, theorem 6.1], which is called the *algebraic Bott sequence*. Now assume that $n \neq 0 \mod 4$. Then $\mathrm{W}_0^{[n]}(k) = 0$ by [4, theorem 5.6], so that the hyperbolic map $H_n : \mathrm{K}_0(k) \to \mathrm{GW}_0^{[n]}(k)$ is surjective. Since $\mathrm{K}_0(k) = \mathbb{Z}$, it follows that $\mathrm{GW}_0^{[n]}(k)$ is generated by the hyperbolic form $H_n(k)$.

The class of k in $\mathrm{K}_0(k)$ is $F_0([k,1])$, so $H_1(k) = 0$ by the above exact sequence. Hence $\mathrm{GW}_0^{[1]} = 0$, so H_2 is injective and $\mathrm{GW}_0^{[2]}(k) = \mathbb{Z}H_2(k)$. Finally, note that the image of $H_2(k)$ under $F_2 : \mathrm{GW}_0^{[2]}(k) \to \mathrm{K}_0(k)$ is $2[k]$, so that $\mathrm{GW}_0^{[3]}(k) = (\mathbb{Z}/2\mathbb{Z})H_3(k)$. In summary,

$$\begin{array}{ll} \mathrm{GW}_0^{[0]}(k) = \mathbb{Z} & \qquad \mathrm{GW}_0^{[1]}(k) = 0 \\ \mathrm{GW}_0^{[2]}(k) = \mathbb{Z} & \qquad \mathrm{GW}_0^{[3]}(k) = \mathbb{Z}/2\mathbb{Z}. \end{array}$$

Since shifted Grothendieck-Witt groups are four-periodic, this completes the computation of the shifted Grothendieck-Witt groups of k.

For the definition of higher Grothendieck-Witt groups, we need to go one step beyond triangulated categories, into the realm of dg categories with duality.

Definition 6.1.10. Let $\mathcal{A}$ be a dg category. The *opposite dg category* $\mathcal{A}^{\mathrm{op}}$ is the dg category with the same objects as $\mathcal{A}$, mapping complexes $\mathcal{A}^{\mathrm{op}}(A,B) = \mathcal{A}(B,A)$ and composition $f^{\mathrm{op}} \circ g^{\mathrm{op}} = (-1)^{|f||g|}(g \circ f)^{\mathrm{op}}$, where f^{op} and g^{op} are homogeneous maps in $\mathcal{A}^{\mathrm{op}}$ corresponding to composable homogeneous maps f and g in $\mathcal{A}$.

Composition in $\mathcal{A}^{\mathrm{op}}$ is given by the commutative diagram

$$\begin{tikzcd} \mathcal{A}^{\mathrm{op}}(B,C) \otimes \mathcal{A}^{\mathrm{op}}(A,B) \arrow[rr] \arrow[d, "\mathrm{op}"] & & \mathcal{A}^{\mathrm{op}}(A,C) \\ \mathcal{A}(C,B) \otimes \mathcal{A}(B,A) \arrow[r, "\tau"] & \mathcal{A}(B,A) \otimes \mathcal{A}(C,B) \arrow[r, "\circ"] & \mathcal{A}(C,A), \arrow[u, "\mathrm{op}"'] \end{tikzcd}$$

which shows that the sign appearing in the composition of morphisms in the opposite category comes from the switch map τ.

Definition 6.1.11. A *dg category with duality* $(\mathcal{A}, \vee, \mathrm{can})$ is a dg category $\mathcal{A}$ together with a dg functor $\vee : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$ and a natural transformation of dg functors $\mathrm{can} : \mathrm{Id}_{\mathcal{A}} \to \vee \circ \vee^{\mathrm{op}}$ called *double dual identification* such that $\mathrm{can}_A^{\vee} \circ \mathrm{can}_{A^{\vee}} = \mathrm{id}_{A^{\vee}}$ for all $A \in \mathcal{A}$.

Definition 6.1.12. Let $(\mathcal{A}, \vee, \mathrm{can})$ and $(\mathcal{B}, \vee, \mathrm{can})$ be dg categories with duality. A *dg form functor* $(F, \eta) : \mathcal{A} \to \mathcal{B}$ is a functor $F : \mathcal{A} \to \mathcal{B}$ together with a natural transformation $\eta : F \circ \vee \to \vee \circ F^{\mathrm{op}}$, whose components are called *duality compatibility morphisms*, such that $\eta_A^{\vee} \mathrm{can}_{F(A)} = \eta_{A^{\vee}} F(\mathrm{can}_A)$ for all objects A of $\mathcal{A}$.

Definition 6.1.13. Let $(\mathcal{A}, \vee, \mathrm{can})$ be a dg category with duality. A *symmetric form* in $\mathcal{A}$ is a closed morphism $\phi : A \to A^{\vee}$ of degree zero such that $\phi^{\vee} \mathrm{can}_A = \phi$. When ϕ is an isomorphism in $H^0\mathcal{A}$, the symmetric form is called *nondegenerate*.

Proposition 6.1.14. *Let* $(\mathcal{A}, \vee, \mathrm{can})$ *and* $(\mathcal{B}, \vee, \mathrm{can})$ *be dg categories with duality and* $(F, \eta) : \mathcal{A} \to \mathcal{B}$ *a dg form functor. If* $\phi : A \to A^{\vee}$ *is a symmetric form, then* $\eta_A F(\phi) : F(A) \to F(A)^{\vee}$ *is a symmetric form.*

Proof. The diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(\mathrm{can}_A)"] \arrow[d, "\mathrm{can}_{F(A)}"'] & F(A^{\vee\vee}) \arrow[r, "F(\phi^\vee)"] \arrow[d, "\eta_{A^\vee}"] & F(A^\vee) \arrow[d, "\eta_A"] \\ F(A)^{\vee\vee} \arrow[r, "\eta_A^\vee"] & F(A^\vee)^\vee \arrow[r, "F(\phi)^\vee"] & F(A)^\vee \end{tikzcd}$$

commutes, because (F, η) is a dg form functor. Furthermore, as ϕ is a symmetric form, $F(\phi^\vee)F(\mathrm{can}_A) = F(\phi^\vee \mathrm{can}_A) = F(\phi)$. Hence

$$(\eta_A F(\phi))^\vee \mathrm{can}_{F(A)} = F(\phi)^\vee \eta_A^\vee \mathrm{can}_{F(A)} = \eta_A F(\phi),$$

as was to be shown. □

The above proposition allows us to define the category **dgCatD** of dg categories with duality, whose morphisms are dg form functors. Furthermore, **dgCatD** can be equipped with a closed symmetric monoidal structure similar to that of **dgCat**, c.f. [72, section 1.10].

As for triangulated categories with duality, we can shift dualities on pretriangulated dg categories with duality. More generally, we can shift a duality by an object in $\mathscr{C}_k$.

Definition 6.1.15. Let A be an object of $\mathscr{C}_k$. The *duality on $\mathscr{C}_k$ defined by A* consists of the following functor $\vee_A : \mathscr{C}_k^{\mathrm{op}} \to \mathscr{C}_k$ and natural transformation can^A : $\mathrm{id} \to \vee_A \circ \vee_A^{\mathrm{op}}$:

(i) on objects, $\vee_A$ is given by $B \mapsto [B, A]$;
(ii) on mapping complexes, the component $[B, C] \to [[C, A], [B, A]]$ of $\vee_A$ is given by
$$f^{\vee_A} = (g \mapsto (-1)^{|f||g|} gf)$$
for homogeneous $f \in [B, C]$ and $g \in [C, A]$; and
(iii) the canonical double dual identification $\mathrm{can}_B^A : B \to [[B, A], A]$ on an object B is given by
$$\mathrm{can}_F^A(x) = (f \mapsto (-1)^{|x||f|} f(x)).$$

The category $\mathscr{C}_k$ equipped with the duality defined by A is denoted by

$$(\mathscr{C}_k, \vee_A, \mathrm{can}^A),$$

or shortly $\mathscr{C}_k^{[A]}$. For $n \in \mathbb{Z}$, $\mathscr{C}_k^{[n]}$ abbreviates $\mathscr{C}_k^{[\Sigma^n]}$, and $\vee_{\Sigma^n}$ is called the *n-th shifted duality on $\mathcal{C}_k$*. For an arbitrary dg category $\mathcal{A}$ with duality, we define $\mathcal{A}^{[A]} = \mathscr{C}_k^{[A]} \otimes \mathcal{A}$.

Note that, for any $A \in \mathscr{C}_k$, one also obtains the dg category with duality $(\mathscr{C}_k, \vee_A, -\mathrm{can}^A)$, the duality of which is sometimes called the *skew-duality* defined by A. Almost all dualities we will consider in this thesis are induced by objects. In particular, for a scheme X, the pretriangulated dg category $\mathrm{Perf}(X)$ of perfect complexes of $\mathcal{O}_X$-modules has a duality of the form $[-, \mathcal{L}]$ for each line bundle $\mathcal{L}$ on X. This is the main example to keep in mind until it is introduced in detail in section 7.1.

We need shifted dualities in order to define a *Grothendieck-Witt spectrum*, the higher homotopy groups of which form the higher Grothendieck-Witt groups. The definition of this spectrum involves simplicial construction akin to Waldhausen's $S_\bullet$-construction [89, section 1.3]. We will only sketch this construction here, a full account can be found in [72, section 4.6], culminating in [72, definition 5.4].

Let $\mathcal{A}$ be a pretriangulated dg category with duality. Recall that the equivalences of $\mathcal{A}$ are the morphisms of $Z^0\mathcal{A}$ that become isomorphisms in $H^0\mathcal{A}$. For $n \in \mathbb{N}$, let

$$[\underline{n}] = \{-n < -n+1 < \cdots < 0 < \cdots < n\}$$

be the category with objects $-n \leq i \leq n$ and a unique morphism $i \to j$ for all $i \leq j$. This category has a strict duality $* : [\underline{n}]^{\mathrm{op}} \to [\underline{n}]$ given by the sign change $(i \leq j) \mapsto (-j \leq -i)$. The category of these $[\underline{n}]$ with order-preserving maps between them is denoted Δ_e, since it is equivalent to the category of edgewise subdivisions of objects in the simplex category Δ. Furthermore, there is a canonical functor $\Delta \to \Delta_e$ given by $[n] \mapsto [\underline{n}]$. Then $\mathbf{Ar}([\underline{n}])$ has pairs (i, j) with $i \leq j$ as its objects and commutative squares

$$\begin{tikzcd} i \arrow[r] \arrow[d] & j \arrow[d] \\ i' \arrow[r] & j' \end{tikzcd}$$

with $i \leq j \leq j'$ and $i \leq i' \leq j'$ as its morphisms, and inherits duality from $[\underline{n}]$.

Define $\mathcal{R}_n\mathcal{A}$ as the following full dg subcategory of the pretriangulated dg category with duality $\mathbf{Fun}(\mathbf{Ar}([\underline{n}]), \mathcal{A})$. Let $A \in \mathbf{Fun}(\mathbf{Ar}([\underline{n}]), \mathcal{A})$. Then $A \in \mathcal{R}_n\mathcal{A}$ if and only if

(i) for all $-n \leq i \leq n$, $A_{i,i} = 0$; and
(ii) for all $i \leq j \leq k$ in $[\underline{n}]$, the sequence

$$0 \longrightarrow A_{i,j} \longrightarrow A_{i,k} \longrightarrow A_{j,k} \longrightarrow 0$$

in $Z^0\mathcal{A}$ is exact.

Then $\mathcal{R}_n\mathcal{A}$ is a pretriangulated dg category with duality. Hence we obtain a simplicial pretriangulated dg category with duality $\mathcal{R}_\bullet\mathcal{A} : \Delta_e^{\mathrm{op}} \to \mathbf{dgCatD}$. For each $n \in \mathbb{N}$, we define $w\mathcal{R}_n\mathcal{A}$ as the category of equivalences in $\mathcal{R}_n\mathcal{A}$, thus obtaining $w\mathcal{R}_\bullet\mathcal{A}$. Note that $w\mathcal{R}_n\mathcal{A}$ consists of all the morphisms in $Z^0\mathcal{R}_n\mathcal{A}$ that become isomorphisms in $H^0\mathcal{R}_n\mathcal{A}$, reducing us to the study of exact categories with weak equivalences and duality as in [74].

Let $\mathcal{S}_\bullet\mathcal{A}$ be Waldhausen's $\mathcal{S}_\bullet$-construction (replace $[\underline{n}]$ with $[n]$ in the above construction) and $(w\mathcal{R}_\bullet\mathcal{A})_h$ the category of nondegenerate symmetric forms in $w\mathcal{R}_\bullet\mathcal{A}$. Then there is a natural composition

$$(w\mathcal{R}_\bullet\mathcal{A})_h \longrightarrow w\mathcal{R}_\bullet\mathcal{A} \longrightarrow w\mathcal{S}_\bullet\mathcal{A},$$

where the first map is the forgetful map $(A, \phi) \mapsto A$ and the second map is induced by the natural inclusion $[n] \to [\underline{n}]$ for each $n \in \mathbb{N}$. Hence we obtain a map

$$|(w\mathcal{R}_\bullet\mathcal{A})_h| \longrightarrow |w\mathcal{S}_\bullet\mathcal{A}|$$

of pointed topological spaces.

Definition 6.1.16. The *Grothendieck-Witt space* $\mathrm{GW}(\mathcal{A})$ *of* $\mathcal{A}$ is the homotopy fiber of the natural map

$$|(w\mathcal{R}_\bullet\mathcal{A})_h| \longrightarrow |w\mathcal{S}_\bullet\mathcal{A}|.$$

Using a multi-simplicial construction, we can now define the Grothendieck-Witt spectrum of the pretriangulated dg category with duality $\mathcal{A}$. For $m_1, \ldots, m_n \in \mathbb{N}$, the pretriangulated dg category with duality $\mathbf{Fun}(\mathbf{Ar}([\underline{m_1}]) \times \cdots \times \mathbf{Ar}([\underline{m_n}]), \mathcal{A})$ consists of functors

$$A : \mathbf{Ar}([\underline{m_1}]) \times \cdots \times \mathbf{Ar}([\underline{m_n}]) \to \mathcal{A},$$

for which $A(i_1 \leq j_1, \ldots, i_n \leq j_n)$ will be denoted by $A_{i_1,j_1;\ldots;i_n,j_n}$. Let $\mathcal{R}^{(n)}_{m_1,\ldots,m_n}\mathcal{A}$ be the full subcategory of $\mathbf{Fun}(\mathbf{Ar}([\underline{m_1}]) \times \cdots \times \mathbf{Ar}([\underline{m_n}]), \mathcal{A})$ consisting of the functors A such that

(i) whenever $i_k = j_k$ for some $1 \leq k \leq n$, $A_{i_1,j_1;\ldots;i_n,j_n} = 0$;
(ii) for every object $(i_1 \leq j_1, \ldots, i_n \leq j_n)$ of $\mathbf{Ar}([\underline{m_1}]) \times \cdots \times \mathbf{Ar}([\underline{m_n}])$, every $1 \leq r \leq n$, and every $j_r \leq k \leq m_r$, the sequence

$$0 \longrightarrow A_{i_1,j_1;\ldots;i_n,j_n} \longrightarrow A_{i_1,j_1;\ldots;i_r,k;\ldots;i_n,j_n} \longrightarrow A_{i_1,j_1;\ldots;j_r,k;\ldots;i_n,j_n} \longrightarrow 0$$

is exact.

We obtain a multi-simplicial pretriangulated dg category with duality $\mathcal{R}^{(n)}_{\bullet,\ldots,\bullet}\mathcal{A}$, and we define $\mathcal{R}^{(n)}_\bullet$ to be the diagonal of this multisimplicial construction. Then $\mathcal{R}^{(n)}_\bullet\mathcal{A}$ has a number of useful categorical properties [72, section 4.6]. In particular, there is a map

$$|(w\mathcal{R}^{(n)}_\bullet\mathcal{A}^{[n]})_h| \to \Omega|(w\mathcal{R}^{(n+1)}_\bullet\mathcal{A}^{[n+1]})_h|$$

for each $n \geq 0$, which is a homotopy equivalence for each $n \geq 1$. Here, $\mathcal{A}^{[n]}$ is n-th shifted pretriangulated dg category with duality as in definition 6.1.15.

Definition 6.1.17. The *Grothendieck-Witt spectrum* $\mathrm{GW}(\mathcal{A})$ *of* $\mathcal{A}$ is the spectrum whose n-th space is given by $|(w\mathcal{R}^{(n)}_\bullet\mathcal{A}^{[n]})_h|$ and whose bonding maps are induced by the maps

$$|(w\mathcal{R}^{(n)}_\bullet\mathcal{A}^{[n]})_h| \to \Omega|(w\mathcal{R}^{(n+1)}_\bullet\mathcal{A}^{[n+1]})_h|.$$

In particular, $\mathrm{GW}(\mathcal{A})$ is an Ω-spectrum in degrees $n \geq 1$.

See [72, section 5.2] for details on the bonding maps of the Grothendieck-Witt spectrum. It is also shown there that $\mathrm{GW}(\mathcal{A})$ is a module spectrum over the ring spectrum $\mathrm{GW}(k)$, where k is the base ring of the dg category $\mathcal{A}$. The infinite loop space $\Omega^\infty \mathrm{GW}(\mathcal{A})$ of the Grothendieck-Witt spectrum is the Grothendieck-Witt space of definition 6.1.16 and $\pi_0 \mathrm{GW}(\mathcal{A})$ is isomorphic to the Grothendieck-Witt group $\mathrm{GW}_0(H^0\mathcal{A})$ of definition 6.1.6 by [72, proposition 5.6].

6.2 Fundamental results

Localization and additivity of algebraic K-theory are two of its main computational tools. For example, localization can be used to relate the K-theory of a scheme to the K-theory of a closed subscheme and that of the open complement via a fiber sequence of K-theory spectra, and additivity states that whenever the category being studied has a semi-orthogonal decomposition, its K-theory splits as a direct sum of the K-theory of each of the semi-orthogonal components; in other words, K-theory makes semi-orthogonal decompositions orthogonal.

Grothendieck-Witt theory also satisfies localization [72, theorem 6.6, theorem 8.10] and additivity [72, proposition 6.8], albeit in a slightly different form to account for duality phenomena. Because of their fundamental importance, we restate these results here.

Theorem 6.2.1 (localization for GW). *Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be a quasi-exact sequence of dg form functors between pretriangulated dg categories with duality. Then there is a homotopy fibration of Grothendieck-Witt spectra*

$$\mathrm{GW}^{[n]}(\mathcal{A}) \longrightarrow \mathrm{GW}^{[n]}(\mathcal{B}) \longrightarrow \mathrm{GW}^{[n]}(\mathcal{C}).$$

The following is a slightly more general version of additivity for GW-theory, c.f. [90, theorem 2.5] and [72, proposition 6.8].

Theorem 6.2.2 (additivity for GW). *Let $(\mathcal{A}, \vee, \mathrm{can})$ be a pretriangulated dg category with duality equipped with a semi-orthogonal decomposition $\langle \mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_r \rangle$, such that $\mathcal{A}_i^\vee \subset \mathcal{A}_{r-i}$.*

(i) The duality $\vee : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$ induces equivalences $\mathcal{A}_i^{\mathrm{op}} \simeq \mathcal{A}_{r-i}$

(ii) Suppose r is odd. Let $q = (r-1)/2$. Then the functor

$$\prod_{i=0}^{q} H\mathcal{A}_i \longrightarrow \mathcal{A}$$
$$\prod_{i=0}^{q} (A_i, B_i) \longmapsto \bigoplus_{i=0}^{q} A_i \oplus B_i^\vee,$$

where $H\mathcal{A}_i$ is the hyperbolic category with duality associated to $\mathcal{A}_i$, induces a stable equivalence of spectra

$$\bigoplus_{i=0}^{q} \mathrm{K}(\mathcal{A}_i) \longrightarrow \mathrm{GW}(\mathcal{A}).$$

(iii) Suppose r is even. Let $q = r/2$. Then the functor

$$\mathcal{A}_q \times \prod_{i=0}^{q-1} H\mathcal{A}_i \longrightarrow \mathcal{A}$$
$$A_q \times \prod_{i=0}^{q-1} (A_i, B_i) \longmapsto A_q \oplus \left(\bigoplus_{i=0}^{q-1} A_i \oplus B_i^\vee \right)$$

induces a stable equivalence of spectra

$$\mathrm{GW}(\mathcal{A}_q) \oplus \bigoplus_{i=0}^{q-1} \mathrm{K}(\mathcal{A}_i) \longrightarrow \mathrm{GW}(\mathcal{A}),$$

which identifies (up to stable equivalence) the homotopy fiber of the forgetful maps $F : \mathrm{GW}(\mathcal{A}) \to \mathrm{K}(\mathcal{A})$ and $F' : \mathrm{GW}(\mathcal{A}_q) \to \mathrm{K}(\mathcal{A}_q)$.

Proof. Since $\mathcal{A}_i^\vee \subset \mathcal{A}_{r-i}$, and since there is an equivalence $\vee : \mathcal{A}_i^{\mathrm{op}} \to \mathcal{A}_{r-i}$ sending a map $A \to B$ in $\mathcal{A}_i^{\mathrm{op}}$ to the dual map $A^\vee \to B^\vee$, which lies in $\mathcal{A}_{r-i}$ by assumption, (i) holds.

Next, (ii) will be proved. Using the notation in the statement of the theorem, let $\mathcal{A}_- = \langle \mathcal{A}_0, \ldots, \mathcal{A}_q \rangle$ and $\mathcal{A}_+ = \langle \mathcal{A}_{q+1}, \ldots, \mathcal{A}_r \rangle$. Then $\mathcal{A} = \langle \mathcal{A}_-, \mathcal{A}_+ \rangle$ is a semi-orthogonal decomposition of $\mathcal{A}$, so there is an exact sequence of pretriangulated dg categories

$$\mathcal{A}_- \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}_+.$$

By additivity [72, proposition 6.8] for GW-theory, it holds that the hyperbolic functor of the statement of (ii) induces a stable equivalence of spectra

$$K(\mathcal{A}_-) \xrightarrow{\sim} \mathrm{GW}(\mathcal{A})$$

Since $\langle \mathcal{A}_0, \ldots, \mathcal{A}_q \rangle$ is a semi-orthogonal decomposition of $\mathcal{A}_-$, additivity for connective K-theory [73, theorem 3.3.7] [1] yields an equivalence

$$\bigoplus_{i=0}^{q} \mathrm{K}(\mathcal{A}_i) \xrightarrow{\sim} \mathrm{K}(\mathcal{A}_-)$$

and the proof of (ii) is finished by composing these two equivalences.

Finally, (iii) will be proved. Let $\langle \mathcal{A}_-, \mathcal{A}_0, \mathcal{A}_+ \rangle$ be the semi-orthogonal decomposition of $\mathcal{A}$ with $\mathcal{A}_- = \langle \mathcal{A}_0, \ldots, \mathcal{A}_{q-1} \rangle$ and $\mathcal{A}_+ = \langle \mathcal{A}_{q+1}, \ldots, \mathcal{A}_r \rangle$. Then it holds that $\mathcal{A}_-^\vee = \mathcal{A}_+$ and $\mathcal{A}_0^\vee = \mathcal{A}_0$. Hence [93, theorem 3.5.6] yields a stable equivalence of spectra

$$\mathrm{GW}(\mathcal{A}_q) \oplus \mathrm{K}(\mathcal{A}_-) \xrightarrow{\sim} \mathrm{GW}(\mathcal{A}).$$

One obtains the desired stable equivalence of spectra with another application of the additivity of connective K-theory. It remains to show that there is a stable equivalence of homotopy fibers $\mathrm{hofib}(F) \simeq \mathrm{hofib}(F')$. There is a commutative diagram of spectra

$$\begin{tikzcd}
\mathrm{hofib}(F' \oplus \mathrm{id}) \arrow[r] \arrow[d, dashed, "\wr"] & \mathrm{GW}(\mathcal{A}_q) \oplus \mathrm{K}(\mathcal{A}_-) \arrow[r, "F' \oplus \mathrm{id}"] \arrow[d, "\wr"] & \mathrm{K}(\mathcal{A}_q) \oplus K(\mathcal{A}_-) \arrow[d, "\wr"] \\
\mathrm{hofib}(F) \arrow[r] & \mathrm{GW}(\mathcal{A}) \arrow[r, "F"] & \mathrm{K}(\mathcal{A})
\end{tikzcd}$$

where the vertical arrows are stable equivalences. Thus it holds that

$$\mathrm{hofib}(F) \simeq \mathrm{hofib}(F' \oplus \mathrm{id}) \simeq \mathrm{hofib}(F') \oplus \mathrm{hofib}(\mathrm{id}),$$

but hofib(id) is contractible and the result follows. □

Corollary 6.2.3. *Let* $(\mathcal{A}, \vee, \mathrm{can})$ *be a pretriangulated dg category with duality.*

(i) If there is a semi-orthogonal decomposition

$$\mathcal{A} = \langle \ldots, \mathcal{A}_{-n}, \ldots, \mathcal{A}_{-1}, \mathcal{A}_1, \ldots, \mathcal{A}_n, \ldots \rangle$$

[1]This theorem is stated and proved for triangulated categories, but holds for dg categories *mutatis mutandis* using lemma 5.6.4.

such that $\mathcal{A}_n^\vee = \mathcal{A}_{-n}$*, then the functor*

$$\bigoplus_i H\mathcal{A}_i \longrightarrow \mathcal{A}$$
$$\bigoplus_i (A_i, B_i) \longmapsto \bigoplus_i A_i \oplus B_i^\vee,$$

induces a stable equivalence of spectra

$$\bigoplus_i \mathrm{K}(\mathcal{A}_i) \longrightarrow \mathrm{GW}(\mathcal{A}).$$

(ii) If there is a semi-orthogonal decomposition

$$\mathcal{A} = \langle \dots, \mathcal{A}_{-n}, \dots, \mathcal{A}_{-1}, \mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_n, \dots \rangle$$

such that $\mathcal{A}_n^\vee = \mathcal{A}_{-n}$*, then the functor*

$$\mathcal{A}_0 \times \bigoplus_{i \neq 0} H\mathcal{A}_i \longrightarrow \mathcal{A}$$
$$A_0 \times \bigoplus_{i \neq 0} (A_i, B_i) \longmapsto \bigoplus_{i \neq 0} A_i \oplus B_i^\vee,$$

induces a stable equivalence of spectra

$$\mathrm{GW}(\mathcal{A}_0) \oplus \bigoplus_{i \neq 0} \mathrm{K}(\mathcal{A}_i) \longrightarrow \mathrm{GW}(\mathcal{A}).$$

Proof. Let $\mathcal{A}_+ = \langle \mathcal{A}_1, \mathcal{A}_2, \dots \rangle$ and $\mathcal{A}_- = \langle \dots, \mathcal{A}_{-2}, \mathcal{A}_{-1} \rangle$, apply additivity, and note that $\mathrm{K}(\mathcal{A}_+)$ decomposes as an infinite direct sum by additivity for K-theory. □

6.3 Equivariant Grothendieck-Witt theory

One side of the isomorphism in the Atiyah-Segal completion theorem is concerned with equivariant cohomology. This section develops the basic theory of equivariant Grothendieck-Witt theory, which can be thought of as a refinement of equivariant algebraic K-theory.

First, we will recall some of the theory of equivariant objects, following [85, section 3.8]. Let $p : \mathcal{S} \to \mathcal{C}$ be a fibered category. Let $G : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ be a presheaf of groups on $\mathcal{C}$ and X an object of $\mathcal{C}$ with an action $\alpha : G \times h_X \to h_X$ of G.

Definition 6.3.1. A *G-equivariant object of* $\mathcal{S}_X$ is an object $\mathcal{F}$ of $\mathcal{S}_X$, together with an action $a' : (G \circ p) \times h_{\mathcal{F}} \to h_{\mathcal{F}}$ of $G \circ p$ on $h_{\mathcal{F}}$ such that the diagram

$$\begin{tikzcd} (G \circ p) \times h_{\mathcal{F}} \arrow[r, "a'"] \arrow[d] & h_{\mathcal{F}} \arrow[d] \\ (G \circ p) \times (h_X \circ p) \arrow[r, "\alpha'"] & h_X \circ p \end{tikzcd}$$

in $\mathbf{PSh}(\mathcal{S})$ commutes, where the vertical maps are induced by p and where the map $\alpha'_{\mathcal{G}} = \alpha_Y$ for each $Y \in \mathcal{C}$ and $\mathcal{G} \in \mathcal{S}_Y$. A morphism $\mathcal{F} \to \mathcal{G}$ of G-equivariant objects is called *G-equivariant* if $h_{\mathcal{F}} \to h_{\mathcal{G}}$ is $G \circ p$-equivariant. The category of G-equivariant objects of $\mathcal{S}_X$ with G-equivariant morphisms is denoted $\mathcal{S}_X^G$.

Now assume that G is a group object of $\mathcal{C}$, acting on an object X of $\mathcal{C}$ via an action $\alpha : G \times X \to X$. In this case, there are several equivalent ways of characterizing equivariant objects; we only state [85, proposition 3.49]. We define:

(i) $m : G \times G \to G$ the multiplication map on G;
(ii) $\mathrm{pr}_2 : G \times X \to X$ the projection map;
(iii) $\mathrm{pr}_3 : G \times G \times X \to X$ the projection map;
(iv) $\mathrm{pr}_{23} : G \times G \times X \to G \times X$ the projection map;
(v) $A : G \times G \times X \to X$ the map $\alpha(m \times \mathrm{id}_X)$; and
(vi) $B : G \times G \times X \to X$ the map $\mathrm{pr}_2(\mathrm{id}_G \times \alpha)$.

Proposition 6.3.2. *To give an object $\mathcal{F}$ of $\mathcal{S}_X$ a G-equivariant structure is the same as giving an isomorphism $\phi : \mathrm{pr}_2^* \mathcal{F} \to \alpha^* \mathcal{F}$ in $\mathcal{S}_{G \times X}$ such that the diagram*

$$\begin{tikzcd}
\mathrm{pr}_3^* \mathcal{F} \arrow[rr, "(m\times \mathrm{id}_X)^*\phi"] \arrow[dr, "\mathrm{pr}_{23}^* \phi"'] & & A^*\mathcal{F} \\
& B^*\mathcal{F} \arrow[ur, "(\mathrm{id}_G \times\alpha)^*\phi"'] &
\end{tikzcd}$$

commutes.

Proposition 6.3.2 holds because $\mathcal{S}$ is fibered over $\mathcal{C}$ and therefore pullbacks exist.

Let X be a scheme with an action of a group scheme G. Since **QCoh** $\to$ **Sch** is a fibered category, proposition 6.3.2 gives a definition of G-equivariant quasi-coherent $\mathcal{O}_X$-modules.

Definition 6.3.3. A *G-equivariant locally free $\mathcal{O}_X$-module* is a locally free sheaf on X which is G-equivariant as a quasi-coherent $\mathcal{O}_X$-module.

Let $\mathrm{Perf}^G(X)$ be the dg category of perfect complexes of G-equivariant locally free $\mathcal{O}_X$-modules up to quasi-isomorphism, see remark 5.5.6 and definition 2.3.3.

Definition 6.3.4. The *G-equivariant Grothendieck-Witt spectrum of X* is defined as $\mathrm{GW}^{[n]}_G(X) = \mathrm{GW}^{[n]}(\mathrm{Perf}^G(X))$.

Example 6.3.5. Let $X = \operatorname{Spec} k$, where k is a field. Let $G = \mathbb{G}_m$. Then $R_G = \mathrm{K}_0^G(X)$ is the ring of k-representations of G. There is a nontrivial 1-dimensional G-representation M, whose action $G \times M \to M$ is given by $(\lambda, a) \mapsto \lambda a$. Note that the action of G on $M^{\otimes n}$ is then given by $(\lambda, a) \mapsto \lambda^n a$. A result in representation theory states that $R_G \cong \mathbb{Z}[x, x^{-1}]$, where x corresponds to M. The *augmentation map* $a : R_G \to \mathbb{Z}$ is given by $\sum a_i x^i \mapsto \sum a_i$ and the *augmentation ideal* is defined to be $I_G = \ker a$, which is generated by the element $(x-1)$. Then the completion of R_G with respect to I_G is $\widehat{R_G} \cong \mathbb{Z}[\![y]\!]$, where y corresponds to the class $[M]-1$, since $\mathbb{Z}[x, x^{-1}]/(x-1)^n \cong \mathbb{Z}[y]/y^n$. Using the projective bundle formula for K-theory, it can be shown that $K_0(\mathrm{B}\mathbb{G}_m) \cong \mathbb{Z}[\![y]\!]$.

7 Projective bundle formula

One of the first computations for any cohomology theory, whether topological or algebro-geometric, is the computation of the cohomology of projective spaces and projective bundles. It is simultaneously a proving ground for the computability of the theory, a toolkit for further computations, and a guide for the general behaviour of the theory; the projective space is often the example that contains the seeds of generality, if one knows where to look for them.

In this chapter, we will prove a projective bundle formula for Grothendieck-Witt theory [70], generalizing a result from [90], following [72, remark 9.11]. A less general version of this result was proven independently in [51].

We will use the theory of pretriangulated dg categories with duality as exhibited in the previous chapters. Throughout, we let X be a scheme with the resolution property, that is, satisfying $\mathrm{Perf}(X) \simeq \mathrm{sPerf}(X)$ as in remark 5.5.6. Note that any quasi-compact quasi-separated schemes with an ample family of line bundles satisfies this property. We let $\mathcal{O}_X$ be the structure sheaf of X and define $\mathbf{dgMod}_{\mathcal{O}_X}$ as in section 5.1. All the dg categories considered in this chapter will be dg categories over $\mathcal{O}_X$, the main example being the pretriangulated dg category $\mathrm{Perf}(X)$ of perfect complexes of $\mathcal{O}_X$-modules of remark 5.5.6, see also section 2.3.

7.1 Duality on the dg category of perfect complexes

The sign conventions used will be the same as those in chapter 5.2. All tensor products in this section are taken over the structure sheaf $\mathcal{O} = \mathcal{O}_X$ of the scheme X, unless indicated otherwise. Let $\mathrm{Perf}(X)$ be the usual closed symmetric monoidal pretriangulated dg category of perfect complexes of $\mathcal{O}$-modules, where the monoidal structure is given by the tensor product of complexes and the monoidal unit is the perfect complex $\mathcal{O}$, concentrated in cohomological degree 0. It is useful to think of $\mathrm{Perf}(X)$ as a subcategory of $\mathbf{dgMod}_{\mathcal{O}}$.

We recall some of our conventions:

(i) complexes of $\mathcal{O}$-modules are cohomologically graded;
(ii) for an object A of $\mathrm{Perf}(X)$ with differential d, the i-th differential is $d^i : A^i \to A^{i+1}$;
(iii) for $n \in \mathbb{Z}$, the n-th shift of A is denoted either by $\Sigma^n A$ or by $A[n]$, with $(\Sigma^n A)^i = A^{i+n}$;
(iv) we have objects $\mathbb{1}$, Σ and Γ in $\mathrm{Perf}(X)$ corresponding to the monoidal unit $\mathcal{O}$, the shift $\mathcal{O}[1]$ and the complex $\mathrm{id} : \mathcal{O} \to \mathcal{O}$ concentrated in degrees -1 and 0;
(v) a homogeneous morphism $f : A \to B$ in $\mathrm{Perf}(X)$ has degree j if its components are $f^i : A^i \to B^{i+j}$ for all $i \in \mathbb{Z}$; and
(vi) since $\mathrm{Perf}(X) \simeq \mathrm{sPerf}(X)$ by assumption, an object A of $\mathrm{Perf}(X)$ will always be assumed to be a cohomologically bounded complex of locally free $\mathcal{O}$-modules, which gives a small model of $\mathrm{sPerf}(X)$.

For a line bundle $\mathcal{L}$ on X, the duality $\vee_{\mathcal{L}[n]}$ on $\mathrm{Perf}(X)$ is closely related to the duality $\vee_{\mathcal{L}}$ on $\mathrm{Vect}(X)$, given by

$$\mathcal{F}^{\vee_{\mathcal{L}}} = \mathscr{H}\!om(\mathcal{F}, \mathcal{L}),$$

whose canonical double dual identification is the evaluation map $\mathrm{ev}_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^{\vee_{\mathcal{L}}\vee_{\mathcal{L}}}$. Fix a line bundle $\mathcal{L}$ on X, let $n \in \mathbb{Z}$ and $L = \mathcal{L}[n]$, and equip $\mathrm{Perf}(X)$ and $\mathrm{Vect}(X)$ with the dualities $\vee_L$ and $\vee = \vee_{\mathcal{L}}$, respectively. Let A be an object of $\mathrm{Perf}(X)$ with differential d. Let B be another object of $\mathrm{Perf}(X)$ and let $f \in [A, B]_j$. The following identities are forced by definition 6.1.15:

$$\begin{array}{rcl} (A^{\vee_L})^i & = & (A^{-i-n})^\vee \\ (d^{\vee_L})^i & = & (-1)^{i+1}(d^{-i-1-n})^\vee \\ (f^{\vee_L})^i & = & (-1)^{ij}(f^\vee)^{-i-j-n} \\ (\mathrm{can}^L_A)^i & = & (-1)^{i(n+1)}\,\mathrm{ev}_{A^i}\,. \end{array} \tag{7.1.1}$$

The following lemma from algebraic geometry, [33, proposition 7.7], plays an important role in the theory of duality on $\mathrm{Perf}(X)$. Indeed, corollary 7.1.5 is crucial in the proof of proposition 7.1.6, which shows that tensoring by a symmetric form in $\mathrm{Perf}(X)^{[L]}$ preserves duality.

Lemma 7.1.2. *Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be $\mathcal{O}$-modules. If $\mathcal{F}$ or $\mathcal{H}$ is finite locally free, then the canonical map*

$$\phi : \mathcal{G} \otimes_{\mathcal{O}} \mathscr{H}\!om(\mathcal{F}, \mathcal{H}) \to \mathscr{H}\!om(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}} \mathcal{H})$$

given by $s \otimes f \mapsto (t \mapsto s \otimes f(t))$ is an isomorphism.

Corollary 7.1.3. *Let M and N be in $\mathrm{Perf}(X)$, $n \in \mathbb{Z}$, and $\mathcal{L}$ a line bundle on X. Then there is a natural isomorphism*

$$\phi : N \otimes [M, \mathcal{L}[n]] \longrightarrow [M, N \otimes \mathcal{L}[n]]$$

given by $s \otimes f \mapsto (x \mapsto s \otimes f(x))$.

Proof. Note that ϕ is the composition

$$N \otimes [M, \mathcal{L}[n]] \xrightarrow{\nabla \otimes 1} [\mathcal{L}[n], N \otimes \mathcal{L}[n]] \otimes [M, \mathcal{L}[n]] \xrightarrow{\ \circ\ } [M, N \otimes \mathcal{L}[n]],$$

and therefore a natural morphism. Here, $\nabla : N \to [\mathcal{L}[n], N \otimes \mathcal{L}[n]]$ is the unit of the tensor-hom adjunction for $\mathcal{L}[n]$. Each component

$$\phi^i : (N \otimes [M, \mathcal{L}[n]])^i \longrightarrow [M, N \otimes \mathcal{L}[n]]^i$$

of ϕ is a map

$$\phi^i : \bigoplus_p N^{i-p} \otimes \mathscr{H}\!om(M^{-p-n}, \mathcal{L}) \longrightarrow \bigoplus_q \mathscr{H}\!om(M^{q-n}, N^{i+q} \otimes \mathcal{L})$$

given by $s \otimes f \mapsto (x \mapsto s \otimes f(x))$. Thus ϕ_i is a direct sum of isomorphisms as in lemma 7.1.2, which ultimately yields that ϕ is a natural isomorphism. □

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be line bundles on X and $m, n \in \mathbb{Z}$. Set $L_1 = \mathcal{L}_1[m]$, $L_2 = \mathcal{L}_2[n]$ and $L_1L_2 = (\mathcal{L}_1 \otimes \mathcal{L}_2)[m+n]$, and denote simple tensors of elements similarly.

Remark 7.1.4. The map

$$\mathcal{O}[m] \otimes \mathcal{L}_1 \otimes \mathcal{O}[n] \otimes \mathcal{L}_2 \xrightarrow{1 \otimes \tau \otimes 1} \mathcal{O}[m] \otimes \mathcal{O}[n] \otimes \mathcal{L}_1 \otimes \mathcal{L}_2$$

induces a natural isomorphism $L_1 \otimes L_2 \to L_1L_2$ given by

$$1_{-m} \otimes x \otimes 1_{-n} \otimes y \longmapsto 1_{-m-n} \otimes xy,$$

where, for $i \in \mathbb{Z}$, $1_{-i} \in \mathcal{O}[i]_{-i}$ is the multiplicative unit.

Corollary 7.1.5. *Let $M, N \in \mathrm{Perf}(X)$. Then there is natural isomorphism*

$$\phi : [M, L_1] \otimes [N, L_2] \to [M \otimes N, L_1L_2]$$

given by $\phi(f \otimes g)(x \otimes y) = (-1)^{|x||g|}(f(x) \otimes g(y))$.

Proof. The map ϕ is the composition

$$\begin{tikzcd}
{[M,L_1] \otimes [N,L_2]} \arrow[r, "\tau"] & {[N,L_2] \otimes [M,L_1]} \arrow[r, "\alpha"] & {[M,[N,L_2] \otimes L_1]} \arrow[dll, "{[1,\tau]}"'] & \\
{[M, L_1 \otimes [N,L_2]]} \arrow[r, "\beta"'] & {[M,[N,L_1 \otimes L_2]]} \arrow[r] & {[M \otimes N, L_1L_2]\,,}
\end{tikzcd}$$

where α and β are natural isomorphisms as in corollary 7.1.3, and the final map is the tensor-hom adjunction combined with the natural isomorphism of remark 7.1.4. □

The following proposition provides a useful tool for constructing dg form functors (cf. [72, remark 1.32]). Indeed, most of the form functors considered in the rest of this paper will be of this type.

Proposition 7.1.6. *Let $\phi : M \to [M, L_1]$ be a symmetric form in $\mathrm{Perf}(X)^{[L_1]}$. Then tensoring by M defines a dg form functor*

$$(M, \phi) \otimes - : \mathrm{Perf}(X)^{[L_2]} \longrightarrow \mathrm{Perf}(X)^{[L_1L_2]}$$

with duality compatibility morphisms

$$\eta_N : M \otimes [N, L_2] \longrightarrow [M, L_1] \otimes [N, L_2] \longrightarrow [M \otimes N, L_1L_2],$$

where the first arrow is $\phi \otimes \mathrm{id}$ and the second arrow is the natural isomorphism of corollary 7.1.5.

Proof. The assignment $N \mapsto M \otimes N$ certainly defines a functor, which leaves to be shown that it defines a form functor with the provided compatibility morphisms, or equivalently, that the square

$$\begin{tikzcd}
M \otimes N \arrow[r, "\mathrm{can}^{L_1L_2}"] \arrow[d, "1\otimes\mathrm{can}^{L_2}"] & {[[M \otimes N, L_1L_2]\,, L_1L_2]} \arrow[d, "\eta_N^{\vee_{L_1L_2}}"] \\
{M \otimes [[N, L_2]\,, L_2]} \arrow[r, "\eta_{[N,L_2]}"] & {[M \otimes [N, L_2]\,, L_1L_2]}
\end{tikzcd} \tag{7.1.7}$$

commutes for all N in $\mathrm{Perf}(X)^{[L_2]}$; this amounts to a computation using the definitions of all the morphisms. □

Remark 7.1.8. Linear maps $\phi : M \otimes M \to N$ such that $\phi\tau = \phi$ are called *symmetric* correspond to symmetric forms via the tensor-hom adjunction. Given a symmetric linear map $\phi : M \otimes M \to N$, the corresponding symmetric form $\tilde{\phi} : M \to [M, N]$ is the composition

$$M \xrightarrow{\nabla} [M, M \otimes M] \xrightarrow{[1,\phi]} [M, N],$$

so that $\tilde{\phi}(x) = (y \mapsto \phi(x \otimes y))$. The morphism $\nabla : [M, M \otimes M]$ is the coevaluation map $x \mapsto (y \mapsto x \otimes y)$. A symmetric linear map is called *nondegenerate* if its corresponding symmetric form is nondegenerate, that is, if it is a quasi-isomorphism.

One consequence of proposition 7.1.6 is that skew-symmetric forms can be transformed into symmetric forms.

Remark 7.1.9. For the purpose of this remark, let $L = \mathcal{L}[m]$, where $\mathcal{L}$ is a line bundle on X and $m \in \mathbb{Z}$, and let $\epsilon \in \{\pm 1\}$ and $i \in \mathbb{Z}$. Consider a symmetric linear map $\phi : M \otimes M \to L$ in $(\mathrm{Perf}(X), \vee_L, \epsilon\,\mathrm{can}^L)$. Then $\phi\tau = \epsilon\phi$. The multiplication map $\mu : \mathcal{O}[i] \otimes \mathcal{O}[i] \to \mathcal{O}[2i]$ given by $x \otimes y \mapsto xy$ satisfies $\mu\tau = (-1)^i\mu$, as witnessed by the identity

$$\mu(\tau(x \otimes y)) = (-1)^{i^2} yx = (-1)^i xy = (-1)^i \mu(x \otimes y).$$

Thus $\mu\tau \otimes \phi\tau = (-1)^i\epsilon(\mu \otimes \phi)$. Since the diagram

$$\begin{array}{ccc}
\mathcal{O}[1] \otimes M \otimes \mathcal{O}[1] \otimes M & \xrightarrow{1\otimes\tau\otimes 1} & \mathcal{O}[1] \otimes \mathcal{O}[1] \otimes M \otimes M \\
\big\downarrow{\scriptstyle \tau} & & \big\downarrow{\scriptstyle \tau\otimes\tau} \\
\mathcal{O}[1] \otimes M \otimes \mathcal{O}[1] \otimes M & \xrightarrow{1\otimes\tau\otimes 1} & \mathcal{O}[1] \otimes \mathcal{O}[1] \otimes M \otimes M
\end{array}$$

commutes, as can be seen from a direct computation, it follows that the composition

$$\psi : \mathcal{O}[1] \otimes M \otimes \mathcal{O}[1] \otimes M \xrightarrow{1\otimes\tau\otimes 1} \mathcal{O}[1] \otimes \mathcal{O}[1] \otimes M \otimes M \xrightarrow{\mu\otimes\phi} \mathcal{O}[2] \otimes L$$

satisfies $\psi\tau = (-1)^i\epsilon\psi$. This yields an equivalence of dg categories with duality

$$(\mathcal{O}[i], \mu) \otimes - : (\mathrm{Perf}(X), \vee_L, \epsilon\,\mathrm{can}^L) \longrightarrow (\mathrm{Perf}(X), \vee_{L[2i]}, (-1)^i\epsilon\,\mathrm{can}^{L[2i]}),$$

which in particular shows how to turn skew-symmetric forms into symmetric ones by taking $\epsilon = -1$ and $i = \pm 1$.

Another consequence is that the tensor product of two symmetric forms is another symmetric form.

Corollary 7.1.10. *Let $\phi : M \otimes M \to L_1$ and $\psi : N \otimes N \to L_2$ be symmetric linear maps in $\mathrm{Perf}(X)^{[L_1]}$ and $\mathrm{Perf}(X)^{[L_2]}$, respectively. Then the composition*

$$M \otimes N \otimes M \otimes N \xrightarrow{1\otimes\tau\otimes 1} M \otimes M \otimes N \otimes N \xrightarrow{\phi\otimes\psi} L_1L_2$$

is a symmetric linear map in $\mathrm{Perf}(X)^{[L_1L_2]}$.

Proof. The symmetric linear maps ϕ and ψ define symmetric forms $\tilde{\phi} : M \to [M, L_1]$ and $\tilde{\psi} : N \to [N, L_2]$. Let $F = (M, \tilde{\phi}) \otimes -$ be the form functor of proposition 7.1.6 with duality compatibility morphisms η_A. Then $\eta_N F(\tilde{\psi})$ is a symmetric form by proposition 6.1.14, and the symmetric linear map it induces is precisely that of the statement of the lemma. □

Corollary 7.1.11. *Let $L = \mathcal{L}[m]$, where $\mathcal{L}$ is a line bundle on X and $m \in \mathbb{Z}$, equipped with the trivial symmetric form $\mu : L \otimes L \to L^{\otimes 2}$ Then tensoring by L induces an equivalence*

$$(L, \mu) \otimes - : \mathrm{Perf}(X)^{[0]} \longrightarrow \mathrm{Perf}(X)^{[L^{\otimes 2}]}$$

of pretriangulated dg categories with duality.

Proof. It is immediate from proposition 7.1.6 that tensoring by L gives a dg form functor

$$F : \mathrm{Perf}(X)^{[0]} \longrightarrow \mathrm{Perf}(X)^{[L^{\otimes 2}]}$$

whose duality compatibility morphisms are isomorphisms. Furthermore, tensoring by L is an equivalence; an inverse is given by tensoring with $[L, \mathcal{O}]$. It follows that F is an equivalence of pretriangulated dg categories with duality. □

7.2 Constructing symmetric forms from Koszul complexes

Let X be a scheme satisfying the resolution property and let $\mathcal{E}$ be a finite locally free $\mathcal{O}_X$-module of rank $r+1$. Let $\mathbb{P} = \mathbb{P}(\mathcal{E})$ be the projective bundle over X associated to $\mathcal{E}$, with projection map $\pi : \mathbb{P} \to X$. Set $s = \lceil r/2 \rceil$ and $\mathcal{O} = \mathcal{O}_{\mathbb{P}}$, so $\mathcal{O}$ no longer denotes $\mathcal{O}_X$ for notational convenience in this section. By the construction of the projective bundle, there is a canonical surjection

$$\pi^* \mathcal{E} \longrightarrow \mathcal{O}(1),$$

giving rise to the Koszul complex K

$$K : \quad 0 \to \ldots \to \Lambda^i \pi^* \mathcal{E}(-i) \to \ldots \to \Lambda^1 \pi^* \mathcal{E}(-1) \to \mathcal{O} \to 0$$

with $K^{-i} = \Lambda^i \pi^* \mathcal{E} \otimes \mathcal{O}(-i)$ in cohomological degree $-i$, which is acyclic by [80, section 4.6]. It also holds that $\Lambda^{r+1} \pi^* \mathcal{E} \cong \pi^* \det \mathcal{E}$ and $\Lambda^1 \pi^* \mathcal{E} = \pi^* \mathcal{E}$. View K as a differential graded algebra with $\Lambda^1 \pi^* \mathcal{E}(-1)$ in degree -1, so that the cohomological degree and the degree of the grading coincide, and write $|x|$ for the degree of a homogeneous element $x \in K$. For ease of notation, fix $\Delta = \det \pi^* \mathcal{E}(-r-1) = K^{-r-1}$.

The rest of this section is dedicated to the construction of a symmetric form (H, ψ) such that the cone of ψ is the Koszul complex, which can be seen as a generalization of [8, section 4]. Two cases are distinguished, r is even and r is odd, the construction in the first case being easier than that in the second case. Such a symmetric form (H, ψ) is a quasi-isomorphism in $\mathrm{Perf}(\mathbb{P})$ because the Koszul complex is acyclic, and therefore defines an element of the Grothendieck-Witt group $\mathrm{GW}_0^{[r]}(\mathbb{P}, \Delta)$, which is a key ingredient in the proof of theorem 7.4.10.

The following proposition is an adaptation of the well-known and useful fact that the Koszul complex is self-dual.

Proposition 7.2.1. *The linear map $\mu : K \otimes K \to \Delta[r+1]$ given by the composition*

$$K \otimes K \xrightarrow{\wedge} K \xrightarrow{\mathrm{pr}} \Delta[r+1],$$

where the last map is the projection map, is symmetric and nondegenerate.

Proof. Symmetry can be checked by a direct computation involving the graded commutativity of K and the sign change on the twist map $\tau : K \otimes K \to K \otimes K$.

Non-degeneracy holds because the components of the induced symmetric form $\tilde{\mu} : K \to [K, \Delta[r+1]]$ are locally isomorphisms of free $\mathcal{O}$-modules and therefore isomorphisms. □

Fix an integer ℓ such that $-r-1 \leq \ell \leq -1$ and let $M = K^{\leq \ell}$ be the naive truncation of K:

$$\begin{array}{ccccccccccccccc}
M: & K^{-r-1} & \xrightarrow{d} & K^{-r} & \xrightarrow{d} & \dots & \xrightarrow{d} & K^{\ell} & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 \\
\downarrow \iota & \| & & \| & & & & \| & & \downarrow & & & & \downarrow \\
K: & K^{-r-1} & \xrightarrow{d} & K^{-r} & \xrightarrow{d} & \dots & \xrightarrow{d} & K^{\ell} & \xrightarrow{d} & K^{\ell+1} & \xrightarrow{d} & \dots & \xrightarrow{d} & K^0.
\end{array}$$

Since K is a differential graded $\mathcal{O}$-algebra, there is a multiplication map $\wedge : K \otimes K \to K$ given by the wedge product. Let φ be the composition

$$M \otimes M \xrightarrow{d\iota \otimes \iota} K \otimes K \xrightarrow{\mu} \Delta[r+2],$$

where μ is the symmetric linear map of proposition 7.2.1. In a formula,

$$\varphi(x \otimes y) = \begin{cases} d(x) \wedge y & \text{if } |x| + |y| = -r-2 \\ 0 & \text{otherwise.} \end{cases}$$

Now φ is a symmetric linear map, which will be molded in such a way that its cone becomes the Koszul complex.

Proposition 7.2.2. *The map*

$$\varphi : M \otimes M \longrightarrow \Delta[r+2]$$

defines a symmetric form $\phi : M[-1] \to [M[-1], \Delta[r]]$ in $\mathrm{Perf}(\mathbb{P})^{[\Delta[r]]}$ given by $x \mapsto (y \mapsto (-1)^{|x|}\varphi(x \otimes y))$.

Proof. Let $\tau : M \otimes M \to M \otimes M$ be the switch map $x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$. By [72, remark 1.31], φ defines a skew-symmetric form $\phi : M \to [M, \Delta[r+2]]$ if φ satisfies $\varphi = -\varphi\tau$, which is what will be shown. By construction,

$$\varphi(\tau(x \otimes y)) = 0 = -\varphi(x \otimes y)$$

if $|x| + |y| \neq -r-2$. Therefore, let $x \otimes y \in M \otimes M$ with $|x| + |y| = -r-2$. Then a direct computation involving the graded Leibniz rule shows that

$$\varphi(\tau(x \otimes y)) = (-1)^{|y|+1}(-1)^{|y|} d(x) \wedge y = -d(x) \wedge y = -\varphi(x \otimes y).$$

By remark 7.1.9, tensoring φ with the skew-symmetric form $\mathcal{O}[-1] \otimes \mathcal{O}[-1] \to \mathcal{O}[-2]$ yields the desired symmetric form ϕ. □

For the remainder of this section, let ϕ be the symmetric form of proposition 7.2.2. Note that ϕ is not necessarily a quasi-isomorphism in $\mathrm{Perf}(\mathbb{P})$ and therefore does not necessarily define an element of $\mathrm{GW}^{[r]}(\mathbb{P}, \Delta)$. However, the ideas of [8, section 4] can be combined with the technique of the proof of [90, theorem 1.5] to obtain a symmetric form (H, ψ) having the Koszul complex, or a related acyclic complex, as its cone.

Proposition 7.2.3. *Assume that r is even and let $(H, \psi) = (M, \phi)$ with $\ell = -s-2$. Then ψ is a quasi-isomorphism.*

Proof. By proposition 7.2.1, ψ becomes isomorphic to the map of complexes

$$\begin{tikzcd}
K^{-r-1} \arrow[r, "-d"] & \dots \arrow[r, "-d"] & K^{-s-2} \arrow[r] \arrow[d, "d"] & 0 & & \\
& 0 \arrow[r] & K^{-s-1} \arrow[r, "d"] & \dots \arrow[r, "d"] & K^{0},
\end{tikzcd}$$

concentrated in cohomological degrees $[-r, 0]$. Therefore, the cone $\operatorname{cone}(\psi)$ of ψ is isomorphic to the Koszul complex K, which is acyclic. Consequently, ψ is a quasi-isomorphism, as was to be shown. $\square$

Now suppose that r is odd. Taking $\ell = -s$, the cone $\operatorname{cone}(\phi)$ becomes isomorphic to the complex $K \oplus K^{-s}[s]$, where $K^{-s} = \Lambda^s \pi^* \mathcal{E}(-s)$ is viewed as a complex concentrated in degree zero. The middle terms of $\operatorname{cone}(\phi)$ are

$$\dots \longrightarrow K^{-s-1} \longrightarrow (K^{-s})^{\oplus 2} \longrightarrow K^{-s+1} \longrightarrow \dots,$$

where the differential $K^{-s-1} \to (K^{-s})^{\oplus 2}$ is given by $x \mapsto (-dx, dx)$ and the differential $(K^{-s})^{\oplus 2} \to K^{-s+1}$ is given by $(x, y) \mapsto dx + dy$. The idea presents itself that there might be a symmetric form whose cone is an acyclic complex that is closely related to the Koszul complex, as long as K^{-s} "splits into two dual parts". The remainder of this section is dedicated to constructing such a symmetric form, with a suitable assumption on $\Lambda^s \mathcal{E}$.

For the moment, consider the exact category with duality $\operatorname{Vect}(X)^{[\det \mathcal{E}]}$ of locally free sheaves on X, where the duality is denoted by $\natural$. The reason for this excursion to $\operatorname{Vect}(X)$ is that the symmetric form (H, ψ) in $\operatorname{sPerf}(\mathbb{P})^{[\Delta[r]]}$, which is being constructed for the splitting of the homotopy fibration in theorem 7.4.10(ii), needs to have a specific image in $\operatorname{sPerf}(X)^{[0]}$.

Lemma 7.2.4. *The restriction of the wedge product $\wedge : \Lambda\mathcal{E} \otimes \Lambda\mathcal{E} \to \Lambda\mathcal{E}$ to $\Lambda^s\mathcal{E} \subset \Lambda\mathcal{E}$ induces a $(-1)^s$-symmetric form*

$$\nu' : \Lambda^s \mathcal{E} \longrightarrow (\Lambda^s \mathcal{E})^\natural$$

in $\operatorname{Vect}(X)^{[\det \mathcal{E}]}$, and in particular defines an element $\nu' \in W^{[r+1]}(X, \det \mathcal{E})$.

Proof. Since $x \wedge y = (-1)^{s^2} y \wedge x = (-1)^s y \wedge x$ for $x, y \in \Lambda^s \mathcal{E}$, ν' is a $(-1)^s$-symmetric form, which is also an isomorphism. As $2s = r + 1$, it follows that ν' defines an element in $W^{[r+1]}(X, \det \mathcal{E})$. $\square$

Now assume that the element ν' of lemma 7.2.4 vanishes in $W^{[r+1]}(X, \det \mathcal{E})$. Then $(\Lambda^s \mathcal{E}, \nu')$ is stably metabolic, so by [5, remark 29] there exists a metabolic space $(\mathcal{N}', \sigma')$ such that $(\Lambda^s \mathcal{E}, \nu') \perp (\mathcal{N}', \sigma')$ is split metabolic and even hyperbolic by [5, example 21], because 2 is invertible; let $(\mathcal{N}', \sigma')$ be such a metabolic space and fix a split exact sequence

$$\begin{tikzcd}
\mathcal{P}' \arrow[r, tail, shift left, "\iota_{\mathcal{P}'}"] & \Lambda^s \mathcal{E} \oplus \mathcal{N}' \arrow[l, two heads, shift left, "\mathrm{pr}_{\mathcal{P}'}"] \arrow[r, two heads, shift left, "\iota_{\mathcal{P}'}^\natural (\nu' \oplus \sigma')"] & \mathcal{P}'^\natural, \arrow[l, tail, shift left, "(\nu' \oplus \sigma')^{-1} \mathrm{pr}_{\mathcal{P}'}^\natural"]
\end{tikzcd} \tag{7.2.5}$$

where $\mathcal{P}'$ is a split Lagrangian of $\Lambda^s\mathcal{E} \oplus \mathcal{N}'$ with orthogonal complement $\mathcal{P}'^\natural$. It is useful to think of this construction as extending $\Lambda^s\mathcal{E}$ by $\mathcal{N}'$ such that it can be chopped into two halves $\mathcal{P}'$ and $\mathcal{P}'^\natural$ which are dual to each other.

Additionally, fix a short exact sequence

$$\mathcal{S}' \xhookrightarrow{\alpha} \mathcal{N}' \xrightarrow{\alpha^\natural\sigma} \mathcal{S}'^\natural,$$

which exists since $\mathcal{N}'$ is metabolic.

Note that $\pi^* : \mathrm{Vect}(X)^{[\det\mathcal{E}]} \to \mathrm{Vect}(\mathbb{P})^{[\det\pi^*\mathcal{E}]}$ is an exact duality-preserving functor. Furthermore, the symmetric form $\mathrm{id} : \mathcal{O}(-s) \to \mathcal{O}(-s)$ in $\mathrm{Vect}(\mathbb{P})^{[\mathcal{O}(-r-1)]}$ yields a duality-preserving equivalence

$$(\mathcal{O}(-s), \mathrm{id}) \otimes - : \mathrm{Vect}(\mathbb{P})^{[\det\pi^*\mathcal{E}]} \longrightarrow \mathrm{Vect}(\mathbb{P})^{[\Delta]}.$$

Let F be the composition

$$F:\ \mathrm{Vect}(X)^{[\det\mathcal{E}]} \xrightarrow{\quad\pi^*\quad} \mathrm{Vect}(\mathbb{P})^{[\det\pi^*\mathcal{E}]} \xrightarrow{(\mathcal{O}(-s),\mathrm{id})\otimes-} \mathrm{Vect}(\mathbb{P})^{[\Delta]},$$

and let $\mathcal{P} = F(\mathcal{P}')$, $\mathcal{N} = F(\mathcal{N}')$, $\mathcal{S} = F(\mathcal{S}')$, $\nu = F(\nu')$ and $\sigma = F(\sigma')$. Note that ν is the $(-1)^s$-symmetric form

$$\nu : \Lambda^s\pi^*\mathcal{E}(-s) \longrightarrow \Lambda^s\pi^*\mathcal{E}^\natural(-s)$$

induced by the restriction of the wedge product $\wedge : K \otimes K \to K$ of the Koszul complex to the middle term K^{-s}. The image of the exact sequence (7.2.5) under F is another split exact sequence

$$\mathcal{P} \underset{\mathrm{pr}_{\mathcal{P}}}{\overset{\iota_{\mathcal{P}}}{\rightleftarrows}} K^{-s} \oplus \mathcal{N} \underset{(\nu\oplus\sigma)^{-1}\,\mathrm{pr}_{\mathcal{P}}^\natural}{\overset{\iota_{\mathcal{P}}^\natural(\nu\oplus\sigma)}{\rightleftarrows}} \mathcal{P}^\natural \tag{7.2.6}$$

in $\mathrm{Vect}(\mathbb{P})^{[\Delta]}$, where the duality on $\mathrm{Vect}(\mathbb{P})$ induced by Δ is also denoted by $\natural$. The fact that $\mathcal{P}$, $\mathcal{N}$ and $\mathcal{S}$ lie in the image of F will be crucial in the proof of theorem 7.4.10(ii). The following two technical lemmas construct the central square of (H, ψ). The takeaway is that this square is symmetric when appropriately embedded in $\mathrm{sPerf}(\mathbb{P})$.

Lemma 7.2.7. *The square*

$$\begin{tikzcd}
K^{-s-1} \oplus \mathcal{S} \arrow[r, "d\oplus\alpha"] \arrow[d, "d\oplus\alpha"'] & K^{-s} \oplus \mathcal{N} \arrow[r, two heads, "\mathrm{pr}_{\mathcal{P}}"] & \mathcal{P} \arrow[d, hook, "(\nu\oplus\sigma)\iota_{\mathcal{P}}"] \\
K^{-s} \oplus \mathcal{N} \arrow[d, two heads, "\iota_{\mathcal{P}}^\natural(\nu\oplus\sigma)"'] & & (K^{-s})^\natural \oplus \mathcal{N}^\natural \arrow[d, "(d\oplus\alpha)^\natural"] \\
\mathcal{P}^\natural \arrow[r, hook, "\mathrm{pr}_{\mathcal{P}}^\natural"'] & (K^{-s})^\natural \oplus \mathcal{N}^\natural \arrow[r, "(d\oplus\alpha)^\natural"'] & (K^{-s-1})^\natural \oplus \mathcal{S}^\natural
\end{tikzcd} \tag{7.2.8}$$

anti-commutes.

Proof. The split exact sequence (7.2.6) yields

$$\iota_{\mathcal{P}} \operatorname{pr}_{\mathcal{P}} + (\nu \oplus \sigma)^{-1} \operatorname{pr}_{\mathcal{P}}^{\natural} \iota_{\mathcal{P}}^{\natural} (\nu \oplus \sigma) = \mathrm{id}_{(\mathrm{K}^{-s} \oplus \mathcal{N})},$$

which becomes

$$(\nu \oplus \sigma)\iota_{\mathcal{P}} \operatorname{pr}_{\mathcal{P}} + \operatorname{pr}_{\mathcal{P}}^{\natural} \iota_{\mathcal{P}}^{\natural} (\nu \oplus \sigma) = (\nu \oplus \sigma)$$

when composed with $(\nu \oplus \sigma)$. Furthermore, note that $d^\natural \nu d : K^{-s-1} \to (K^{-s-1})^\natural$ is given by $x \mapsto (y \mapsto d(x) \wedge d(y))$, but $d(x) \wedge d(y) = d(x \wedge d(y)) = d(0) = 0$, so $d^\natural \nu d = 0$. Thus the sum of the two paths of the square satisfies

$$\begin{aligned}
&(d \oplus \alpha)^\natural (\nu \oplus \sigma)\iota_{\mathcal{P}} \operatorname{pr}_{\mathcal{P}} (d \oplus \alpha) + (d \oplus \alpha)^\natural \operatorname{pr}_{\mathcal{P}}^{\natural} \iota_{\mathcal{P}}^{\natural} (\nu \oplus \sigma)(d \oplus \alpha) \\
&\quad = (d \oplus \alpha)^\natural ((\nu \oplus \sigma)\iota_{\mathcal{P}} \operatorname{pr}_{\mathcal{P}} + \operatorname{pr}_{\mathcal{P}}^{\natural} \iota_{\mathcal{P}}^{\natural} (\nu \oplus \sigma))(d \oplus \alpha) \\
&\quad = (d \oplus \alpha)^\natural (\nu \oplus \sigma)(d \oplus \alpha) \\
&\quad = (d^\natural \nu d \oplus \alpha^\natural \sigma \alpha) \\
&\quad = 0,
\end{aligned}$$

which proves the result. □

The next lemma applies the previous one in the context of $\mathrm{sPerf}(\mathbb{P})^{[\Delta[r]]}$, with the duality $\vee = \vee_{\Delta[r]}$. It is essentially an application of the identities (7.1.1).

Lemma 7.2.9. *The map of complexes ψ given by*

$$\begin{tikzcd}
K^{-s-1} \oplus \mathcal{S} \arrow[rr, "\operatorname{pr}_{\mathcal{P}}(d \oplus \alpha)"] \arrow[d, "\iota_{\mathcal{P}}^{\natural}(\nu \oplus \sigma)(d \oplus \alpha)"'] & & \mathcal{P} \arrow[d, "(-1)^s (d \oplus \alpha)^\natural (\nu \oplus \sigma)\iota_{\mathcal{P}}"] \\
\mathcal{P}^\natural \arrow[rr, "(-1)^{s+1}(d \oplus \alpha)^\natural \operatorname{pr}_{\mathcal{P}}^{\natural}"'] & & (K^{-s-1})^\natural \oplus \mathcal{S}^\natural
\end{tikzcd}$$

concentrated in cohomological degrees $[-s, -s+1]$ is symmetric in the pretriangulated dg category with duality $(\mathrm{sPerf}(\mathbb{P}), \vee, \mathrm{can})$.

Proof. By lemma 7.2.7, the square commutes. For notational convenience, the subscript of the evaluation map $\mathrm{ev}_{\mathcal{F}}$ is suppressed. It remains to be shown that $\psi = \psi^\vee \mathrm{can}$. Note that $(\psi^\vee)_{-s+1} = \psi^\natural_{s-1-r} = \psi^\natural_{-s}$. Since $(\nu \oplus \sigma)$ is $(-1)^s$-symmetric,

$$\begin{aligned}
(\iota_{\mathcal{P}}^{\natural} (\nu \oplus \sigma)(d \oplus \alpha))^\natural \mathrm{ev} &= (d \oplus \alpha)^\natural (\nu \oplus \sigma)^\natural \iota_{\mathcal{P}}^{\natural\natural} \mathrm{ev} \\
&= (d \oplus \alpha)^\natural (\nu \oplus \sigma)^\natural \mathrm{ev}\, \iota_{\mathcal{P}} \\
&= (-1)^s (d \oplus \alpha)^\natural (\nu \oplus \sigma)\iota_{\mathcal{P}}.
\end{aligned}$$

A similar computation shows that

$$\Big((-1)^s (d \oplus \alpha)^\natural (\nu \oplus \sigma)\iota_{\mathcal{P}}\Big)^\natural \mathrm{ev} = \iota_{\mathcal{P}}^{\natural} (\nu \oplus \sigma)(d \oplus \alpha),$$

which concludes the proof. □

Let H be the following complex, concentrated in degrees $[-r, -s+1]$:

$$K^{-r-1} \xrightarrow{d} \dots \xrightarrow{d} K^{-s-2} \xrightarrow{d} K^{-s-1} \oplus \mathcal{S} \xrightarrow{\mathrm{pr}_{\mathcal{P}}(d\oplus\alpha)} \mathcal{P},$$

where $d : K^{-s-2} \to K^{-s-1} \oplus \mathcal{S}$ is the composition of the differential $d : K^{-s-2} \to K^{-s-1}$ and the canonical inclusion $K^{-s-1} \to K^{-s-1} \oplus \mathcal{S}$. In some sense, H is "half" of the Koszul complex, with some surgical alterations at the end to ease the conditions under which it can be constructed. Piecing together the various results obtained thus far yields the following theorem.

Theorem 7.2.10. *Assume that $\nu' = 0$ in $W^{[r+1]}(X, \det \mathcal{E})$. Let $\psi : H \to H^\vee$ be the chain map in $\mathrm{sPerf}(\mathbb{P})^{[\Delta[r]]}$ given by*

$$\begin{tikzcd}
H^{-r} \arrow[r] \arrow[d] & \dots \arrow[r] \arrow[d] & H^{-s-1} \arrow[r] \arrow[d] & H^{-s} \arrow[r] \arrow[d] & H^{-s+1} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \dots \arrow[r] & 0 \arrow[r] & (H^{-s+1})^\natural \arrow[r] & (H^{-s})^\natural \arrow[r] & (H^{-s-1})^\natural \arrow[r] & \dots \arrow[r] & (H^{-r})^\natural,
\end{tikzcd}$$

where the central square is that of lemma 7.2.9. Then ψ is symmetric and a quasi-isomorphism.

Proof. In this proof, let d' denote the differential of H. The central square commutes and is symmetric by lemma 7.2.9. The square directly left of the central square commutes since

$$\psi_{-s} d'_{-s-1} = \iota_{\mathcal{P}}^\natural (\nu \oplus \sigma)(d \oplus \alpha) d'_{-s-1} = 0,$$

and similarly for the square directly right of the central square. It follows that ψ is symmetric and it remains to be shown that ψ is a quasi-isomorphism, or equivalently, that the cone $\mathrm{cone}(\psi)$ of ψ is acyclic. Note that $\mathrm{cone}(\psi)$ is the complex

$$K^{-r-1} \to \dots \to K^{-s-1} \oplus \mathcal{S} \to K^{-s} \oplus \mathcal{N} \to (K^{-s-1})^\natural \oplus \mathcal{S}^\natural \to \dots \to (K^{-r-1})^\natural,$$

which is isomorphic to the Koszul complex away from the middle degrees $[-s-2, -s+2]$ by proposition 7.2.1. In the middle degrees, $\mathrm{cone}(\phi)$ is given as

$$K^{-s-2} \xrightarrow{d} K^{-s-1} \oplus \mathcal{S} \xrightarrow{d\oplus\alpha} K^{-s} \oplus \mathcal{N} \xrightarrow{d\oplus\alpha^\natural\sigma} (K^{-s-1})^\natural \oplus \mathcal{S}^\natural \xrightarrow{d} (K^{-s-2})^\natural.$$

Thus $\mathrm{cone}(\phi)$ is the direct sum of the acyclic koszul complex K and the exact sequence $\mathcal{S} \to \mathcal{N} \to \mathcal{S}^\natural$, seen as an acyclic complex concentrated in degrees $[-s-1, -s+1]$. Therefore, $\mathrm{cone}(\phi)$ itself is acyclic, as was to be shown. □

Theorem 7.2.10 finishes the construction of the symmetric form (H, ψ), but it is not a priori clear when the condition on ν' holds. The following lemma provides a useful criterion.

Lemma 7.2.11. *If $\mathcal{E}$ admits a quotient bundle of odd rank, then ν vanishes in $W^{[r+1]}(X, \det \mathcal{E})$.*

Proof. This is [90, proposition 8.1]. Note that if $\mathcal{E}$ is trivial, it certainly admits a quotient bundle of odd rank. □

In the case of a trivial projective bundle $\mathbb{P}^r_X$ of odd dimension, it is possible to construct an alternative symmetric form in $\mathrm{Perf}(\mathbb{P}^r_X)^{[r]}$, which is more concrete than the one obtained in theorem 7.2.10. Let

$$\mathcal{E} = \bigoplus_{i=0}^{r} \mathcal{O}_X T_i$$

be the free sheaf on X of even rank $r+1$. Set $\mathbb{P} = \mathbb{P}(\mathcal{E})$, $\mathcal{O} = \mathcal{O}_\mathbb{P}$ and $s = (r+1)/2$. Let $\pi : \mathbb{P} \to \mathcal{E}$ be the projection map. For each $i = 0, \dots, r$ there is a complex of locally free sheaves

$$T_i : \mathcal{O}(-1) \to \mathcal{O},$$

given by multiplication with T_i, with $\mathcal{O}(-1)$ in cohomological degree -1. Let M be the complex

$$M = \mathcal{O}(s-1) \otimes \bigotimes_{i=0}^{r-1} \left(\mathcal{O}(-1) \xrightarrow{T_i} \mathcal{O} \right)$$

in $\mathrm{Perf}(\mathbb{P})$. This is the complex

$$\mathcal{O}(-s) \to \dots \to \mathcal{O}(-s+i)^{\oplus \binom{r}{i}} \to \dots \to \mathcal{O}(s-2)^{\oplus r} \to \mathcal{O}(s-1)$$

with $\mathcal{O}(-s)$ in cohomological degree $-r$. Furthermore, there is an isomorphism

$$M^\vee \cong \mathcal{O}(-s+1) \otimes \bigotimes_{i=0}^{r-1} \left(\mathcal{O} \xrightarrow{T_i} \mathcal{O}(1) \right),$$

which is a complex

$$\mathcal{O}(-s+1) \to \dots \to \mathcal{O}(-s+i+1)^{\oplus \binom{r}{i}} \to \dots \to \mathcal{O}(s-1)^{\oplus r} \to \mathcal{O}(s)$$

with $\mathcal{O}(-s+1)$ in cohomological degree 0. Multiplication by T_r induces a chain map

$$\begin{array}{ccccccc}
\mathcal{O}(-s) & \longrightarrow & \mathcal{O}(-s+1)^{\oplus r} & \longrightarrow & \dots & \longrightarrow & \mathcal{O}(s-1) \\
\downarrow{\scriptstyle T_r} & & \downarrow{\scriptstyle T_r} & & \downarrow & & \downarrow{\scriptstyle T_r} \\
\mathcal{O}(-s+1) & \longrightarrow & \mathcal{O}(-s+2)^{\oplus r} & \longrightarrow & \dots & \longrightarrow & \mathcal{O}(s)
\end{array}$$

from M to $M^\vee[r]$, which will be shown to be a symmetric form in $\mathrm{Perf}(\mathbb{P})^{[r]}$.

Proposition 7.2.12. *The map $T_r : M \to M^\vee[r]$ defines an element $\mu \in \mathrm{GW}_0^{[r]}(\mathbb{P})$.*

Proof. By [77, Tag 0628], the cone of T_r is (a twist of) the Koszul complex of $\mathcal{E}$, which is acyclic. Therefore, T_r is a quasi-isomorphism, and $\mu = [M, T_r]$ is an element of $\mathrm{GW}_0^{[r]}(\mathbb{P})$, as was to be shown. □

7.3 Cutting the Koszul complex in half abstractly

Here is an abstract argument that shows we can cut the Koszul complex in half in specific cases, suggested to me by Marcus Zibrowius. The Koszul complex K on X comes equipped with an isomorphism $\mu : K \to [K, \Delta[r+1]]$, induced by the wedge product. Consider the pretriangulated dg category $\mathcal{B} = \mathrm{Perf}(\mathbb{P})'$ of perfect complexes of $\mathcal{O}_{\mathbb{P}}$-modules (before localizing with respect to the acyclic complexes), equipped with the duality induced by $\Delta[r+1]$. Let $\mathcal{A} \subset \mathcal{B}$ the dg subcategory of acyclic complexes and let $\mathcal{C} = \mathcal{B}/\mathcal{A}$ be the quotient dg category. Then there is a quasi-exact sequence

$$\mathcal{A} \xrightarrow{a} \mathcal{B} \xrightarrow{b} \mathcal{C}.$$

Taking homotopy categories, this yields an exact sequence of triangulated categories.

Since K is acyclic and μ is an isomorphism, the Koszul form (K, μ) is a symmetric form in $\mathcal{A}$. By [3, theorem 5.2], there is a 12-term localization exact sequence

$$\ldots \longrightarrow \mathrm{W}^{[n]}(\mathcal{A}) \longrightarrow \mathrm{W}^{[n]}(\mathcal{B}) \longrightarrow \mathrm{W}^{[n]}(\mathcal{C}) \xrightarrow{\partial_n} \mathrm{W}^{[n+1]}(\mathcal{A}) \longrightarrow \ldots,$$

where $\partial_n([P, \psi]) = [\mathrm{cone}(P, \psi)]$ for (P, ψ) a symmetric form in $\mathcal{C}$ with ψ a quasi-isomorphism. The element $[\mathrm{cone}(P, \psi)]$ is not trivial to define; the notation here is merely meant to suggest that ∂_n is given by taking the cone of a symmetric form and then putting some canonical symmetric form on that space. The fact that triangular Witt groups are four-periodic explains why the localization sequence wraps around after twelve terms. Note that $[K, \mu] \in \mathrm{W}^{[0]}(\mathcal{A})$.

Assume that $a([K, \mu]) = 0$ in $\mathrm{W}^{[0]}(\mathcal{B})$. Then there exists a class $[H, \psi] \in \mathrm{W}^{[-1]}(\mathcal{C})$ such that $\mathrm{cone}(\psi) = K$ and $[H, \psi] \in \mathrm{GW}_0^{[r]}(\mathrm{Perf}(\mathbb{P}), \Delta)$, which is the Koszul complex cut in half. This argument does not tell us what it is concretely, but gives a philosophical reason for its existence.

7.4 Grothendieck-Witt spectra of projective bundles

In this section, formulae for the Grothendieck-Witt spectra of general projective bundles are stated and proven. Let X be a quasi-compact quasi-separated scheme over $\operatorname{Spec} \mathbb{Z}[1/2]$ with the resolution property, i.e. $\mathrm{sPerf}(X) \simeq \mathrm{Perf}(X)$. As noted before, schemes with an ample family of line bundles satisfy the resolution property. Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules of rank $r+1$ and write $\mathbb{P} = \mathbb{P}(\mathcal{E})$. Also set $s = \lceil r/2 \rceil$. Let $\pi : \mathbb{P} \to X$ be the associated projective bundle and let $\mathcal{O} = \mathcal{O}_{\mathbb{P}}$. Write $\mathcal{A}$ for $\mathrm{Perf}(\mathbb{P})$. Let $\mathcal{L}$ be a line bundle on X and let $L = (\mathcal{O}(m) \otimes \pi^*\mathcal{L})[0]$, with $m \in \mathbb{Z}$, be the object of $\mathcal{A}$ consisting of a single copy of $\mathcal{O}(m) \otimes \pi^*\mathcal{L}$ concentrated in cohomological degree 0. Let $\mathcal{A}^{[L]}$ be the pretriangulated dg category with duality given by the mapping complex $[-, L]$; write $\natural$ for this duality on $\mathcal{A}$ and reserve the symbol $\vee$ for the standard duality on $\mathrm{Vect}(X)$, $\mathrm{Vect}(\mathbb{P})$ and their respective categories of perfect complexes. By corollary 7.1.11, $\mathcal{O}(m)$ may be replaced by $\mathcal{O}(m+2i)$ for any $i \in \mathbb{Z}$ in the definition of L without affecting the Grothendieck-Witt spectrum. Therefore, m can be chosen freely up to parity in the proofs below.

The following theorem is contained in the proof of [73, theorem 3.5.1]; a version of this result in the context of stable ∞-categories is [54, theorem B].

Theorem 7.4.1. *The following statements hold.*

(i) For each $k \in \mathbb{Z}$, the assignment $\mathcal{F} \mapsto \mathcal{O}(k) \otimes p^*\mathcal{F}$ defines a fully faithful functor $\mathrm{Perf}(X) \to \mathcal{A}$. The essential image of such a functor will be denoted $\mathcal{A}(k)$.

(ii) For each $i \in \mathbb{Z}$, $\langle \mathcal{A}(i-r), \ldots, \mathcal{A}(i) \rangle$ is a semi-orthogonal decomposition of $\mathcal{A}$.

Proposition 7.4.2. *With notation as in theorem 7.4.1, the essential image of $\mathcal{A}(k)$ under the duality $\natural$ is $\mathcal{A}(m-k)$ for all $k \in \mathbb{Z}$.*

Proof. First note that π^* can be made into a dg form functor (cf. [72, section 9.3], which discusses the functoriality of π^*). Any object of $\mathcal{A}(k)$ can be written as $\pi^*M \otimes \mathcal{O}(k)$ with $M \in \mathrm{Perf}(X)$. The dual of such an object satisfies

$$\begin{aligned}(\pi^*M \otimes \mathcal{O}(k))^\natural &\cong [\pi^*M \otimes \mathcal{O}(k), L]\\ &\cong [\pi^*M, [\mathcal{O}(k), L]]\\ &\cong L \otimes [\pi^*M, \mathcal{O}(-k)]\\ &\cong \mathcal{O}(m) \otimes \pi^*\mathcal{L}[0] \otimes \pi^*[M, \mathcal{O}_X] \otimes \mathcal{O}(-k)\\ &\cong \pi^*[M, \mathcal{L}[0]] \otimes \mathcal{O}(m-k),\end{aligned}$$

where the isomorphisms are given by various results of section 7.1, as well as standard properties of the pullback π^*. Hence $(\pi^*M \otimes \mathcal{O}(k))^\natural$ is an object of $\mathcal{A}(m-k)$. It follows that $\mathcal{A}(k)^\natural \subset \mathcal{A}(m-k)$ and $\mathcal{A}(m-k)^\natural \subset \mathcal{A}(k)$. As $\natural$ is an equivalence, the proof is done. □

The following theorem is the important projective bundle formula for m and r of equal parity. Its proof is an application of additivity for Grothendieck-Witt spectra.

Theorem 7.4.3. *Recall that $s = \lceil r/2 \rceil$. The following statements hold for $n \in \mathbb{Z}$.*

(i) If m and r are even, then there is a stable equivalence of spectra

$$\mathrm{GW}^{[n]}(X, \mathcal{L}) \oplus \mathrm{K}(X)^{\oplus s} \longrightarrow \mathrm{GW}^{[n]}(\mathbb{P}, \pi^*\mathcal{L}(m)).$$

(ii) If m and r are odd, then there is a stable equivalence of spectra

$$\mathrm{K}(X)^{\oplus s} \longrightarrow \mathrm{GW}^{[n]}(\mathbb{P}, \pi^*\mathcal{L}(m)).$$

Proof. Without loss of generality, assume that $m = -r$. By theorem 7.4.1, there is a semi-orthogonal decomposition

$$\langle \mathcal{A}(-r), \mathcal{A}(-r+1), \ldots, \mathcal{A}(0) \rangle$$

of $\mathcal{A}$. The duality maps $\mathcal{A}(i)$ to $\mathcal{A}(-r-i)$ for all $i \in \mathbb{Z}$ by proposition 7.4.2. Hence the additivity theorem 6.2.2 applies and yields the desired result. □

This covers the easy cases of the projective bundle formula. Now set $m = -r-1$. By theorem 7.4.1, there is a semi-orthogonal decomposition

$$\langle \mathcal{A}(-r-1), \mathcal{A}(-r), \ldots, \mathcal{A}(-1) \rangle$$

of $\mathcal{A}$. By proposition 7.4.2, $\mathcal{A}(i)^\natural \subset \mathcal{A}(-r-1-i)$ for all $i \in \mathbb{Z}$. Let

$$\mathcal{A}_0 = \langle \mathcal{A}(-r), \ldots, \mathcal{A}(-1) \rangle.$$

Then the constituents of $\mathcal{A}_0$ are exchanged by the duality, so that $\mathrm{GW}(\mathcal{A}_0)$ may be computed using the additivity theorem 6.2.2. Furthermore, there is a quasi-exact sequence of pretriangulated dg categories with duality

$$\mathcal{A}_0^{[L]} \longrightarrow \mathcal{A}^{[L]} \longrightarrow (\mathcal{A}/\mathcal{A}_0)^{[L]}. \tag{7.4.4}$$

Thus understanding $\mathrm{GW}^{[n]}((\mathcal{A}/\mathcal{A}_0)^{[L]})$ is paramount to understanding $\mathrm{GW}^{[n]}(\mathcal{A}^{[L]})$. Fix $\Delta = \det \pi^*\mathcal{E}(-r-1)$.

Lemma 7.4.5. *There is a quasi-equivalence of pretriangulated dg categories*

$$F : \mathrm{Perf}(X) \longrightarrow \mathcal{A}/\mathcal{A}_0$$

given by $M \mapsto \Delta[r] \otimes \pi^* M$.

Proof. The functor F is the composition

$$\mathrm{Perf}(X) \xrightarrow{\pi^*} \mathcal{A} \xrightarrow{\Delta[r]\otimes -} \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{A}_0.$$

Denote the composition $\mathrm{Perf}(X) \to \mathcal{A}$ of the first two maps by F'. Then

$$\begin{aligned} F'(\det \mathcal{E}^\vee[-r]) &= \Delta[r] \otimes \pi^* \det \mathcal{E}^\vee[-r] \\ &\cong \det \pi^*\mathcal{E}[r] \otimes \pi^* \det \mathcal{E}^\vee[-r] \otimes \mathcal{O}(-r-1) \\ &\cong \mathcal{O}(-r-1). \end{aligned}$$

As tensoring with $\det \mathcal{E}^\vee[-r]$ gives a self-equivalence of $\mathrm{Perf}(X)$, the essential image of F' consists of objects of the form $\pi^*M \otimes \mathcal{O}(-r-1)$ and is therefore the subcategory $\mathcal{A}(-r-1)$ of $\mathcal{A}$ of theorem 7.4.1. In particular, $F' : \mathrm{Perf}(X) \to \mathcal{A}(-r-1)$ is a quasi-equivalence. Hence F factors through the canonical map $F'' : \mathcal{A}(-r-1) \to \mathcal{A}/\mathcal{A}_0$. Since $\langle \mathcal{A}(-r-1), \mathcal{A}_0 \rangle$ is a semi-orthogonal decomposition of $\mathcal{A}$, it follows that F'' is a quasi-equivalence. Therefore F, being a composition of quasi-equivalences, is also a quasi-equivalence, as was to be shown. □

For the next lemma, it will be convenient to use the machinery of triangulated categories. Let w be the class of quasi-isomorphisms in $\mathcal{A}$ and denote by $\mathcal{TA}$ the triangulated category $w^{-1}H^0\mathcal{A}$, which is equivalent to the bounded derived category $D^b(\mathrm{Vect}(\mathbb{P}))$. By [72, lemma 3.6], any duality on $\mathcal{A}$ is inherited by $\mathcal{TA}$. The Verdier quotient $\mathcal{TA}/\mathcal{TA}_0$ is the triangulated category $v^{-1}H^0\mathcal{A}$, where v is the class of morphisms in $\mathcal{A}$ whose cone lies in $\mathcal{A}_0$.

Let $M[-1] = K^{\leq -1}[-1]$. Consider the chain map $\alpha : M[-1] \to [M[-1], \Delta[r]]$ given by

$$\begin{array}{ccccccccc} K^{-r-1} & \xrightarrow{-d} & K^{-r} & \xrightarrow{-d} & \dots & \xrightarrow{-d} & K^{-2} & \xrightarrow{-d} & K^{-1} \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle d} \\ K^{-r} & \xrightarrow{-d} & K^{-r+1} & \xrightarrow{-d} & \dots & \xrightarrow{-d} & K^{-1} & \xrightarrow{-d} & K^{0} \end{array} \tag{7.4.6}$$

concentrated in degrees $[-r, 0]$. The bottom complex is identified with $[M[-1], \Delta[r]]$ via the perfect pairings $\nu_i : K^i \to [K^{-r-1-i}, \Delta]$ given by $x \mapsto (y \mapsto x \wedge y)$ for $-r-1 \leq i \leq 0$. Note that α factors as

$$\begin{array}{ccccccccc} K^{-r-1} & \xrightarrow{-d} & K^{-r} & \xrightarrow{-d} & \dots & \xrightarrow{-d} & K^{-2} & \xrightarrow{-d} & K^{-1} \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle d} \\ 0 & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & \mathcal{O} \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle 1} \\ K^{-r} & \xrightarrow{-d} & K^{-r+1} & \xrightarrow{-d} & \dots & \xrightarrow{-d} & K^{-1} & \xrightarrow{-d} & K^{0}, \end{array}$$

where the map $M[-1] \to \mathcal{O}$ is a quasi-isomorphism in $\mathcal{A}$ and the cokernel of the inclusion $\mathcal{O} \to [M[-1], \Delta[r]]$ lies in $\mathcal{A}_0$. Therefore, α is a weak equivalence in $\mathcal{A}/\mathcal{A}_0$. It will be shown that α is in fact a nondegenerate symmetric form in $\mathcal{TA}/\mathcal{TA}_0$.

Lemma 7.4.7. *The chain map $\alpha : M[-1] \to [M[-1], \Delta[r]]$ defined above defines a nondegenerate symmetric form in the Verdier quotient $\mathcal{TA}/\mathcal{TA}_0$. In particular, α defines an element $[\Delta[r], \beta] \in \mathrm{GW}_0((\mathcal{A}/\mathcal{A}_0)^{[\Delta[r]]})$.*

Proof. It has already been shown that α is a weak equivalence in $\mathcal{A}/\mathcal{A}_0$, so it remains to be shown that α is a symmetric form in $\mathcal{TA}/\mathcal{TA}_0$ with the duality $\vee$ induced by $\Delta[r]$, and it will suffice to show that α is a symmetric form in $\mathcal{TA}$ with the same duality $\vee$.

Using the identities (7.1.1) and the fact that K is a differential graded algebra, one computes $\alpha^\vee$ can to be the chain map

$$\begin{tikzcd}
K^{-r-1} \arrow[r, "-d"] \arrow[d, "(-1)^r d"] & K^{-r} \arrow[r, "-d"] \arrow[d, "0"] & \dots \arrow[r, "-d"] & K^{-2} \arrow[r, "-d"] \arrow[d, "0"] & K^{-1} \arrow[d, "0"] \\
K^{-r} \arrow[r, "-d"] & K^{-r+1} \arrow[r, "-d"] & \dots \arrow[r, "-d"] & K^{-1} \arrow[r, "-d"] & K^{0}.
\end{tikzcd}$$

The homogeneous map $h \in [M[-1], [M[-1], \Delta[r]]]_1$ of degree 1 given by

$$\begin{tikzcd}
K^{-r-1} \arrow[r, "-d"] \arrow[d] & K^{-r} \arrow[r, "-d"] \arrow[d, "(-1)^r"] & K^{-r+1} \arrow[r, "-d"] \arrow[d, "(-1)^{r-1}"] & \dots \arrow[r, "-d"] & K^{-2} \arrow[r, "-d"] \arrow[d, "(-1)^2"] & K^{-1} \arrow[r] \arrow[d, "-1"] & 0 \arrow[d] \\
0 \arrow[r] & K^{-r} \arrow[r, "-d"] & K^{-r+1} \arrow[r, "-d"] & \dots \arrow[r, "-d"] & K^{-2} \arrow[r, "-d"] & K^{-1} \arrow[r, "-d"] & K^{0}
\end{tikzcd}$$

defines a chain homotopy from α to $\alpha^\vee$ can in $\mathcal{A}$ since the sum of the compositions of the sides of the leftmost square is $(-1)^{r+1}d = (\alpha - \alpha^\vee)_{-r}$, the sum of the compositions of the sides of the rightmost square is $d = (\alpha - \alpha^\vee)_0$, and the sum of the compositions of the sides is $0 = (\alpha - \alpha^\vee)_i$ for all the inner squares with top left corner K^i, $-r \le i \le -2$. Therefore, $\alpha = \alpha^\vee$ can in $\mathcal{TA}$.

Note that $M[-1] \cong \Delta[r]$ in $\mathcal{TA}/\mathcal{TA}_0$, since the kernel of the natural projection $M[-1] \to \Delta[r]$ lies in $\mathcal{A}_0$. By [72, proposition 3.8], α defines an element $[\Delta[r], \beta]$ of $\mathrm{GW}_0((\mathcal{A}/\mathcal{A}_0)^{[\Delta[r]]})$, as was to be shown. □

Proposition 7.4.8. *The quasi-equivalence $F : \mathrm{sPerf}(X) \to \mathcal{A}/\mathcal{A}_0$ of proposition 7.4.5 can be made into a nondegenerate dg form functor*

$$(F, \eta) : \mathrm{sPerf}(X)^{[0]} \to (\mathcal{A}/\mathcal{A}_0)^{[\Delta[r]]}.$$

In particular, there is a quasi-equivalence of pretriangulated dg categories with weak equivalences and duality

$$\mathrm{sPerf}(X)^{[\det \mathcal{E}^\vee \otimes \mathcal{L}[-r]]} \simeq (\mathcal{A}/\mathcal{A}_0)^{[L]}.$$

Proof. Let β be a symmetric form in $\mathcal{A}^{[\Delta[r]]}$ whose image in $(\mathcal{A}/\mathcal{A}_0)^{[\Delta[r]]}$ corresponds to the element $[\Delta[r], \beta]$ of lemma 7.4.7, by abuse of notation. Although β itself might be degenerate, its image in $(\mathcal{A}/\mathcal{A}_0)^{[\Delta[r]]}$ is nondegenerate.

The quasi-equivalence F of proposition 7.4.8 is equivalent to the composition

$$F : \quad \mathrm{sPerf}(X)^{[0]} \xrightarrow{\pi^*} \mathcal{A}^{[0]} \xrightarrow{(\Delta[r],\beta)\otimes -} \mathcal{A}^{[\Delta[r]]} \longrightarrow (\mathcal{A}/\mathcal{A}_0)^{[\Delta[r]]},$$

now ornamented with the dualities of each category. Note that π^* is a nondegenerate dg form functor by [72, section 9.3], and the composition $\mathcal{A}^{[0]} \to (\mathcal{A}/\mathcal{A}_0)^{[\Delta[r]]}$ is a nondegenerate dg form functor by proposition 7.1.6 and lemma 7.4.7. Thus, equipping F with the duality compatibility morphism η induced by β yields a quasi-equivalence (F, η) of pretriangulated dg categories with weak equivalences and duality.

Finally, twisting the duality in $\mathrm{sPerf}(X)^{[0]}$ by the invertible complex $\det \mathcal{E}^\vee \otimes \mathcal{L}[-r]$ gives the desired quasi-equivalence

$$\mathrm{sPerf}(X)^{[\det \mathcal{E}^\vee \otimes \mathcal{L}[-r]]} \simeq (\mathcal{A}/\mathcal{A}_0)^{[L]},$$

and the proof is done. □

The following technical lemma will be instrumental in the construction of the splitting of the homotopy fibrations of theorem 7.4.10.

Lemma 7.4.9. *If r is even, let $[H, \psi] \in \mathrm{GW}_0(\mathcal{A}^{[\Delta[r]]})$ be the element of proposition 7.2.3. If r is odd and the element ν' of lemma 7.2.4 vanishes in $\mathrm{W}_0^{[r+1]}(X, \det \mathcal{E})$, let $[H, \psi] \in \mathrm{GW}_0(\mathcal{A}^{[\Delta[r]]})$ be the element of theorem 7.2.10. Let $[\Delta[r], \beta]$ be the element in $\mathrm{GW}_0((\mathcal{A}/\mathcal{A}_0)^{[\Delta[r]]})$ of lemma 7.4.7.*

In both cases,

$$(-1)^s[H, \psi] = [\Delta[r], \beta]$$

in $\mathrm{GW}_0((\mathcal{A}/\mathcal{A}_0)^{[\Delta[r]]})$.

Proof. Let $\vee$ denote the duality on $\mathcal{A}$ induced by $\Delta[r]$. First assume that r is even. Consider the element $[H, \psi] \in \mathrm{GW}_0(\mathcal{A}^{[\Delta[r]]})$ of proposition 7.2.3. The chain map $M[-1] \to H$ given by

$$\begin{array}{ccccccccccc}
K^{-r-1} & \longrightarrow & \dots & \longrightarrow & K^{-s-1} & \longrightarrow & K^{-s} & \longrightarrow & \dots & \longrightarrow & K^{-1} \\
\downarrow{\scriptstyle 1} & & & & \downarrow{\scriptstyle 1} & & \downarrow & & & & \\
K^{-r-1} & \longrightarrow & \dots & \longrightarrow & K^{-s-1} & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0
\end{array}$$

is a weak equivalence in $\mathcal{A}/\mathcal{A}_0$, so $[H, \psi] = [M[-1], \psi']$. Recall from lemma 7.4.7 that $[\Delta[r], \beta] = [M[-1], \alpha]$. Hence, it suffices to show that $(-1)^s\psi'$ and α are chain homotopy equivalent. The homogeneous map $h \in [M[-1], [M[-1], \Delta[r]]]_1$ of degree 1 given by

$$\begin{array}{ccccccccccccccccc}
K^{-r-1} & \xrightarrow{-d} & K^{-r} & \xrightarrow{-d} & \dots & \xrightarrow{-d} & K^{-s-1} & \xrightarrow{-d} & K^{-s} & \xrightarrow{-d} & \dots & \xrightarrow{-d} & K^{-2} & \xrightarrow{-d} & K^{-1} & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle 0} & & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle (-1)^s} & & & & \downarrow{\scriptstyle (-1)^2} & & \downarrow{\scriptstyle -1} & & \downarrow \\
0 & \longrightarrow & K^{-r} & \xrightarrow{-d} & \dots & \xrightarrow{-d} & K^{-s-1} & \xrightarrow{-d} & K^{-s} & \xrightarrow{-d} & \dots & \xrightarrow{-d} & K^{-2} & \xrightarrow{-d} & K^{-1} & \xrightarrow{-d} & K^0
\end{array}$$

defines the required chain homotopy from α to $(-1)^s\psi'$.

Now assume that r is odd and that the element ν' of lemma 7.2.4 vanishes in $\mathrm{W}_0^{[r+1]}(X, \det \mathcal{E})$. Let $[H, \psi] \in \mathrm{GW}_0(\mathcal{A}^{[\Delta[r]]})$ be the element of theorem 7.2.10. There is a canonical chain map $\gamma : M[-1] \to H$ given by the identity maps $1 : K^{-i-1} \to K^{-i-1}$ in degrees $-r \leq i \leq -s-1$, the canonical injection $K^{-s-1} \to K^{-s-1} \oplus \mathcal{S}$ in degree $-s$, the composition $\rho : K^{-s} \to K^{-s} \oplus \mathcal{N} \to \mathcal{P}$ in degree $-s+1$, and the zero map in all other degrees. The canonical weak equivalence $M[-1] \to \Delta[r]$ in $\mathcal{A}/\mathcal{A}_0$ factors through γ, and the canonical projection $H \to \Delta[r]$ is a weak equivalence since $\mathcal{N}$, $\mathcal{P}$ and $\mathcal{S}$ lie in $\mathcal{A}_0$ by construction of H, so γ is a weak equivalence in $\mathcal{A}/\mathcal{A}_0$ by two out of three. Let $\psi' = \gamma^\vee \psi \gamma$. Then $[H, \psi] = [M[-1], \psi']$ in $\mathrm{GW}_0((\mathcal{A}/\mathcal{A}_0)^{[\Delta[r]]})$. As before, it suffices to show that α is chain homotopic to $(-1)^s \psi'$. For this, it will be useful to zoom in on the central square of H, defined in lemma 7.2.7. Let $\natural$ be the duality on $\mathrm{Vect}(\mathbb{P})$ induced by Δ. In degrees $[-s-1, -s+2]$, ψ' is given by the following commutative diagram

$$\begin{tikzcd}
K^{-s-2} \arrow[r, "-d"] \arrow[dddd, "0"] & K^{-s-1} \arrow[rr, "-d"] \arrow[d] & & K^{-s} \arrow[r, "-d"] \arrow[dl, "\upsilon"'] \arrow[d, "\rho"] & K^{-s+1} \arrow[dddd, "0"] \\
& K^{-s-1} \oplus \mathcal{S} \arrow[r, "-(d\oplus\alpha)"] \arrow[d, "d\oplus\alpha"'] & K^{-s} \oplus \mathcal{N} \arrow[r, two heads, "\mathrm{pr}_{\mathcal{P}}"] \arrow[dl, "-1"] & \mathcal{P} \arrow[d, hook, "(-1)^s(\nu\oplus\sigma)\iota_{\mathcal{P}}"] & \\
& K^{-s} \oplus \mathcal{N} \arrow[d, two heads, "\iota_{\mathcal{P}}^{\natural}(\nu\oplus\sigma)"'] & & (K^{-s})^{\natural} \oplus \mathcal{N}^{\natural} \arrow[dl, "(-1)^s"'] \arrow[d, "(d\oplus\alpha)^{\natural}"] & \\
& \mathcal{P}^{\natural} \arrow[r, hook, "\mathrm{pr}_{\mathcal{P}}^{\natural}"'] \arrow[d, "\rho^{\natural}"'] & (K^{-s})^{\natural} \oplus \mathcal{N}^{\natural} \arrow[r, "(-1)^s(d\oplus\alpha)^{\natural}"'] \arrow[dl, "\upsilon^{\natural}"] & (K^{-s-1})^{\natural} \oplus \mathcal{S}^{\natural} \arrow[d] & \\
(K^{-s+1})^{\natural} \arrow[r, "(-1)^{s+1}d^{\natural}"'] & (K^{-s})^{\natural} \arrow[rr, "(-1)^s d^{\natural}"'] & & (K^{-s-1})^{\natural} \arrow[r, "(-1)^{s-1}d^{\natural}"'] & (K^{-s-2})^{\natural}
\end{tikzcd}$$

Note that the vertical map $K^{-s-1} \to (K^{-s})^\natural$ is $\upsilon^\natural \,\mathrm{pr}_{\mathcal{P}}^\natural \,\iota_{\mathcal{P}}^\natural(\nu \oplus \sigma)\upsilon d$. Furthermore, it follows from the proof of lemma 7.2.7 that

$$\begin{aligned}
d^\natural \upsilon^\natural (\nu \oplus \sigma)\iota_{\mathcal{P}} \,\mathrm{pr}_{\mathcal{P}}\, \upsilon &= d^\natural \upsilon^\natural (1 - \mathrm{pr}_{\mathcal{P}}^\natural \,\iota_{\mathcal{P}}^\natural)(\nu \oplus \sigma)\upsilon \\
&= d^\natural \upsilon^\natural (\nu \oplus \sigma)\upsilon - d^\natural \upsilon^\natural \,\mathrm{pr}_{\mathcal{P}}^\natural \,\iota_{\mathcal{P}}^\natural(\nu \oplus \sigma)\upsilon \\
&= d^\natural \nu - d^\natural \upsilon^\natural \,\mathrm{pr}_{\mathcal{P}}^\natural \,\iota_{\mathcal{P}}^\natural(\nu \oplus \sigma)\upsilon.
\end{aligned}$$

For $-r-1 \le i \le 0$, let $\nu_i : K^i \to (K^{-r-1-i})^\natural$ be the usual perfect pairing and note that $\nu = \nu_{-s}$. Then $d^\natural \nu_i = (-1)^{i+1}\nu_{i+1} d$ for all $-r-1 \le i \le -1$ by an application of the graded Leibniz rule. Define a homogeneous map

$$h \in [M[-1], [M[-1], \Delta[r]]]_1$$

of degree 1, with the following components $h_i : K^{i-1} \to (K^{-r-i})^\natural$ in degrees $-r \le i \le 0$:

$$h_i = \begin{cases} 0 & \text{if } -r \le i \le -s; \\ -\upsilon^\natural \,\mathrm{pr}_{\mathcal{P}}^\natural \,\iota_{\mathcal{P}}^\natural(\nu \oplus \sigma)\upsilon & \text{if } i = -s+1; \text{ and} \\ (-1)^{i+s}\nu_{i-1} & \text{if } -s+2 \le i \le 0. \end{cases}$$

By construction,

$$(-1)^{-i-1} d^\natural h_i + h_{i+1}(-d) = \begin{cases} \psi_i & \text{if } -s \le i \le -s+1; \\ 0 & \text{if } -s+2 \le i \le -1; \text{ and} \\ (-1)^{s+1} d^\natural \nu_{-1} & \text{if } i = 0. \end{cases}$$

Note that $(-1)^{s+1}d^\natural\nu_{-1} = (-1)^{s+1}\nu_0 d = (-1)^{s+1}\alpha_0$. It follows that h is a chain homotopy from ψ' to $(-1)^s\alpha$, as was to be shown. $\square$

Now there are no more obstacles to the computation of the Grothendieck-Witt spectrum of projective bundles when the parities of r and m differ.

Theorem 7.4.10. *The following statements hold.*

(i) If r is even and m is odd, then there is a split homotopy fibration

$$\mathrm{K}(X)^{\oplus s} \longrightarrow \mathrm{GW}^{[n]}(\mathbb{P}, \pi^*\mathcal{L}(m)) \longrightarrow \mathrm{GW}^{[n-r]}(X, \det\mathcal{E}^\vee \otimes \mathcal{L}).$$

(ii) If r is odd and m is even, then there is a homotopy fibration

$$\mathrm{GW}^{[n]}(X,\mathcal{L}) \oplus \mathrm{K}(X)^{\oplus s-1} \to \mathrm{GW}^{[n]}(\mathbb{P}, \pi^*\mathcal{L}(m)) \to \mathrm{GW}^{[n-r]}(X, \det\mathcal{E}^\vee \otimes \mathcal{L}),$$

which splits if the element ν' of lemma 7.2.4 vanishes in $\mathrm{W}_0^{[r+1]}(X, \det\mathcal{E})$. The condition for the splitting is satisfied e.g. if $\mathcal{E}$ is a trivial bundle.

Proof. Without loss of generality, one may assume $m = -r-1$ in both cases. The quasi-exact sequence (7.4.4) gives rise to a homotopy fibration of Grothendieck-Witt spectra

$$\mathrm{GW}^{[n]}(\mathcal{A}_0^{[L]}) \longrightarrow \mathrm{GW}^{[n]}(\mathcal{A}^{[L]}) \longrightarrow \mathrm{GW}^{[n]}((\mathcal{A}/\mathcal{A}_0)^{[L]})$$

by [72, theorem 6.6]. Note that $\mathrm{GW}^{[n]}(\mathcal{A}^{[L]})$ is just a different way of writing $\mathrm{GW}^{[n]}(\mathbb{P}, \pi^*\mathcal{L}(m))$. As already remarked, the additivity theorem 6.2.2 gives a formula for the first term $\mathrm{GW}^{[n]}(\mathcal{A}_0^{[L]})$ of both (i) and (ii). Thus it suffices to show that there is a quasi-equivalence

$$F : \mathrm{sPerf}(X)^{[\det\mathcal{E}^\vee\otimes\mathcal{L}[-r]]} \longrightarrow (\mathcal{A}/\mathcal{A}_0)^{[L]},$$

but this holds by proposition 7.4.8. This yields both claimed homotopy fibrations.

If r is even or r is odd and the element ν' of lemma 7.2.4 vanishes in the Witt group $W_0^{[r+1]}(X, \det\mathcal{E})$, either proposition 7.2.3 or theorem 7.2.10 constructs an element $[H,\psi] \in \mathrm{GW}_0(\mathcal{A}^{[\Delta[r]]})$. Thus the composition

$$F' : \quad \mathrm{sPerf}(X)^{[0]} \xrightarrow{\pi^*} \mathcal{A}^{[0]} \xrightarrow{(-1)^s(H,\psi)\otimes -} \mathcal{A}^{[\Delta[r]]}$$

induces a map $(-1)^s[H,\psi]\cup - : \mathrm{GW}^{[-r]}(X, \det\mathcal{E}^\vee\otimes\mathcal{L}) \to \mathrm{GW}^{[0]}(\mathbb{P}, \pi^*\mathcal{L}(m))$ of Grothendieck-Witt spectra. To show that this map splits the homotopy fibration, it suffices to show that the triangle of pretriangulated dg categories with duality

$$\begin{tikzcd} & \mathcal{A}^{[\Delta[r]]} \arrow[dr] & \\ \mathrm{sPerf}(X)^{[0]} \arrow[ur, "F'"] \arrow[rr, "F"] & & (\mathcal{A}/\mathcal{A}_0)^{[\Delta[r]]} \end{tikzcd}$$

commutes up to natural weak equivalence, but this follows from lemma 7.4.9, which states that $(-1)^s[H,\psi] = [\Delta[r],\beta]$.

Note that lemma 7.2.11 provides a sufficient condition for ν' to vanish, and that this condition holds in particular if $\mathcal{E}$ is a trivial bundle. $\square$

8 Atiyah-Segal completion for split tori

The time has come to take the proverbial bull by the horns and prove an important special case of the Atiyah-Segal completion theorem. The proof for split tori leans heavily on the projective bundle formula for Grothendieck-Witt theory that was proven in section 7. Atiyah-Segal completion for Grothendieck-Witt theory is a first step to other generalizations of Atiyah-Segal completion for non-oriented cohomology theories.

8.1 Equivariance for split tori

Representations of split tori over a field k correspond to multi-graded vector spaces. This phenomenon is not limited to representations, but extends to equivariant sheaves when the base scheme is not the spectrum of some field k anymore. This significantly simplifies computations of equivariant cohomology, which is why results are often first proved for split tori, cf. [2] and [57].

Fix a base scheme S and let T be a split torus of rank t over S, see example 3.1.5. Let X be a scheme over S with the resolution property and equip it with the trivial T-action. Since T-equivariant sheaves of $\mathcal{O}_X$-modules correspond to $\mathbb{Z}^t$-graded sheaves of $\mathcal{O}_X$-modules (c.f. [22, proposition 1.1.17]), a finite locally free $\mathcal{O}_X$-module $\mathcal{F}$ equipped with a T-equivariant structure decomposes as a direct sum

$$\mathcal{F} = \bigoplus_{\lambda \in \mathbb{Z}^t} \mathcal{F}_\lambda$$

of finite locally free $\mathcal{O}_X$-modules $\mathcal{F}_\lambda$, where all but finitely many of the $\mathcal{F}_\lambda$ are zero. For such $\mathcal{F}$, denote by $W_\mathcal{F} \subset \mathbb{Z}^t$ the subset of all $\lambda \in \mathbb{Z}^t$ such that $\mathcal{F}_\lambda \neq 0$. These λ are called the *weights of* $\mathcal{F}$. Note that the trivial action induces the trivial $\mathbb{Z}^t$-grading on $\mathcal{O}_X$, so that it is concentrated in multi-degree $(0, \ldots, 0) \in \mathbb{Z}^t$.

Write $\mathrm{Vect}^T(X)$ for the exact category of T-equivariant finite locally free $\mathcal{O}_X$-modules, and $\mathcal{A} = \mathrm{Perf}^T(X)$ for the corresponding dg category of perfect complexes of T-equivariant finite locally free $\mathcal{O}_X$-modules. If $S = \operatorname{Spec} k$ for some field k, then a T-equivariant finite locally free sheaf on S is a representation of T, which is equivalent to $\mathbb{Z}^t$-graded vector space.

The following proposition gives a semi-orthogonal decomposition of $\mathcal{A}$ to facilitate computations of its K-theory and GW-theory.

Proposition 8.1.1. *For $\lambda \in \mathbb{Z}^t$, let $\mathcal{A}_\lambda$ be the pretriangulated dg subcategory of $\mathcal{A}$ consisting of perfect complexes of T-equivariant locally free $\mathcal{O}_X$-modules concentrated in multi-degree λ. The following statements hold:*

(i) for each $\lambda \in \mathbb{Z}^t$, $\mathcal{A}_\lambda$ is equivalent to $\mathrm{Perf}(X)$*;*

(ii) for $\mu, \lambda \in \mathbb{Z}^t$ such that $\mu \neq \lambda$, $M \in \mathcal{A}_\mu$ and $L \in \mathcal{A}_\lambda$,

$$[M, L]^T = 0 \qquad \textit{and} \qquad [L, M]^T = 0,$$

where $[-,-]^T$ is the internal mapping complex of $\mathcal{A}$; and

(iii) there is a semi-orthogonal decomposition $\mathcal{A} = \langle \mathcal{A}_\lambda \mid \lambda \in \mathbb{Z}^t \rangle$.

Proof. For $\lambda \in \mathbb{Z}^t$, the functor $\mathrm{Perf}(X) \to \mathcal{A}_\lambda$ sending a locally free $\mathcal{O}_X$-module $\mathcal{F}$ to the T-equivariant locally free $\mathcal{O}_X$-module $\mathcal{F}(\lambda)$ such that $W_{\mathcal{F}(\lambda)} = \{\lambda\}$ and $\mathcal{F}(\lambda)_\lambda = \mathcal{F}$ is an equivalence.

Furthermore, T-equivariant maps $\mathcal{F} \to \mathcal{G}$ of T-equivariant locally free $\mathcal{O}_X$-modules respect the induced $\mathbb{Z}^t$-gradings on $\mathcal{F}$ and $\mathcal{G}$. So if $W_\mathcal{F} \cap W_\mathcal{G} = \emptyset$, then $\mathscr{H}om^T(\mathcal{F}, \mathcal{G}) = 0$, and this extends to perfect complexes.

Lastly, every object M of $\mathcal{A}$ decomposes as a direct sum

$$M = \bigoplus_{\lambda \in \mathbb{Z}^t} M_\lambda$$

with M_λ in $\mathcal{A}_\lambda$ for all $\lambda \in \mathbb{Z}^t$. □

This yields the following generalization of an important result in the representation theory of split tori.

Corollary 8.1.2. *There is an isomorphism of* $\mathrm{K}_0(X)$*-algebras*

$$\mathrm{K}_*(\mathcal{A}) \simeq \mathrm{K}_*(X) \otimes_{\mathrm{K}_0(X)} \mathrm{K}_0(\mathcal{A})$$

In particular, there is an isomorphism of rings

$$\mathrm{K}_0(\mathcal{A}) \cong \frac{\mathrm{K}_0(X)[x_1, \dots, x_t, y_1, \dots, y_t]}{(x_i y_i + x_i + y_i \mid i = 1, \dots, t)}.$$

Proof. By additivity for K-theory and proposition 8.1.1, there are isomorphisms of $\mathrm{K}_0(X)$-modules

$$\mathrm{K}_i(X) \otimes_{\mathrm{K}_0(X)} \left(\bigoplus_{\lambda \in \mathbb{Z}^t} \mathrm{K}_0(X) \right) \longrightarrow \mathrm{K}_i(\mathcal{A}),$$

which form an isomorphism of graded $\mathrm{K}_0(X)$-algebras

$$K_*(\mathcal{A}) \simeq \mathrm{K}_*(X) \otimes_{\mathrm{K}_0(X)} \mathrm{K}_0(\mathcal{A}).$$

For $\lambda \in \mathbb{Z}^t$, let $\mathcal{O}_X(\lambda)$ be the T-equivariant $\mathcal{O}_X$-module with $W_{\mathcal{O}_X(\lambda)} = \{\lambda\}$ and $\mathcal{O}_X(\lambda)_\lambda = \mathcal{O}_X$. For $1 \leq i \leq n$, let $e_i \in \mathbb{Z}^t$ be the i-th unit vector, and write x_i and y_i for the K-theory classes $[\mathcal{O}_X(e_i)] - 1$ and $[\mathcal{O}_X(-e_i)] - 1$, respectively. Then $x_i y_i + x_i + y_i = 0$ for all i. Hence

$$\mathrm{K}_0(\mathcal{A}) \cong \frac{\mathrm{K}_0(X)[x_1, \dots, x_t, y_1, \dots, y_t]}{(x_i y_i + x_i + y_i \mid i = 1, \dots, t)},$$

as was to be shown. □

Remark 8.1.3. The K-theory of $\mathrm{Proj}(R)$ where R was a strongly $\mathbb{Z}$-graded ring is computed in [43], and the methods of the computation can likely be extended to compute its Grothendieck-Witt theory.

Let $(\mathbb{Z}^t - \{0\})/\{\pm\}$ be the quotient of $\mathbb{Z}^t - \{0\}$ by the sign involution. A useful system of representatives C of this is given by nonzero $(a_1, \dots, a_t) \in \mathbb{Z}^t$ such that the first nonzero entry a_i is positive; if $t = 1$, C consists of the positive integers.

Corollary 8.1.4. *Let* $\mathrm{K}(X)_\lambda = \mathrm{K}(\mathcal{A}_\lambda)$ *for all* $\lambda \in C$. *For each* $i, n \in \mathbb{Z}$, *the map of* $\mathrm{GW}_0^{[0]}(X)$*-modules*

$$\mathrm{GW}_i^{[n]}(X) \oplus \bigoplus_{\lambda \in C} \mathrm{K}_i(X)_\lambda \longrightarrow \mathrm{GW}_i^{[n]}(\mathcal{A}) \tag{8.1.5}$$

induced by the dg form functor

$$\mathcal{A}_0 \times \bigoplus_{\lambda \in C} H\mathcal{A}_\lambda \longrightarrow \mathcal{A}$$
$$A_0 \times \bigoplus_{\lambda \in C} (A_\lambda, B_\lambda) \longrightarrow A_0 \oplus \left(\bigoplus_{\lambda \in C} A_\lambda \oplus B_\lambda^\vee \right)$$

is an isomorphism. In particular, the map

$$\begin{aligned} \mathrm{GW}_0^{[0]}(X) \oplus \bigoplus_{\lambda \in C} \mathrm{K}_0(X)_\lambda &\longrightarrow \mathrm{GW}_0^{[0]}(\mathcal{A}) \\ (a, (b_\lambda)_\lambda) &\longmapsto a(0) + \sum_{\lambda \in C} (H_0(b_\lambda(\lambda)) - 2) \end{aligned} \tag{8.1.6}$$

is an isomorphism.

Proof. Let $\lambda \in \mathbb{Z}^t$. If $\lambda = 0$, then $\mathcal{A}_\lambda$ is fixed by the standard duality on $\mathcal{A}$. Otherwise, $(\mathcal{A}_\lambda)^\vee = \mathcal{A}_{-\lambda}$. Thus, letting $\mathcal{A}_+ = \langle \mathcal{A}_\lambda \mid \lambda \in C \rangle$, there is a semi-orthogonal decomposition $\mathcal{A} = \langle \mathcal{A}_+^\vee, \mathcal{A}_0, \mathcal{A}_+ \rangle$, and the result follows from proposition 8.1.1 and corollary 6.2.3.

Furthermore, there is an automorphism on

$$\mathrm{GW}_0^{[0]}(X) \oplus \bigoplus_{\lambda \in C} \mathrm{K}_0(X)_\lambda,$$

which is the identity on $\mathrm{GW}_0^{[0]}(X)$ and given by $[\mathcal{O}_X(\lambda)] \mapsto [\mathcal{O}_X(\lambda)] - 1$ for $\lambda \in C$. By composing the map of $\mathrm{GW}_0^{[0]}(X)$-modules induced by (8.1.5) with this automorphism, one obtains the -2 term on the right hand side of (8.1.6). $\square$

The following two definitions are instrumental in the statement and proof of Atiyah-Segal completion for split tori.

Definition 8.1.7. Let $R_O = \mathrm{GW}_0^{[0]}(\mathcal{A})$. The map $\alpha : R_O \to \mathrm{GW}_0^{[0]}(X)$ which forgets the equivariant structure is called the *Hermitian augmentation map*. Its kernel $I_O = \ker \alpha$ is called the *Hermitian augmentation ideal*.

Definition 8.1.8. For any $n, i \in \mathbb{Z}$, the kernel of the map $\mathrm{GW}_i^{[n]}(\mathcal{A}) \to \mathrm{GW}_i^{[n]}(X)$ is the *reduced Grothendieck-Witt group* $\widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A})$. The groups $\widetilde{K}_i(\mathcal{A})$ and $\widetilde{\mathrm{W}}^{[n]}(\mathcal{A})$ are defined similarly.

By corollary 8.1.4,

$$I_O \cong \bigoplus_{\lambda \in C} \mathrm{K}_0(X)$$

under the isomorphism 8.1.6.

For $R = \mathrm{K}_0(\mathcal{A})$, the augmentation map $\alpha : R \to \mathrm{K}_0(X)$ has kernel I, which is given by $(x_1, \ldots, x_t, y_1, \ldots, y_t)$ under the isomorphism of corollary 8.1.2. For $\lambda, \mu \in \mathbb{N}^t$, let

$$\boldsymbol{x}^\lambda \boldsymbol{y}^\mu = \prod_{i=1}^{t} x_i^{\lambda_i} y_i^{\mu_i}.$$

The identity $x_i y_i = -(x_i + y_i)$ on R shows that R is generated as a $\mathrm{K}_0(X)$-module by monomials $\boldsymbol{x}^\lambda \boldsymbol{y}^\mu$ with $\lambda, \mu \in \mathbb{N}^t$ such that, for each i, either $\lambda_i = 0$ or $\mu_i = 0$.

The standard duality on $\mathcal{A}$ induces an involution $* : R \to R$, given by $x_i \mapsto y_i$. Let $F_0 : \mathrm{GW}_0^{[0]}(\mathcal{A}) \to \mathrm{K}_0(\mathcal{A})$ be the forgetful map. Note that F_0 restricts to a map $F_0 : I_O \to I$. The following lemma shows that R_O splits as $\mathrm{GW}_0^{[0]}(X)$ and the $*$-fixed points of I.

Lemma 8.1.9. *Let I_+ be the $*$-fixed points of I. The map $F_0 : I_O \to I$ is injective with image I_+.*

Proof. Consider the following diagram

$$\begin{array}{ccccccc}
\widetilde{\mathrm{GW}}_0^{[3]}(\mathcal{A}) & \xrightarrow{F_3} & \widetilde{\mathrm{K}}_0(\mathcal{A}) & \xrightarrow{H_0} & \widetilde{\mathrm{GW}}_0^{[0]}(\mathcal{A}) & \longrightarrow & \widetilde{\mathrm{W}}_0^{[0]}(\mathcal{A}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{GW}_0^{[3]}(\mathcal{A}) & \xrightarrow{F_3} & \mathrm{K}_0(\mathcal{A}) & \xrightarrow{H_0} & \mathrm{GW}_0^{[0]}(\mathcal{A}) & \longrightarrow & \mathrm{W}^{[0]}(\mathcal{A}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{GW}_0^{[3]}(X) & \xrightarrow{F_3} & \mathrm{K}_0(X) & \xrightarrow{H_0} & \mathrm{GW}_0^{[0]}(X) & \longrightarrow & \mathrm{W}^{[0]}(X)
\end{array}$$

of Karoubi sequences [72, theorem 6.1] of R_O-modules. Note that $\widetilde{\mathrm{K}}_0(\mathcal{A}) = I$ and $\widetilde{\mathrm{GW}}_0^{[0]}(\mathcal{A}) = I_O$. The image of the forgetful map $F_0 : I_O \to I$ is necessarily contained in I_+. Furthermore, I_+ is generated as a $\mathrm{K}_0(X)$-module by elements of the form $\boldsymbol{x}^\lambda \boldsymbol{y}^\mu + \boldsymbol{x}^\mu \boldsymbol{y}^\lambda$, where $\lambda, \mu \in \mathbb{N}^t$. Define a map $G_0 : I_+ \to I_O$ by $\boldsymbol{x}^\lambda \boldsymbol{y}^\mu + \boldsymbol{x}^\mu \boldsymbol{y}^\lambda \mapsto H_0(\boldsymbol{x}^\lambda \boldsymbol{y}^\mu)$. This map is well-defined since $H_0(\boldsymbol{x}^\lambda \boldsymbol{y}^\mu) = H_0(\boldsymbol{x}^\mu \boldsymbol{y}^\lambda)$.

Since $\mathrm{W}^{[n]}(\mathcal{A}) \cong \mathrm{W}^{[n]}(X)$ for $n \in \mathbb{Z}$, it follows that $\widetilde{\mathrm{W}}^{[0]}(\mathcal{A}) = 0$. Therefore, the hyperbolic map $H_0 : I \to I_O$ is surjective, and one may write $F_0 : H_0(I) \to I_+$ and $G_0 : I_+ \to H_0(I)$. Upon inspection, F_0 and G_0 are inverse to each other, thus concluding the proof. $\square$

8.2 Atiyah-Segal completion for split tori

Let k be a field of characteristic other than 2, and let T be a split torus over k of rank $t \geq 1$. For ease of notation, let $\mathcal{A} = \mathrm{Perf}^T(k)$ be the category of perfect complexes of T-representations, equipped with the standard duality. Recall the construction of the motivic classifying space $\mathrm{B_{gm}}T$ from definition 3.6.2 and example 3.6.14. The results from the previous sections will be applied to obtain a formula for $\mathrm{GW}^{[n]}(\mathrm{B_{gm}}T)$. More precisely, it will be shown that $\mathrm{GW}_i^{[n]}(\mathrm{B_{gm}}T)$ is the completion of $\mathrm{GW}_i^{[n]}(\mathcal{A})$ with respect to the Hermitian augmentation ideal $I_O \subset \mathrm{GW}_0^{[0]}(\mathcal{A})$. The proof consists of the following steps:

(i) show that $\mathrm{GW}_i^{[n]}(\mathrm{B_{gm}}T)$ is the limit of Grothendieck-Witt groups of products of projective spaces;

(ii) pass to the *reduced* Grothendieck-Witt groups of definition 8.1.8;
(iii) use the isomorphism of lemma 8.1.9 to show that the reduced Karoubi fundamental sequences induce short exact sequences

$$0 \longrightarrow \widetilde{\mathrm{GW}}_0^{[n]}(\mathcal{A}) \longrightarrow \widetilde{\mathrm{K}}_0(\mathcal{A}) \longrightarrow \widetilde{\mathrm{GW}}_0^{[n+1]}(\mathcal{A}) \longrightarrow 0,$$

and that they remain exact after taking the quotient by I_O^r for $r \geq 1$;
(iv) show that the exact sequences in degree 0 extend to a long exact sequence of pro-R_O-modules by tensoring with $\mathrm{K}_i(k)$;
(v) show that I_O-adic completion of $\mathrm{K}^T(k)$ agrees with the I-adic completion;
(vi) construct a surjective map of long exact sequences of pro-R_O-modules from the pro-Karoubi sequence of the previous step to the pro-Karoubi sequence associated with $(\mathbb{P}_k^{2r})^t$ for $r \geq 1$; and
(vii) use an induction argument on the diagram from the previous step to prove the claim.

Step (i) will now be done by carefully considering the construction of $\mathrm{B_{gm}}T$. Let $(\mathbb{A}_k^r, U_r, f_r)_{r\in\mathbb{N}}$ be the admissible gadget over k with nice T-action from theorem 3.6.9. In this case, $U_r = \mathbb{A}_k^{rt} \setminus \{0\}$ is a punctured affine space over k and $U_r/T \cong (\mathbb{P}_k^{r-1})^t$. Then the geometric classifying space $\pi : \mathrm{B_{gm}}T \to \operatorname{Spec} k$ is defined as

$$\mathrm{B_{gm}}T = \operatorname*{colim}_r U_r/ = \operatorname*{colim}_r (\mathbb{P}_k^{r-1})^t.$$

For the computation of $\mathrm{GW}(\mathrm{B_{gm}}T)$, it is therefore important to understand the Grothendieck-Witt theory of products of projective spaces. The following result provides this understanding for the even-dimensional case.

Proposition 8.2.1. *Let $\mathbb{P} = (\mathbb{P}_k^{2r})^t$ be a product of even-dimensional projective spaces over S with projections $\pi : \mathbb{P} \to S$ and $\pi_i : \mathbb{P} \to \mathbb{P}_k^{2r}$ for each $1 \leq i \leq t$. Let L be the set $\{-r, \dots, r\}^t$ and let L' be a set of representatives of $(L - \{0\})/\{\pm\}$. For nonzero $\lambda = (\lambda_1, \dots, \lambda_t) \in L$, let $\lambda \in L'$ be the image of λ in L'. Then there is an equivalence of spectra*

$$\mathrm{GW}^{[n]}(k) \oplus \bigoplus_{\lambda\in L'} \mathrm{K}(k) \longrightarrow \mathrm{GW}^{[n]}(\mathbb{P})$$

induced by the functor

$$\begin{aligned} \mathrm{Perf}(k)^{[n]} \times \bigoplus_{\lambda\in L'} \mathrm{Perf}(k) &\longrightarrow \mathrm{Perf}(\mathbb{P})^{[n]} \\ (A, (B_\lambda)_{\lambda\in L'}) &\longmapsto \pi^* A + \sum_{\lambda\in L'} H(\pi^* b_\lambda(\lambda)), \end{aligned}$$

where $\pi^* b_\lambda(\lambda)$ is the invertible sheaf on $\mathbb{P}$ given by the tensor product $\pi_1^* b_{\lambda_1}(\lambda_1) \otimes \cdots \otimes \pi_t^* b_{\lambda_t}(\lambda_t)$. In particular, there is an equivalence of spectra

$$\mathrm{GW}^{[n]}((\mathbb{P}_k^{2r})^t) \simeq \mathrm{GW}^{[n]}(k) \oplus \mathrm{K}(k)^{\oplus \frac{(2r+1)^t - 1}{2}}.$$

Proof. The dg category $\mathrm{Perf}(\mathbb{P})$ has a semi-orthogonal decomposition into $(2r+1)^t$ pieces by iterated application of theorem 7.4.1. These pieces are labeled $\mathcal{A}(\lambda)$ with $\lambda \in L$, and the semi-orthogonal decomposition puts them in lexicographical order. One can think of this decomposition as a t-dimensional cube of points with $2r+1$ points along each edge, each of which represents a copy of $\mathrm{Perf}(k)$. The standard duality on $\mathrm{Perf}(\mathbb{P})$ is an involution that sends $\mathcal{A}(\lambda)$ to $\mathcal{A}(-\lambda)$, and therefore fixes $\mathcal{A}(0,\dots,0)$. Hence, additivity theorem 6.2.2 gives the desired equivalence

$$\mathrm{GW}^{[n]}(k) \oplus \bigoplus_{\lambda \in L'} \mathrm{K}(k) \longrightarrow \mathrm{GW}^{[n]}((\mathbb{P}).$$

For the explicit description of this morphism, note that each equivalence $\mathrm{Perf}(k) \to \mathcal{A}(\lambda)$ is given by

$$\mathcal{F} \mapsto \pi^*\mathcal{F} \otimes \pi_1^*\mathcal{O}_{\mathbb{P}^{2r}_k}(\lambda_1) \otimes \cdots \otimes \pi_t^*\mathcal{O}_{\mathbb{P}^{2r}_k}(\lambda_t),$$

as a result of the repeated use of theorem 7.4.1. Additionally, note that $\#L' = ((2r+1)^t - 1)/2$, so that

$$\mathrm{GW}^{[n]}((\mathbb{P}^{2r}_k)^t) \simeq \mathrm{GW}^{[n]}(k) \oplus \mathrm{K}(k)^{\oplus \frac{(2r+1)^t-1}{2}},$$

which finishes the proof. $\square$

Next, it is shown that computing the limit of Grothendieck-Witt groups of products of projective spaces is a viable method for computing $\mathrm{GW}(\mathrm{B_{gm}}T)$, which completes step (i).

Proposition 8.2.2. *The pro-group*

$$\text{“}\lim_{r\in\mathbb{N}}\text{”}\ \mathrm{GW}^{[n]}_i((\mathbb{P}^{2r}_k)^t)$$

satisfies the Mittag-Leffler condition for all $i \in \mathbb{Z}$. In particular,

$$\mathrm{GW}^{[n]}_i(\mathrm{B_{gm}}T) \cong \lim_{r\in\mathbb{N}} \mathrm{GW}^{[n]}_i((\mathbb{P}^{2r}_k)^t)$$

for all $i \in \mathbb{Z}$.

Proof. Note that

$$\operatorname*{colim}_r (\mathbb{P}^{r-1}_k)^t \cong \operatorname*{colim}_r (\mathbb{P}^{2r}_k)^t$$

as simplicial sheaves, so it may and will be assumed that

$$\mathrm{B_{gm}}T = \operatorname*{colim}_r (\mathbb{P}^{2r}_k)^t,$$

to leverage the previous result on products of even-dimensional projective spaces. Let $\iota_r : (\mathbb{P}^{2r}_k)^t \to (\mathbb{P}^{2r+2}_k)^t$ be given by inclusion on the first r coordinates for each copy of $\mathbb{P}^{2r}$. There is a commutative diagram

$$\begin{tikzcd} (\mathbb{P}^{2r})^t \arrow[r, "\iota_r"] \arrow[dr, "\pi_{2r}"'] & (\mathbb{P}^{2r+2})^t \arrow[d, "\pi_{2r+2}"] \\ & S, \end{tikzcd}$$

so $\pi_{2r}^* \cong \iota_r^* \pi_{2r+2}^*$. Furthermore, $\iota_r^* \mathcal{O}_{(\mathbb{P}^{2r+2})^t}(\lambda) = \mathcal{O}_{(\mathbb{P}^{2r})^t}(\lambda)$ for all $\lambda \in \mathbb{Z}^t$. For $r \in \mathbb{N}$, let $L_r = \{-r, \ldots, r\}^t$ and $L'_r = (L_r - \{(0, \ldots, 0)\})/\{\pm\}$. There is an equivalence

$$\mathrm{GW}^{[n]}(k) \oplus \bigoplus_{\bar{\lambda} \in L'_r} \mathrm{K}(k) \longrightarrow \mathrm{GW}^{[n]}((\mathbb{P}_k^{2r})^t)$$

which is given explicitly in terms of π_{2r}^* and the hyperbolic functor by proposition 8.2.1, and a similar equivalence exists for $(\mathbb{P}_k^{2r+2})^t$. It follows that the map on Grothendieck-Witt spectra

$$\iota_r^* : \mathrm{GW}^{[n]}((\mathbb{P}_k^{2r+2})^t) \longrightarrow \mathrm{GW}^{[n]}((\mathbb{P}_k^{2r})^t)$$

is equivalent to a map which splits via the inclusion

$$\mathrm{GW}^{[n]}(k) \oplus \bigoplus\nolimits_{\bar{\lambda} \in L'_r} \mathrm{K}(k) \longrightarrow \mathrm{GW}^{[n]}(k) \oplus \bigoplus\nolimits_{\bar{\lambda} \in L'_{r+1}} \mathrm{K}(k)$$

given by $a \mapsto a$ for $a \in \mathrm{GW}^{[n]}(k)$ and $b_{\bar{\lambda}} \mapsto b_{\overline{f(\lambda)}}$ for $b_{\bar{\lambda}}$ in the copy of $\mathrm{K}(k)$ corresponding to $\bar{\lambda}$, where $f : L'_r \to L'_{r+1}$ is the canonical inclusion. Therefore, on the level of stable homotopy groups, ι_r^* is a surjection onto $\mathrm{GW}_i^{[n]}((\mathbb{P}_k^{2r})^t)$ for all $i \in \mathbb{Z}$. Hence, the pro-group

$$\left(\mathrm{GW}_i^{[n]}((\mathbb{P}_k^{2r})^t)\right)_{r \in \mathbb{N}}$$

satisfies the Mittag-Leffler condition for all $i \in \mathbb{Z}$, so the $\lim^1$-term of the Milnor short exact sequence (4.2.6)

$$\lim_{r \in \mathbb{N}}{}^1 \mathrm{GW}_{i-1}^{[n]}((\mathbb{P}_k^{2r})^t) \longrightarrow \mathrm{GW}_i^{[n]}(\mathrm{B_{gm}}T) \longrightarrow \lim_{r \in \mathbb{N}} \mathrm{GW}_i^{[n]}((\mathbb{P}_k^{2r})^t)$$

vanishes and it ultimately follows that

$$\mathrm{GW}_i^{[n]}(\mathrm{B_{gm}}T) \cong \lim_{r \in \mathbb{N}} \mathrm{GW}_i^{[n]}((\mathbb{P}_k^{2r})^t)$$

for all $i \in \mathbb{Z}$. □

Corollary 8.2.3. *There is an isomorphism*

$$\mathrm{GW}_i^{[n]}(\mathrm{B_{gm}}T) \cong \lim_{r \in \mathbb{N}} \mathrm{GW}_i^{[n]}(k) \oplus \mathrm{K}_i(k)^{\oplus \frac{(2r+1)^t - 1}{2}}$$

for all $i \in \mathbb{Z}$.

Proof. The result follows from proposition 8.2.2 and proposition 8.2.1. □

Step (i) is finished, and $\mathrm{GW}_i^{[n]}(\mathrm{B_{gm}}T)$ can be described as a pro-group by taking the formal limit of Grothendieck-Witt groups of products of projective spaces.

Since $(\mathbb{P}_k^{2r})^t \cong U_r/T$ for some smooth scheme U_r, there is an isomorphism $\mathrm{GW}_i^{[n]}((\mathbb{P}_k^{2r})^t) \cong \mathrm{GW}_{T,i}^{[n]}(U_r)$, as well as a surjective forgetful map

$$\mathrm{GW}_i^{[n]}((\mathbb{P}_k^{2r})^t) \longrightarrow \mathrm{GW}_i^{[n]}(U_r).$$

Let α' be a map

$$\mathrm{GW}_i^{[n]}((\mathbb{P}_k^{2r})^t) \longrightarrow \mathrm{GW}_i^{[n]}(k)$$

induced by pulling back to a k-valued point of $(\mathbb{P}_k^{2r})^t$. Note that α' does not depend on the choice of this k-valued point, and that α' is surjective. The kernel of α' is denoted by $\widetilde{\mathrm{GW}}_i^{[n]}((\mathbb{P}_k^{2r})^t)$ and is also called the i-th reduced Grothendieck-Witt group. Now it will be shown that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A}) \arrow[r] \arrow[d] & \mathrm{GW}_i^{[n]}(\mathcal{A}) \arrow[r, "\alpha"] \arrow[d] & \mathrm{GW}_i^{[n]}(k) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \widetilde{\mathrm{GW}}_i^{[n]}((\mathbb{P}_k^{2r})^t) \arrow[r] & \mathrm{GW}_i^{[n]}((\mathbb{P}_k^{2r})^t) \arrow[r, "\alpha'"] & \mathrm{GW}_i^{[n]}(k) \arrow[r] & 0,
\end{tikzcd}$$

where the map α is the Hermitian augmentation map induced by the canonical forgetful dg form functor $\mathrm{Perf}^T(k)^{[n]} \to \mathrm{Perf}(k)^{[n]}$. It suffices to show that the composition

$$\mathrm{GW}_i^{[n]}(\mathcal{A}) \longrightarrow \mathrm{GW}_i^{[n]}((\mathbb{P}_k^{2r})^t) \xrightarrow{\alpha'} \mathrm{GW}_i^{[n]}(k)$$

is the map α, because in that case the map between the reduced GW-groups can simply be taken to be the map between kernels. Let $\iota : \operatorname{Spec} k \to (\mathbb{P}_k^{2r})^t$ be a k-valued point. For $\lambda \in \mathbb{Z}^t$, $\iota^*\mathcal{O}_{(\mathbb{P}_k^{2r})^t}(\lambda) = \mathcal{O}_{\operatorname{Spec} k}$. Hence, the composition $\mathrm{Perf}(\mathcal{A}) \to \mathrm{Perf}((\mathbb{P}_k^{2r})^t) \to \mathrm{Perf}(k)$ is the same as the forgetful map $\mathrm{Perf}(\mathcal{A}) \to \mathrm{Perf}(k)$, and the diagram commutes. This observation makes it plausible that results for the reduced Grothendieck-Witt groups extend to results for the Grothendieck-Witt groups themselves, as claimed in step (ii), but the actual proof of this is deferred to the last moment.

For $n \in \mathbb{Z}$, consider the diagram

$$\begin{tikzcd}
\dots \arrow[r] & \widetilde{\mathrm{GW}}_{i+1}^{[n+1]}(\mathcal{A}) \arrow[r] \arrow[d] & \widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A}) \arrow[r] \arrow[d] & \widetilde{\mathrm{K}}_i(\mathcal{A}) \arrow[r] \arrow[d] & \widetilde{\mathrm{GW}}_i^{[n+1]}(\mathcal{A}) \arrow[r] \arrow[d] & \widetilde{\mathrm{GW}}_{i-1}^{[n]}(\mathcal{A}) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \mathrm{GW}_{i+1}^{[n+1]}(\mathcal{A}) \arrow[r] \arrow[d] & \mathrm{GW}_i^{[n]}(\mathcal{A}) \arrow[r] \arrow[d] & \mathrm{K}_i(\mathcal{A}) \arrow[r] \arrow[d] & \mathrm{GW}_i^{[n+1]}(\mathcal{A}) \arrow[r] \arrow[d] & \mathrm{GW}_{i-1}^{[n]}(\mathcal{A}) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \mathrm{GW}_{i+1}^{[n+1]}(k) \arrow[r] & \mathrm{GW}_i^{[n]}(k) \arrow[r] & \mathrm{K}_i(k) \arrow[r] & \mathrm{GW}_i^{[n+1]}(k) \arrow[r] & \mathrm{GW}_{i-1}^{[n]}(k) \arrow[r] & \dots
\end{tikzcd}$$

where the rows are the long exact Karoubi sequences of R_O-modules coming from [72, theorem 6.1], and where the columns are short exact. The top exact sequence will be called the *reduced Karoubi sequence.*

Let $R = \mathrm{K}_0(\mathcal{A})$ be the representation ring of T and $R_O = \mathrm{GW}_0^{[0]}(\mathcal{A})$ the Hermitian representation ring as before. Note that $I = \widetilde{\mathrm{K}}_0(\mathcal{A})$ and $I_O = \widetilde{\mathrm{GW}}_0^{[0]}(\mathcal{A})$ are the augmentation ideal and the Hermitian augmentation ideal, respectively. Recall that $\mathrm{K}_0(k) \cong \mathbb{Z}$.

The image of the forgetful map $R_O \to R$ is contained in the $*$-fixed points R_+ of R. By corollary 8.1.2, there is an isomorphism

$$R \cong \frac{\mathbb{Z}[x_1, \dots, x_t, y_1, \dots, y_t]}{\langle x_i y_i + x_i + y_i \mid i = 1, \dots, t \rangle},$$

where $x_i = [\mathcal{O}_{\operatorname{Spec} k}(e_i)] - 1$ and $y_i = [\mathcal{O}_{\operatorname{Spec} k}(-e_i)] - 1$, with $e_i \in \mathbb{Z}^t$ the i-th unit vector. It follows that R is an integral domain. Furthermore, there is an isomorphism $I_O \cong I_+$ by lemma 8.1.9. As a result of this, step (iii) can be initiated.

Lemma 8.2.4. *For $n \in \mathbb{Z}$, the reduced Karoubi sequence*

$$\widetilde{\mathrm{GW}}_0^{[n]}(\mathcal{A}) \xrightarrow{F_n} \widetilde{\mathrm{K}}_0(\mathcal{A}) \xrightarrow{H_{n+1}} \widetilde{\mathrm{GW}}_0^{[n+1]}(\mathcal{A}) \longrightarrow \widetilde{\mathrm{W}}^{[n+1]}(\mathcal{A})$$

is isomorphic to

$$\begin{aligned} I_+ &\longrightarrow I \longrightarrow I/I_+ \longrightarrow 0 \qquad &&\text{if } n \text{ is even; and} \\ I/I_+ &\xrightarrow{1-*} I \xrightarrow{1+*} I_+ \longrightarrow 0 \qquad &&\text{if } n \text{ is odd,} \end{aligned}$$

where $* : K_0(\mathcal{A}) \to K_0(\mathcal{A})$ denotes the involution induced by the standard duality, as in section 8.1. Moreover, the forgetful maps F_n are all injective, so the sequence

$$0 \longrightarrow \widetilde{\mathrm{GW}}_0^{[n]}(\mathcal{A}) \xrightarrow{F_n} \widetilde{\mathrm{K}}_0(\mathcal{A}) \xrightarrow{H_{n+1}} \widetilde{\mathrm{GW}}_0^{[n+1]}(\mathcal{A}) \longrightarrow 0 \tag{8.2.5}$$

is short exact.

Proof. As $\widetilde{\mathrm{W}}^{[n]}(\mathcal{A}) = 0$ for all $n \in \mathbb{Z}$, the hyperbolic maps H_n are all surjective, so $\widetilde{\mathrm{GW}}_0^{[n]}(\mathcal{A}) = H_n(I)$. Thus the reduced Karoubi sequences are of the form

$$H_n(I) \xrightarrow{F_n} I \xrightarrow{H_{n+1}} H_{n+1}(I) \longrightarrow 0.$$

By the same argument as in the proof of lemma 8.1.9, $H_2(I) \cong I_+$. Hence, F_n is injective for even n and the sequence (8.2.5) is isomorphic to the short exact sequence

$$0 \longrightarrow I_+ \longrightarrow I \xrightarrow{1-*} I/I_+ \longrightarrow 0,$$

and it follows that $H_{n+1}(I)$ is isomorphic to I/I_+, the isomorphism $I/I_+ \to H_{n+1}(I)$ being given by $\bar{a} \mapsto H_{n+1}(a)$ for $a \in I$.

Now suppose that n is odd. Note that $F_n(H_n(I)) = \ker H_{n+1}$ consists of elements of the form $a - a^*$, with $a \in I$. Let $a \in \ker F_n H_n$. Then $a = a^*$, so $a \in I_+ \cong F_{n-1}(H_{n-1}(I))$. As $H_n F_{n-1} = 0$, it follows that $H_n(a) = 0$. Thus $\ker F_n = 0$ and F_n is injective, and the sequence (8.2.5) is isomorphic to the short exact sequence

$$0 \longrightarrow I/I_+ \xrightarrow{1-*} I \xrightarrow{1+*} I_+ \longrightarrow 0,$$

as was to be shown. □

To complete step (iii), I_+ must be analyzed in detail. Let I_- be the image of the inclusion $1 - * : I/I_+ \to I$. Since the forgetful map $R_O \to R$ is a ring map whose image lies in the subring $R_+ \subset R$, powers of I_O correspond to powers of $I_+ \subset R_+$.

It will be useful to first collect some properties of I_+ by exploiting the explicit descriptions of R and R_+, which are then used in the construction of the pro-Karoubi sequence.

As a free abelian group, the augmentation ideal I is generated by monomials $\boldsymbol{x}^\lambda \boldsymbol{y}^\mu$ with $\lambda, \mu \in \mathbb{N}^t$ and, for all i, either $\lambda_i = 0$ and $\mu_i \neq 0$, or $\mu_i = 0$ and $\lambda_i \neq 0$, as can be seen from the relation $x_i y_i = -(x_i + y_i)$ in R. Consequently, the Hermitian augmentation ideal I_+ is generated as a free abelian group by elements of the form $\boldsymbol{x}^\lambda \boldsymbol{y}^\mu + \boldsymbol{x}^\mu \boldsymbol{y}^\lambda$ with $\boldsymbol{x}^\lambda \boldsymbol{y}^\mu$ one of the generators of I, up to the ordering of μ and λ. The following lemma gives a finite set of ideal generators of $I_+ \subset R_+$.

Lemma 8.2.6. *As an ideal of R_+, I_+ is generated by elements of the form $\boldsymbol{x}^\gamma + \boldsymbol{y}^\gamma$, where $\gamma \in \{0,1\}^t - \{0\}^t$. In particular, I_+ is a finitely generated ideal with a generating set of $2^t - 1$ elements.*

Proof. Let J be the ideal generated by of the form $\boldsymbol{x}^\gamma + \boldsymbol{y}^\gamma$, where $\gamma \in \{0,1\}^t - \{0\}^t$. Since I_+ is generated as a free abelian group by elements of the form $\boldsymbol{x}^\lambda \boldsymbol{y}^\mu + \boldsymbol{x}^\mu \boldsymbol{y}^\lambda$ with $\lambda, \mu \in \mathbb{N}^t$, it suffices to show that J contains these elements.

First, it will be shown by induction on n that J contains all elements of the form $\boldsymbol{x}^\lambda + \boldsymbol{y}^\lambda$, where $\lambda \in \{0, \dots, n\}^t$ for arbitrary $n \in \mathbb{N}$. For $n = 1$, this follows directly from the definition of J. Now assume that $\boldsymbol{x}^\lambda + \boldsymbol{y}^\lambda \in J$ for all $\lambda \in \{0, \dots, n\}^t$ and let $\lambda \in \{0, \dots, n+1\}^t$. Then $\lambda = \mu + \gamma$ with $\mu \in \{0, \dots, n\}^t$ and $\gamma \in \{0,1\}^t$ such that $\gamma_i = 1$ if and only if $\lambda_i = n + 1$. Note that

$$(\boldsymbol{x}^\mu + \boldsymbol{y}^\mu)(\boldsymbol{x}^\gamma + \boldsymbol{y}^\gamma) = \boldsymbol{x}^\lambda + \boldsymbol{y}^\lambda + \boldsymbol{x}^\mu \boldsymbol{y}^\gamma + \boldsymbol{x}^\gamma \boldsymbol{y}^\mu,$$

and $(\boldsymbol{x}^\mu + \boldsymbol{y}^\mu)(\boldsymbol{x}^\gamma + \boldsymbol{y}^\gamma) \in J$. Moreover, $\mu - \gamma \in \{0, \dots, n\}^t$, so

$$\boldsymbol{x}^\mu \boldsymbol{y}^\gamma + \boldsymbol{x}^\gamma \boldsymbol{y}^\lambda = \boldsymbol{x}^{\mu-\gamma} \boldsymbol{x}^\gamma \boldsymbol{y}^\gamma + \boldsymbol{x}^\gamma \boldsymbol{y}^\gamma \boldsymbol{y}^{\mu-\gamma} = (\boldsymbol{x}^{\mu-\gamma} + \boldsymbol{y}^{\mu-\gamma}) \prod_{i=1}^t (-(x_i + y_i))^{\gamma_i}$$

is contained in J. By induction, it follows that $\boldsymbol{x}^\lambda + \boldsymbol{y}^\lambda \in J$ for all $\lambda \in \mathbb{N}^t$, as claimed.

Now let $\lambda, \mu \in \mathbb{N}^t$ and consider $\boldsymbol{x}^\lambda \boldsymbol{y}^\mu + \boldsymbol{x}^\mu \boldsymbol{y}^\lambda$. Note that

$$(\boldsymbol{x}^\lambda + \boldsymbol{y}^\lambda)(\boldsymbol{x}^\mu + \boldsymbol{y}^\mu) = \boldsymbol{x}^{\lambda+\mu} + \boldsymbol{y}^{\lambda+\mu} + \boldsymbol{x}^\lambda \boldsymbol{y}^\mu + \boldsymbol{x}^\mu \boldsymbol{y}^\lambda$$

is contained in J, and since $\boldsymbol{x}^{\lambda+\mu} + \boldsymbol{y}^{\lambda+\mu} \in J$, this yields $\boldsymbol{x}^\lambda \boldsymbol{y}^\mu + \boldsymbol{x}^\mu \boldsymbol{y}^\lambda \in J$. ☐

One useful corollary of this shows that the quotients I_+/I_+^m are finitely generated abelian groups.

Corollary 8.2.7. *For $\lambda \in \mathbb{N}^t$ with $n = \max(\lambda_1, \dots, \lambda_t)$ and $m = \lceil \frac{n}{2} \rceil$, the element $\boldsymbol{x}^\lambda + \boldsymbol{y}^\lambda$ is contained in I_+^m. In particular, I_+/I_+^m is finitely generated as an abelian group.*

Proof. If $n = 1$, then $m = 1$ and $\boldsymbol{x}^\lambda + \boldsymbol{y}^\lambda \in I_+$. Assume that the result holds for all $n' \leq n$ and let $\lambda \in \{0, \dots, n+1\}^t - \{0, \dots, n\}^t$. Let $m = \lceil \frac{n+1}{2} \rceil$. The proof of lemma 8.2.6 gives

$$\boldsymbol{x}^\lambda + \boldsymbol{y}^\lambda = (\boldsymbol{x}^\mu + \boldsymbol{y}^\mu)(\boldsymbol{x}^\gamma + \boldsymbol{y}^\gamma) - (\boldsymbol{x}^{\mu-\gamma} + \boldsymbol{y}^{\mu-\gamma}) \prod_{i=1}^t (-(x_i + y_i))^{\gamma_i} \tag{8.2.8}$$

with $\mu \in \{0, \dots, n\}^t - \{0, \dots, n-1\}^t$, $\gamma \in \{0,1\}^t$ and $\lambda = \mu + \gamma$. As $\boldsymbol{x}^\mu + \boldsymbol{y}^\mu, \boldsymbol{x}^{\mu-\gamma} + \boldsymbol{y}^{\mu-\gamma} \in I_+^{m-1}$ by assumption, $\boldsymbol{x}^\lambda + \boldsymbol{y}^\lambda \in I_+^m$, and the desired result follows by induction. ☐

Another useful corollary shows that the quotients I_+/I_+^m are moreover *free* abelian groups.

Corollary 8.2.9. *For all $f \in R_+$, $n \in \mathbb{Z}$ nonzero and $m \geq 1$, $nf \in I_+^m$ if and only if $f \in I_+^m$. In particular, I_+/I_+^m is a finitely generated free abelian group.*

Proof. If $f \in I_+^m$, then $nf \in I_+^m$. Therefore, suppose that $nf \in I_+^m$. If n is a unit, $f \in I_+^m$, so assume that n is not a unit. Applying (8.2.8) iteratively, nf can be written as a sum

$$nf = \sum a_i g_i,$$

where $a_i \in \mathbb{Z}$ and the g_i are distinct products of at least m generators of I_+. Since n does not divide any of the g_i, it follows that $n \mid a_i$ for all i, which yields $f \in I_+^m$. Thus I_+/I_+^m is torsion-free, as well as finitely generated by corollary 8.2.7. Consequently, I_+/I_+^m is a finitely generated free abelian group, as was to be shown. □

The following lemma concludes step (iii), and paves the way for step (iv), which is the construction of the long exact pro-Karoubi sequence.

Lemma 8.2.10. *For each $r \in \mathbb{N}$, the quotient of the reduced Karoubi sequence (8.2.5) by I_O^r is isomorphic to one of the following two short exact sequences:*

$$0 \longrightarrow \frac{I_+}{I_+^{r+1}} \longrightarrow \frac{I}{I_+^r I} \xrightarrow{1-*} \frac{I_-}{I_+^r I_-} \longrightarrow 0$$

$$0 \longrightarrow \frac{I_-}{I_+^r I_-} \longrightarrow \frac{I}{I_+^r I} \xrightarrow{1+*} \frac{I_+}{I_+^{r+1}} \longrightarrow 0.$$

Moreover, the terms in these sequences are finitely generated free abelian groups.

Proof. Applying $- \otimes_{R_O} R_O/I_O^r$ to the reduced Karoubi sequence (8.2.5) (or equivalently $- \otimes_{R_O} R_+/I_+^r$) yields exact sequences of R_O-modules

$$\frac{I_+}{I_+^{r+1}} \longrightarrow \frac{I}{I_+^r I} \xrightarrow{1-*} \frac{I_-}{I_+^r I_-} \longrightarrow 0$$

$$\frac{I_-}{I_+^r I_-} \longrightarrow \frac{I}{I_+^r I} \xrightarrow{1+*} \frac{I_+}{I_+^{r+1}} \longrightarrow 0.$$

Thus it remains to show that the first map in each of these two sequences is injective. Let $f \in I_+ \cap I_+^r I$. Then $f = f^*$, so $f + f^* = 2f$. Since $1 + * : I \to I_+$ maps $I_+^r I$ to I_+^r, $2f \in I_+^{r+1}$. It follows from 8.2.9 that $f \in I_+^{r+1}$, so the first sequence is exact. A similar argument shows that the second sequence is exact.

Again by corollary 8.2.9, I_+/I_+^{r+1} is a finitely generated free abelian group. The fact that $I/I_+^r I$ is a finitely generated free abelian group is proven similarly. Then $I_-/I_+^r I_-$, being a subgroup of a finitely generated free abelian group, is also a finitely generated free abelian group. □

For all i, $\mathrm{K}_i(k) = \mathrm{K}_0(k) \otimes_{\mathrm{K}_0(k)} \mathrm{K}_i(k)$. By corollary 8.1.4,

$$\widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A}) \cong \bigoplus_{\lambda \in C} \mathrm{K}_i(k).$$

Similarly, $\widetilde{\mathrm{K}}_i(\mathcal{A})$ is a direct sum of copies of $\mathrm{K}_i(k)$. Hence, using $\mathrm{K}_0(k) = \mathbb{Z}$,

$$\begin{aligned}\widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A}) &\cong \widetilde{\mathrm{GW}}_0^{[n]}(\mathcal{A}) \otimes_{\mathbb{Z}} \mathrm{K}_i(k)\\ \widetilde{\mathrm{K}}_i(\mathcal{A}) &\cong \widetilde{\mathrm{K}}_0(\mathcal{A}) \otimes_{\mathbb{Z}} \mathrm{K}_i(k).\end{aligned}$$

The diagram

$$\begin{tikzcd}
\widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A}) \arrow[r, "F_n"] \arrow[d] & \widetilde{\mathrm{K}}_i(\mathcal{A}) \arrow[r, "H_{n+1}"] \arrow[d] & \widetilde{\mathrm{GW}}_i^{[n+1]}(\mathcal{A}) \arrow[d] \\
\widetilde{\mathrm{GW}}_0^{[n]}(\mathcal{A}) \otimes_{\mathbb{Z}} \mathrm{K}_i(k) \arrow[r] & \widetilde{\mathrm{K}}_0(\mathcal{A}) \otimes_{\mathbb{Z}} \mathrm{K}_i(k) \arrow[r] & \widetilde{\mathrm{GW}}_0^{[n+1]}(\mathcal{A}) \otimes_{\mathbb{Z}} \mathrm{K}_i(k)
\end{tikzcd}$$

commutes, so these isomorphisms are compatible with the reduced Karoubi sequence. Together with this identification, lemma 8.2.10 is used to construct the long exact pro-karoubi sequence.

Lemma 8.2.11 (Reduced pro-karoubi sequence)**.** *For all $r \in \mathbb{N}$, the quotient of the reduced Karoubi sequence*

$$\cdots \longrightarrow \frac{\widetilde{\mathrm{GW}}_{i+1}^{[n+1]}(\mathcal{A})}{I_O^r \widetilde{\mathrm{GW}}_{i+1}^{[n+1]}(\mathcal{A})} \longrightarrow \frac{\widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A})}{I_O^r \widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A})} \longrightarrow \frac{\widetilde{\mathrm{K}}_i(\mathcal{A})}{I_O^r \widetilde{\mathrm{K}}_i(\mathcal{A})} \longrightarrow \frac{\widetilde{\mathrm{GW}}_i^{[n+1]}(\mathcal{A})}{I_O^r \widetilde{\mathrm{GW}}_i^{[n+1]}(\mathcal{A})} \longrightarrow \cdots$$

by I_O^r is a long exact sequence of R_O-modules. In particular, these sequences induce a levelwise long exact sequence of pro-R_O-modules, called the reduced pro-karoubi sequence.

Proof. The short exact sequence

$$0 \longrightarrow I_+^r I \longrightarrow I \longrightarrow I/I_+^r I \longrightarrow 0$$

remains short exact after tensoring with $K_i(k)$ since all the terms are free abelian groups. Hence, by lemma 8.2.4,

$$\frac{\widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A})}{I_O^r \widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A})} \cong \frac{I_n}{I_+^r I_n} \otimes_{\mathbb{Z}} \mathrm{K}_i(k) \qquad \text{and} \qquad \frac{\widetilde{\mathrm{K}}_i(\mathcal{A})}{I_O^r \widetilde{\mathrm{K}}_i(\mathcal{A})} \cong \frac{I}{I_+^r I} \otimes_{\mathbb{Z}} \mathrm{K}_i(k)$$

for all $n, i \in \mathbb{Z}$ and $r \in \mathbb{N}$, where $I_n = I_+$ if n is even and $I_n = I_-$ otherwise. Additionally, the short exact sequences

$$0 \longrightarrow \frac{I_+}{I_+^{r+1}} \longrightarrow \frac{I}{I_+^r I} \xrightarrow{1-*} \frac{I_-}{I_+^r I_-} \longrightarrow 0$$

$$0 \longrightarrow \frac{I_-}{I_+^r I_-} \longrightarrow \frac{I}{I_+^r I} \xrightarrow{1+*} \frac{I_+}{I_+^{r+1}} \longrightarrow 0$$

remain short exact after tensoring with $K_i(k)$, again because all the terms are free abelian groups. Therefore, there is an isomorphism of long exact sequences

$$
\begin{tikzcd}
\cdots \arrow[r] & \dfrac{\widetilde{\mathrm{GW}}^{[n+1]}_{i+1}(\mathcal{A})}{I^r_O\widetilde{\mathrm{GW}}^{[n+1]}_{i+1}(\mathcal{A})} \arrow[r] \arrow[d] & \dfrac{\widetilde{\mathrm{GW}}^{[n]}_{i}(\mathcal{A})}{I^r_O\widetilde{\mathrm{GW}}^{[n]}_{i}(\mathcal{A})} \arrow[r] \arrow[d] & \dfrac{\widetilde{\mathrm{K}}_i(\mathcal{A})}{I^r_O\widetilde{\mathrm{K}}_i(\mathcal{A})} \arrow[r] \arrow[d] & \dfrac{\widetilde{\mathrm{GW}}^{[n+1]}_{i}(\mathcal{A})}{I^r_O\widetilde{\mathrm{GW}}^{[n+1]}_{i}(\mathcal{A})} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \dfrac{I_{n+1}}{I^r_+ I_{n+1}} \otimes \mathrm{K}_{i+1}(k) \arrow[r, "0"] & \dfrac{I_n}{I^r_+ I_n} \otimes \mathrm{K}_i(k) \arrow[r] & \dfrac{I}{I^r_+ I} \otimes \mathrm{K}_i(k) \arrow[r] & \dfrac{I_{n+1}}{I^r_+ I_{n+1}} \otimes \mathrm{K}_i(k) \arrow[r, "0"] & \cdots,
\end{tikzcd}
$$

which proves the result. □

Step (v) boils down to the following lemma, which is useful for comparing completions. In particular, it can be used to show that the middle term of the pro-Karoubi sequence is the I-adic completion of $\mathrm{K}_0^T(k)$.

For a linear algebraic group G, let $\mathcal{B} = \mathrm{Perf}^T(k)$ be the dg category of perfect complexes of T-representations, and let $R_G = \mathrm{K}_0(\mathcal{B})$ be the representation ring, $R_{G,O} = \mathrm{GW}_0^{[0]}(\mathcal{B})$ the Hermitian representation ring, $I_G = \ker(\mathrm{K}_0(\mathcal{B}) \to \mathrm{K}_0(k))$ the augmentation ideal and $I_{G,O} = \ker(\mathrm{GW}_0^{[0]}(\mathcal{B}) \to \mathrm{GW}_0^{[0]}(k))$ the Hermitian augmentation ideal.

Lemma 8.2.12 (Filtration lemma)**.** *For a linear algebraic group G over k, the I_G-adic and $I_{G,O}$-adic topologies on R_G coincide.*

Proof. Fix an embedding $\iota : G \to H$ with $H = \mathrm{SO}_{2m+1}(k)$ for some $m \in \mathbb{N}$, which can be realized as the composition of embeddings

$$G \longrightarrow \mathrm{GL}_m \longrightarrow \mathrm{SO}_{2m} \longrightarrow \mathrm{SO}_{2m+1}\,.$$

Let $F_G : R_{G,O} \to R_G$ be the forgetful map. It will be shown that

$$\iota^*(I_H)R_G \subset F_G(I_{G,O})R_G \subset I_G R_G, \tag{8.2.13}$$

after which it suffices to show that the $\iota^*(I_H)$-adic and I_G-adic topologies on R_G coincide.

The second inclusion of (8.2.13) follows from the commutativity of

$$
\begin{tikzcd}
R_{G,O} \arrow[r, "F_G"] \arrow[d] & R_G \arrow[d] \\
\mathrm{GW}_0(k) \arrow[r, "F"] & \mathrm{K}_0(k).
\end{tikzcd}
$$

Since H is split reductive and all irreducible representations of H are symmetric by [94, lemma 3.14], the forgetful map $F_H : R_{H,O} \to R_H$ is surjective. The trivial map $\phi : H \to H$ given by $g \mapsto 1$ induces morphisms $\phi^* : R_H \to R_H$ and $\phi^* : R_{H,O} \to R_{H,O}$ which replace any H-representation by the trivial one. As elements of I_H are of the form $a - b$ with $\phi^* a = \phi^* b$, the map $1 - \phi^* : R_H \to I_H$ splits the inclusion $I_H \to R_H$. Similarly, $1-\phi^* : R_{H,O} \to I_{H,O}$ splits $I_{H,O} \to R_{H,O}$. Thus the commutative diagram

$$\begin{array}{ccc} R_{H,O} & \xrightarrow{F_H} & R_H \\ {\scriptstyle 1-\phi^*}\downarrow & & \downarrow{\scriptstyle 1-\phi^*} \\ I_{H,O} & \xrightarrow{F_H} & I_H \end{array}$$

shows that $F_H : I_{H,O} \to I_H$ is surjective. Consequently, $\iota^*(I_H) = \iota^*(F_H(I_{H,O}))$. Therefore, the commutativity of

$$\begin{array}{ccccc} I_{H,O} & \longrightarrow & & \longrightarrow & I_H \\ & \searrow & & & \downarrow \\ \downarrow & & R_{H,O} & \xrightarrow{F_H} & R_H \\ & & \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \iota^*} \\ I_{G,O} & \longrightarrow & R_{G,O} & \xrightarrow{F_G} & R_G \end{array}$$

shows that $\iota^*(I_H) \subset F_G(I_{G,O})$, which proves (8.2.13). Hence, the $\iota^*(I_H)$-adic and I_G-adic topologies on R_G coincide by [29, corollary 6.1], and the result follows. $\square$

There is a canonical morphism $\mathrm{GW}_i^{[n]}(\mathcal{A}) \to \mathrm{GW}_i^{[n]}((\mathbb{P}_k^{2r})^t)$ for all $r \in \mathbb{N}$, because $(\mathbb{P}_k^{2r})^t = U_r/T$ is a geometric quotient and therefore

$$\mathrm{GW}_{T,i}^{[n]}(U_r) \cong \mathrm{GW}_i^{[n]}((\mathbb{P}_k^{2r})^t)$$

for all $n, i \in \mathbb{Z}$.

Step (vi), the construction of a surjective map of pro-Karoubi sequences, uses the description of $\mathrm{GW}_i^{[n]}((\mathbb{P}_k^{2r})^t)$ as a quotient of $\mathrm{GW}_i^{[n]}(\mathcal{A})$. Let $\mathbb{P} = \mathbb{P}_k^{2r}$ for some $r \geq 1$ and let $\widetilde{\mathrm{GW}}_i^{[n]}(\mathbb{P}^t)$ be the kernel of the surjective map $\mathrm{GW}_i^{[n]}(\mathbb{P}^t) \to \mathrm{GW}_i^{[n]}(k)$. Additionally, for $n \in \mathbb{Z}$, consider the commutative diagram

$$\begin{array}{ccccccc} \dots \longrightarrow & J_n & \longrightarrow & J & \longrightarrow & J_{n+1} & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ \dots \longrightarrow & \widetilde{\mathrm{GW}}_0^{[n]}(\mathcal{A}) & \longrightarrow & \widetilde{\mathrm{K}}_0(\mathcal{A}) & \longrightarrow & \widetilde{\mathrm{GW}}_0^{[n+1]}(\mathcal{A}) & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ \dots \longrightarrow & \widetilde{\mathrm{GW}}_0^{[n]}(\mathbb{P}^t) & \longrightarrow & \widetilde{\mathrm{K}}_0(\mathbb{P}^t) & \longrightarrow & \widetilde{\mathrm{GW}}_0^{[n+1]}(\mathbb{P}^t) & \longrightarrow 0 \end{array}$$

where the rows are exact and the columns are short exact. Then $J_n \to J$ is injective, as $\widetilde{\mathrm{GW}}_0^{[n]}(\mathcal{A}) \to \widetilde{\mathrm{K}}_0(\mathcal{A})$ is injective. Furthermore, $\widetilde{\mathrm{GW}}_0^{[n]}(\mathbb{P}^t) \to \widetilde{\mathrm{K}}_0(\mathbb{P}^t)$ is injective as a result of the projective bundle formulae for GW and K.

Lemma 8.2.14. *Fix $n \in \mathbb{Z}$. For $m \in \mathbb{N}$ large enough, $I_O^m \widetilde{\mathrm{GW}}_0^{[n]}(\mathcal{A}) \subset J_n$.*

Proof. As $J_n = J \cap \widetilde{\mathrm{GW}}_0^{[n]}(\mathcal{A})$, it suffices to show that $I_O^m \subset J$ for m large enough. Since $x_i \in I$ is given by the class $[\mathcal{O}(e_i)] - 1$, the image of x_i^{2r+1} is the class of an exact Koszul complex on $\mathbb{P}^t$, whence $x_i^{2r+1} \in J$. For nonzero $\gamma \in \{0,1\}^t$,

$$(\boldsymbol{x}^\gamma + \boldsymbol{y}^\gamma)^{4r+2} = \sum_{j=0}^{4r+2} \binom{4r+2}{j} (\boldsymbol{x}^\gamma)^{4r+2-j} (\boldsymbol{y}^\gamma)^j,$$

so $j \geq 2r+1$ or $4r+2-j \geq 2r+1$. Hence, $(\boldsymbol{x}^\gamma + \boldsymbol{y}^\gamma)^{4r+2} \in J$. Consequently, as I_O is finitely generated by elements of the form $\boldsymbol{x}^\gamma + \boldsymbol{y}^\gamma$ with nonzero $\gamma \in \{0,1\}^t$ via the isomorphism $I_O \cong I_+$, the pigeonhole principle forces $I_O^m \subset J$ for m large enough. $\square$

It is now possible to construct the crucial map of exact sequences of pro-R_O-modules. By [25], the pro-category of an abelian category is abelian. Between any two pro-objects, there is an obvious zero map. See section A.3 for details.

Corollary 8.2.15. *There is a natural surjective map of long exact sequences of pro-R_O-modules*

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \text{“}\lim_r\text{”}\, \dfrac{\widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A})}{I_O^r \widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A})} & \longrightarrow & \text{“}\lim_r\text{”}\, \dfrac{\widetilde{\mathrm{K}}_i(\mathcal{A})}{I_O^r \widetilde{\mathrm{K}}_i(\mathcal{A})} & \longrightarrow & \text{“}\lim_r\text{”}\, \dfrac{\widetilde{\mathrm{GW}}_i^{[n+1]}(\mathcal{A})}{I_O^r \widetilde{\mathrm{GW}}_i^{[n+1]}(\mathcal{A})} & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & \text{“}\lim_r\text{”}\, \widetilde{\mathrm{GW}}_i^{[n]}((\mathbb{P}_k^{2r})^t) & \longrightarrow & \text{“}\lim_r\text{”}\, \widetilde{\mathrm{K}}_i((\mathbb{P}_k^{2r})^t) & \longrightarrow & \text{“}\lim_r\text{”}\, \widetilde{\mathrm{GW}}_i^{[n+1]}((\mathbb{P}_k^{2r})^t) & \longrightarrow \cdots \end{array}$$

Moreover, the middle vertical arrow is an isomorphism for each $i \in \mathbb{Z}$.

Proof. By lemma 8.2.14, for $r \in \mathbb{N}$, the natural surjective map

$$\widetilde{\mathrm{GW}}_0^{[n]}(\mathcal{A}) \longrightarrow \widetilde{\mathrm{GW}}_0^{[n]}((\mathbb{P}_k^{2r})^t)$$

factors through $(\widetilde{\mathrm{GW}}_0^{[n]}(\mathcal{A}))/(I_O^m \widetilde{\mathrm{GW}}_0^{[n]}(\mathcal{A}))$ for large enough m. This induces a natural surjective map of pro-R_O-modules

$$\text{“}\lim_r\text{”}\, \frac{\widetilde{\mathrm{GW}}_0^{[n]}(\mathcal{A})}{I_O^r \widetilde{\mathrm{GW}}_0^{[n]}(\mathcal{A})} \longrightarrow \text{“}\lim_r\text{”}\, \widetilde{\mathrm{GW}}_0^{[n]}((\mathbb{P}_k^{2r})^t)$$

for each $n \in \mathbb{Z}$. Furthermore, for all $r \geq 1$ and $n, i \in \mathbb{Z}$,

$$\frac{\widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A})}{I_O^r \widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A})} \cong \frac{\widetilde{\mathrm{GW}}_0^{[n]}(\mathcal{A})}{I_O^r \widetilde{\mathrm{GW}}_0^{[n]}(\mathcal{A})} \otimes_{\mathbb{Z}} \mathrm{K}_i(k)$$
$$\widetilde{\mathrm{GW}}_i^{[n]}((\mathbb{P}_k^{2r})^t) \cong \widetilde{\mathrm{GW}}_0^{[n]}((\mathbb{P}_k^{2r})^t) \otimes_{\mathbb{Z}} \mathrm{K}_i(k),$$

which allows the construction of canonical surjective maps

$$\text{“}\lim_r\text{”}\, \frac{\widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A})}{I_O^r \widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A})} \longrightarrow \text{“}\lim_r\text{”}\, \widetilde{\mathrm{GW}}_i^{[n]}((\mathbb{P}_k^{2r})^t)$$

for all $n, i \in \mathbb{Z}$. Finally, by the filtration lemma 8.2.12 and the proof of [57, lemma 8.3], there is a composition of natural isomorphisms

$$\text{“}\lim_r\text{”}\, \frac{\widetilde{\mathrm{K}}_i(\mathcal{A})}{I_O^r \widetilde{\mathrm{K}}_i(\mathcal{A})} \longrightarrow \text{“}\lim_r\text{”}\, \frac{\widetilde{\mathrm{K}}_i(\mathcal{A})}{I^r \widetilde{\mathrm{K}}_i(\mathcal{A})} \longrightarrow \text{“}\lim_r\text{”}\, \widetilde{\mathrm{K}}_i((\mathbb{P}_k^{2r})^t).$$

It remains to be shown that the resulting diagram of pro-R_O-modules commutes. This follows from the fact that the diagram

$$\begin{tikzcd}
\frac{\widetilde{\mathrm{GW}}_0^{[n]}(\mathcal{A})}{I_O^m \widetilde{\mathrm{GW}}_0^{[n]}(\mathcal{A})} \otimes_{\mathbb{Z}} \mathrm{K}_i(k) \arrow[r] \arrow[d] & \frac{\widetilde{\mathrm{K}}_0(\mathcal{A})}{I_O^m \widetilde{\mathrm{K}}_0(\mathcal{A})} \otimes_{\mathbb{Z}} \mathrm{K}_i(k) \arrow[r] \arrow[d] & \frac{\widetilde{\mathrm{GW}}_0^{[n+1]}(\mathcal{A})}{I_O^m \widetilde{\mathrm{GW}}_0^{[n+1]}(\mathcal{A})} \otimes_{\mathbb{Z}} \mathrm{K}_i(k) \arrow[d] \\
\widetilde{\mathrm{GW}}_0^{[n]}((\mathbb{P}_k^{2r})^t) \otimes_{\mathbb{Z}} \mathrm{K}_i(k) \arrow[r] & \widetilde{\mathrm{K}}_0((\mathbb{P}_k^{2r})^t) \otimes_{\mathbb{Z}} \mathrm{K}_i(k) \arrow[r] & \widetilde{\mathrm{GW}}_0^{[n+1]}((\mathbb{P}_k^{2r})^t) \otimes_{\mathbb{Z}} \mathrm{K}_i(k)
\end{tikzcd}$$

commutes, where $r, m \in \mathbb{N}$ and m is large enough for the vertical maps to exist. □

The Atiyah-Segal completion theorem for algebraic K-theory, the reduced pro-Karoubi sequence and the filtration lemma can be combined to prove step (vii), the Atiyah-Segal completion theorem for the higher Grothendieck-Witt groups of classifying spaces of split tori.

Theorem 8.2.16. *For all $i, n \in \mathbb{Z}$, the canonical morphism*

$$\mathrm{GW}_i^{[n]}(\mathcal{A}) \longrightarrow \mathrm{GW}_i^{[n]}(\mathrm{B}_{\mathrm{gm}}T)$$

is the completion of the R_O-module $\mathrm{GW}_i^{[n]}(\mathcal{A})$ with respect to the augmentation ideal I_O.

Proof. Let

$$\begin{tikzcd}
\dots \arrow[r] & A_{i+1}^{[n+1]} \arrow[r] \arrow[d] & A_i^{[n]} \arrow[r] \arrow[d] & B_i \arrow[r] \arrow[d] & A_i^{[n+1]} \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & A_{i+1}'^{[n+1]} \arrow[r] & A_i'^{[n]} \arrow[r] & B_i' \arrow[r] & A_i'^{[n+1]} \arrow[r] & \dots
\end{tikzcd}$$

be the surjective map of long exact sequences of pro-R_O-modules of corollary 8.2.15, where $B_i \to B_i'$ is the isomorphism in K-theory. To show that each $A_i^{[n]} \to A_i'^{[n]}$ is an isomorphism, a method is employed which could be called *pro-Karoubi induction* since it closely resembles the classical Karoubi induction of [72, lemma 6.4]. For each $n \in \mathbb{Z}$, the diagram terminates on the right as

$$\begin{tikzcd}
\dots \arrow[r] & A_1^{[n+1]} \arrow[r] \arrow[d] & A_0^{[n]} \arrow[r] \arrow[d] & B_0 \arrow[r] \arrow[d] & A_0^{[n+1]} \arrow[r] \arrow[d] & 0 \\
\dots \arrow[r] & A_1'^{[n+1]} \arrow[r] & A_0'^{[n]} \arrow[r] & B_0' \arrow[r] & A_0'^{[n+1]} \arrow[r] & 0
\end{tikzcd}$$

since the reduced Witt groups all vanish. Thus by the four-lemma, $A_0^{[n+1]} \to A_0'^{[n+1]}$ is injective, and therefore an isomorphism, for each $n \in \mathbb{Z}$. Now fix $i > 0$ and assume that $A_j^{[n+1]} \to A_j'^{[n+1]}$ is an isomorphism for all $j < i$ and $n \in \mathbb{Z}$. Then applying the four-lemma to

$$\begin{tikzcd}
\dots \arrow[r] & A_i^{[n]} \arrow[r] \arrow[d] & B_i \arrow[r] \arrow[d] & A_i^{[n+1]} \arrow[r] \arrow[d] & A_{i-1}^{[n]} \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & A_i'^{[n]} \arrow[r] & B_i' \arrow[r] & A_i'^{[n+1]} \arrow[r] & A_{i-1}'^{[n]} \arrow[r] & \dots
\end{tikzcd}$$

shows that $A_i^{[n+1]} \to A_i'^{[n+1]}$ is injective, and therefore an isomorphism, for all $n \in \mathbb{Z}$. Hence, by induction, the canonical maps

$$\text{“}\lim_r\text{”} \frac{\widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A})}{I_O^r \widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A})} \longrightarrow \text{“}\lim_r\text{”}\, \widetilde{\mathrm{GW}}_i^{[n]}((\mathbb{P}_k^{2r})^t)$$

are isomorphisms of pro-groups for all $i \in \mathbb{N}$ and $n \in \mathbb{Z}$. Passing to the actual limits, one sees that the canonical morphism

$$\lim_r \frac{\widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A})}{I_O^r \widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A})} \longrightarrow \widetilde{GW}_i^{[n]}(\mathrm{B_{gm}}T)$$

is an isomorphism. The exact sequence

$$0 \longrightarrow \widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A}) \longrightarrow \mathrm{GW}_i^{[n]}(\mathcal{A}) \longrightarrow \mathrm{GW}_i^{[n]}(k) \longrightarrow 0$$

splits, so it remains exact when taking the quotient by I_O^r. Note that $I_O^r\,\mathrm{GW}_i^{[n]}(k) = 0$, because $I_O\,\mathrm{GW}_i^{[n]}(k) \subset \widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A})$. Then the five-lemma yields that the middle vertical arrow in the following commutative diagram is an isomorphism:

$$\begin{tikzcd}
0 \arrow[r] & \lim_r \frac{\widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A})}{I_O^r \widetilde{\mathrm{GW}}_i^{[n]}(\mathcal{A})} \arrow[r] \arrow[d] & \lim_r \frac{\mathrm{GW}_i^{[n]}(\mathcal{A})}{I_O^r \mathrm{GW}_i^{[n]}(\mathcal{A})} \arrow[r] \arrow[d] & \mathrm{GW}_i^{[n]}(k) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \widetilde{GW}_i^{[n]}(\mathrm{B_{gm}}T) \arrow[r] & GW_i^{[n]}(\mathrm{B_{gm}}T) \arrow[r] & \mathrm{GW}_i^{[n]}(k) \arrow[r] & 0,
\end{tikzcd}$$

which shows that the canonical morphism

$$\mathrm{GW}_i^{[n]}(\mathcal{A}) \longrightarrow \mathrm{GW}_i^{[n]}(\mathrm{B_{gm}}T)$$

is the completion of the R_O-module $\mathrm{GW}_i^{[n]}(\mathcal{A})$ with respect to the augmentation ideal I_O for all $i, n \in \mathbb{Z}$, which concludes the proof of the Atiyah-Segal completion theorem. □

9 Connected split reductive groups

We have already investigated the equivariant Grothendieck-Witt theory of split tori in section 8.1. In this section, we turn our attention to the equivariant Grothendieck-Witt theory of the larger class of *connected split reductive groups*. Throughout this chapter, G will be a connected split reductive group over a field k of characteristic zero, which encompasses general linear groups, special linear groups and other important examples. The representation theory of connected split reductive groups is well-understood in characteristic zero, and we will apply this understanding to G-equivariant Grothendieck-Witt theory.

9.1 A semi-orthogonal decomposition of the derived category of representations

First, it is necessary to determine how the standard semi-orthogonal decomposition of $\mathrm{Perf}^G(k)$ behaves under duality. Let k be a field of characteristic zero. The notation of [22] will be used. Let (G, B, T) be a triple consisting of a connected split reductive group G over k with $T \subset B \subset G$, where B is a Borel subgroup and T is a maximal torus of rank t. Let $X(T) \cong \mathbb{Z}^t$ be the character lattice of T. Let $\Phi = \Phi(G, T)$ be a root system and let $\Phi^+ = \Phi(B, T)$ be the system of positive roots associated to B as in [22, proposition 1.4.4] and let $\Delta \subset \Phi^+$ be the set of simple positive roots. Let $W = W_G(T) = W(\Phi)$ be the Weyl group, with longest element $w_0 \in W$, which is the unique element such that $w_0(\Phi^+) = -\Phi^+$. Finally, let

$$C = \left\{\lambda \in X(T) \mid \langle \lambda, a^\vee \rangle \geq 0 \text{ for all } a \in \Delta\right\}$$

be a closed Weyl chamber, that is, a set of dominant weights. The orbit of C under the action of W is $X(T)$.

Proposition 9.1.1. *The pretriangulated dg category $\mathcal{A} = \mathrm{Perf}^G(k)$ has a semi-orthogonal decomposition*

$$\mathcal{A} = \langle \mathcal{A}_\lambda \mid \lambda \in C \rangle,$$

where $\mathcal{A}_\lambda = \langle M_\lambda \rangle$ is generated by the irreducible representation M_λ of highest weight λ. Furthermore, $\mathcal{A}_\lambda \simeq \mathrm{Perf}(k)$ as dg categories.

Proof. By the proof of [76, lemme 6] (or the theorem of the highest weight [22, theorem 1.5.6]), isomorphism classes of irreducible representations of G correspond bijectively to C. Therefore, let $\{M_\lambda \mid \lambda \in C\}$ be a set of representatives, and $\mathcal{A}_\lambda = \langle M_\lambda \rangle$ the full triangulated subcategory of $\mathcal{A}$ generated by M_λ. Let

$$F_\lambda : \mathrm{Perf}(k) \longrightarrow \mathcal{A}_\lambda$$

be given by $F_\lambda(N) = N \otimes_k M_\lambda$, where N is equipped with the trivial G-action. The proof of [76, lemme 5] shows that M_λ is absolutely irreducible, so $\mathrm{End}_k^G(M_\lambda) \cong k$. It follows that F_λ is fully faithful and essentially surjective. Furthermore, F_λ is exact. Hence $\mathcal{A}_\lambda \simeq \mathrm{Perf}(k)$ as dg categories.

Let $\lambda, \mu \in \mathbb{Z}^t/W$ such that $\lambda \neq \mu$ and let $f : M_\lambda \to M_\mu$ be a morphism in $\mathcal{A}$. By Schur's lemma $f = 0$, since M_λ and M_μ are not isomorphic. Hence $\mathrm{Hom}(\mathcal{A}_\lambda, \mathcal{A}_\mu) = 0$. Furthermore, since every representation of G splits into irreducible representations, $\mathrm{Ext}^i(M_\lambda, M_\mu) = 0$ for all $i > 0$.

It remains to be shown that the $\mathcal{A}_\lambda$ generate $\mathcal{A}$. Let M be a finite dimensional G-representation. Then each irreducible G-representation in the decomposition of M is contained in one of the $\mathcal{A}_a$, whence $M \in \langle \mathcal{A}_a \mid a \in \mathbb{Z}^t/W \rangle$, which finishes the proof. □

Example 9.1.2. Here is an example (c.f. [14, example 1.2]) that shows that $\mathrm{Ext}^1(A, B) = 0$ does not always hold for irreducible representations A and B of G when k is a field of nonzero characteristic. Let $k = \mathbb{F}_2$, $G = \mathrm{GL}_2$, and let V be the standard two-dimensional representation of G. Then $\Lambda^2 V$ and $\mathrm{Sym}^2 V$ are non-isomorphic irreducible G-representations, and the exact sequence

$$0 \longrightarrow \Lambda^2 V \longrightarrow V \otimes V \longrightarrow \mathrm{Sym}^2 V \longrightarrow 0$$

does not split. Hence $\operatorname{Ext}^1(\operatorname{Sym}^2 V, \Lambda^2 V) \neq 0$.

The non-vanishing of Ext-groups of irreducible representations is the most important obstruction to a proof of proposition 9.1.1 in arbitrary characteristic.

Corollary 9.1.3. *The G-equivariant* K*-theory of k is given by*

$$\mathrm{K}_i^G(k) \cong \bigoplus_{\lambda \in C} K_i(k) \cong K_0^G(k) \otimes_{\mathbb{Z}} K_i(k)$$

for all $i \in \mathbb{N}$.

Proof. This follows from additivity for K-theory and the semi-orthogonal decomposition of proposition 9.1.1. □

Recall the theorem of the highest weight [22, theorem 1.5.6], which states that every irreducible G-representation has a unique highest weight. The following lemma is folklore.

Lemma 9.1.4. *Let V be an irreducible G-representation with highest weight λ. Then the dual representation V^* has highest weight $-w_0\lambda$.*

Proof. Let $\Omega_V \subset X(T)$ be the set of weights of V. Note that $\Omega_{V^*} = -\Omega_V$. As λ is the highest weight of V and Ω_{V^*} is W-invariant, all weights in Ω_{V^*} are of the form

$$-w_0\lambda + \sum_{b \in w_0(\Delta)} m_b b$$

with $m_b \in \mathbb{Z}_{\geq 0}$. Note that $w_0(\Delta) \subset -\Phi^+$, so for $b \in w_0(\Delta)$ and $m_b \in \mathbb{Z}_{\geq 0}$,

$$m_b b = -\sum_{a \in \Delta} n_a a$$

with $n_a \in \mathbb{Z}_{\geq 0}$, as Δ is a base for Φ^+. Hence all weights in Ω_{V^*} are of the form

$$-w_0\lambda - \sum_{a \in \Delta} n_a a,$$

and it follows that $-w_0\lambda$ is the highest weight of V^*, as was to be shown. □

The following corollary is a direct consequence of the above lemma.

Corollary 9.1.5. *The duality functor $* : \mathcal{A} \to \mathcal{A}$ on $\mathcal{A} = \operatorname{Perf}^G(k)$ sends $\mathcal{A}_a$ to $\mathcal{A}_{-w_0 a}$, where $a \in C$.*

Let $C^+ \subset C$ be the set of dominant weights fixed by $-w_0$, and let D be a set of representatives for the set of orbits $(C - C^+)/(-w_0)$ of elements of C that are not fixed by $-w_0$.

Example 9.1.6. Let $G = \mathrm{GL}_n$. The maximal torus of diagonal matrices has weight lattice $\mathbb{Z}^n$ with standard unit vectors e_i. The roots $\Phi = \Phi(G,T)$ of G are all elements of the form $e_i - e_j$ with $1 \leq i, j \leq n$ and $i \neq j$. Letting $B \subset G$ be the Borel subgroup of upper triangular matrices, the positive roots $\Phi^+ = \Phi(B,T)$ are all elements $e_i - e_j$ with $i < j$, and the simple roots Δ are the elements $e_i - e_{i+1}$. The Weyl group W

is generated by the reflections $s_{ij} : \mathbb{Z}^n \to \mathbb{Z}^n$ that switch e_i and e_j. The longest element $w_0 \in W$ with respect to Φ^+ is the product

$$w_0 = \prod_{i<j} s_{ij}.$$

For example, if $n = 3$, then $w_0 = s_{12}s_{13}s_{23} = s_{13}$, and if $n = 4$, then $w_0 = s_{14}s_{23}$. In general, there is an isomorphism $W \to S_n$, $s_{ij} \mapsto (i\ j)$, and under this isomorphism $w_0 = (1\ n)(2\ n-1)\cdots(\lceil n/2\rceil\ \lfloor n/2\rfloor + 1)$, so w_0 can be thought of as swapping the linear order on $\{1, \dots, n\}$, sending the lowest number to the highest number, the second lowest to the second highest, and so on. If n is odd, there is a middle number which is fixed by w_0. The dominant weights of G are given by

$$C = \left\{ \sum_{i=1}^{n} m_i e_i \mid m_i \in \mathbb{Z}, m_i \geq m_{i+1} \right\}.$$

Then the $-w_0$-fixed points $C^+ \subset C$ are weights of the form

$$\lambda^+ = \sum_{i=1}^{\lceil n/2\rceil} m_i e_i - \sum_{i=\lfloor n/2\rfloor+1}^{n} m_{n+1-i} e_i,$$

still satisfying $m_i \geq m_{i+1}$.

Proposition 9.1.7. *With notation as before, the G-equivariant* GW*-theory of k is given by*

$$\mathrm{GW}^{[0]}_{G,i}(k) \cong \bigoplus_{\lambda^+ \in C^+} \mathrm{GW}^{[0]}_i(k) \oplus \bigoplus_{\lambda \in D} \mathrm{K}_i(k).$$

Proof. This follows from corollary 9.1.5 and corollary 6.2.3. □

10 Grothendieck-Witt spectra of Grassmannians

After computing the cohomology of projective spaces, one can take a few different directions for further computations in algebraic geometry, but perhaps the most natural generalization is to consider Grassmannians. Projective spaces are themselves edge cases of Grassmannians, and their combinatorics generalizes to other Grassmannians through the theory of *Young diagrams*.

In this chapter, we compute the Grothendieck-Witt spectra of a certain class of Grassmannians. This will in turn enable us to compute the higher Grothendieck-Witt groups of the geometric classifying spaces $\mathrm{B_{gm}GL}_n$ of general linear groups, which is an important step in proving a general Atiyah-Segal completion theorem for Grothendieck-Witt theory.

10.1 Semi-orthogonal decompositions for Grassmannians

Let k be a field of characteristic zero. Let $n = d + e$ be some positive integer. Let $X = \mathrm{Gr}(d, n)$ be the Grassmannian of d-dimensional subspaces in k^n, with

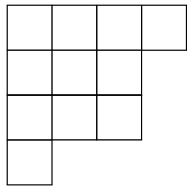


Figure 10.1.1: Young diagram for the partition $(4, 3, 3, 1)$.

tautological bundle $\mathcal{U}$ of rank d and dual bundle $\mathcal{T}$ of rank e. These bundles fit together in an exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{O}_X^{\oplus n} \longrightarrow \mathcal{U} \longrightarrow 0.$$

Let $\Delta = \det \mathcal{U}$ be the determinant of the tautological bundle.

We can replace $\operatorname{Spec} k$ and k^n with a base scheme S over k, and k^n with a vector bundle $\mathcal{V}$ over S. For now, we refrain from stating results in such generality to ensure a clear exposition.

We will follow [14], and eventually expand on the results obtained there by introducing duality. Since we are working in characteristic zero, we could have followed [49] instead. Write X as the homogeneous space G/P, where $G = \mathrm{GL}_n$ and P is the parabolic subgroup

$$\begin{pmatrix} \mathrm{GL}_d & * \\ 0 & \mathrm{GL}_e \end{pmatrix},$$

which contains both the Levi subgroup $H = \mathrm{GL}_d \times \mathrm{GL}_e$ and the Borel subgroup B of upper triangular matrices. As usual, let $T \subset B$ be the maximal split torus of diagonal matrices. From now on, we will write $G_1 = \mathrm{GL}_d$ and $G_2 = \mathrm{GL}_e$, as well as $T_i = G_i \cap T$ for $i = 1, 2$. The theory of vector bundles on X is intimately related to the representation theory of G_1, which we will exploit. The character lattice of $T_1 = T \cap G_1$ is $\mathbb{Z}^d$, and its positive roots are of the form

$$(0, \dots, 0, 1, 0, \dots, 0, -1, 0 \dots, 0).$$

There is a natural partial order $\leq$ on T_1 for which $\lambda \leq \mu$ if and only if $\mu - \lambda$ is a sum of positive roots. The dominant weights α of G_1 are non-increasing tuples $(\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d$. Dominant weights with non-negative entries correspond to *partitions*, which can be visualized as Young diagrams. The Young diagram of a partition $(\alpha_1, \dots, \alpha_d)$ has d rows of blocks, where the i-th row consists of α_i blocks, see figure 10.1.1 for an example.

Young diagrams, in turn, correspond to *Schubert cells*, which are subschemes of Grassmannians that form cellular decompositions, and the intersection theory of these Schubert cells is called *Schubert calculus.* The Young diagrams considered here do *not* correspond to the usual Schubert cells, but to twisted versions of these; here, the generator of $\mathcal{A}$ corresponding to a partition α is the Schur functor of α applied to the standard representation of GL_d, twisted by a power of the determinant of the standard representation, as we will see later.

The *degree* $|\alpha|$ of a weight $\alpha \in \mathbb{Z}^d$ is the sum $\sum \alpha_i$ of its entries. A representation has degree m if all its weights have degree m, and a representation is called *polynomial* if all its weights are partitions. Let V be the standard representation of G_1 of dimension d. The determinant $\det V = \Lambda^d V$ of V is a irreducible representation with highest weight $(1,\dots,1)$. For a partition α and a vector space (or G_1-representation) W, let

$$\Lambda^\alpha W = \bigotimes_{i=1}^{d} \Lambda^{\alpha_i} W,$$

and similarly when replacing W with a vector bundle $\mathcal{E}$ on X. Furthermore, each partition α allows the definition of the *Schur functor* L^α and the *Weyl functor* K^α, both of which are functors $\mathrm{Vect}(k) \to \mathbf{Rep}(G_1)$.

In particular, $L^\alpha V$ is the induced representation with highest weight α, and since k has characteristic zero, it is irreducible. For the explicit constructions of the Schur and Weyl functors, see [14, section 2]. There is an equivalence of categories $\mathcal{L}_X : \mathbf{Rep}(P) \to \mathrm{Vect}^G(X)$ by [19, theorem 2.7]. Composing $\mathcal{L}_X$ with the inclusion $\mathbf{Rep}(G_1) \subset \mathbf{Rep}(P)$ induced by the projection $P \to G_1$ allows us to study G-equivariant vector bundles on X in terms of G_1-representations, which are more easily understood.

In representation theory and Schubert calculus, it seems to be customary to focus on polynomial representations, which is often justified by the remark that representations of GL_d can be tensored with the determinant representation with highest weight $(1,\dots,1)$ to obtain polynomial representations, but the dual of a polynomial representation need not be polynomial. We explain here how the determinant representation interacts with duality and how we can use this to obtain a satisfying theory of duality on Young diagrams, which makes a computation of the Grothendieck-Witt theory of Grassmannians possible.

Let $P_{d,e} \subset \mathbb{Z}^d$ be the set of partitions whose Young diagrams have at most d rows and at most e columns. Such partitions correspond to the Schubert cells of X and are visualized by Young diagrams that fit in a $(d \times e)$-frame. For $\alpha \in P_{d,e}$, write $\alpha = (\alpha_1,\dots,\alpha_d)$ with $\alpha_i \geq \alpha_{i+1}$ for all $i = 1,\dots,d-1$. For any $m \in \mathbb{Z}$ and $\alpha \in P_{d,e}$, we write $\underline{m} + \alpha$ for $(m+\alpha_1,\dots,m+\alpha_d) \in P_{d,e+m}$.

Definition 10.1.2. Let $\alpha \in P_{d,e}$ be a partition. The *transpose* α^T *of* α is the partition $\alpha^T \in P_{e,d}$ obtained by transposing the Young diagram of α.

Since X is a projective scheme with an ample line bundle, we define

$$\mathcal{A} = \mathrm{Perf}(X) = \mathrm{Ch}^b(\mathrm{Vect}(X))/\,\mathrm{Ac}^b(\mathrm{Vect}(X))$$

as in remark 5.5.6. Note that the homotopy category $H^0\mathcal{A}$ is the usual bounded derived category $D^b(X)$ of X. For $i \in \mathbb{N}$, we let $\mathcal{C}_i \subset \mathbf{Rep}(G_1)$ be the full subcategory of the category of finite dimensional polynomial G_1-representations consisting of those representations whose irreducible components have highest weight $\alpha \in P_{d,e}$ such that $|\alpha| = i$. Let $(P_{d,e})_i \subset P_{d,e}$ be the subset of partitions of degree i. The bounded derived dg category $\mathrm{Perf}(\mathcal{C}_i)$ is generated as a pretriangulated dg category by the irreducible representations with highest weight $\alpha \in P_{d,e}$ with $|\alpha| = i$. By pulling back G_1-representations along the projection $P \to G_1$, we can define, for each $i \in \mathbb{N}$, a fully faithful dg functor

$$\Phi_i : \mathrm{Perf}(\mathcal{C}_i) \to \mathcal{A} \tag{10.1.3}$$

which sends M to $\mathcal{L}_X(M)$, see [14, p. 6, 11] and note that this can really be constructed as a dg functor. By [14, theorem 5.6], there is a semi-orthogonal decomposition

$$\mathcal{A} = \langle \mathcal{A}_0, \ldots, \mathcal{A}_{de} \rangle, \tag{10.1.4}$$

where $\mathcal{A}_i$ is the quasi-essential image of the dg functor Φ_i, which is the full dg subcategory of $\mathcal{A}$ on objects quasi-isomorphic to objects in the image of Φ_i. For each dominant weight $\lambda \in \mathbb{Z}^d$ of G_1, fix a representative M_λ for the isomorphism class of irreducible representations of weight λ. Then $\mathrm{Perf}(\mathcal{C}_i)$ is generated by the irreducible representations M_α with highest weight $\alpha \in (P_{d,e})_i$. For $\alpha, \beta \in (P_{d,e})_i$ such that $\alpha \neq \beta$, Schur's lemma ensures that $\mathrm{Hom}(M_\alpha, M_\beta) = 0$ in the category of G_1-representations. Since the base field k has characteristic zero, every G_1-representation splits into irreducible representations and it follows that

$$\mathrm{Ext}^i(M_\alpha, M_\beta) = \mathrm{Hom}(M_\alpha, M_\beta[i]) = 0$$

in the triangulated category $H^0 \,\mathrm{Perf}(\mathcal{C}_i)$. Hence, there is a semi-orthogonal decomposition

$$\mathrm{Perf}(\mathcal{C}_i) = \langle \langle M_\alpha \rangle \mid \alpha \in (P_{d,e})_i \rangle, \tag{10.1.5}$$

where the set of partitions $P_{d,e}$ can be equipped with any linear order to fulfill the conditions of definition 5.6.10. The semi-orthogonal decompositions (10.1.4) and (10.1.5) enable a computation of the K-theory of X: by additivity for K-theory

$$\mathrm{K}(\mathcal{A}) \simeq \bigoplus_{i=0}^{de} \mathrm{K}(\mathcal{A}_i) \simeq \bigoplus_{\alpha \in P_{d,e}} \mathrm{K}(k),$$

where the final equivalence follows from the fact that $\langle M_\alpha \rangle \simeq \mathrm{Perf}(k)$ for each $\alpha \in P_{d,e}$. Note that $|P_{d,e}| = \binom{n}{d}$.

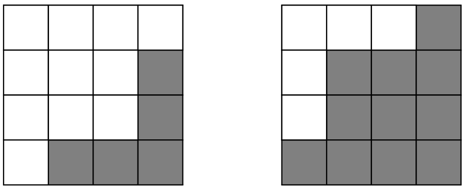
10.2 Duality on Young diagrams

We would like to use the semi-orthogonal decompositions from the previous section for the computation of Witt groups and Grothendieck-Witt groups, but there is a problem: the standard duality on $\mathcal{A}$ does not permute the factors of the decompositions, and we cannot use the relevant additivity theorems. We will now present a way to overcome this problem when either d or e is even. The inspiration for this solution came from a combination of the solution for projective spaces as presented in section 7.4, and the semi-orthogonal decomposition of proposition 9.1.1, which *is* permuted by the duality by corollary 9.1.5.

The case when both d and e are odd remains open, though the author expects the existence of an approach that unifies all the cases, in a similar fashion to [7, theorem 6.1], the results of which we will compare with that of our approach in due course.

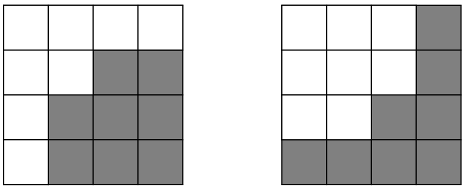
First, we must understand the duality on $\mathcal{A}$. Since, by [14, lemma 5.1], $\mathcal{A}$ is generated by exterior powers $\Lambda^{\alpha^T}\mathcal{U}$ with $\alpha \in P_{d,e}$ where $\alpha^T \in P_{e,d}$ is the transpose of $\alpha \in P_{d,e}$, it suffices to understand the duals of generators.

Recall from example 9.1.6 that the longest element w_0 of the Weyl group of GL_e inverts the ordering of a dominant weight $\lambda \in \mathbb{Z}^e$, when seen as an ordered



$\alpha = (4,3,3,1)$ $\qquad$ $\alpha^\vee = (3,1,1,0)$

Figure 10.2.2: the dual of a Young diagram.



$\alpha = (4,2,1,1)$ $\qquad$ $\alpha^\vee = (3,3,2,0)$

Figure 10.2.3: an example of a half partition and its dual.

e-tuple. Also recall that $\mathcal{U}$ is the tautological bundle associated with the standard d-dimensional representation of G_1 and $\Delta = \det \mathcal{U}$ is the line bundle associated with the determinant representation with highest weight $(1, \dots, 1)$.

Lemma 10.2.1. *Let $\alpha \in P_{d,e}$. Then*

$$(\Lambda^{\alpha^T}\mathcal{U})^\vee = \Lambda^{\underline{d}-w_0\alpha^T}\mathcal{U} \otimes \Delta^{-e},$$

where $\Delta^{-e} = (\Delta^\vee)^e$.

Proof. Because the duality $\vee : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$ commutes with tensor products and $\Lambda^{\alpha^T}\mathcal{U}$ is invariant under permutations of the entries of α^T, it suffices to show

$$(\Lambda^i\mathcal{U})^\vee = \Lambda^{d-i}\mathcal{U} \otimes \Delta^{-1}$$

for $0 \leq i \leq d$, but this follows from the perfect pairing $\Lambda^i\mathcal{U} \otimes \Lambda^{d-i}\mathcal{U} \to \Delta$: it gives an isomorphism

$$\Lambda^i\mathcal{U} \longrightarrow \mathrm{Hom}(\Lambda^{d-i}\mathcal{U}, \Delta),$$

and taking the dual on both sides gives the desired result. $\square$

Inspired by the above lemma, we make a few definitions to help us organize our data.

Definition 10.2.4. Let $\alpha \in P_{d,e}$. Let w_0 be the longest element of the Weyl group of GL_d. The *dual partition $\alpha^\vee$ of α in $P_{d,e}$* is the partition $\underline{e} - w_0\alpha$. Note that if $|\alpha| = i$, then $|\alpha^\vee| = de - i$. Therefore, we call α a *half partition of $P_{d,e}$* if α has weight de', so that $|\alpha| = |\alpha^\vee|$.

Note that if $|\alpha| = i$, then $|\alpha^\vee| = de - i$, and that the degree of a weight is invariant under w_0. Additionally, note that the dual of the transpose $\alpha^T \in P_{e,d}$ of some $\alpha \in P_{d,e}$ is $w_0(\underline{d} - \alpha^T) \in P_{e,d}$, where w_0 is the longest element of the Weyl group of G_2. This duality on partitions is best described pictorially with Young diagrams, see figure 10.2.2. The dual of a Young diagram is obtained by rotating the diagram 180 degrees and swapping the filled part with the empty part of the $(d \times e)$-frame. For an example of a half partition in a 4×4-frame, see figure 10.2.3.

10.3 The Grothendieck-Witt spectrum of an even Grassmannian

We call the Grassmannian $X = \mathrm{Gr}(d, d+e)$ *even* if de is even. From here on, we assume that e is even (so the Grassmannian is even) and let $e' = e/2$. We return to the semi-orthogonal decomposition of $\mathcal{A}$. The functor $- \otimes \Delta : \mathcal{A} \to \mathcal{A}$ is an equivalence of pretriangulated dg categories. Then we obtain the twisted semi-orthogonal decomposition

$$\mathcal{A} = \langle \mathcal{A}_0 \otimes \Delta^{-e'}, \ldots, \mathcal{A}_{de} \otimes \Delta^{-e'} \rangle. \tag{10.3.1}$$

The reason for this twist is explained by the following results.

Lemma 10.3.2. *The standard duality $\vee : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$ acts on the semi-orthogonal decomposition (10.3.1) as*

$$(\mathcal{A}_i \otimes \Delta^{-e'})^\vee = \mathcal{A}_{de-i} \otimes \Delta^{-e'}$$

for each $i \in \{0, \ldots, de\}$.

Proof. Let $\alpha \in P_{d,e}$ be a partition of degree i. Then $\Lambda^{\alpha^T}\mathcal{U} \otimes \Delta^{-e'}$ is a generator of $\mathcal{A}_i \otimes \Delta^{-e'}$. Combining lemma 10.2.1 with definition 10.2.4, we obtain

$$(\Lambda^{\alpha^T}\mathcal{U} \otimes \Delta^{-e'})^\vee = \Lambda^{(\alpha^T)^\vee}\mathcal{U} \otimes \Delta^{-e'},$$

which is a generator of $\mathcal{A}_{de-i} \otimes \Delta^{-e'}$ since $|(\alpha^T)^\vee| = de - i$. □

From now on, let $\mathcal{B}_i = \mathcal{A}_i \otimes \Delta^{-e'}$. As an immediate corollary, we obtain a partial calculation of the Grothendieck-Witt spectrum $\mathrm{GW}(X)$. This calculation holds in arbitrary characteristic due to the results of [14].

Corollary 10.3.3. *There is a natural equivalence*

$$\mathrm{GW}^{[n]}(X) \simeq \mathrm{GW}^{[n]}(\mathcal{B}_{de'}) \oplus \bigoplus_{i=0}^{de'-1} \mathrm{K}(\mathcal{B}_i).$$

In particular,

$$\mathrm{W}^{[n]}(X) \simeq \mathrm{W}^{[n]}(\mathcal{B}_{de'}).$$

Proof. This is an application of the additivity theorem 6.2.2 for Grothendieck-Witt theory since $(\mathcal{B}_i)^\vee = \mathcal{B}_{de-i}$ for all $1 \le i \le de'$; in particular, $(\mathcal{B}_{de'})^\vee = \mathcal{B}_{de'}$. □

As $\mathrm{K}(\mathcal{B}_i)$ splits into copies of $\mathrm{K}(k)$, it remains to compute $\mathrm{GW}^{[n]}(\mathcal{B}_{de'})$. For $0 \le i \le de$, define $\mathcal{D}_i = \mathrm{Perf}(\mathcal{C}_i) \otimes (\det V)^{-e'}$ as a pretriangulated dg subcategory of $\mathrm{Perf}(\mathbf{Rep}(G_1))$. By [14, theorem 5.8] and [19, section 3.1.4], the dg functor

$$\Phi' : \mathcal{D}_{de'} \longrightarrow \mathcal{B}_{de'}$$

given by $M \otimes (\det V)^{-e'} \mapsto \mathcal{L}_X(M) \otimes \Delta^{-e'}$ is a duality-preserving equivalence. The twisted version of the semi-orthogonal decomposition (10.1.5) is

$$\mathcal{D}_{de'} = \langle \langle M_{\alpha - \underline{e'}} \rangle \mid \alpha \in (P_{d,e})_{de'} \rangle, \tag{10.3.4}$$

using the isomorphism $M_\alpha \otimes \det V^{-e'} \cong M_{\alpha - \underline{e'}}$ for $\alpha \in P_{d,e}$. Note that this semi-orthogonal decomposition does not exist in prime characteristic, because the irreducible representations M_α do not form an exceptional collection, see example 9.1.2.

Lemma 10.3.5. *For a partition $\alpha \in P_{d,e}$,*

$$M^*_{\alpha - \underline{e'}} \cong M_{\alpha^\vee - \underline{e'}}.$$

Proof. By lemma 9.1.4,

$$M^*_{\alpha - \underline{e'}} \cong M_{-w_0(\alpha - \underline{e'})}.$$

Since $w_0(\underline{m}) = \underline{m}$ for all $m \in \mathbb{Z}$, $-w_0(\alpha - \underline{e'}) = -w_0\alpha + \underline{e'}$. On the other hand, definition 10.2.4 yields

$$\alpha^\vee - \underline{e'} = \underline{e} - w_0\alpha - \underline{e'} = -w_0\alpha + \underline{e'},$$

which concludes the proof. □

In particular, if α is a half partition, then $M_{\alpha - \underline{e'}}$ and its dual have highest weights of the same degree. Hence, to study how $\mathcal{D}_{de'}$ behaves under duality, we need to understand the combinatorics of half partitions. We will distinguish between two kinds of half partitions.

Definition 10.3.6. A *symmetric half partition* $\alpha \in P_{d,e}$ is a half partition (see definition 10.2.4) such that $\alpha^\vee = \alpha$. A half partition is called *asymmetric* if it is not symmetric. The Young diagram of a symmetric partition is also called *symmetric*.

Lemma 10.3.7. *Let $d' = \lfloor \frac{d}{2} \rfloor$. There are $\binom{d'+e'}{e'}$ symmetric half partitions in $P_{d,e}$.*

Proof. Note that Young diagrams in the $(d \times e)$-frame correspond to binary sequences of length $d + e$, containing d zeroes and e ones. Given such a binary sequence, if we think of a Young diagram as a path starting in the lower left corner of the frame, each zero in the sequence means going up one row, and each one means going one column to the right. For example, the Young diagram $(4, 3, 3, 1)$ in the (4×4)-frame corresponds to the binary sequence 10110010, read from left to right. A Young diagram is symmetric if and only if the corresponding binary sequence is a palindrome. If d is odd, it follows that the middle bit in the binary sequence of a symmetric Young diagram must be a one, since e is even by assumption. Thus a symmetric Young diagram is completely determined by the first $d' + e'$ bits of the corresponding binary sequence, e' of which must be zero. It follows that the number of different symmetric Young diagrams in the $(d \times e)$-frame is

$$\binom{d'+e'}{e'},$$

as was to be shown. □

The total number of asymmetric half partitions in $P_{d,e}$ is harder to determine, but can be expressed using a recursive formula. Ultimately, we are more interested in the symmetric half partitions because the asymmetric half partitions only contribute to the hyperbolic part of the Grothendieck-Witt spectrum. For $i \in \mathbb{N}$, $|(P_{d,e})_i|$ is the total number of partitions of degree i in $P_{d,e}$.

Lemma 10.3.8. *The total number of half partitions $|(P_{d,e})_{de'}|$ is even, and given by the recursive formula*

$$|(P_{d,e})_{de'}| = \sum_{j=1}^{e} |(P_{d-1,j})_{de'-j}|.$$

Proof. An $\alpha \in (P_{d,e})_{de'}$ is given by a d-tuple $(\alpha_1, \dots, \alpha_d)$ such that $0 \leq \alpha_i \leq e$ and $\alpha_i \geq \alpha_{i+1}$ for all $0 \leq i \leq d$. Note that $\alpha_1 \neq 0$ since $de' \neq 0$ by assumption. If $\alpha_1 = j$ for some $j \in \mathbb{N}$, then $\alpha_i \leq j$ for all $i \geq 2$. Hence $(\alpha_2, \dots, \alpha_d) \in (P_{d-1,j})_{de'-j}$, and the recursive formula follows. □

Finally, here is a computation of the Grothendieck-Witt spectrum of $\mathcal{D}_{de'}$. Let $A_{d,e}$ be the number of asymmetric half partitions of $P_{d,e}$.

Lemma 10.3.9. *There is an equivalence*

$$\mathrm{GW}^{[n]}(\mathcal{D}_{de'}) \simeq \bigoplus_{i=1}^{\binom{d'+e'}{e'}} \mathrm{GW}^{[n]}(k) \oplus \bigoplus_{j=1}^{\frac{A_{d,e}}{2}} \mathrm{K}(k).$$

Proof. On the one hand, the duality $\vee : (P_{d,e})_{de'} \to (P_{d,e})_{de'}$ sends asymmetric half partitions to asymmetric half partitions, leaving no asymmetric half partition fixed. Hence, if α is an asymmetric half partition, $M_{\alpha-\underline{e'}}$ is different from its dual and contributes a copy of $\mathrm{K}(\langle M_{\alpha-\underline{e'}}\rangle)$ to the direct sum decomposition of $\mathrm{GW}^{[n]}(\mathcal{D}_{de'})$ by additivity.

On the other hand, if α is a symmetric half partition, $M_{\alpha-\underline{e'}}$ is self-dual and contributes a copy of $\mathrm{GW}^{[n]}(\langle M_{\alpha-\underline{e'}}\rangle)$ to the direct sum decomposition.

By proposition 9.1.1, $\langle M_{\alpha-\underline{e'}}\rangle \simeq \mathrm{Perf}(k)$, which finishes the proof. □

Putting everything together, we obtain our main result for Grothendieck-Witt spectra of Grassmannians.

Theorem 10.3.10. *Let $p = \binom{d'+e'}{e'}$ and $q = \frac{1}{2}(\binom{d+e}{e} - p)$. There is an equivalence*

$$\mathrm{GW}^{[n]}(X) \simeq \bigoplus_{i=1}^{p} \mathrm{GW}^{[n]}(k) \oplus \bigoplus_{j=1}^{q} \mathrm{K}(k).$$

Proof. By corollary 10.3.3, copies of $\mathrm{GW}^{[n]}(k)$ are contributed solely by

$$\mathrm{GW}^{[n]}(\mathcal{D}_{de'}),$$

and lemma 10.3.9 tells us that there exactly p of such copies.

Furthermore, the exceptional collection of $\mathcal{A}$ consisting of bundles $L^{\alpha}\mathcal{U}$ has $\binom{d+e}{e}$ members, one for each partition in $P_{d,e}$. Only p of these generators are self-dual, and the rest occur in dual pairs. Each such dual pair contributes exactly one copy of $\mathrm{K}(k)$ by the additivity theorem 6.2.2 and the fact that $\langle M_{\alpha-\underline{e'}}\rangle \simeq \mathrm{Perf}(k)$. The result follows. □

10.4 The Grothendieck-Witt theory of the classifying space of a general linear group

We shift our attention to the classifying space of the general linear group $G = \mathrm{GL}_d$ over a field k of characteristic zero. The Grassmannians $X_r = \mathrm{Gr}(d, r)$ form an admissible gadget for $\mathrm{B_{gm}}G$ by example 3.6.15. Since we have computations of the Grothendieck-Witt spectrum of X_r when $r - d$ is even, we define

$$\mathrm{B_{gm}}G = \operatorname*{colim}_r X_{d+2r},$$

where the inclusion $f_r : X_{d+2r} \to X_{d+2r+2}$ is induced by the embedding $k^{d+2r} \subset k^{d+2r+2}$ sending the standard unit vector e_i to e_{i+1}. In other words, we pad the ambient space k^{d+2r} with a coordinate on the left and on the right.

To show that, for $i, n \in \mathbb{Z}$,

$$\mathrm{GW}_i^{[n]}(\mathrm{B_{gm}}G) \cong \lim_r \mathrm{GW}_i^{[n]}(X_{d+2r}),$$

it must be shown that the tower of $\mathrm{GW}_0^{[0]}(k)$-modules $\{\mathrm{GW}_{i+1}^{[n]}(X_{d+2r})\}_{r\in\mathbb{N}}$ satisfies the Mittag-Leffler condition since this will force the $\lim^1$-term of the Milnor exact sequence (4.2.6) to vanish. We have the following stronger result.

Lemma 10.4.1. *For each $n \in \mathbb{Z}$ and $r \in \mathbb{N}$, the pullback map*

$$f_r^* : \mathrm{GW}^{[n]}(X_{d+2r+2}) \longrightarrow \mathrm{GW}^{[n]}(X_{d+2r})$$

is a split epimorphism of spectra, up to homotopy. In particular, f_r^* is split surjective on stable homotopy groups.

Proof. For $r \in \mathbb{N}$, let $\mathcal{A}_r = \mathrm{Perf}(X_{d+2r})$, $\mathcal{U}_r$ the universal bundle of X_{d+2r} and $\Delta_r = \det \mathcal{U}_r$. We must construct a map of spectra

$$g : \mathrm{GW}^{[n]}(X_{d+2r}) \to \mathrm{GW}^{[n]}(X_{d+2r+2})$$

for each $r \in \mathbb{N}$, such that $f_r^* g$ is the identity.

Fix $r \in \mathbb{N}$. Then $\mathcal{A}_r$ has a semi-orthogonal decomposition

$$\mathcal{A}_r = \langle \mathcal{A}_{r,0}, \ldots, \mathcal{A}_{r,2dr} \rangle$$

as in (10.1.4). By twisting the above semi-orthogonal decomposition by Δ_r^{-r}, we obtain a semi-orthogonal decomposition

$$\mathcal{A}_r = \langle \mathcal{B}_{r,0}, \ldots, \mathcal{B}_{r,2dr} \rangle,$$

where $\mathcal{B}_{r,i} = \mathcal{A}_{r,i} \otimes \Delta_r^{-r}$ as in (10.3.1).

By corollary 10.3.3,

$$\mathrm{GW}^{[n]}(X_{d+2r}) \simeq \mathrm{GW}^{[n]}(\mathcal{B}_{r,dr}) \oplus \bigoplus_{i=0}^{dr-1} \mathrm{K}(\mathcal{B}_{r,i})$$

$$\mathrm{GW}^{[n]}(X_{d+2r+2}) \simeq \mathrm{GW}^{[n]}(\mathcal{B}_{r+1,d(r+1)}) \oplus \bigoplus_{i=0}^{d(r+1)-1} \mathrm{K}(\mathcal{B}_{r+1,i}).$$

We construct a map $g : \mathrm{GW}^{[n]}(X_{d+2r}) \to \mathrm{GW}^{[n]}(X_{d+2r+2})$ on each of the summands by constructing corresponding functors of pretriangulated dg categories. For $0 \le i \le dr$, let $G_i : \mathcal{B}_{r,i} \to \mathcal{B}_{r+1,i+1}$ be the dg functor given by $L^\alpha\mathcal{U}_r \otimes \Delta_r^{-r} \mapsto L^{\alpha+\underline{1}}\mathcal{U}_{r+1} \otimes \Delta_{r+1}^{-r-1}$. For $\alpha, \beta \in (P_{d,2r})_i$ and $j \neq 0$,

$$\begin{aligned} H^j\mathcal{B}_{r,i}(L^\alpha\mathcal{U}_r, L^\beta\mathcal{U}_r) &\cong H^j\mathcal{B}_{r,i}(L^\alpha\mathcal{U}_r \otimes \Delta_r^{-r}, L^\beta\mathcal{U}_r \otimes \Delta_r^{-r}) \\ &\cong H^j\mathcal{B}_{r+1,i+1}(L^{\alpha+\underline{1}}\mathcal{U}_{r+1} \otimes \Delta_{r+1}^{-r-1}, L^{\beta+\underline{1}}\mathcal{U}_{r+1} \otimes \Delta_{r+1}^{-r-1}) \\ &\cong H^j\mathcal{B}_{r+1,i+1}(L^{\alpha+\underline{1}}\mathcal{U}_{r+1}, L^{\beta+\underline{1}}\mathcal{U}_{r+1}) \\ &\cong 0 \end{aligned}$$

since higher ext groups between irreducibles vanish by [49, lemma 3.2]. Furthermore

$$\begin{aligned} \delta_{\alpha,\beta}k &\cong H^0\mathcal{B}_{r,i}(L^\alpha\mathcal{U}_r, L^\beta\mathcal{U}_r) \\ &\cong H^0\mathcal{B}_{r,i}(L^\alpha\mathcal{U}_r \otimes \Delta_r^{-r}, L^\beta\mathcal{U}_r \otimes \Delta_r^{-r}) \\ &\cong H^0\mathcal{B}_{r+1,i+1}(L^{\alpha+\underline{1}}\mathcal{U}_{r+1} \otimes \Delta_{r+1}^{-r-1}, L^{\beta+\underline{1}}\mathcal{U}_{r+1} \otimes \Delta_{r+1}^{-r-1}) \\ &\cong H^0\mathcal{B}_{r+1,i+1}(L^{\alpha+\underline{1}}\mathcal{U}_{r+1}, L^{\beta+\underline{1}}\mathcal{U}_{r+1}), \end{aligned}$$

where $\delta_{\alpha,\beta}$ is the Kronecker delta of α and β, by the Littlewood-Richardson rule (see [49, section 3.3]). It follows that $G_i : \mathcal{B}_{r,i} \to \mathcal{B}_{r+1,i+1}$ is a quasi-fully faithful dg functor for each i, and $f_r^*|_{\mathcal{B}_{r+1,i+1}} G_i$ is the identity on $\mathcal{B}_{r,i}$ by construction since

$$f_r^*(L^{\alpha+\underline{1}}\mathcal{U}_{r+1} \otimes \Delta_{r+1}^{-r-1}) = f_r^*(L^\alpha\mathcal{U}_{r+1} \otimes \Delta_{r+1}^{-r}) = L^\alpha\mathcal{U}_r \otimes \Delta_r^{-r}.$$

Moreover, g_{dr} is a dg form functor as a consequence of lemma 10.3.5. Hence we obtain maps

$$\begin{aligned} g_i : \mathrm{K}(\mathcal{B}_{r,i}) &\longrightarrow \mathrm{K}(\mathcal{B}_{r+1,i+1}) \\ g_{dr} : \mathrm{GW}^{[n]}(\mathcal{B}_{r,dr}) &\longrightarrow \mathrm{GW}^{[n]}(\mathcal{B}_{r+1,d(r+1)}) \end{aligned}$$

whose sum g splits the pullback map $f_r^* : \mathrm{GW}^{[n]}(X_{d+2r+2}) \to \mathrm{GW}^{[n]}(X_{d+2r})$, as was to be shown. □

As a corollary, we obtain a computation of $\mathrm{GW}_i^{[n]}(\mathrm{B_{gm}}G)$.

Corollary 10.4.2. *For $i, n \in \mathbb{Z}$,*

$$\mathrm{GW}_i^{[n]}(\mathrm{B_{gm}}G) \cong \lim_r \mathrm{GW}_i^{[n]}(X_{d+2r}).$$

Proof. By lemma 10.4.1, the map

$$f_r^* : \mathrm{GW}_i^{[n]}(X_{d+2r+2}) \to \mathrm{GW}_i^{[n]}(X_{d+2r})$$

is a split surjection for all $i, n \in \mathbb{Z}$. Hence, the tower $\{\mathrm{GW}_{i+1}^{[n]}(X_{d+2r})\}$ satisfies the Mittag-Leffler condition for all $i, n \in \mathbb{Z}$, and the $\lim^1$-term of the Milnor exact sequence (4.2.6) vanishes. The result follows. □

A Appendix

In this appendix, we compute the total Grothendieck-Witt ring of the projective plane over a quadratically closed field.

We also prove a lifting criterion for closed immersions of schemes and we show that the category of pro-objects on an abelian category is itself abelian.

A.1 The total Grothendieck-Witt ring of the projective plane

The results in this section are joint work with Thomas Hudson.

In this thesis, we mostly consider the additive structure of Grothendieck-Witt theory, but it also possesses a natural ring structure, as already noted after definition 6.1.17 of the Grothendieck-Witt spectrum. Computing the ring structure of a cohomology theory, even in special cases, significantly increases our understanding of the theory and is often closely related to intersection theory.

Let k be a quadratically closed field of characteristic not two and X be a scheme of finite type over k. As in chapter 7, we consider the Grothendieck-Witt groups of the pretriangulated dg category $\mathrm{Perf}(X)$ with duality induced by $\mathcal{L}[n]$ where $\mathcal{L}$ is a line bundle over X and $n \in \mathbb{Z}$.

Definition A.1.1. The *total Grothendieck-Witt ring* $\mathrm{GW}^{\mathrm{tot}}(X)$ *of* X is the ring

$$\mathrm{GW}^{\mathrm{tot}}(X) = \bigoplus \mathrm{GW}_0^{[i]}(X, \mathcal{L}),$$

where the sum runs over $i \in \mathbb{Z}$ and $\mathcal{L} \in \mathrm{Pic}(X)$, with sum induced by the direct sum and product induced by the tensor product of symmetric forms, up to lax similitude as defined in [6].

Since we will only consider the classical triangular Grothendieck-Witt groups $\mathrm{GW}_0^{[n]}$ in this section, we will drop the subscript 0 from the notation. We use the *Euler class* in Grothendieck-Witt theory as defined in [31, section 2.4] for computations in the ring $\mathrm{GW}^{\mathrm{tot}}$. These classes can be used in computations involving the multiplicative structure, since they satisfy a Whitney formula. For a vector bundle $V \to X$ of rank r over X, the Euler class $e(V)$ of V is an element of $\mathrm{GW}^{[r]}(X, \det V^\vee)$.

Another useful tool will be the *twisted hyperbolic map*

$$H_n^{\mathcal{L}} : \mathrm{K}_0(X) \to \mathrm{GW}_0^{[n]}(X, \mathcal{L}), \quad [M] \mapsto \left[M \oplus [M, M^\vee[n] \otimes \mathcal{L}]], \begin{pmatrix} 0 & 1 \\ \mathrm{can} & 0 \end{pmatrix} \right].$$

When X is a projective space $\mathbb{P}^n_k$, we write H_n^i for the twisted hyperbolic map $H_n^{\mathcal{O}(i)}$, and when $\mathcal{L} = \mathcal{O}$, we write H_n instead of $H_n^{\mathcal{O}}$.

First, we recall the computation of the additive structure of $\mathrm{GW}^{\mathrm{tot}}(k)$ from example 6.1.9. The shifted Grothendieck-Witt groups of k are

$$\begin{aligned} \mathrm{GW}^{[0]}(k) &= \mathbb{Z} & \qquad \mathrm{GW}^{[1]}(k) &= 0 \\ \mathrm{GW}^{[2]}(k) &= \mathbb{Z} & \qquad \mathrm{GW}^{[3]}(k) &= \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

We show how to organize them into a ring. The additive generator $\alpha \in \mathrm{GW}^{[2]}(k)$ is the hyperbolic form $H_2(k[1])$, where $k[1]$ is the usual shift of k regarded as a complex concentrated in degree zero, and $\alpha^2 = 4$ in $\mathrm{GW}^{[0]}(k)$. The only nonzero element $\beta \in \mathrm{GW}^{[3]}(k)$ is the hyperbolic form $H_3(k[1])$. Note that $H_3(k[1]) = H_3(k[2])$, so since H_3 is additive, $-H_3(k) = H_3(k)$, whence $2\beta = 0$. Furthermore, $\beta^2 \in \mathrm{GW}^{[6]}(k)$ is a class of the form

$$\begin{tikzcd} k \arrow[r] \arrow[d] & k^2 \arrow[r] \arrow[d] & k \arrow[d] \\ k \arrow[r] & k^2 \arrow[r] & k, \end{tikzcd}$$

sitting in degrees $[-4,-2]$. Hence $\beta^2 = H_6(k[2] + k[3]) = 0$. Since $\mathrm{GW}_0^{[5]}(k) \cong \mathrm{GW}_0^{[1]}(k) = 0$, $\alpha\beta = 0$.

Summarizing, we see that

$$\mathrm{GW}^{\mathrm{tot}}(k) \cong \mathbb{Z}[\alpha,\beta]/(\alpha^2 - 4, \beta^2, \alpha\beta, 2\beta).$$

Next, we prove a result about the vanishing of powers of $e(\mathcal{O}(1))$ for projective spaces. Let $\mathbb{P} = \mathbb{P}_k^n$ and $\mathcal{O} = \mathcal{O}_{\mathbb{P}}$.

Proposition A.1.2. *The class $e(\mathcal{O}(1))$ in $\mathrm{GW}^{[1]}(\mathbb{P}, \mathcal{O}(-1))$ satisfies $e(\mathcal{O}(1))^{n+1} = 0$.*

Proof. The Euler sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}^{\oplus n+1} \longrightarrow \mathcal{Q} \longrightarrow 0$$

is exact ($\mathcal{Q}$ denotes the universal quotient bundle), and it remains exact if we tensor it by $\mathcal{O}(1)$. Hence $e(\mathcal{O}(1)^{\oplus n+1})$ vanishes by [31, proposition 22], and by repeated application of the Whitney formula [31, theorem 21], $e(\mathcal{O}(1)^{\oplus n+1}) = e(\mathcal{O}(1))^{n+1}$, which concludes the proof. $\square$

We can also compute the square of the Euler class, as follows.

Proposition A.1.3. *The square of the euler class $e(\mathcal{O}(1))$ in $\mathrm{GW}_0^{[2]}(\mathbb{P}, \mathcal{O})$ is given by $e^2(\mathcal{O}(1)) = H_2^0(\mathcal{O}(1)) - H_2^0(\mathcal{O})$.*

Proof. By the Whitney formula [31, theorem 21] for quadratic Euler classes,

$$e^i(\mathcal{O}(1)) = e(\mathcal{O}(1)^{\oplus i})$$

for all $i \in \mathbb{N}$. Hence, [31, proposition 14] yields

$$e^2(\mathcal{O}(1)) = H_2^0(\mathcal{O}(1)) + [\mathcal{O}[1]^{\oplus 2}, -\phi],$$

where $\phi : \mathcal{O}^{\oplus 2} \to [\mathcal{O}^{\oplus 2}, \mathcal{O}]$ is the canonical map induced by the determinant pairing $\mathcal{O}^{\oplus 2} \otimes \mathcal{O}^{\oplus 2} \to \mathcal{O}$ given by $(a_1, a_2) \otimes (b_1, b_2) \mapsto a_1 b_2 - a_2 b_1$. There is an isomorphism $\mathcal{O}[1] \to [\mathcal{O}[1], \mathcal{O}[2]]$ given by $1 \mapsto (1 \mapsto 1)$. One checks that $-\phi$ is isometric to the map

$$\begin{pmatrix} 0 & 1 \\ -\operatorname{ev}_{\mathcal{O}} & 0 \end{pmatrix} : \mathcal{O}[1] \oplus [\mathcal{O}[1], \mathcal{O}[2]] \longrightarrow [\mathcal{O}[1], \mathcal{O}[2]] \oplus [[\mathcal{O}[1], \mathcal{O}[2]], \mathcal{O}[2]],$$

so that $[\mathcal{O}[1]^{\oplus 2}, -\phi] = H_2^0(\mathcal{O}[1])$. It follows that

$$e^2(\mathcal{O}(1)) = H_2^0(\mathcal{O}(1) - \mathcal{O}),$$

as was to be shown. $\square$

Now assume that $n = 2s$. Then the projective bundle formula 7.4.3 yields

$$\mathrm{GW}_0^{[n]}(\mathbb{P}, \mathcal{O}) \cong \mathrm{GW}_0^{[n]}(k) \oplus \mathrm{K}_0(k)^{\oplus s},$$

where the embedding $\pi^* : \mathrm{GW}_0^{[n]}(k) \to \mathrm{GW}^{[n]}(\mathbb{P}, \mathcal{O})$ is the pullback along the structure map $\pi : \mathbb{P} \to \operatorname{Spec} k$ and $\mathrm{K}_0(k)^{\oplus s} \to \mathrm{GW}^{[n]}(\mathbb{P}, \mathcal{O})$ is given by

$$(\mathcal{F}_1, \ldots, \mathcal{F}_s) \mapsto \sum_{i=1}^{s} H_n^0(\mathcal{F}_i \otimes \mathcal{O}(i)).$$

The twisted Grothendieck-Witt groups $\mathrm{GW}_0^{[n]}(\mathbb{P}, \mathcal{O}(1))$ can be described similarly, but the pullback must be twisted by a special element $\mu : \mathrm{GW}_0^{[2s]}(\mathbb{P}, \mathcal{O}(1))$. The exact Euler sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}^{2s+1} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

that we saw in the proof of the above proposition, is an exact sequence of finite locally free sheaves on $\mathbb{P}$. We call $\mathcal{Q}$ the *quotient bundle* on $\mathbb{P}$. Note that $\mathcal{Q}$ has rank $2s$, and therefore $e(\mathcal{Q}) \in \mathrm{GW}^{[2s]}(\mathbb{P}, \det \mathcal{Q}^\vee)$. Since

$$\det \mathcal{Q} \otimes \det \mathcal{O}(-1) \cong \det \mathcal{O}^{2s+1},$$

it follows that $\det \mathcal{Q} \cong \mathcal{O}(1)$. By [31, proposition 14],

$$e(\mathcal{Q}) = [\Lambda^s \mathcal{Q}^\vee[s], (-1)^s \phi_s] + H_{2s}^1\left(\sum_{i=0}^{s-1} (-1)^i [\Lambda^i \mathcal{Q}^\vee]\right),$$

where $\phi_s : \Lambda^s \mathcal{Q}^\vee[s] \to [\Lambda^s \mathcal{Q}^\vee[s], \mathcal{O}(-1)[2s]]$ is the canonical isomorphism. Now let $\mu = [\Lambda^s \mathcal{Q}^\vee[s], (-1)^s \phi_s]$, which lives in $\mathrm{GW}^{[2s]}(\mathbb{P}, \mathcal{O}(-1))$.

Note that the class μ is (up to sign) a twist of the canonical symmetric form $\Lambda^s \Omega_{\mathbb{P}/k} \otimes \Lambda^s \Omega_{\mathbb{P}/k} \to \omega_{\mathbb{P}/k}$.

When we write $\mu \in \mathrm{GW}^{[2s]}(\mathbb{P}, \mathcal{O}(1))$, we understand this to be an element lax similar to μ, obtained by multiplying μ with the canonical element of $\mathrm{GW}^{[0]}(\mathbb{P}, \mathcal{O}(2))$ induced by the symmetric bilinear form $\mathcal{O}(1) \otimes \mathcal{O}(1) \to \mathcal{O}(2)$, which is itself lax similar to $1 \in \mathrm{GW}^{[0]}(\mathbb{P})$. Similarly, we may write $e(\mathcal{O}(1)) \in \mathrm{GW}^{[1]}(\mathbb{P}, \mathcal{O}(1))$, when strictly speaking $e(\mathcal{O}(1))$ is an element of $\mathrm{GW}^{[1]}(\mathbb{P}, \mathcal{O}(-1))$.

By theorem 7.4.10, there is an isomorphism

$$\mathrm{GW}_0^{[n]}(\mathbb{P}, \mathcal{O}(1)) \cong \mathrm{GW}_0^{[n-2s]}(k) \oplus \mathrm{K}_0(k)^{\oplus s},$$

where $\mathrm{K}_0(k)^{\oplus s} \to \mathrm{GW}_0^{[n]}(\mathbb{P}, \mathcal{O}(1))$ is given as before, and the map

$$\mathrm{GW}_0^{[n-2s]}(k) \to \mathrm{GW}_0^{[n]}(\mathbb{P}, \mathcal{O}(1))$$

is given by $a \mapsto \mu \pi^* a$.

We further restrict our attention to the projective plane. Let $\mathbb{P} = \mathbb{P}_k^2$ with projection map $\pi : \mathbb{P} \to \operatorname{Spec} k$ and $s = 1$. Let $1 = [\mathcal{O}_{\mathbb{P}}, 1] \in \mathrm{GW}^{[0]}(\mathbb{P})$ be the unit, or an element lax similar to the unit. We keep in mind the following table to keep track of each component of the total Grothendieck-Witt ring. By abuse of notation, we simply denote $\pi^* \alpha$ and $\pi^* \beta$ by α and β, respectively.

i	0		1	
component	$\mathrm{K}_0(k)$	$\mathrm{GW}_0^{[n]}(k)$	$\mathrm{K}_0(k)$	$\mathrm{GW}_0^{[n-2]}(k)$
$\mathrm{GW}_0^{[0]}(\mathbb{P},\mathcal{O}(i))$	$\mathbb{Z}H_0^0(\mathcal{O}(1))$	$\mathbb{Z}1$	$\mathbb{Z}H_0^1(\mathcal{O}(1))$	$\mathbb{Z}\alpha\mu$
$\mathrm{GW}_0^{[1]}(\mathbb{P},\mathcal{O}(i))$	$\mathbb{Z}H_1^0(\mathcal{O}(1))$	0	$\mathbb{Z}H_1^1(\mathcal{O}(1))$	$(\mathbb{Z}/2)\beta\mu$
$\mathrm{GW}_0^{[2]}(\mathbb{P},\mathcal{O}(i))$	$\mathbb{Z}H_2^0(\mathcal{O}(1))$	$\mathbb{Z}\alpha$	$\mathbb{Z}H_2^1(\mathcal{O}(1))$	$\mathbb{Z}\mu$
$\mathrm{GW}_0^{[3]}(\mathbb{P},\mathcal{O}(i))$	$\mathbb{Z}H_3^0(\mathcal{O}(1))$	$(\mathbb{Z}/2)\beta$	$\mathbb{Z}H_3^1(\mathcal{O}(1))$	0

Note that $\alpha = -H_2^0(\mathcal{O})$ and $\beta = H_3^0(\mathcal{O})$. Let $\varepsilon = e(\mathcal{O}(1)) = H_1^1(\mathcal{O}(1))$. We have already computed $\varepsilon^2 - \alpha = H_2^0(\mathcal{O}(1))$ and $\varepsilon^3 = 0$ in proposition A.1.3 and proposition A.1.2. Furthermore, we have already shown that $\alpha^2 = 4$, $\beta^2 = 0$ and $\alpha\beta = 0$. Using the explicit definition of the Euler class and the exact Karoubi sequences for GW, we compute

$$\begin{aligned}
\alpha\varepsilon &= H_2^0(\mathcal{O}[1])H_1^1(\mathcal{O}(1)) = 2H_3^1(\mathcal{O}(1)[1]) = -2H_3^1(\mathcal{O}(1)) \\
\alpha\varepsilon^2 &= 4 - 2H_0^0(\mathcal{O}(1)) \\
\beta\varepsilon &= 0 \\
\mu\varepsilon &= H_3^0(\mathcal{O}(1)+\mathcal{O}) = H_3^0(\mathcal{O}(1)) + \beta \\
\mu\varepsilon^2 &= \alpha\mu - 2H_0^1(\mathcal{O}(1)) \\
\alpha\mu\varepsilon &= 2H_1^0(\mathcal{O}(1)) \\
\alpha\mu\varepsilon^2 &= 4H_2^1(\mathcal{O}(1)) + 4\mu \\
\mu^2 &= 10 - 3H_0^0(\mathcal{O}(1)) \\
\mu H_0^0(\mathcal{O}(1)) &= H_2^1(\mathcal{Q}(1)[1]) = -H_2^1(3\mathcal{O}(1) - \mathcal{O}) = -2H_2^1(\mathcal{O}(1)).
\end{aligned} \tag{A.1.4}$$

These equations identify the ring structure on $\mathrm{GW}^{\mathrm{tot}}(\mathbb{P})$.

Theorem A.1.5. *The total Grothendieck-Witt ring* $\mathrm{GW}^{\mathrm{tot}}(\mathbb{P})$ *is given as*

$$\mathrm{GW}^{\mathrm{tot}}(\mathbb{P}) \cong \mathrm{GW}^{\mathrm{tot}}(k)[a_0,a_1,a_2,a_3,b_0,b_1,b_2,b_3]/I,$$

where I is the ideal generated by the relations (A.1.4), and where $a_i = H_i^0(\mathcal{O}(1))$ and $b_i = H_i^1(\mathcal{O}(1))$ for $i = 0,1,2,3$.

Proof. This follows from the addititive structure of $\mathrm{GW}^{\mathrm{tot}}(\mathbb{P})$ combined with the relations (A.1.4). ☐

Furthermore, we obtain the following expressions for the additive generators of $\mathrm{GW}_0^{\mathrm{tot}}(\mathbb{P})$:

$$\begin{aligned}
H_0^0(\mathcal{O}(1)) &= \frac{4-\alpha\varepsilon^2}{2} = \frac{10-\mu^2}{3}\\
H_1^0(\mathcal{O}(1)) &= \frac{\alpha\mu\varepsilon}{2}\\
H_2^0(\mathcal{O}(1)) &= \varepsilon^2-\alpha\\
H_3^0(\mathcal{O}(1)) &= \mu\varepsilon+\beta\\
H_0^1(\mathcal{O}(1)) &= \frac{4-\alpha\varepsilon^2}{2}\\
H_1^1(\mathcal{O}(1)) &= \frac{\mu(\alpha-\varepsilon^2)}{2}\\
H_2^1(\mathcal{O}(1)) &= \frac{\mu(\alpha\varepsilon^2-4)}{4}\\
H_3^1(\mathcal{O}(1)) &= \frac{-\alpha\varepsilon}{2}.
\end{aligned}$$

Also note that $e(\mathcal{O}(2)) = H_1^0(\mathcal{O}(1))$, so $2e(\mathcal{O}(2)) = \alpha\mu\varepsilon$, and

$$H_0^0(\mathcal{O}(1)-\mathcal{O}) = -\tfrac{1}{2}\alpha\varepsilon^2,$$

so that $H_0^0(\mathcal{O}(1)-\mathcal{O})^2 = 0$, since $\varepsilon^4 = 0$. This recovers a result from [95, example 5.3].

A.2 A lifting criterion for closed immersions of schemes

The *valuative criteria* in algebraic geometry are an important tool to determine separatedness and universal closedness of morphisms of schemes. For an explanation of the valuative criteria, see [77, Tag 01KA] and [77, Tag 01KY], or the original results [35, proposition 7.2.3 and théorème 7.3.8].

In this appendix, we prove a variation for closed immersions of the valuative criteria, which can be useful in proving directly that a morphism of schemes is a closed immersion, rather than proving that it is both proper and a monomorphism.

Theorem A.2.1 (extensive criterion)**.** *Let $f : X \to Y$ be a quasi-compact morphism of schemes. Then f is a closed immersion if and only if it is right orthogonal to all ring extensions, that is, if for all solid commutative diagrams*

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r, "a"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
\operatorname{Spec} B \arrow[r, "b"] \arrow[ur, dashed, "h"] & Y,
\end{tikzcd}$$

where g corresponds to an extension of rings $B \to A$, there exists a unique dotted arrow $h : \operatorname{Spec} B \to X$ such that $hg = a$ and $fh = b$.

The concept of orthogonality is a useful categorical notion, closely related to factorization systems. Here is the definition of orthogonality in an arbitrary category.

Definition A.2.2. Let $f : A \to B$ be a morphism in a category $\mathcal{C}$. Let S be a collection of morphisms in $\mathcal{C}$. Then f is

(i) left orthogonal to S if, for each $s \in S$ and each commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & S \arrow[d, "s"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & T, \end{tikzcd}$$

there exists a unique lift h making the triangles commute; and

(ii) right orthogonal to S if, for each $s \in S$ and each commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d, "s"'] & A \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed, "h"] & B, \end{tikzcd}$$

there exists a unique lift h making the triangles commute.

Let **Ring** be the category of commutative rings. Every ring homomorphism factors as a surjective ring homomorphism followed by an injective one, by passing through the image. The following proposition shows that this defines an *orthogonal factorization system* on **Ring**. Several other such factorization systems are studied in [1].

Proposition A.2.3. *Let $\phi : A \to B$ be a morphism in* **Ring**. *Then*

(i) ϕ is surjective if and only if ϕ is left orthogonal to every injective ring homomorphism; and

(ii) ϕ is injective if and only if every surjective ring homomorphism is left orthogonal to ϕ.

Proof. We only prove (i), the proof of (ii) being similar. First, suppose that ϕ is surjective and let $\psi : C \to D$ be an injective map of rings. Given a commutative square

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\phi"'] & C \arrow[d, "\psi"] \\ B \arrow[r, "\beta"'] \arrow[ur, dashed, "\gamma"] & D, \end{tikzcd}$$

there exists a lift $\gamma : B \to C$ given by $\gamma(b) = \alpha(a)$, where $a \in A$ is such that $\phi(a) = b$. This map is a well-defined and unique lift, since if $a, a' \in A$ satisfy $\phi(a) = \phi(a')$, then $\psi(\alpha(a)) = \psi(\alpha(a'))$ by the commutativity of the diagram, and by the injectivity of ψ, $\alpha(a) = \alpha(a')$.

Conversely, suppose that ϕ is left orthogonal to every injective ring homomorphism. Then the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\phi"'] & \phi(A) \arrow[d] \\ B \arrow[r, equal] \arrow[ur, dashed] & B \end{tikzcd}$$

admits a unique lift $B \to \phi(A)$ which is both surjective and injective, and therefore an isomorphism. It follows that ϕ is surjective. $\square$

Note that the above proof only used properties of **Set**, namely the existence of images, and surjective and injective maps. Therefore this factorization system exists on any category $\mathcal{C}$ with a forgetful functor to **Set** in which the set-theoretic image is well-defined.

By using the anti-equivalence between the category of affine schemes and the category of commutative rings, one obtains the *extensive criterion for closed immersions of affine schemes*, or simply the *affine extensive criterion.*

Proposition A.2.4 (affine extensive criterion)**.** *A morphism $f : X \to Y$ of affine schemes is a closed immersion if and only if it is right orthogonal to all ring extensions.*

Proof. This statement is a formal equivalent of proposition A.2.3, since closed immersions of affine schemes correspond to surjective ring homomorphisms. □

The *affinification* of a scheme X is the affine scheme that best approximates X. It is a universal construction that turns out to be useful for studying morphisms of schemes that are left orthogonal to some class of affine morphisms. For the remainder of this section, let X be a scheme.

Definition A.2.5. The *affinification* of the scheme X is the canonical map $\operatorname{aff}_X : X \to \operatorname{Spec} \mathcal{O}_X(X)$ of [33, remark 3.7], which corresponds to the identity on $\mathcal{O}_X(X)$ under the adjunction $\Gamma \dashv \operatorname{Spec}$ between $\operatorname{Spec} : \mathbf{Ring} \to \mathbf{Aff}$ and the global section functor $\Gamma : \mathbf{Aff} \to \mathbf{Ring}$.

Sometimes we will drop the subscript X from the notation of aff_X, if no ambiguity arises. We collect some elementary properties of the canonical map aff_X, which will play a role in the proof of theorem A.2.1.

Lemma A.2.6. *If aff_X is a split monomorphism, then X is an affine scheme.*

Proof. Write $S = \operatorname{Spec} \mathcal{O}_X(X)$. Let $s : S \to X$ be a morphism such that $s \operatorname{aff}_X = \mathrm{id}_X$. Then

$$\operatorname{aff}_X s \operatorname{aff}_X = \mathrm{id}_S \operatorname{aff}_X$$

and since both $\operatorname{aff}_X s$ and id_S are morphisms of affine schemes, it follows that $\operatorname{aff}_X s = \mathrm{id}_S$ by [59, proposition 2.3.25]. Hence aff_X is an isomorphism and X is an affine scheme, as was to be shown. □

Proposition A.2.7. *The largest quasi-coherent $\mathcal{O}_{\operatorname{Spec} \mathcal{O}_X(X)}$-module contained in the kernel of the canonical morphism of sheaves $\phi : \mathcal{O}_{\operatorname{Spec} \mathcal{O}_X(X)} \to \operatorname{aff}_* \mathcal{O}_X$ is zero. If X is quasi-compact, then ϕ is injective. If, moreover, X is quasi-separated, then ϕ is an isomorphism.*

Proof. Let $M \subset \mathcal{O}_X(X)$ be the $\mathcal{O}_X(X)$-module corresponding to the largest quasi-coherent $\mathcal{O}_{\operatorname{Spec} \mathcal{O}_X(X)}$-module contained in the kernel of ϕ. By definition,

$$\phi(\widetilde{M}(D(f))) = \phi(M[\tfrac{1}{f}]) = 0$$

for all $f \in \mathcal{O}_X(X)$. In particular, the above equation holds for $f = 1$, and since $\phi : \mathcal{O}_{\operatorname{Spec} \mathcal{O}_X(X)}(\operatorname{Spec} \mathcal{O}_X(X)) \to \operatorname{aff}_* \mathcal{O}_X(\operatorname{Spec} \mathcal{O}_X(X))$ is simply the identity on

$\mathcal{O}_X(X)$, it follows that $M = 0$. Note that this is equivalent to the affinification morphism $\mathrm{aff} : X \to \operatorname{Spec} \mathcal{O}_X(X)$ being scheme-theoretically surjective.

If X is quasi-compact, then $\ker \phi$ is quasi-coherent by [77, Tag 01R8], so ϕ is injective. In this case, aff is a dominant morphism.

If X is quasi-compact quasi-separated, then ϕ is an isomorphism by [59, proposition 2.3.12]. □

We are now ready to prove the extensive criterion for closed immersions.

proof of theorem A.2.1. First assume that f is right orthogonal to ring extensions. Being a closed immersion is a local condition, so Y may be assumed affine. In particular, Y is quasi-compact and therefore X is quasi-compact. Hence there exists a finite affine cover $\{X_i\}_{i \in I}$ of X. Then the source of the epimorphism

$$\pi : \coprod_{i \in I} X_i \longrightarrow X$$

is affine. Hence $\mathrm{aff}\, \pi$ is the morphism of affine schemes corresponding to the ring extension

$$\mathcal{O}_X(X) \longrightarrow \prod_{i \in I} \mathcal{O}_X(X_i).$$

Write $f' : \operatorname{Spec} \mathcal{O}_X(X) \to Y$ for the unique morphism satisfying $f' \mathrm{aff} = f$. By assumption, there exists a unique lift h in the diagram

$$\begin{tikzcd} \coprod_{i \in I} X_i \arrow[r, "\pi"] \arrow[d, "\mathrm{aff}\, \pi"'] & X \arrow[d, "f"] \\ \operatorname{Spec} \mathcal{O}_X(X) \arrow[r, "f'"] \arrow[ur, dashed, "h"] & Y. \end{tikzcd}$$

As π is an epimorphism, it follows that $h\, \mathrm{aff} = \mathrm{id}_X$. Hence aff is a split monomorphism, so X is affine by lemma A.2.6. Consequently, f is a closed immersion by proposition A.2.4.

Now assume that f is a closed immersion. Let $\alpha : A \to B$ be a ring extension. Since being a closed immersion and being a ring extension are both local properties, we reduce to the case that Y is affine. By applying the global sections functor to the commutative diagram

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r, "b"] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r, "a"] \arrow[ur, dashed, "h"] & Y, \end{tikzcd}$$

we obtain a commutative diagram

$$\begin{tikzcd} B & \mathcal{O}_X(X) \arrow[l] \arrow[dl, dashed] \\ A \arrow[u] & \mathcal{O}_Y(Y), \arrow[l] \arrow[u] \end{tikzcd}$$

in which a unique dotted arrow exists by proposition A.2.3, and this arrow corresponds to a unique lift $h : \operatorname{Spec} A \to X$. Hence f is right orthogonal to ring extensions, as was to be shown. □

Using the above theorem and proposition A.2.7, it can be shown that the scheme-theoretic image defines an orthogonal factorization system on the category of quasi-compact quasi-separated schemes.

Remark A.2.8. Stefan Schröer pointed out to me an alternative proof of one implication of theorem A.2.1, which uses the valuative criterion. Assume that $f : X \to Y$ is right orthogonal to all ring extensions. For each field k, each commutative diagram

$$\begin{tikzcd} \operatorname{Spec} k \times \operatorname{Spec} k \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} k \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

admits a unique lift. Here, $\operatorname{Spec} k \times \operatorname{Spec} k \to \operatorname{Spec} k$ corresponds to the diagonal morphism $k \to k \times k$. In other words, $X(k) \to Y(k)$ is injective for each field k, so f is universally injective. Then f is separated by [77, Tag 05VE]. By [77, Tag 01KF], f is also universally closed. From Serre's criterion, it follows that X is affine. Hence, by proposition A.2.4, f is a closed immersion.

A.3 Pro-abelian categories

See [25] and [45]. Let $\mathcal{A}$ be a category. The category $\mathbf{Pro}(\mathcal{A})$ of pro-objects in $\mathcal{A}$ is defined as the category of functors $X : I \to \mathcal{A}$, where I is a cofiltered category. For $X : I \to \mathcal{A}$ and $Y : J \to \mathcal{A}$ in $\mathbf{Pro}(\mathcal{A})$, the morphism set $\mathbf{Pro}(\mathcal{A})(X, Y)$ is defined as the limit of the functor $I^{\mathrm{op}} \times J : \mathcal{A}(X(-), Y(-)) \to \mathbf{Set}$. One usually writes X_i for $X(i)$ with $i \in I$. Here is a useful characterization of cofiltered categories.

Lemma A.3.1. *Let I be a category with a terminal object $*$. The following are equivalent.*

(i) The category I is cofiltered.

(ii) Every solid diagram

$$\begin{tikzcd} k \arrow[r, dashed] \arrow[d, dashed] & j \arrow[d] \\ j' \arrow[r] & i \end{tikzcd}$$

in I can be completed to a commutative square.

(iii) Every non-empty finite diagram in I admits a cone.

Proof. If each finite diagram in I admits a cone, then I is obviously cofiltered, so (iii) implies (i).

If (ii) holds, then taking $i = *$ in the diagram yields the first condition for I to be cofiltered. Now suppose given $f, g : j \to i$. Then there is a commutative diagram

$$\begin{tikzcd} k \arrow[r, "h_1"] \arrow[d, "h_2"] & j \arrow[d, "f"] \\ j \arrow[r, "g"] & i \end{tikzcd}$$

and another commutative diagram

$$\begin{tikzcd} k' \arrow[r, "h_1'"] \arrow[d, "h_2'"] & k \arrow[d, "h_1"] \\ k \arrow[r, "h_2"] & j. \end{tikzcd}$$

Hence $h = h_2 h_2' = h_1 h_1'$ and $fh = gh$. Thus I is cofiltered, and (ii) implies (i).

Now suppose that (i) holds and that $F : D \to I$ is a finite diagram in I. Let $i \in I$ such that there exists an arrow $i \to F(d)$ in I for each $d \in I$, which is possible by iterated application of (i), since D has finitely many objects. Fix $d_0 \in D$. There are finitely many arrows $d \to d_0$ in D and so a finite number of compositions $i \to F(d) \to F(d_0)$. Hence there exists $i_0 \to i$ such that all compositions $i_0 \to i \to F(d)$ and $i_0 \to i \to F(d) \to F(d_0)$ are equal. Replace i by i_0 and fix a new $d_0 \in D$. Continuing iteratively, this yields a cone over F in I, and it follows that (i) implies (iii), which finishes the proof. $\square$

Lemma A.3.2. *Let I and J be cofiltering categories. Then $I \times J$ is cofiltering.*

Proof. Let (i, j) and (i', j') be objects of $I \times J$. Then there exist $i'' \in I$ and $j'' \in J$ such that

$$\begin{tikzcd}[row sep=small] & i & & & j \\ i'' \arrow[ur] \arrow[dr] & & & j'' \arrow[ur] \arrow[dr] & \\ & i' & & & j' \end{tikzcd}$$

by assumption, whence

$$\begin{tikzcd}[row sep=small] & (i, j) \\ (i'', j'') \arrow[ur] \arrow[dr] & \\ & (i', j'). \end{tikzcd}$$

Now suppose given a diagram

$$\begin{tikzcd} (i, j) \arrow[r, shift left, "f_i \times f_j"] \arrow[r, shift right, "g_i \times g_j"'] & (i', j') \end{tikzcd}$$

in $I \times J$. Then there exist maps $h_i : i'' \to i$ and $h_j : j'' \to j$ such that $f_i h_i = g_i h_i$ and $f_j h_j = g_j h_j$. Hence $(f_i \times f_j)(h_i \times h_j) = (g_i \times g_j)(h_i \times h_j)$. It follows that $I \times J$ is cofiltering. $\square$

Let $X : I \to \mathcal{A}$ and $Y : J \to \mathcal{A}$ be pro-objects, and A an object of $\mathcal{A}$. Then

$$\mathbf{Pro}(\mathcal{A})(X, A) = \operatorname*{colim}_i \mathcal{A}(X_i, A),$$

so a map $f : X \to cA$ is nothing but a choice of $i \in I$ and a map $X_i \to A$. Two maps $X_i \to A$ and $X_{i'} \to A$ determine the same map $X \to cA$ if there exist $i'' \in I$ and maps $i'' \to i$ and $i'' \to i'$ such that the diagram

$$\begin{tikzcd} X_{i''} \arrow[r] \arrow[d] & X_i \arrow[d] \\ X_{i'} \arrow[r] & A \end{tikzcd}$$

commutes. Then a map $f : X \to Y$ is given concretely as follows. Since

$$\mathbf{Pro}(\mathcal{A})(X, Y) = \lim_j \operatorname*{colim}_i \mathcal{A}(X_i, Y_j) = \lim_j \mathbf{Pro}(\mathcal{A})(X, cY_j),$$

a map $f : X \to Y$ is a map of sets $\phi : J \to I$ together with a family $(f_j)_{j \in J}$ of maps $f_j : X_{\phi j} \to Y_j$ such that for every map $j' \to j$ in J, there exist $i \in I$ and maps $i \to \phi(j)$ and $i \to \phi(j')$ such that the diagram

$$\begin{tikzcd} & X_i \arrow[dl] \arrow[dr] & \\ X_{\phi(j')} \arrow[d] & & X_{\phi(j)} \arrow[d] \\ Y_{j'} \arrow[rr] & & Y_j \end{tikzcd}$$

commutes. Furthermore, given maps of pro-objects $f : X \to Y$ and $g : Y \to X$, $gf = \mathrm{id}_X$ if and only if each $g_{\psi(i)} f_i : X_{\phi(\psi(i))} \to X_i$ factors through a structure map of X.

The explicit description of arrows in $\mathbf{Pro}(\mathcal{A})$ is often useful when proving fundamental properties of $\mathbf{Pro}(\mathcal{A})$.

Definition A.3.3. A functor $F : I \to J$ of cofiltered categories is *cofinal* if for each $j \in J$, there exist $i \in I$ and an arrow $F(i) \to j$.

Lemma A.3.4. *Let $X : J \to \mathcal{A}$ be a pro-object in $\mathcal{A}$ and let $F : I \to J$ be a cofinal functor of cofiltered categories. Then the pro-object $X' : I \to J \to \mathcal{A}$ is canonically isomorphic to X.*

Proof. For $i \in I$, $X'_i = X_{F(i)}$. Hence there is a canonical map $f : X \to X'$ given by the underlying map of sets of $F : I \to J$ and $f_i : X_{F(i)} \to X'_i$ for each $i \in I$, where f_i is the identity map of $X_{F(i)}$.

Construct a map $g : X' \to X$ as follows. For each $j \in J$, choose a $G(j)$ in I such that there exists an arrow $FG(j) \to j$, and choose such an arrow, which is possible by the cofinality of F; this is the crux of the proof. Then $G : J \to I$ is a map of sets. For each $j \in J$, let $g_j : X'_{G(i)} \to X_j$ be the chosen map $X(FG(j)) \to j)$. Then $g_{F(i)} f_i : X'_{GF(i)} \to X'_i$ is an arrow $X(FGF(i) \to F(i))$, and $f_{G(j)} g_j : X_{FG(j)} \to X_j$ is an arrow $X(FG(j) \to j)$. It follows that f is an isomorphism with inverse g. $\square$

From now on, $\mathcal{A}$ is an abelian category. This section presents a self-contained proof of the fact that $\mathbf{Pro}(\mathcal{A})$ is also abelian [25].

For a pro-object $X : I \to \mathcal{A}$, the cofiltered category I may always be assumed to have a final object 0, which is mapped to the zero object of $\mathcal{A}$, since the inclusion $I \to I_+$ is cofinal and X is the composition $I \to I_+ \to \mathcal{A}$, so that $X : I \to \mathcal{A}$ and $X_+ : I_+ \to \mathcal{A}$ are canonically isomorphic. Therefore from now on, all pro-objects $X : I \to \mathcal{A}$ are assumed to have small pointed cofiltered index category I, whose base point is mapped to the zero object of $\mathcal{A}$.

There is a constant functor $c : \mathcal{A} \to \mathbf{Pro}(\mathcal{A})$ which sends an object A of $\mathcal{A}$ to the constant pro-object with value A. The zero object 0 of $\mathcal{A}$ is mapped to the zero object $c0$ of $\mathbf{Pro}(\mathcal{A})$.

Lemma A.3.5. *The coproduct $X \amalg Y$ of pro-objects X and Y (indexed by I and J, respectively) in $\mathcal{A}$ exists, and is equal to $X \times Y : I \times J \longrightarrow \mathcal{A}$ given by $(i,j) \mapsto X_i \times Y_j$. The inclusion map $s : X \to X \amalg Y$ is given by the family $(s_{i,j})_{(i,j)\in I\times J}$ where $s_{i,j} : X_i \to X_i \amalg Y_j$ is the canonical inclusion. The other inclusion map is defined similarly.*

Proof. It needs to be shown that $X \amalg Y$ satisfies the universal property for coproducts. Let T be a pro-object in $\mathcal{A}$ indexed by a small pointed cofiltered category K, together with maps $f : X \to T$ and $g : Y \to T$, and let $\phi : K \to I$ and $\psi : K \to J$ be their respective maps of index sets. Define $h : X \amalg Y \to T$ as the family $(f_k \amalg g_k)_{k\in K}$. Then $(hs)_k = (f_k \amalg g_k)s_k = f_k$ and $(ht)_k = g_k$, so $hs = f$ and $ht = g$.

Suppose that $h' : X \amalg Y \to T$ is another map satisfying both $h's = f$ and $h't = g$, given by a map $(\alpha, \beta) : K \to I \times J$ and a family $h'_k : X_{\alpha(k)} \amalg Y_{\beta(k)} \to T_k$. As $h's = f$, $h'_k s_{\alpha(k)} = f_k$ as maps of pro-objects, so there exist $i \in I$ and maps $i \to \phi(k)$ and $i \to \alpha(k)$ such that the diagram

$$\begin{tikzcd} X_i \arrow[r] \arrow[d] & X_{\phi(k)} \arrow[d, "f_k"] \\ X_{\alpha(k)} \arrow[r, "h'_k s_{\alpha(k)}"] & T_k \end{tikzcd}$$

commutes. Similarly, there exists a commutative diagram

$$\begin{tikzcd} Y_j \arrow[r] \arrow[d] & Y_{\psi(k)} \arrow[d, "g_k"] \\ Y_{\beta(k)} \arrow[r, "h'_k t_{\beta(k)}"] & T_k. \end{tikzcd}$$

Hence there is a commutative diagram

$$\begin{tikzcd} X_i \amalg Y_j \arrow[r] \arrow[d] & X_{\phi(k)} \amalg Y_{\psi(k)} \arrow[d, "h_k"] \\ X_{\alpha(k)} \amalg Y_{\beta(k)} \arrow[r, "h'_k"] & T_k, \end{tikzcd}$$

and it follows that $h_k = h'_k$ as maps of pro-objects. Hence $h = h'$, as was to be shown. $\square$

Lemma A.3.6. *The category $\mathbf{Pro}(\mathcal{A})$ is additive.*

Proof. Let $X, Y \in \mathbf{Pro}(\mathcal{A})$, indexed by small cofiltering categories I and J, respectively. By definition

$$\mathbf{Pro}(\mathcal{A})(X,Y) = \lim_j \operatorname*{colim}_i \mathcal{A}(X_i, Y_j).$$

Since $\mathcal{A}$ is abelian, and $\mathbf{Ab}$ has all small limits and colimits, it follows that the morphism set $\mathbf{Pro}(\mathcal{A})(X,Y)$ is an abelian group. Hence $\mathbf{Pro}(\mathcal{A})$ is pre-additive.

Note that

$$\mathbf{Pro}(\mathcal{A})(c0, X) = \lim_i \mathcal{A}(0, X_i) = 0.$$

Thus $c0$ is a zero object of $\mathbf{Pro}(\mathcal{A})$.

By lemma A.3.5, $\mathbf{Pro}(\mathcal{A})$ has finite coproducts, so it follows from [77, Tag 09SE] that $\mathbf{Pro}(\mathcal{A})$ has finite products. Hence $\mathbf{Pro}(\mathcal{A})$ is additive. $\square$

The following result is key for the construction of kernels and cokernels of morphisms of pro-objects.

Lemma A.3.7. *Let $f : X \to Y$ be a map in $\mathbf{Pro}(\mathcal{A})$ given as a map of index sets $\phi : J \to I$ together with a family $f_j : X_{\phi(j)} \to Y_j$. Then the category K whose objects are compositions*

$$X_i \longrightarrow X_{\phi(j)} \longrightarrow Y_j$$

and whose morphisms are commutative diagrams

$$\begin{tikzcd} X_{i'} \arrow[r] \arrow[d] & X_{\phi(j')} \arrow[r] & Y_{j'} \arrow[d] \\ X_i \arrow[r] & X_{\phi(j)} \arrow[r] & Y_j \end{tikzcd}$$

is cofiltered.

Proof. Given $j, j' \in J$, there exists $j'' \in J$ lying over both as J is cofiltered. This yields

$$\begin{tikzcd} X_{i'} \arrow[r] & X_{\phi(j')} \arrow[r] & Y_{j'} \\ X_{i''} \arrow[u, dashed] \arrow[d, dashed] \arrow[r, dashed] & X_{\phi(j'')} \arrow[r] & Y_{j''} \arrow[u] \arrow[d] \\ X_i \arrow[r] & X_{\phi(j)} \arrow[r] & Y_j, \end{tikzcd}$$

and an $i'' \in I$ and dashed arrows making both rectangles commute exist by the fact that f is a map of pro-objects and lemma A.3.1(iii). A similar argument shows that there is a cone over each parallel pair of arrows $k' \rightrightarrows k$ in K. $\square$

Lemma A.3.8. *Let $f : X \to Y$ be a map of pro-objects in $\mathcal{A}$, given by $\phi : J \to I$ and $f_j : X_{\phi(j)} \to Y_j$ as usual. Then* $\ker f$ *exists and its pro-object is given by*

$$\ker f : K \longrightarrow \mathcal{A}, \qquad k \longmapsto \ker k$$

where K is as in lemma A.3.7. The structure maps of $\ker f$ *are those induced by commutative diagrams*

$$\begin{tikzcd} \ker k' \arrow[r] \arrow[d, dashed] & X_{i'} \arrow[r, "k'"] \arrow[d] & Y_{j'} \arrow[d] \\ \ker k \arrow[r] & X_i \arrow[r, "k"] & Y_j. \end{tikzcd}$$

The canonical map $\ker f \to X$ *is given by* $\mathrm{id} : X_i \to X_i$, *where the source is identified with the kernel of $X_i \to X_0 \to Y_0$.*

Proof. It must be shown that $\ker f$ is really a kernel of f that satisfies the relevant universal property.

Let $Z : L \to \mathcal{A}$ be a pro-object and $\psi : I \to L$ a map of sets. Let $g : Z \to X$ be a map of pro-objects given by a family $g_i : Z_{\psi(i)} \to X_i$ such that $fg = 0$. It will be shown that g factors uniquely through $\ker f \to X$. Let $k \in K$ be a composition $X_i \to X_{\phi(j)} \to Y_j$. Then there are an $l \in L$ and a diagram

$$\begin{tikzcd}
 & Z_l \arrow[d] \arrow[dr] \arrow[ddl, dashed, "h_k"'] & & \\
 & Z_{\psi(i)} \arrow[d] & Z_{\psi(\phi(j))} \arrow[d, "g_{\phi(j)}"] & \\
\ker k \arrow[r] & X_i \arrow[r] & X_{\phi(j)} \arrow[r, "f_j"] & Y_j
\end{tikzcd}$$

such that the pentagon commutes and the composition $Z_l \to Y_j$ is zero. Hence a unique dashed arrow h_k exists by the universal property of $\ker k$. The map h_k does not depend (as a map of pro-objects) on the choice of $l \in L$. If $l' \in L$ is another object such that $Z_{l'} \to Z_{\psi(i)} \to X_i \to X_{\phi(j)} \to Y_j$ is zero, then there exists $l'' \in L$ such that

$$\begin{tikzcd}
Z_{l''} \arrow[r] \arrow[d] & Z_l \arrow[d] \\
Z_{l'} \arrow[r] & Z_{\psi(i)}
\end{tikzcd}$$

commutes, but then

$$\begin{tikzcd}
Z_{l''} \arrow[r] \arrow[d] & Z_l \arrow[d, "h_k"] \\
Z_{l'} \arrow[r, "h'_k"] & \ker k
\end{tikzcd}$$

also commutes by the uniqueness of the map $Z_{l''} \to \ker k$, so that $h_k = h'_k$ as maps of pro-objects. Therefore, this construction defines a map of pro-objects $h : Z \to W$. Note that the composition $Z \to W \to X$ is given, for each i, by a map $Z_l \to X_i$ that factors through $g_i : Z_{\psi(i)} \to X_i$ and is thus the same as g.

Since the only choice involved in the construction of h was the choice of an l for each $k \in K$, and it was shown that h does not depend on this choice, it follows that h is the unique map through which g factors. $\square$

Example A.3.9. Here is an explicit characterization of the condition $X = 0$ for a pro-object $X : I \to \mathcal{A}$. The canonical maps $0 \to X$ and $X \to 0$ must be inverse isomorphisms. This means that for each $i \in I$, there must exist a morphism $i' \to i$ in I such that the structure map $X_{i'} \to X_i$ is the zero map in $\mathcal{A}$. In other words, the subcategory $I' \subset I$ spanned by morphisms $i' \to i$ such that $X_{i'} \to X_i$ is zero must be cofiltered and cofinal in I.

Lemma A.3.10. *Let $f : X \to Y$ be a morphism of $\mathbf{Pro}(\mathcal{A})$, given by $f_j : X_{\phi(j)} \to Y_j$. Then* $\operatorname{coker} f$ *exists and is given by*

$$\operatorname{coker} f : K \longrightarrow \mathcal{A}, \qquad k \longmapsto \operatorname{coker} k$$

where K is as in lemma A.3.7. The structure maps of $\operatorname{coker} f$ *are the unique maps* *induced by commutative diagrams*

$$\begin{tikzcd} X_{i'} \arrow[r] \arrow[d] & X_{\phi(j')} \arrow[r, "f_{j'}"] & Y_{j'} \arrow[r] \arrow[d] & \operatorname{coker} k' \arrow[d, dashed] \\ X_i \arrow[r] & X_{\phi(j)} \arrow[r, "f_j"] & Y_j \arrow[r] & \operatorname{coker} k. \end{tikzcd}$$

The canonical map $Y \to \operatorname{coker} f$ *has components* $Y_j \to \operatorname{coker} k$*, where* k *is a composition* $X_i \to X_{\phi(j)} \to Y_j$*.*

Proof. The proof is analogous to that of lemma A.3.8. □

The following lemma gives a useful criterion for checking injectivity and surjectivity of morphisms in $\mathbf{Pro}(\mathcal{A})$.

Lemma A.3.11. *Let* $f : X \to Y$ *be a morphism in* $\mathbf{Pro}(\mathcal{A})$*. Let* I*,* J*,* $\phi : J \to I$ *and* K *be as in lemma A.3.7.*

(i) If, for all $k \in K$*, there exists a map* $k' \to k$ *in* K *such that* $\ker k' = 0$*, then* $\ker f = 0$*.*
(ii) If, for all $k \in K$*, there exists a map* $k' \to k$ *in* K *such that* $\operatorname{coker} k' = 0$*, then* $\operatorname{coker} f = 0$*.*

Proof. Let $K' \subset K$ be the full subcategory spanned by $k' \in K$ such that $\ker k' = 0$. Then K' is cofiltered and cofinal in K by assumption. Hence $\ker f$ is isomorphic to the composition $K' \to K \to \mathcal{A}$ by lemma A.3.4, which is the zero object of $\mathbf{Pro}(\mathcal{A})$, as was to be shown.

The proof of (ii) is analogous. □

Lemma A.3.12. *Let* $f : X \to Y$ *be a morphism in* $\mathbf{Pro}(\mathcal{A})$*.*

(i) The pro-object $\ker \operatorname{coker} f$ *is canonically isomorphic to*

$$\operatorname{im} f : K \longrightarrow \mathcal{A}, \qquad k \longmapsto \operatorname{im} k$$

where K *is as in lemma A.3.7. The canonical maps* $X \to \operatorname{im} f$ *and* $\operatorname{im} f \to Y$ *are given by families* $X_i \to \operatorname{im} k$ *and* $\operatorname{im} f_j \to Y_j$*, respectively, where* k *is a composition* $X_i \to X_{\phi(j)} \to Y_j$ *and* f_j *is the composition* $X_{\phi(j)} = X_{\phi(j)} \to Y_j$*.*

(ii) The pro-object $\operatorname{coker} \ker f$ *is canonically isomorphic to*

$$\operatorname{coim} f : K \longrightarrow \mathcal{A}, \qquad k \longmapsto \operatorname{coim} k$$

where K *is as in lemma A.3.7. The canonical maps* $X \to \operatorname{coim} f$ *and* $\operatorname{coim} f \to Y$ *are given by families* $X_i \to \operatorname{coim} k$ *and* $\operatorname{coim} f_j \to Y_j$*, respectively, where* k *and* f_j *are as above.*

(iii) The canonical map $\operatorname{coim} f \to \operatorname{im} f$ *given by* $\operatorname{coim} k \to \operatorname{im} k$ *for each* $k \in K$ *is an isomorphism.*

Proof. Note that $\operatorname{coker} f : K \to \mathcal{A}$ has the index category K of lemma A.3.7, so $\ker \operatorname{coker} f : L \to \mathcal{A}$ has index category L whose objects are compositions

$$Y_{j'} \longrightarrow Y_j \longrightarrow \operatorname{coker} k,$$

where k is a composition $X_i \to X_{\phi(j)} \to Y_j$. Note that $\operatorname{im} f$ is the composition $K \to L \to \mathcal{A}$, where the functor $F : K \to L$ is given by

$$\begin{array}{ccc} k' & & \\ \downarrow & \longmapsto & \\ k & & \end{array} \quad \begin{tikzcd} Y_{j'} \arrow[r, equal] \arrow[d] & Y_{j'} \arrow[r] & \operatorname{coker} k' \arrow[d] \\ Y_j \arrow[r, equal] & Y_j \arrow[r] & \operatorname{coker} k. \end{tikzcd}$$

It now suffices to show that F is cofinal by lemma A.3.4. Let $l \in L$ be a composition $Y_{j'} \to Y_j \to \operatorname{coker} k$. As J is cofiltered and f is a map of pro-objects, there exist $j'' \in J$, $i'' \in I$ and a commutative diagram

$$\begin{tikzcd} X_{i''} \arrow[r] \arrow[dd] & X_{\phi(j'')} \arrow[r] & Y_{j''} \arrow[d] \\ & & Y_{j'} \arrow[d] \\ X_i \arrow[r] & X_{\phi(j)} \arrow[r] & Y_j. \end{tikzcd}$$

Let $k'' \in K$ be the top row of the diagram, and note that the bottom row is k, so that the diagram represents a map $k'' \to k$ in K, which induces a unique map $\operatorname{coker} k'' \to \operatorname{coker} k$. Then the diagram

$$\begin{tikzcd} Y_{j''} \arrow[r, equal] \arrow[d] & Y_{j''} \arrow[r] & \operatorname{coker} k'' \arrow[d] \\ Y_{j'} \arrow[r] & Y_j \arrow[r] & \operatorname{coker} k \end{tikzcd}$$

commutes, so it gives a map $F(k'') \to l$. Hence F is cofinal, as was to be shown. This proves (i), and the proof of (ii) is similar.

Finally, (iii) follows from the fact that $\mathcal{A}$ is abelian, so that the canonical morphism $\operatorname{coim} f \to \operatorname{im} f$ is a level isomorphism. $\square$

This yields the following useful theorem [25].

Theorem A.3.13. *Let $\mathcal{A}$ be an abelian category. Then* $\mathbf{Pro}(\mathcal{A})$ *is a abelian.*

Proof. By lemmas A.3.8 and A.3.10, $\mathbf{Pro}(\mathcal{A})$ has kernels and cokernels, and by lemma A.3.12(iii), images and coimages coincide in $\mathbf{Pro}(\mathcal{A})$. Hence $\mathbf{Pro}(\mathcal{A})$ is abelian. $\square$

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