Bergische Universität Wuppertal & Universität Paris-Saclay
Fachbereich Mathematik und Naturwissenschaften
Institut für Mathematik

BERGISCHE UNIVERSITÄT WUPPERTAL

Algebraic cobordism of spherical varieties

université PARIS-SACLAY

Binationale Dissertation

zur Erlangung des akademischen Grades des Doktors in Naturwissenschaften Dr. rer. nat

eingereicht von

Henry July

im August 2022

Thesis Jury:

Prof. Dr. Matthias Wendt	President of the jury
Prof. Dr. Amalendu Krishna	Reviewer & Examiner
Prof. Dr. Marc Levine	Reviewer & Examiner
Prof. Dr. Jens Hornbostel	Thesis director & Examiner
Prof. Dr. Nicolas Perrin	Thesis director & Examiner
Prof. Dr. Dimitri Zvonkine	Examiner

If the statement is not true, one simply has to change the assumptions.

Henry July

Acknowledgements

I am deeply indebted to my advisors Jens Hornbostel and Nicolas Perrin for choosing the topic, and conducting the project even though Nicolas Perrin had moved to the university of Versailles. Further, I would like to thank them for their patient support, valuable feedback and corrections as well as many helpful discussions. Lastly, I am grateful to them for helping me organising the process of the cotutelle project between Wuppertal and Paris-Saclay, and for their advice and assistance concerning the procedure of my research publication.

I am grateful to Michel Brion for providing many details of some of his results which were generalised in this thesis.

I am grateful to Boris Pasquier for some very helpful explanations concerning the classification of smooth projective horospherical varieties of Picard number two.

I am grateful to Adeel Khan for being of great help clarifying some of the details concerning the localisation sequence in equivariant cobordism.

I would like to thank Herman for being the best office mate during the first years of my PhD time, for discussing the beautiful world of algebraic geometry with me and for always having an open ear whenever I needed someone to talk to. Further, I would like to thank them for proofreading parts of my thesis.

I would like to thank Christoph for his useful explanations during many discussions concerning representation theory and root systems.

I would like to thank Thomas for being of great help when I started studying algebraic cobordism during the first year of my PhD time. Further, I am grateful to him for many discussions concerning formal group laws and its properties.

I would like to thank Jule for being a good friend at the office, for always sharing her cookies with me and for proofreading parts of my thesis.

I would like to thank Moritz for being at my side during the whole time of my studies in Mainz where I discovered the beauty of mathematics. Additionally, I am grateful to him for proofreading my thesis.

I would like to thank Jan for having a great time at the conference in Cambridge and for proofreading parts of my thesis.

I would like to thank my parents for their continous support throughout my studies at various places all over the world.

Lisa, I would like to thank you for your relentless faith in me and for always having my back during the hard times of the last three years. Without you, this would not have been possible.

The author is supported by the DFG Research Training Group 2240: *Algebro-Geometric Methods in Algebra, Arithmetic and Topology.* This research was conducted at the Bergische Universität Wuppertal and the university Paris-Saclay at the mathematical institute of Versailles (LMV).

Abstract

In this work, we study the equivariant algebraic cobordism rings for the action of a torus T on smooth varieties over an algebraically closed field of characteristic zero. Algebraic cobordism is the universal oriented cohomology theory which generalises Chow groups and connective K-theory. Due to its universality, one needs more sophisticated frameworks in order to describe the algebraic cobordism rings using the universal formal group law.

This work is split into five main parts. In the first part, we recall the construction of algebraic cobordism and define equivariant algebraic cobordism as well as horospherical and spherical varieties which are the objects of main interest throughout this work. These objects are illustrated by presenting several relevant computations which provide the necessary tools being subsequently used in the following chapters.

In the second part, the current state of research related to computations in equivariant cobordism is reviewed. Furthermore, we prove a relation in equivariant cobordism which will be essential in the sequel of this work. Additionally, we prove that the localisation at T-fixed points also holds for a refinement of coefficients. Lastly, for a connected reductive algebraic group G and a maximal torus T in G, we provide a proof of a Künneth formula in T-equivariant cobordism for smooth projective G-varieties X and Y if $X \times Y$ has finitely many T-fixed points with respect to the diagonal action. The last two results of course imply similar statements for equivariant Chow groups and equivariant connective K-theory the first of which is a new result even for Chow groups.

The third part is dedicated to the main theorem of this work. First, we review the proof techniques for the computations of rational T-equivariant Chow groups of smooth projective spherical G-varieties which we then adapt to rational equivariant cobordism making the necessary modifications. Using the relation from the second part, we can prove the structure theorem which describes the rational T-equivariant cobordism rings of smooth projective spherical G-varieties.

In the fourth part, we provide a plethora of computations in order to apply the main theorem of this work. We describe the geometry of smooth projective horospherical varieties of Picard number one including the odd symplectic Grassmannians and compute their rational T-equivariant cobordism rings. Additionally, we extend these results to some specific smooth projective horospherical varieties of Picard number two. These results of course imply similar statements for e.g. rational T-equivariant Chow groups. Furthermore, the given computations were previously unknown even in the Chow group case.

The last part of this work provides computations of classes in rational equivariant cobordism. To be more precise, we can determine the ring structure for certain smooth T-filtrable schemes by describing the generators of the rational equivariant cobordism module in the rational equivariant cobordism ring. This develops the state of research because we can now multiply certain classes inside the rational T-equivariant cobordism ring. This will then be applied to the example of the odd symplectic Grassmannian $\mathrm{IG}(2,5)$ in which we describe the generators of the rational T-equivariant cobordism module geometrically and then compute their classes in the rational T-equivariant cobordism ring. As above, similar results for rational T-equivariant Chow groups can be deduced from the previous result.

Résumé

Dans ce travail, nous étudions les anneaux de cobordisme algébrique équivariant pour l'action d'un tore T sur des variétés lisses sur un corps algébriquement clos de caractéristique nulle. Le cobordisme algébrique est la théorie cohomologique orientée universelle qui généralise les groupes de Chow et la K-théorie. En raison de son universalité, on a besoin de techniques plus sophistiquées pour décrire les anneaux de cobordisme algébrique en utilisant la loi de groupe formelle universelle.

Cette thèse est divisée en cinq parties. Dans la première partie, nous rappelons la construction du cobordisme algébrique et définissons le cobordisme algébrique équivariant ainsi que les variétés horosphériques et sphériques qui sont les principaux objets d'intérêt tout au long de ce travail. Ces objets sont illustrés par la présentation de plusieurs calculs pertinents qui fournissent des outils nécessaires qui seront utilisés dans les chapitres suivants.

Dans la deuxième partie, nous présentons l'état actuel des recherches aux calculs en cobordisme équivariant. De plus, nous prouvons une relation en cobordisme équivariant qui est essentielle dans la suite de ce travail. Nous prouvons aussi que la localisation aux points fixes de T reste valide pour un raffinement des coefficients. Enfin, pour un groupe algébrique réductif connexe G et un tore maximal T dans G, nous donnons une preuve d'une formule de Künneth en cobordisme T-équivariant pour des G-variétés projectives lisses X et Y si $X \times Y$ a un nombre fini de points fixes de T par rapport à l'action diagonale. Les deux derniers résultats bien sûr impliquent des résultats similaires pour les groupes de Chow équivariants et la K-théorie équivariant. Le résultat sur le raffinement des coefficients est nouveau même pour les groupes de Chow.

La troisième partie est consacrée au théorème principal de cette thèse. Dans un premier temps, nous passons en revue les techniques de preuve pour les calculs des groupes de Chow rationnels T-équivariants des G-variétés sphériques projectives lisses que nous étendons ensuite pour le cobordisme rationnel T-équivariant. En utilisant la relation de la deuxième partie, nous prouvons le théorème principal qui décrit les anneaux de cobordisme rationnels T-équivariants des G-variétés sphériques projectives lisses.

Dans la quatrième partie, nous effectuons de nombreaux calculs qui illustrent le théorème principal de ce travail. Nous décrivons la géométrie des variétés horosphériques projectives lisses de nombre de Picard 1, y compris les grassmanniennes symplectiques impaires, et calculons leurs anneaux de cobordisme rationnels T-équivariants. De plus, nous étendons ces résultats à certaines variétés horosphériques projectives lisses de nombre de Picard 2. Ces résultats se spécialisent au cas des groupes de Chow rationnels T-équivariants et sont déjà nouveaux dans le cas des groupes de Chow.

La dernière partie de ce travail fournit des calculs de classes de nature géométrique en cobordisme rationnel T-équivariant. Comme on peut déterminer la structure d'anneau de certaines variétés lisses T-filtrable en décrivant les générateurs du module de cobordisme rationnel T-équivariant et en localisant aux points fixes, on peut désormais multiplier ces classes géométriques dans l'anneau de cobordisme rationnel T-équivariant. Nous appliquons ces techniques au cas de la grassmannienne symplectique impaire $\mathrm{IG}(2,5)$ dans laquelle nous décrivons les générateurs du module de cobordisme rationnel T-équivariant de manière géométrique puis calculons leurs classes dans l'anneau de cobordisme rationnel T-équivariant. Comme ci-dessus, un résultat similaire pour les groupes de Chow rationnels T-équivariant peut être déduit du résultat précedent.

Kurzfassung

In dieser Dissertation berechnen wir äquivariante Kobordismusringe für die Wirkung eines Torus T auf glatten Varietäten über einem algebraisch abgeschlossenem Körper von Charakteristik null. Algebraischer Kobordismus ist die universelle orientierte Kohomologietheorie und verallgemeinert aufgrund dessen Chowgruppen und konnektive K-Theorie. Die Beschreibung von Kobordismusringen benötigt weitergehende Methoden, weil das universelle Gruppengesetz für Berechnungen der Ringstrukturen verwendet werden muss.

Diese Arbeit ist in fünf Kapitel gegliedert. Im ersten Kapitel wird die Konstruktion von algebraischem Kobordismus wiederholt und äquivarianter Kobordismus definiert. Des Weiteren werden sphärische und horosphärische Varietäten eingeführt, welche die wichtigsten Beispielklassen für die dargelegten Berechnungen darstellen. Diese Objekte werden anhand von einigen Beispielen am Ende des ersten Kapitels veranschaulicht.

Im zweiten Kapitel wird der aktuelle Forschungsstand hinsichtlich der Berechnungen von äquivariantem Kobordismus von Varietäten dargelegt. Weiterhin wird eine Relation in äquivariantem Kobordismus bewiesen, welche im weiteren Verlauf der Arbeit mehrfach verwendet wird. Zudem wird ein Lokalisierungstheorem für einen verfeinerten Koeffizientenring formuliert und abschließend eine Künneth Formel für T-äquivarianten Kobordismus gezeigt, wobei T ein maximaler Torus ist. Die letzten beiden Resultate implizieren ähnliche Resultate für äquivariante Chowgruppen und äquivariante konnektive K-Theorie, wobei das verfeinerte Lokalisierungstheorem sogar ein neues Resultat für Chowgruppen darstellt.

Das dritte Kapitel befasst sich mit dem Hauptsatz der vorliegenden Arbeit. Zuerst werden die Beweistechniken für die Berechnungen von rationalen äquivarianten Chowringen von glatten, projektiven und sphärischen Varietäten dargestellt, welche im Laufe des Kapitels für rationalen äquivarianten Kobordismus erweitert werden, um den Hauptsatz zu beweisen. Hierbei ist die beschriebene Relation aus Kapitel zwei von besonderer Relevanz. Letztere ermöglicht die Berechnung von rationalen T-äquivarianten Kobordismusringen von glatten, projektiven und sphärischen Varietäten.

Im vierten Kapitel werden unter Verwendung des Hauptsatzes zahlreiche explizite Rechnungen durchgeführt. Zunächst wird die Geometrie von glatten, projektiven und horosphärischen Varietäten von Picard-Rang eins analysiert, um danach deren rationale T-äquivariante Kobordismusringe zu berechnen. Unter anderem werden diese Ringe für ungerade symplektische Grassmann-Varietäten berechnet und des Weiteren auch für einige glatte, projektive und horosphärische Varietäten von Picard-Rang zwei. Wir erhalten dadurch auch neue Ergebnisse für rationale T-äquivariante Chowringe von glatten, projektiven und horosphärischen Varietäten.

Im letzten Kapitel der Arbeit werden die Klassen der Erzeuger im rationalen T-äquivarianten Kobordismusring berechnet. Die Ringstruktur der rationalen T-äquivarianten Kobordismusringe kann beschrieben werden, indem die Erzeuger des Moduls in der Algebra identifiziert werden. Dies erweitert den aktuellen Forschungsstand, da nun für glatte und T-filtrierbare Varietäten die Klassen der Erzeuger miteinander multipliziert werden können. Diese Theorie wird dann am Beispiel der ungeraden symplektischen Grassmann-Varietät $\mathrm{IG}(2,5)$ angewendet, indem die Erzeuger des Moduls geometrisch beschrieben werden, was für die Berechnung der Klassen notwendig ist. Dadurch kann der rationale T-äquivariante Kobordismusring von $\mathrm{IG}(2,5)$ beschrieben werden, was erneut ähnliche Resultate für rationale T-äquivariante Chowringe liefert.

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1 Introductions

1.1 Introduction (English version)

1.1.1 Motivation

Cohomology theories have been always of great interest in algebraic geometry, let it be the notions of Chow groups, K-theory or quantum cohomology. One needs to distiguish between non-oriented and oriented cohomology theories and our main focus will lie on the latter ones. The orientation basically means that the cohomology theory has the additional structure of Chern classes for complex vector bundles. These cohomology theories are primarily used in order to understand the geometric intersection theory of varieties, the latter being the main part of Hilbert's 15^{th} problem.

The original motivation to understand algebraic cobordism was to find an analogue to the cobordism of differentiable manifolds introduced in the fundamental paper of Quillen (cf. [48]). Quillen observed that the complex cobordism theory $\mathrm{MU}^*(X)$ is the universal complex oriented cohomology theory on the category of differentiable manifolds. Later on, Levine and Morel (cf. [37]) were able to prove analogues of MU^* for the category of smooth k-schemes for any field k of characteristic zero. They called the resulting universal oriented cohomology theory Ω^* algebraic cobordism and furthermore, they gave applications and examples explaining the relations between Ω^* and the functor K_0 of Grothendieck groups or the Chow ring functor CH^*.

Equivariant cohomology theories originally arose because one wanted to understand the ordinary cohomology theory of classifying spaces. Subsequently, equivariant cohomology theories were studied as they include group actions on varieties into their computations. It turned out that those are a very powerful tool in order to describe the ordinary cohomology theories and many computations were made in equivariant cohomology theories (e.g. [1, 7, 8, 9, 11, 12, 17, 20, 30, 31, 32, 34, 50]) for different kinds of varieties. As we are mainly interested in algebraic cobordism, we also mostly investigate equivariant algebraic cobordism of smooth varieties over a field of characteristic zero.

The last objects of interest in this thesis are spherical varieties. Those were studied for example in [6, 16, 29, 39, 41, 42, 43, 44, 45, 46, 47]. This class of varieties includes a wide range of very well known varieties as for example flag varieties, toric varieties, symmetric varieties, wonderful varieties or horospherical varieties. The geometry of the latter was intensively studied for example in [16, 41, 42, 43, 44, 46]. In this thesis our main focus lies on computations of equivariant algebraic cobordism of spherical varieties and on explicit computations for horospherical varieties. In fact, this last class of varieties includes a phletora of interesting examples as flag varieties, toric varieties or the odd symplectic Grassmannian $\mathrm{IG}(k, 2n+1)$ for $n \geq 2$ and $k \in [2, n]$.

The motivation specifically for this work was the state of the research at the time this project was started. Let G be a connected reductive algebraic group with a maximal torus T in G over some algebraically closed field k of characteristic zero. Brion computed the rational T-equivariant Chow rings for smooth projective spherical G-varieties (cf. [7]) and T-equivariant cobordism rings were already computed for e.g. smooth toric varieties (cf. [34]), flag varieties (cf. [28]), wonderful symmetric varieties of minimal rank (cf. [28]) and smooth projective varieties with finitely many T-fixed points and finitely many T-stable curves (cf. [31]). Therefore, a natural question is the generalisation of Brion's results to rational T-equivariant cobordism for any smooth projective spherical G-variety especially because the geometry of spherical varieties is an active field of research.

1.1.2 Main results

In this section, we describe our main results. We start by providing necessary notations and definitions needed in the main theorems of this thesis. First of all, the following lemma illustrates the correspondence between formal group laws and oriented cohomology theories which is one of the fundamental facts of this theory (see Section 2.1 for more details).

Lemma. *[37, Lemma 1.1.3] Let A^* be an oriented cohomology theory on $\mathbf{Sm}_k$ for any field k of characteristic zero. Then there is a unique power series*

$$F_A(u,v) = \sum_{i,j} a_{i,j} u^i v^j \in A^*(k)[[u,v]]$$

with $a_{i,j} \in A^{1-i-j}(k)$ such that for any $X \in \mathbf{Sm}_k$ and any pair of line bundles L and M on X, we have

$$F_A(c_1(L), c_1(M)) = c_1(L \otimes M).$$

Furthermore, the pair $(A^*(k), F_A)$ is a commutative formal group law of rank one.

For example Chow groups are associated to $F_{\mathrm{CH}}(u,v) = u + v$ and K-theory to $F_K(u,v) = u + v - \beta uv$. Algebraic cobordism corresponds to the universal commutative formal group law F_Ω of rank one.

Next, we fix some notation which we use throughout the thesis. Let G be a connected reductive algebraic group G over an algebraically closed field k of characteristic zero. Furthermore, let $B \subseteq G$ be a Borel subgroup, $T \subseteq B$ a maximal torus, W the Weyl group of G corresponding to the maximal torus of B and U the unipotent radical of B. Additionally, we choose the order on weights induced by the Borel subgroup B, i.e. for two weights χ, χ' we have $\chi \geq \chi'$ if $\chi - \chi'$ is a non-negative linear combination of simple roots.

Definition. *Let X be a normal G-variety. We call X* ***spherical*** *if it contains an open B-orbit.*

In the following, we recall that the T-equivariant cobordism ring $\Omega^*_T(X)$ for a smooth variety X is given by an inverse limit of ordinary cobordism rings of mixed quotients, the latter being constructed from the group T (cf. Definition 2.40). We remark that $\Omega^*_T(X)$ is an $\Omega^*_T(k)$-algebra and furthermore, for any smooth projective variety X with finitely many T-fixed points, the generators in the $\Omega^*_T(k)$-module $\Omega^*_T(X)$ are some classes $[f : Y \to X]$ where f is a projective T-equivariant morphism from a smooth variety Y (cf. [31, Corollary 4.8]). From now on we use $S(T) := \Omega^*_T(k)$. We also recall that the T-equivariant cobordism ring $S(T)$ is isomorphic to the graded power series ring $\mathbb{L}[[t_1, ..., t_n]]_{\mathrm{gr}}$ (cf. Construction 3.25) where n denotes the rank of the torus T (cf. [32, Proposition 6.7]) and $\mathbb{L}$ the Lazard ring (cf. Construction 2.8).

In order to proceed to the main results of this thesis, we define the rational T-equivariant cobordism ring $\Omega^*_T(X)_{\mathbb{Q}} := \Omega^*_T(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ for a smooth variety X where this tensor product denotes the graded topological tensor product (cf. Construction 3.30). We remark that $\Omega^*_T(X)_{\mathbb{Q}}$ is an $S(T)_{\mathbb{Q}}$-algebra if X is smooth.

Now, we give the state of the art before this project was conducted. The next proposition is a generalisation by Krishna [31] of classical localisation results (cf. [7, 12, 17]) which is a very powerful tool in order to compute rational T-equivariant cobordism for varieties having finitely many fixed points and stable curves.

Proposition. *[31, Theorem 7.8] Let X be a smooth projective scheme where a torus T acts with finitely many fixed points $x_1, ..., x_p$ and finitely many stable curves. Let $i : X^T \to X$ be the inclusion of the fixed point locus. Then the image of*

$$i^* : \Omega^*_T(X)_{\mathbb{Q}} \to \Omega^*_T(X^T)_{\mathbb{Q}} = S(T)^p_{\mathbb{Q}}$$

is the set of $(f_1, ..., f_p) \in S(T)^p_{\mathbb{Q}}$ such that $f_i \equiv f_j \mod \chi$ whenever x_i and x_j are connected by a stable irreducible curve where T acts through the weight χ.

This proposition already suggests several possible generalisations which are first of all of course extending the statement to smooth projective varieties with a torus action where T acts with finitely many T-fixed points, but infinitely many T-stable curves. One class of potential candidates is the class of smooth projective spherical varieties as those always have finitely many T-fixed points (cf. Lemma 4.8). Therefore, we tried to generalise the statements for rational T-equivariant Chow groups for smooth projective spherical G-varieties. Since there is no presentation of the T-equivariant cobordism module, one cannot run the same proof as for rational T-equivariant Chow groups, but instead one needs different lemmata in order to be able to run a similar strategy. The first technical result is the following lemma (cf. Lemma 3.36).

Lemma. *Let T be a torus of rank n and F be a finite subgroup. Then we have a graded $\mathbb{L}$-algebra isomorphism*

$$\Omega^*_T(k)_{\mathbb{Q}} \cong \Omega^*_{T/F}(k)_{\mathbb{Q}}.$$

This statement ensures that the rational T-equivariant cobordism does not see a difference between actions of maximal tori of SL_2 or PSL_2. These actions occur naturally in Brion's description of fixed point loci of spherical varieties, see below. This lemma is proved only using formal group laws and computations in formal power series. Indeed, this statement also holds if one considers coefficients in $\mathbb{Z}[\frac{1}{p_1}, ..., \frac{1}{p_\ell}]$ where $p_1, ..., p_\ell$ are the primes occurring in the prime factorisation of the order of the group F.

The following result by Brion describes the fixed point loci of spherical varieties.

Proposition. *[7, Proposition 7.1] Let X be a spherical G-variety and let $T' \subseteq T$ be a subtorus of codimension one.*

(i) Each irreducible component of $X^{T'}$ is a spherical $C_G(T')$-variety.

(ii) If T' is regular, then $X^{T'}$ is at most one-dimensional.

(iii) If T' is singular, then $X^{T'}$ is at most two-dimensional. Furthermore, any two-dimensional connected component of $X^{T'}$ is either a rational ruled surface

$$\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$$

where $C_G(T')$ acts through the natural action of SL_2, or the projective plane where $C_G(T')$ acts through the projectivisation of a non-trivial SL_2-module of dimension three.

Before we are able to state the main result of this thesis, we provide some more background regarding formal group laws. We recall the existence of a unique formal graded power series $\chi(u_i) \in \mathbb{L}[[u_1, ..., u_n]]_{\mathrm{gr}}$ which satisfies $F_\Omega(u_i, \chi(u_i)) = 0$ where F_Ω denotes the universal commutative formal group law of rank one. For any positive integer $b \in \mathbb{Z}_{\geq 1}$ we establish the following notations.

$$\begin{aligned} u_i +_{F_\Omega} u_j &:= F_\Omega(u_i, u_j) \in \mathbb{L}[[u_i, u_j]]_{\mathrm{gr}}, \\ [-1]_{F_\Omega} u_i &:= \chi(u_i) \in \mathbb{L}[[u_i]]_{\mathrm{gr}}, \\ u_i -_{F_\Omega} u_j &:= F_\Omega(u_i, \chi(u_j)) \in \mathbb{L}[[u_i, u_j]]_{\mathrm{gr}}, \\ [0]_{F_\Omega} u_i &:= 0, \\ [b]_{F_\Omega} u_i &:= F_\Omega(u_i, [b-1]_{F_\Omega} u_i) \in \mathbb{L}[[u_i]]_{\mathrm{gr}}. \end{aligned}$$

It is clear that $[b]_{F_\Omega} u$ is divisible by u for any $u \in \mathbb{L}[[u_1, ..., u_n]]_{\mathrm{gr}}$ of degree 1.

The next theorem is a replacement for the fact that we do not have any explicit presentation of T-equivariant cobordism as in the case of T-equivariant Chow groups. This result (cf. Theorem 3.39) is crucial for the proof of the structure theorem for the rational T-equivariant cobordism rings of smooth projective spherical G-varieties.

Theorem. *Let X be a smooth T-variety, $[h : Y \to X]$ the equivariant fundamental class of a T-stable cobordism cycle and $f \in k(Y)$ a rational T-eigenfunction with weight χ. Denote by Z_0 and Z_∞ the zeros and poles of f, respectively, and assume that they are smooth. Then the relation*

$$c_1^T(L_\chi) \cdot [Y \to X] = h_* F_\Omega\left([Z_0 \to Y], [-1]_{F_\Omega}[Z_\infty \to Y]\right)$$

holds in $\Omega_*^T(X)$ where F_Ω denotes the universal formal group law and $[-1]_{F_\Omega}$ is the inverse in the universal formal group law.

Next, we provide the last two definitions which are necessary in order to state the main theorem of this thesis (cf. Theorem 4.13).

Definition. *Let $u \in \mathbb{L}[[u_1, ..., u_n]]_{\mathrm{gr}}$ be a homogeneous element of degree 1. Then for $n \in \mathbb{Z}_{\geq 1}$ we define*

$$[-n]_{F_\Omega} u := [-1]_{F_\Omega}\left([n]_{F_\Omega} u\right).$$

Furthermore, if there exists a homogeneous element $u' \in (\mathbb{L}[[u_1, ..., u_n]]_{\mathrm{gr}})_{\mathbb{Q}}$ of degree 1 such that $[m]_{F_\Omega} u' = u$ holds for $m \in \mathbb{Z}_{\geq 1}$, then we define

$$\left[\frac{1}{m}\right]_{F_\Omega} u := u'.$$

Definition. *In the setting of the above definition we define the operator $\rho_{n/m}$ by*

$$\rho_{n/m} u := \frac{[n]_{F_\Omega}\left(\left[\frac{1}{m}\right]_{F_\Omega} u\right)}{u}$$

in $(\mathbb{L}[[u_1, ..., u_n]]_{\mathrm{gr}})_{\mathbb{Q}}$ for any $n \in \mathbb{Z} \setminus \{0\}$ and $m \in \mathbb{Z}_{\geq 1}$.

Now, we established all the necessary tools in order to be able to state the main theorem (cf. Theorem 4.13) of this thesis. The ordering of the T-fixed points in the connected components of $X^{T'}$ for singular codimension one subtori $T' \subseteq T$ will be described in the paragraph below the following theorem.

Theorem. *For any smooth projective and spherical G-variety X, the pullback map*

$$i^* : \Omega_T^*(X)_{\mathbb{Q}} \to \Omega_T^*(X^T)_{\mathbb{Q}}$$

is injective. Moreover, the image of i^* consists of all families $(f_x)_{x\in X^T}$ such that

(i) $f_x \equiv f_y \mod c_1^T(L_\chi)$ *whenever x and y are connected by a T-stable curve where T acts through the weight χ.*

(ii) $(f_x - f_y) + \rho_{1/2} c_1^T(L_\alpha)(f_z - f_x) \equiv 0 \mod c_1^T(L_\alpha)^2$ *whenever α is a positive root of G relative to T, x, y and z lie in a connected component of $X^{\mathrm{Ker}(\alpha)^0}$ isomorphic to a projective plane $\mathbb{P}^2$ and $x \geq y \geq z$ are ordered by their corresponding weights.*

(iii) $f_w - f_x - f_y + f_z \equiv 0 \mod c_1^T(L_\alpha)^2$ *whenever α is a positive root of G relative to T, w, x, y and z lie in a connected component of $X^{\mathrm{Ker}(\alpha)^0}$ isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and $w \geq x, y \geq z$ are ordered by their corresponding weights.*

(iv) $\rho_{-n/2} c_1^T(L_\alpha)(f_y - f_z) + \rho_{n/2} c_1^T(L_\alpha)(f_w - f_x) \equiv 0 \mod c_1^T(L_\alpha)^2$ *whenever α is a positive root of G relative to T, w, x, y and z lie in a connected component of $X^{\mathrm{Ker}(\alpha)^0}$ isomorphic to a rational ruled surface $\mathbb{F}_n$, $n \geq 1$, and $w \geq x \geq y \geq z$ are ordered by their corresponding weights.*

This theorem allows us to describe the rational T-equivariant cobordism rings as long as we can determine the surfaces occurring in the connected components of the fixed point subschemes $X^{T'}$. One more major tool in the proof of the previous theorem is the computation of the rational T-equivariant cobordism ring of projective planes and Hirzebruch surfaces (cf. Proposition 4.16). In order to be able to state this proposition, we describe the irreducible components of $X^{T'}$ for singular codimension one subtori T' coming from the previous Proposition in some more detail.

Therefore, let D be the torus of diagonal matrices in SL_2 and let α be the positive root. First, we want to consider the two cases of $\mathbb{P}(V)$ for a non-trivial SL_2-module V of dimension three. Set $V_{n+1} := \mathrm{Sym}^{n+1}(k^2)$. Let $V = V_0 \oplus V_1$ be the first non-trivial SL_2-module of dimension three. The weights of D in V are $\alpha/2, 0$ and $-\alpha/2$ induced by the given group action of D on V from Example 3.9 (iii). We denote by x, y and z the corresponding fixed points of D in $\mathbb{P}(V)$ which are also described in Example 3.9 (iii). To be more explicit, the corresponding fixed points to the weights $\alpha/2, 0, -\alpha/2$ are $x = [1 : 0 : 0], y = [0 : 1 : 0]$ and $z = [0 : 0 : 1]$, respectively. Therefore, we identify $\Omega_D^*(\mathbb{P}(V)^D)_{\mathbb{Q}}$ with $S(D)_{\mathbb{Q}}^3$.

Similarly, for the second non-trivial SL_2-module $V = V_2 = \mathfrak{sl}_2$ of dimension three, the corresponding weights are $\alpha, 0$ and $-\alpha$ whereas the corresponding fixed points are again $x = [1 : 0 : 0], y = [0 : 1 : 0]$ and $z = [0 : 0 : 1]$, respectively.

Next, we consider the case $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ with D-action given by

$$d \cdot ([a : b], [u : v]) = ([da : d^{-1}b], [du : d^{-1}v]).$$

We denote by w and z the D-fixed points $([1 : 0], [1 : 0])$ and $([0 : 1], [0 : 1])$, respectively. Further, we denote the remaining two D-fixed points $([1 : 0], [0 : 1])$ and $([0 : 1], [1 : 0])$ by x and y, respectively.

Lastly, we have a look at the rational ruled surfaces $\mathbb{F}_n$, $n \geq 1$, which we describe in large detail in Example 3.9 (v). We recall that $\mathbb{F}_n$ has four D-fixed points w, x, y and z with corresponding weights $(n+1)\alpha/2, \alpha/2, -\alpha/2$ and $-(n+1)\alpha/2$, respectively, by the induced D-action on $\mathbb{F}_n$ which is presented in Example 3.9 (v). Therefore, we can identify $\Omega_D^*(\mathbb{F}_n^D)_{\mathbb{Q}}$ with $S(D)_{\mathbb{Q}}^4$.

Now, we are able to state the previously announced proposition (cf. Proposition 4.16).

Proposition. *Let X be a Hirzebruch surface $\mathbb{F}_n$ or a projective plane $\mathbb{P}(V)$.*

(i) The image of the pullback

$$i^* : \Omega_D^*(\mathbb{F}_n)_{\mathbb{Q}} \to S(D)_{\mathbb{Q}}^4$$

consists of all $(f_w, f_x, f_y, f_z) \in S(D)_{\mathbb{Q}}^4$ such that

$$\begin{aligned} f_w \equiv f_x \equiv f_y \equiv f_z \quad &\mathrm{mod}\ c_1^D(L_\alpha) \text{ and} \\ f_w - f_x - f_y + f_z \equiv 0 \quad &\mathrm{mod}\ c_1^D(L_\alpha)^2 \end{aligned}$$

hold for $n = 0$ and of all $(f_w, f_x, f_y, f_z) \in S(D)_{\mathbb{Q}}^4$ such that

$$\begin{aligned} f_w \equiv f_x \equiv f_y \equiv f_z \quad &\mathrm{mod}\ c_1^D(L_\alpha) \text{ and} \\ \rho_{-n/2} c_1^D(L_\alpha)(f_y - f_z) + \rho_{n/2} c_1^D(L_\alpha)(f_w - f_x) \equiv 0 \quad &\mathrm{mod}\ c_1^D(L_\alpha)^2 \end{aligned}$$

hold for $n \geq 1$.

(ii) Moreover, the image of

$$i^* : \Omega_D^*(\mathbb{P}(V))_{\mathbb{Q}} \to S(D)_{\mathbb{Q}}^3$$

consists of all (f_x, f_y, f_z) such that

$$\begin{aligned} f_x \equiv f_y \equiv f_z \quad &\mathrm{mod}\ c_1^D(L_\alpha) \text{ and} \\ (f_x - f_y) + \rho_{1/2} c_1^D(L_\alpha)(f_z - f_x) \equiv 0 \quad &\mathrm{mod}\ c_1^D(L_\alpha)^2 \end{aligned}$$

hold.

The major application of the main Theorem is given by describing the geometry of smooth projective horospherical G-varieties X of Picard number one. These varieties were classified by Pasquier [43] and the classification is given by the following theorem.

Proposition. *[43, Theorem 0.1] Let G be a connected reductive algebraic group. Let X be a smooth projective horospherical G-variety with Picard number one. Then one of the following cases can occur.*

(i) X is homogeneous.

(ii) X is horospherical of rank 1. Its automorphism group is a connected non-reductive linear algebraic group, acting with exactly two orbits.

Moreover, in the second case X is uniquely determined by its two closed G-orbits Y and Z, isomorphic to G/P_Y and G/P_Z, respectively, and (G, P_Y, P_Z) is one of the triples of the following list.

(1) $(B_n, P(\omega_{n-1}), P(\omega_n))$ *for* $n \geq 3$

(2) $(B_3, P(\omega_1), P(\omega_3))$

(3) $(C_n, P(\omega_m), P(\omega_{m-1}))$ *for* $n \geq 2$ *and* $m \in [2, n]$

(4) $(F_4, P(\omega_2), P(\omega_3))$

(5) $(G_2, P(\omega_1), P(\omega_2)$

Here we denote by $P(\omega_i)$ *the maximal parabolic subgroup of* G *corresponding to the dominant weight* ω_i *using the notations from Bourbaki [5].*

Remark. In our notation $P(\omega_i)$ denotes the maximal parabolic subgroup $P_{S \setminus \alpha_i}$ for the simple root α_i associated to the fundamental weight ω_i.

In the sequel, we are only interested in the cases which are not homogeneous because the rational T-equivariant cobordism for the homogeneous varieties can be described using [31, Theorem 7.8]. Therefore, we recall the construction from [16, Section 1.3].

Let X be a smooth projective horospherical but non homogeneous variety of Picard number one with associated triple (G, P_Y, P_Z). In this case, we denote the previous triple also by $(G, P(\omega_Y), P(\omega_Z))$ for the corresponding fundamental weights ω_Y and ω_Z. Furthermore, the dense orbit is given by $G/H = G \cdot [v_Y + v_Z] \subseteq \mathbb{P}(V_Y \oplus V_Z)$ where V_Y and V_Z are the irreducible G-representations with highest weights ω_Y and ω_Z and the corresponding highest weight vectors v_Y and v_Z. We conclude by the construction that P_Y and P_Z are the stabilisers of $[v_Y]$ and $[v_Z]$ in $\mathbb{P}(V_Y)$ and $\mathbb{P}(V_Z)$ and that Y and Z are the G-orbits of $[v_Y]$ and $[v_Z]$ in $\mathbb{P}(V_Y)$ and $\mathbb{P}(V_Z)$, respectively. Lastly, we have that $X = \overline{G \cdot [v_Y + v_Z]} \subseteq \mathbb{P}(V_Y \oplus V_Z)$ is the closure of the G-orbit $G \cdot [v_Y + v_Z]$ in $\mathbb{P}(V_Y \oplus V_Z)$.

The T-fixed points of X are given by the T-fixed points of the two closed G-orbits. Now, we analyse the T-stable curves and the fixed point subschemes $X^{T'}$ for some given X in order to be able to use the main Theorem with the aim of obtaining the rational T-equivariant cobordism of X. It is known how to determine the T-stable curves in the closed orbits G/P_Y and G/P_Z which are flag varieties (see e.g. [15]). Next, we investigate the T-stable curves meeting the dense open orbit G/H for any smooth projective horospherical variety X of Picard number one. We will use the diagram

$$\begin{array}{ccc} & G/H & \\ & \downarrow{\scriptstyle \pi} & \\ & G/(P_Y \cap P_Z) & \\ {\scriptstyle p_Y} \swarrow & & \searrow {\scriptstyle p_Z} \\ G/P_Y & & G/P_Z \end{array} \tag{1.1}$$

where π is a $\mathbb{C}^*$-bundle corresponding to the fact that X is horospherical of rank one.

Additionally, we define $\chi := \omega_Y - \omega_Z$ to be the difference of the two given fundamental weights. Having established these notations, we are able to state another essential result of this thesis (cf. Lemma 5.23).

Lemma. *For any smooth projective horospherical variety X of Picard number one we have the following properties.*

(1) The only T-stable curves in X meeting the open orbit G/H occurring as a connected component of $X^{T'}$ for some codimension one subtorus T' are of the form $\overline{\pi^{-1}(z)}$ where $z \in G/(P_Y \cap P_Z)$ is a T-fixed point.

(2) The surfaces occurring in $X^{T'}$ only arise from codimension one subtori of the form $T' = \mathrm{Ker}(w\alpha)^0 = \mathrm{Ker}(w\chi)^0$ for some positive root α which is a non-zero multiple of χ and some $w \in W$.

This result is the main ingredient for the algorithm which determines the surfaces in the connected components of $X^{T'}$. Here, we refer to Example 5.24 in which we describe all occurring surfaces in the connected components of $X^{T'}$ for all smooth projective horospherical varieties X of Picard number one. From this we can then deduce the structure of the rational T-equivariant cobordism rings of all smooth projective horospherical varieties of Picard number one which are explicitly given in Example 5.25. We emphasise that all of these computations in the mentioned examples also hold for rational T-equivariant Chow groups and that even for Chow groups, those computations were previously unknown.

Next, we investigate some smooth projective horospherical varieties of Picard number two using the same algorithm. Furthermore, in the classification of these varieties, products of varieties are explicitly excluded, but nevertheless it is always a natural question to evaluate the T-equivariant cobordism rings of products of varieties. Therefore, we establish some Künneth formula for T-equivariant cobordism (cf. Proposition 3.63) which also reduces to a Künneth formula for T-equivariant Chow groups.

Proposition. *(Künneth formula) Let X, Y be smooth projective G-varieties such that $X \times Y$ has finitely many T-fixed points with respect to the diagonal action. Then there exists an isomorphism*

$$\Omega^*_T(X) \otimes_{\Omega^*_T(k)} \Omega^*_T(Y) \cong \Omega^*_T(X \times Y).$$

We recall that $S(T)_{\mathbb{Q}}[M^{-1}]$ is the graded ring obtained by inverting all non-zero linear forms $\sum_{j=1}^{n} m_j t_j$ in $S(T)_{\mathbb{Q}}$ which is described more generally in Construction 3.43. For a smooth k-scheme X with a torus action, we denote $\Omega^*_T(X)_{\mathbb{Q}} \otimes_{S(T)_{\mathbb{Q}}} S(T)_{\mathbb{Q}}[M^{-1}]$ by $\Omega^*_T(X)_{\mathbb{Q}}[M^{-1}]$.

The following definition (cf. Definition 6.5) is well-defined because the pushforward map $i_* : \Omega^*_T(X^T)_{\mathbb{Q}} \to \Omega^*_T(X)_{\mathbb{Q}}$ becomes an isomorphism after base change to $S(T)_{\mathbb{Q}}[M^{-1}]$ for a smooth T-filtrable variety X (cf. Corollary 6.4).

Definition. *Let X be a smooth T-filtrable variety with an action of a torus T. Further, let $[Y \to X] \in \Omega^*_T(X)_{\mathbb{Q}}[M^{-1}]$ and $x \in X$ be an isolated T-fixed point. We distinguish between isolated fixed points and connected components $F \subseteq X^T$ which are not an isolated point. For any isolated fixed point we define the* ***equivariant multiplicity***

$$e_{x,X}[Y \to X] \in S(T)_{\mathbb{Q}}[M^{-1}]$$

of X at x to be given by the equality

$$[Y \to X] = i_* \left(\sum_{\substack{x \in X^T \\ isolated}} e_{x,X}[Y \to X][x \to x] + \sum_{F \subseteq X^T} e_F[F' \to F] \right)$$

which holds in $\Omega_T^*(X)_{\mathbb{Q}}[M^{-1}]$ for some $e_F \in S(T)_{\mathbb{Q}}[M^{-1}]$ and $[F' \to F] \in \Omega_T^*(F)_{\mathbb{Q}}$.

Using this definition, we can state the last main result (cf. Proposition 6.7) which is used in order to compute classes in $\Omega_T^*(X)_{\mathbb{Q}}$ for smooth T-filtrable schemes X.

Proposition. *Let X be a smooth T-filtrable scheme with a T-action. Let $x \in X$ be an isolated fixed point and $[f : Y \to X]$ a class in the $S(T)_{\mathbb{Q}}$-algebra $\Omega_T^*(X)_{\mathbb{Q}}$. Assume further that all fixed points in the fibre $f^{-1}(x)$ are isolated. Then we have*

$$e_{x,X}[Y \to X] = \sum_{\substack{y \in Y^T \\ f(y)=x}} e_{y,Y}[Y \to Y].$$

The second obvious generalisation of the first Proposition is to consider some refined coefficient ring which results in mainly proving the localisation theorem (cf. [31, Theorem 7.6]) with a refined coefficient ring. The following result (cf. Theorem 3.59) proves the localisation result for a refined coefficient ring $\mathbb{Z}[S_X^{-1}]$ where S_X denotes some multiplicative set which will be discussed in more detail in Definition 3.58. This theorem also generalises Brion's result [7, Theorem 3.3] for Chow groups because it considers a finer coefficient ring.

Theorem. *Let X be a smooth projective variety where a torus T of rank n acts with finitely many isolated fixed points $x_0, ..., x_p$. Let $i : X^T \to X$ be the inclusion of the fixed point locus. Then the image of $i^* : \Omega_T^*(X)_{\mathbb{Z}[S_X^{-1}]} \to \Omega_T^*(X^T)_{\mathbb{Z}[S_X^{-1}]}$ is the intersection of the images of the restriction maps*

$$i_{T'}^* : \Omega_T^*(X^{T'})_{\mathbb{Z}[S_X^{-1}]} \to \Omega_T^*(X^T)_{\mathbb{Z}[S_X^{-1}]}$$

where T' runs over all subtori of codimension one in T.

This theorem enables us to make almost all of our computations also for this refined coefficient ring. In some cases, we need to invert $p = 2$ additionally, if it is not already inverted in $\mathbb{Z}[S_X^{-1}]$, because the difference of SL_2 and PSL_2, which is a group of order two, plays a major role in the proof of our main Theorem. This result then of course also generalises the description of rational T-equivariant Chow groups of smooth projective G-spherical varieties to the finer coefficient ring $\mathbb{Z}[S_X^{-1}]$ and furthermore, this gives a new result even for Chow groups.

1.1.3 Organisation of the thesis

In this section, we outline the structure of the thesis including the organisation of the five chapters.

In **Chapter 2**, we mainly present established notions and constructions which will be used in the sequel of the thesis. First of all, we recall the construction of *algebraic cobordism* Ω^* due to Levine and Morel [37] in Section 2.2 and discuss its main properties originally provided in [37]. In what follows, we define *equivariant algebraic cobordism* (cf. Definition 2.40) which was independently invented by Krishna [32] and Heller, Malagón-López [20]. We subsequently analyse its most important properties proved in [20, 32].

In the last section of Chapter 2, we discuss *spherical* and *horospherical* varieties (cf. Definition 2.47) and investigate examples which will be essential throughout the thesis.

In **Chapter 3**, we provide necessary notions in order to be able to state main known results for computations in equivariant algebraic cobordism (cf. [20, 28, 31, 34]). In this chapter, we are mainly interested in torus actions on schemes.

We start this chapter by introducing the notion of T*-filtrable* schemes (cf. Section 3.1). Next, we prove that the rational T-equivariant cobordism ring is isomorphic to the rational T/F-equivariant cobordism ring if F is a finite subgroup of T (cf. Lemma 3.36). Indeed, this statement also holds if one only considers coefficients in $\mathbb{Z}[\frac{1}{p_1}, ..., \frac{1}{p_\ell}]$ for appropriate primes $p_1, ..., p_\ell$. In what follows, we prove that T-equivariant cobordism satisfies a certain relation (cf. Theorem 3.39) which reduces to Brion's relation (cf. [7, Theorem 2.1]) in Chow groups described earlier in this chapter.

After having defined the necessary notions, we present the description of the rational T-equivariant cobordism ring of smooth projective varieties with finitely many T-fixed points and finitely many T-stable curves which was proved by Krishna in [31, Theorem 7.8]. One of the main purposes of this chapter is to generalise the previous statement to finer coefficient rings (cf. Proposition 3.60) using the refined localisation theorem (cf. Proposition 3.59).

For later use, we proceed by proving a Künneth formula for T-equivariant cobordism of smooth projective varieties with finitely many T-fixed points (cf. Proposition 3.63).

In **Chapter 4** we describe the rational T-equivariant cobordism rings of smooth projective G-spherical varieties X (cf. Theorem 4.13). The proof of the latter result specifically requires the fact that we are working over an algebraically closed field because one has to take n-th roots of unity in order to describe the connected components of $X^{T'}$ for codimension one subtori $T' \subseteq T$ which can be seen in the proof of [7, Theorem 7.1]. Later on, Theorem 4.13 can be reduced to computing the rational T-equivariant cobordism of projectives planes and Hirzebruch surfaces (cf. Proposition 4.16). The previous results can also be proved over some refined coefficient ring $\mathbb{Z}[\frac{1}{p_1}, ... \frac{1}{p_\ell}]$ (cf. Remark 4.23).

We finish the chapter computing the rational T-equivariant cobordism rings of the odd symplectic Grassmannians $\mathrm{IG}(k, 2n+1)$ for $n \geq 2$ and $k \in [2, n]$ (cf. Example 4.31).

The main purpose of **Chapter 5** is the computation of rational T-equivariant cobordism rings for smooth projective horospherical varieties using Theorem 4.13. This is done by describing the T-stable curves and the T-fixed points of the corresponding varieties.

In the following, we are mainly interested in smooth projective horospherical G-varieties X with $\mathrm{Pic}(X) \cong \mathbb{Z}$. These varieties were classified in [43] and the geometry was described in [16]. We extend the understanding of the geometry of the given varieties and using that, we describe their rational T-equivariant cobordism rings.

To finish this chapter, we compute the rational T-equivariant cobordism of some specific smooth projective horospherical G-varieties X with $\mathrm{Pic}(X) \cong \mathbb{Z}^2$.

In the closing **Chapter 6**, we finally study classes in the rational T-equivariant cobordism rings which were only described combinatorially up to this point. First, we define the notion of *equivariant multiplicities* (cf. Definition 6.5) based on the idea of equivariant multiplicities for equivariant Chow groups (cf. [7]). Using this definition, we investigate the equivariant multiplicities for some classes $[f : Y \to X]$ in the rational T-equivariant cobordism ring of smooth T-filtrable schemes X at isolated fixed points $x \in X$ (cf. Proposition 6.7). This result can be similarly proved for the refined coefficient ring. An essential ingredient in the proof is the self-intersection formula proved by Krishna [31, Proposition 3.1]. The rest of this chapter is mainly dedicated to examples which show the significance of the results in this thesis. In all the computations we can deduce that it is enough to invert finitely many primes in the coefficient ring, and in most cases it is even enough to invert $p = 2$.

Furthermore, we discuss the differences between equivariant cobordism and other oriented equivariant cohomology theories such as equivariant connective K-theory or equivariant Chow groups.

Many results of this thesis are in [26].

1.2 Introduction (version française)

1.2.1 Motivation

Les théories cohomologiques ont toujours été d'un grand intérêt en géométrie algébrique, que ce soit les notions de groupes de Chow, de K-théorie ou de cohomologie quantique. On peut faire la distinction entre les théories cohomologiques orientées et non orientées. Nous nous concentrerons principalement sur les premières. L'orientation signifie essentiellement que la théorie cohomologique dispose d'une notion de classes de Chern pour les fibrés vectoriels complexes. Ces théories cohomologiques sont principalement utilisées pour comprendre la théorie de l'intersection géométrique des variétés qui trouve sa source dans le 15ème problème de Hilbert.

La motivation originale pour comprendre le cobordisme algébrique était de trouver un analogue au cobordisme des variétés différentielles introduit dans l'article fondamental de Quillen (cf. [48]). Quillen a observé que la théorie du cobordisme complexe $\mathrm{MU}^*(X)$ est la théorie cohomologique orientée universelle complexe sur la catégorie des variétés différentielles. Plus tard, Levine et Morel (cf. [37]) ont pu montrer l'existence d'une théorie analogue à MU^* pour la catégorie des k-schémas lisses pour tout corps k de caractéristique nulle. Ils ont appelé cobordisme algébrique la théorie cohomologique orientée universelle Ω^* ainsi obtenue et en outre, ils ont donné des applications et des exemples expliquant les relations entre Ω^* et le foncteur K_0 des groupes de Grothendieck ou le foncteur d'anneaux de Chow CH^*.

Les théories cohomologiques équivariantes sont apparues à l'origine parce qu'on voulait comprendre la théorie cohomologique ordinaire des espaces de classification. Par la suite, les théories cohomologiques équivariantes ont été étudiées car elles incluent des actions de groupe sur les variétés dans leurs calculs. Il s'est avéré que celles-ci sont un outil très puissant pour décrire les théories cohomologiques ordinaires et de nombreux calculs ont été effectués dans des théories cohomologiques équivariantes (par exemple [1, 7, 8, 9, 11, 12, 17, 20, 30, 31, 32, 34, 50]) pour différents types de variétés. Comme nous nous intéressons notamment au cobordisme algébrique, nous étudions aussi principalement le cobordisme algébrique équivariant des variétés lisses sur un corps de caractéristique nulle.

Les derniers objets d'intérêt de cette thèse sont les variétés sphériques. Celles-ci ont été étudiées par exemple dans [6, 16, 29, 39, 41, 42, 43, 44, 45, 46, 47]. Cette classe de variétés comprend un large éventail de variétés très connues comme par exemple les variétés drapeaux, les variétés toriques, les variétés symétriques, les variétés magnifiques ou les variétés horosphériques. La géométrie de ces dernières a été intensivement étudiée par exemple dans [16, 41, 42, 43, 44, 46]. Dans cette thèse, nous nous concentrons principalement sur les calculs du cobordisme algébrique équivariant des variétés sphériques et sur les calculs explicites pour les variétés horosphériques. En fait, cette dernière classe de variétés comprend de nombreux exemples intéressants comme les variétés drapeaux, les variétés toriques ou la grassmannienne symplectique impaire $\mathrm{IG}(k, 2n+1)$ pour $n \geq 2$ et $k \in [2, n]$.

La motivation spécifique pour ce travail était l'état de la recherche au moment où ce projet a été lancé. Soit G un groupe algébrique réductif connexe avec un tore maximal T dans G sur un corps algébriquement clos k de caractéristique nulle. Brion a calculé les anneaux de Chow rationnels T-équivariants pour les G-variétés sphériques projectives lisses (cf. [7]) et des anneaux de cobordisme T-équivariants ont déjà été calculés pour par exemple les variétés toriques lisses (cf. [34]), les variétés de drapeaux (cf. [28]), les compactifications magnifiques des variétés symétriques de rang minimal (cf. [28]) et les variétés projectives lisses avec un nombre fini de points fixes de T et un nombre fini de courbes T-stables (cf. [31]). Par conséquent, une question naturelle est la généralisation des résultats de Brion au cobordisme rationnel T-équivariant pour toute G-variété sphérique projective lisse, d'autant plus que la géométrie des variétés sphériques est un domaine de recherche actif.

1.2.2 Principaux résultats

Dans cette section, nous décrivons nos principaux résultats. Nous commençons par donner les notations et définitions nécessaires aux principaux théorèmes de cette thèse. Tout d'abord, le lemme suivant illustre la correspondance entre les lois formelles de groupes et les théories cohomologiques orientées qui est l'un des faits fondamentaux de cette théorie (voir la Section 2.1 pour plus de détails).

Lemme. *[37, Lemme 1.1.3] Soit A^* une théorie cohomologique orientée sur $\mathbf{Sm}_k$ pour un corps k de caractéristique nulle. Alors il existe une unique série formelle*

$$F_A(u, v) = \sum_{i,j} a_{i,j} u^i v^j \in A^*(k)[[u, v]]$$

avec $a_{i,j} \in A^{1-i-j}(k)$ telle que pour tout $X \in \mathbf{Sm}_k$ et toute paire de fibrés en droites L et M sur X, nous avons

$$F_A(c_1(L), c_1(M)) = c_1(L \otimes M).$$

De plus, le couple $(A^*(k), F_A)$ est une loi de groupe formelle commutative de rang un.

Par exemple, les groupes de Chow sont associés à $F_{\mathrm{CH}}(u, v) = u + v$ et la K-théorie à $F_K(u, v) = u + v - \beta uv$. Le cobordisme algébrique correspond à la loi de groupe formelle commutative universelle F_Ω de rang un.

Ensuite, nous fixons quelques notations que nous utilisons tout au long de la thèse. Soit G un groupe algébrique réductif connexe sur un corps algébriquement clos k de caractéristique nulle. De plus, soit $B \subseteq G$ un sous-groupe de Borel, $T \subseteq B$ un tore maximal, W le groupe de Weyl de G correspondant au tore maximal de B et U le radical unipotent de B. Nous choisissons l'ordre sur les poids induit par le sous-groupe de Borel B, c'est-à-dire que pour deux poids χ, χ' on a $\chi \geq \chi'$ si $\chi - \chi'$ est une combinaison linéaire non négative de racines simples.

Définition. *Soit X une G-variété normale. On dit que X est **sphérique** si elle contient une B-orbite ouverte.*

Dans la suite, nous rappelons que l'anneau de cobordisme T-équivariant $\Omega_T^*(X)$ pour une variété lisse X est donné par une limite inverse des anneaux de cobordisme ordinaires de quotients mixtes, ces derniers étant construits à partir du groupe T (cf. Définition 2.40). Nous remarquons que $\Omega_T^*(X)$ est une $\Omega_T^*(k)$-algèbre et de plus, pour toute variété projective lisse X avec un nombre fini de points fixes de T, les générateurs dans le $\Omega_T^*(k)$-module $\Omega_T^*(X)$ sont certaines classes $[f : Y \to X]$ où f est un morphisme projectif T-équivariant de source une variété lisse Y (cf. [31, Corollaire 4.8]). A partir de maintenant nous utilisons $S(T) := \Omega_T^*(k)$. Nous rappelons également que l'anneau de cobordisme T-équivariant $S(T)$ est isomorphe à l'anneau de séries formelles gradué $\mathbb{L}[[t_1, ..., t_n]]_{\mathrm{gr}}$ (cf. Construction 3.25) où n désigne le rang du tore T (cf. [32, Proposition 6.7]) et $\mathbb{L}$ l'anneau de Lazard (cf. Construction 2.8).

Afin d'énoncer les résultats principaux de cette thèse, nous définissons l'anneau de cobordisme rationnel T-équivariant $\Omega_T^*(X)_{\mathbb{Q}} := \Omega_T^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ pour une variété lisse X où ce produit tensoriel désigne le produit tensoriel topologique gradué (cf. Construction 3.30). Nous remarquons que $\Omega_T^*(X)_{\mathbb{Q}}$ est une $S(T)_{\mathbb{Q}}$-algèbre si X est lisse.

Nous donnons maintenant l'état de l'art avant la réalisation de ce projet. La proposition suivante est une généralisation par Krishna [31] des résultats classiques de localisation (cf. [7, 12, 17]) qui est un outil très puissant pour calculer le cobordisme rationnel T-équivariant pour les variétés ayant un nombre fini de points fixes et de courbes stables.

Proposition. *[31, Théorème 7.8] Soit X un schéma projectif lisse où un tore T agit avec un nombre fini de points fixes $x_1, ..., x_p$ et un nombre fini de courbes stables. Soit $i : X^T \to X$ l'inclusion du lieu des points fixes. Alors l'image de*

$$i^* : \Omega_T^*(X)_{\mathbb{Q}} \to \Omega_T^*(X^T)_{\mathbb{Q}} = S(T)_{\mathbb{Q}}^p$$

est l'ensemble des $(f_1, ..., f_p) \in S(T)_{\mathbb{Q}}^p$ tels que $f_i \equiv f_j \mod \chi$ chaque fois que x_i et x_j sont reliés par une courbe irréductible stable où T agit par le poids χ.

Cette proposition suggère déjà plusieurs généralisations possibles dont une est l'extension de l'énoncé aux variétés projectives lisses avec une action de tore où T agit avec un nombre fini de points fixes de T, mais un nombre infini de courbes T-stables. Une classe de candidats potentiels est la classe des variétés sphériques projectives lisses, car celles-ci ont toujours un nombre fini de points fixes de T (cf. Lemme 4.8). Par conséquent, nous avons essayé de généraliser les résultats de Brion [7] concernant les groupes de Chow rationnels T-équivariants pour les G-variétés sphériques projectives lisses. Puisqu'il n'y a pas de présentation du module de cobordisme T-équivariant, on ne peut pas faire la même preuve que pour les groupes de Chow rationnels T-équivariants, mais on a besoin de différents lemmes afin d'adapter la stratégie. Le premier résultat technique est le lemme suivant (cf. Lemme 3.36).

Lemme. *Soit T un tore de rang n et F un sous-groupe fini. Nous avons alors un isomorphisme des $\mathbb{L}$-algèbres graduées*

$$\Omega_T^*(k)_{\mathbb{Q}} \cong \Omega_{T/F}^*(k)_{\mathbb{Q}}.$$

Cet énoncé assure que le cobordisme rationnel T-équivariant ne voit pas de différence entre les actions des tores maximaux de SL_2 ou de PSL_2. Ces actions apparaissent naturellement dans la description de Brion des lieux de points fixes des variétés sphériques, voir ci-dessous. Ce lemme est prouvé uniquement en utilisant des lois de groupe formelles et des calculs sur les séries formelles. En effet, cette affirmation est également valable si l'on considère des coefficients dans $\mathbb{Z}[\frac{1}{p_1}, ..., \frac{1}{p_\ell}]$ où $p_1, ..., p_\ell$ sont les nombres premiers apparaissant dans la factorisation en nombre premiers de l'ordre du groupe F.

Le résultat suivant de Brion décrit les lieux de points fixes des variétés sphériques.

Proposition. *[7, Proposition 7.1] Soit X un G-variété sphérique et soit $T' \subseteq T$ un sous-tore de codimension un.*

(i) Chaque composante irréductible de $X^{T'}$ est une $C_G(T')$-variété sphérique.

(ii) Si T' est régulier, alors $X^{T'}$ est au plus de dimension un.

(iii) Si T' est singulier, alors $X^{T'}$ est au plus de dimension deux. De plus, toute composante connexe de dimension deux de $X^{T'}$ est soit une surface réglée rationnelle

$$\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$$

où $C_G(T')$ agit par l'action naturelle de SL_2, soit le plan projectif où $C_G(T')$ agit par la projectivisation d'un SL_2-module non trivial de dimension trois.

Avant de pouvoir énoncer le résultat principal de cette thèse, nous fournissons quelques informations supplémentaires sur les lois des groupes formelles. Nous rappelons l'existence d'une unique série formelle graduée $\chi(u_i) \in \mathbb{L}[[u_1, ..., u_n]]_{\mathrm{gr}}$ qui satisfait la relation $F_\Omega(u_i, \chi(u_i)) = 0$. Pour tout entier positif $b \in \mathbb{Z}_{\geq 1}$ nous fixons les notations suivantes.

$$\begin{aligned} u_i +_{F_\Omega} u_j &:= F_\Omega(u_i, u_j) \in \mathbb{L}[[u_i, u_j]]_{\mathrm{gr}}, \\ [-1]_{F_\Omega} u_i &:= \chi(u_i) \in \mathbb{L}[[u_i]]_{\mathrm{gr}}, \\ u_i -_{F_\Omega} u_j &:= F_\Omega(u_i, \chi(u_j)) \in \mathbb{L}[[u_i, u_j]]_{\mathrm{gr}}, \\ [0]_{F_\Omega} u_i &:= 0, \\ [b]_{F_\Omega} u_i &:= F_\Omega(u_i, [b-1]_{F_\Omega} u_i) \in \mathbb{L}[[u_i]]_{\mathrm{gr}}. \end{aligned}$$

On voit que $[b]_{F_\Omega} u$ est divisible par u pour tout $u \in \mathbb{L}[[u_1, ..., u_n]]_{\mathrm{gr}}$ de degré 1.

Le théorème suivant permet de contourner le fait que nous n'avons pas de présentation explicite du cobordisme T-équivariant comme dans le cas des groupes de Chow T-équivariants. Ce résultat (cf. Théorème 3.39) est essentiel pour la preuve du théorème de structure pour les anneaux de cobordisme rationnels T-équivariants des G-variétés sphériques projectives lisses.

Théorème. *Soit X une T-variété lisse, $[h : Y \to X]$ la classe fondamentale équivariante d'un cycle de cobordisme T-stable, $f \in k(Y)$ une T-fonction propre rationnelle de poids χ et soient Z_0 et Z_∞ le lieu des zéros et le lieu des pôles de f. De plus, nous supposons que Z_0 et Z_∞ sont lisses. Alors on a la relation*

$$c_1^T(L_\chi) \cdot [Y \to X] = h_* F_\Omega\left([Z_0 \to Y], [-1]_{F_\Omega}[Z_\infty \to Y]\right)$$

dans $\Omega_*^T(X)$ où F_Ω désigne la loi de groupe formelle universelle et $[-1]_{F_\Omega}$ est l'inverse dans la loi de groupe formelle universelle.

Ensuite, nous donnons les deux dernières définitions qui sont nécessaires pour énoncer le théorème principal de cette thèse (cf. Théorème 4.13).

Définition. *Soit* $u \in \mathbb{L}[[u_1, ..., u_n]]_{\mathrm{gr}}$ *un élément homogène de degré* 1*. Pour* $n \in \mathbb{Z}_{\geq 1}$ *on définit*

$$[-n]_{F_\Omega} u := [-1]_{F_\Omega} \left([n]_{F_\Omega} u\right).$$

De plus, s'il existe un élément homogène $u' \in (\mathbb{L}[[u_1, ..., u_n]]_{\mathrm{gr}})_{\mathbb{Q}}$ *de degré* 1 *tel que* $[m]_{F_\Omega} u' = u$ *est vrai pour* $m \in \mathbb{Z}_{\geq 1}$*, alors nous définissons*

$$\left[\frac{1}{m}\right]_{F_\Omega} u := u'.$$

Définition. *Dans le cadre de la définition ci-dessus, nous définissons l'opérateur* $\rho_{n/m}$ *par*

$$\rho_{n/m} u := \frac{[n]_{F_\Omega} \left(\left[\frac{1}{m}\right]_{F_\Omega} u \right)}{u}$$

dans $(\mathbb{L}[[u_1, ..., u_n]]_{\mathrm{gr}})_{\mathbb{Q}}$ *pour tout* $n \in \mathbb{Z} \setminus \{0\}$ *et* $m \in \mathbb{Z}_{\geq 1}$*.*

Nous avons maintenant établi tous les outils nécessaires afin de pouvoir énoncer le théorème principal (cf. Théorème 4.13) de cette thèse. L'ordre des points fixes de T dans les composantes connexes de $X^{T'}$ sera décrit dans le paragraphe après le théorème suivant pour les sous-tores singuliers $T' \subseteq T$ de codimension un.

Théorème. *Pour toute G-variété sphérique projective lisse X, le tiré en arrière*

$$i^* : \Omega_T^*(X)_{\mathbb{Q}} \to \Omega_T^*(X^T)_{\mathbb{Q}}$$

est injectif. De plus, l'image de i^* *est constituée de toutes les familles* $(f_x)_{x \in X^T}$ *telles que*

(i) $f_x \equiv f_y \mod c_1^T(L_\chi)$ *chaque fois que x et y sont reliés par une courbe T-stable où T agit par le poids χ.*

(ii) $(f_x - f_y) + \rho_{1/2} c_1^T(L_\alpha)(f_z - f_x) \equiv 0 \mod c_1^T(L_\alpha)^2$ *lorsque α est une racine positive de G par rapport à T, x, y et z se trouvent dans une composante connexe de $X^{\mathrm{Ker}(\alpha)^0}$ isomorphe à un plan projectif $\mathbb{P}^2$ et $x \geq y \geq z$ sont ordonnés par leurs poids correspondants.*

(iii) $f_w - f_x - f_y + f_z \equiv 0 \mod c_1^T(L_\alpha)^2$ *lorsque α est une racine positive de G par rapport à T, w, x, y et z se trouvent dans une composante connexe de $X^{\mathrm{Ker}(\alpha)^0}$ isomorphe à $\mathbb{P}^1 \times \mathbb{P}^1$ et $w \geq x, y \geq z$ sont ordonnés par leurs poids correspondants.*

(iv) $\rho_{-n/2} c_1^T(L_\alpha)(f_y - f_z) + \rho_{n/2} c_1^T(L_\alpha)(f_w - f_x) \equiv 0 \mod c_1^T(L_\alpha)^2$ *lorsque α est une racine positive de G par rapport à T, w, x, y et z se trouvent dans une composante connexe de $X^{\mathrm{Ker}(\alpha)^0}$ isomorphe à une surface réglée rationnelle $\mathbb{F}_n$, $n \geq 1$, et $w \geq x \geq y \geq z$ sont ordonnés par leurs poids correspondants.*

Ce théorème nous permet de décrire les anneaux de cobordisme rationnels T-équivariants dès que nous pouvons déterminer les surfaces apparaissant dans les composantes connexes des lieux de points fixes $X^{T'}$. Un autre outil important dans la preuve du théorème précédent est le calcul de l'anneau de cobordisme rationnel T-équivariant des plans projectifs et des surfaces de Hirzebruch (cf. Proposition 4.16). Afin de pouvoir énoncer cette proposition, nous décrivons un peu plus en détail les composantes irréductibles de $X^{T'}$ pour les sous-tores singuliers T' de codimension un issus de la Proposition précédente.

Par conséquent, soit D le tore des matrices diagonales dans SL_2 et soit α la racine positive. Dans un premier temps, nous voulons considérer les deux cas de $\mathbb{P}(V)$ pour un SL_2-module V non trivial de dimension trois. Nous fixons $V_{n+1} := \mathrm{Sym}^{n+1}(k^2)$. Soit $V = V_0 \oplus V_1$ le premier SL_2-module non trivial de dimension trois. Les poids de D dans V sont $\alpha/2, 0$ et $-\alpha/2$ induits par l'action de groupe donnée de D sur V de l'Exemple 3.9 (iii). Nous désignons par x, y et z les points fixes de D correspondants dans $\mathbb{P}(V)$ qui est également décrit dans l'Exemple 3.9 (iii). Pour être plus explicite, les points fixes correspondants aux poids $\alpha/2, 0, -\alpha/2$ sont $x = [1:0:0], y = [0:1:0]$ et $z = [0:0:1]$, respectivement. Par conséquent, nous identifions $\Omega^*_D(\mathbb{P}(V)^D)_\mathbb{Q}$ avec $S(D)^3_\mathbb{Q}$.

De même, pour le second SL_2-module non trivial $V = V_2 = \mathfrak{sl}_2$ de dimension trois, les poids correspondants sont $\alpha, 0$ et $-\alpha$ tandis que les points fixes correspondants sont à nouveau $x = [1:0:0], y = [0:1:0]$ et $z = [0:0:1]$, respectivement.

Ensuite, nous considérons le cas $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ avec l'action de D donnée par

$$d \cdot ([a:b],[u:v]) = ([da : d^{-1}b], [du : d^{-1}v]).$$

Nous désignons par w et z les points fixes de D qui sont donnés par $([1:0],[1:0])$ et $([0:1],[0:1])$, respectivement. De plus, nous désignons les deux points fixes de D restants $([1:0],[0:1])$ et $([0:1],[1:0])$ par x et y, respectivement. Enfin, nous nous intéressons aux surfaces réglées rationnelles $\mathbb{F}_n$, $n \geq 1$, que nous décrivons en détail dans l'Exemple 3.9 (v). Nous rappelons que $\mathbb{F}_n$ a quatre points fixes de D que nous désignons par w, x, y et z avec leurs poids correspondants $(n+1)\alpha/2, \alpha/2, -\alpha/2$ et $-(n+1)\alpha/2$, respectivement, induits par l'action de D sur $\mathbb{F}_n$ qui est présentée dans l'Exemple 3.9 (v). Par conséquent, nous pouvons identifier $\Omega^*_D(\mathbb{F}_n^D)_\mathbb{Q}$ avec $S(D)^4_\mathbb{Q}$.

Maintenant nous sommes en mesure d'énoncer la proposition annoncée précédemment (cf. Proposition 4.16).

Proposition. *Soit X une surface de Hirzebruch $\mathbb{F}_n$ ou un plan projectif $\mathbb{P}(V)$.*

(i) L'image du tiré en arrière $i^* : \Omega^*_D(\mathbb{F}_n)_\mathbb{Q} \to S(D)^4_\mathbb{Q}$ consiste en tous les tuples $(f_w, f_x, f_y, f_z) \in S(D)^4_\mathbb{Q}$ tels que

$$f_w \equiv f_x \equiv f_y \equiv f_z \mod c_1^D(L_\alpha) \text{ et}$$
$$f_w - f_x - f_y + f_z \equiv 0 \mod c_1^D(L_\alpha)^2$$

sont valables pour $n = 0$ et en tous les $(f_w, f_x, f_y, f_z) \in S(D)^4_\mathbb{Q}$ tels que

$$f_w \equiv f_x \equiv f_y \equiv f_z \mod c_1^D(L_\alpha) \text{ et}$$
$$\rho_{-n/2} c_1^D(L_\alpha)(f_y - f_z) + \rho_{n/2} c_1^D(L_\alpha)(f_w - f_x) \equiv 0 \mod c_1^D(L_\alpha)^2$$

sont valables pour $n \geq 1$.

(ii) *De plus, l'image de* $i^* : \Omega_D^*(\mathbb{P}(V))_{\mathbb{Q}} \to S(D)^3_{\mathbb{Q}}$ *consiste en tous les tuples* (f_x, f_y, f_z) *tels que*

$$\begin{aligned} f_x \equiv f_y \equiv f_z &\mod c_1^D(L_\alpha) \text{ et} \\ (f_x - f_y) + \rho_{1/2} c_1^D(L_\alpha)(f_z - f_x) \equiv 0 &\mod c_1^D(L_\alpha)^2. \end{aligned}$$

Nous donnons une application du Théorème principal en considérant les variétés horosphériques projectives lisses X de nombre de Picard un. Ces variétés ont été classées par Pasquier [43] et la classification est donnée par le théorème suivant.

Proposition. *[43, Theorem 0.1] Soit* G *un groupe algébrique réductif connexe. Soit* X *une* G*-variété horosphérique projective lisse de nombre de Picard égal à un. Alors l'un des cas suivants peut se produire.*

(i) X *est homogène.*

(ii) X *est horosphérique de rang 1. Son groupe d'automorphisme est un groupe algébrique linéaire connexe non réductif, agissant avec exactement deux orbites.*

De plus, dans le second cas, X *est déterminé de manière unique par ses deux* G*-orbites fermées* Y *et* Z*, isomorphes à* G/P_Y *et* G/P_Z*, respectivement, et* (G, P_Y, P_Z) *est un des triplets de la liste suivante.*

(1) $(B_n, P(\omega_{n-1}), P(\omega_n))$ *pour* $n \geq 3$.

(2) $(B_3, P(\omega_1), P(\omega_3))$

(3) $(C_n, P(\omega_m), P(\omega_{m-1}))$ *pour* $n \geq 2$ *et* $m \in [2, n]$.

(4) $(F_4, P(\omega_2), P(\omega_3))$

(5) $(G_2, P(\omega_1), P(\omega_2)$

Nous désignons ici par $P(\omega_i)$ *le sous-groupe parabolique maximal de* G *correspondant au poids dominant* ω_i *en utilisant les notations de Bourbaki [5].*

Remarque. Dans notre notation $P(\omega_i)$ désigne le sous-groupe parabolique maximal $P_{S \setminus \alpha_i}$ pour la racine simple α_i associée au poids fondamental ω_i.

Dans la suite, nous ne nous intéressons qu'aux cas qui ne sont pas homogènes car le cobordisme rationnel T-équivariant pour les variétés homogènes peut être décrit à l'aide de [31, Théorème 7.8]. Par conséquent, nous rappelons la construction de [16, Section 1.3].

Soit X une variété horosphérique projective lisse mais non homogène de nombre de Picard un avec le triplet associé (G, P_Y, P_Z). Dans ce cas, nous désignons le triplet précédent également par $(G, P(\omega_Y), P(\omega_Z))$ pour les poids fondamentaux correspondants ω_Y et ω_Z. De plus, l'orbite dense est donnée par $G/H = G \cdot [v_Y + v_Z] \subseteq \mathbb{P}(V_Y \oplus V_Z)$ où V_Y et V_Z sont les représentations irréductibles de G de plus hauts poids ω_Y et ω_Z et les vecteurs de plus hauts poids correspondants v_Y et v_Z. On conclut par construction que P_Y et P_Z sont les stabilisateurs de $[v_Y]$ et $[v_Z]$ dans $\mathbb{P}(V_Y)$ et $\mathbb{P}(V_Z)$ et que Y et Z sont les G-orbites de $[v_Y]$ et $[v_Z]$ dans $\mathbb{P}(V_Y)$ et $\mathbb{P}(V_Z)$, respectivement. Enfin, nous avons que $X = \overline{G \cdot [v_Y + v_Z]} \subseteq \mathbb{P}(V_Y \oplus V_Z)$ est la fermeture de la G-orbite $G \cdot [v_Y + v_Z]$ dans $\mathbb{P}(V_Y \oplus V_Z)$.

Les points fixes de T sont donnés par les points fixes de T dans les G-orbites fermées. Maintenant, nous analysons les courbes T-stables et les sous-schémas des points fixes $X^{T'}$ pour un certain X donné afin de pouvoir utiliser le Théorème principal dans le but d'obtenir le cobordisme rationnel T-équivariant de X. On sait comment déterminer les courbes T-stables dans les orbites fermées G/P_Y et G/P_Z qui sont des variétés drapeaux (voir par exemple [15]). Ensuite, nous étudions les courbes T-stables rencontrant l'orbite ouverte dense G/H pour toute variété horosphérique projective lisse X de nombre de Picard un. Nous utiliserons le diagramme

$$\begin{array}{ccccc} & & G/H & & \\ & & \downarrow{\scriptstyle \pi} & & \\ & & G/(P_Y \cap P_Z) & & \\ & \swarrow^{p_Y} & & \searrow^{p_Z} & \\ G/P_Y & & & & G/P_Z \end{array} \tag{1.2}$$

où π est une $\mathbb{C}^*$-fibration correspondant au fait que X est horosphérique de rang un.

De plus, nous définissons $\chi := \omega_Y - \omega_Z$ comme étant la différence des deux poids fondamentaux donnés. Après avoir établi ces notations, nous sommes en mesure d'énoncer un autre résultat essentiel de cette thèse (cf. Lemma 5.23).

Lemme. *Pour toute variété horosphérique projective lisse X de nombre de Picard un, nous avons les propriétés suivantes.*

(1) Les seules courbes T-stables dans X qui rencontrent l'orbite ouverte G/H apparaissant comme une composante connexe de $X^{T'}$ pour un sous-tore T' de codimension un sont de la forme $\overline{\pi^{-1}(z)}$ où $z \in G/(P_Y \cap P_Z)$ est un point fixe de T.

(2) Les surfaces apparaissant dans $X^{T'}$ ne proviennent que des sous-tores de codimension un de la forme $T' = \mathrm{Ker}(w\alpha)^0 = \mathrm{Ker}(w\chi)^0$ pour une racine positive α qui est un multiple non nul de χ et un élément $w \in W$.

Ce résultat est l'ingrédient principal de l'algorithme qui détermine les surfaces dans les composantes connexes de $X^{T'}$. Nous nous référons ici à l'Exemple 5.24 dans lequel nous décrivons toutes les surfaces apparaissant dans les composantes connexes de $X^{T'}$ pour toutes les variétés horosphériques projectives lisses X de nombre de Picard un. Nous pouvons alors en déduire la structure des anneaux de cobordisme rationnels T-équivariants de toutes les variétés horosphériques projectives lisses de nombre de Picard un, qui est explicitement donnée dans l'Exemple 5.25. Nous soulignons que tous ces calculs dans les exemples mentionnés sont également valables pour les groupes de Chow rationnels T-équivariants et que même pour les groupes de Chow, ces calculs étaient auparavant inconnus.

Ensuite, nous étudions quelques variétés horosphériques projectives lisses de nombre de Picard deux en utilisant le même algorithme. De plus, dans la classification de ces variétés, les produits de variétés sont explicitement exclus, mais néanmoins, il est toujours naturel d'évaluer les anneaux de cobordisme T-équivariants des produits de variétés. Par conséquent, nous établissons une formule de Künneth pour le cobordisme T-équivariant (cf. Proposition 3.63) qui se réduit également à une formule de Künneth pour les groupes de Chow T-équivariants.

Proposition. *(Formule de Künneth) Soit X, Y des G-variétés projectives lisses telles que $X \times Y$ a un nombre fini de points fixes de T par rapport à l'action diagonale. Alors il existe un isomorphisme*

$$\Omega_T^*(X) \otimes_{\Omega_T^*(k)} \Omega_T^*(Y) \cong \Omega_T^*(X \times Y).$$

Nous rappelons que $S(T)_{\mathbb{Q}}[M^{-1}]$ est l'anneau gradué obtenu en inversant toutes les formes linéaires non nulles $\sum_{j=1}^n m_j t_j$ dans $S(T)_{\mathbb{Q}}$ qui est décrit plus généralement dans la Construction 3.43. Pour un k-schéma lisse X avec une action d'un tore T, nous désignons $\Omega_T^*(X)_{\mathbb{Q}} \otimes_{S(T)_{\mathbb{Q}}} S(T)_{\mathbb{Q}}[M^{-1}]$ par $\Omega_T^*(X)_{\mathbb{Q}}[M^{-1}]$.

La définition suivante (cf. Définition 6.5) est bien définie car le poussé en avant $i_* : \Omega_T^*(X^T)_{\mathbb{Q}} \to \Omega_T^*(X)_{\mathbb{Q}}$ devient un isomorphisme après un changement de base vers $S(T)_{\mathbb{Q}}[M^{-1}]$ pour une variété lisse T-filtrable X (cf. Corollaire 6.4).

Définition. *Soit X une variété lisse T-filtrable avec une action d'un tore T. De plus, soient $[Y \to X] \in \Omega_T^*(X)_{\mathbb{Q}}[M^{-1}]$ et $x \in X$ un point fixe de T isolé. Nous distinguons deux types de composantes connexes de X^T : celles qui sont formées d'un point fixe isolé, notées $x \in X^T$ et les autres, notées $F \subseteq X^T$. Pour tout point fixe isolé, nous définissons la* ***multiplicité équivariante***

$$e_{x,X}[Y \to X] \in S(T)_{\mathbb{Q}}[M^{-1}]$$

de X à x comme étant donnée par l'égalité

$$[Y \to X] = i_* \left(\sum_{\substack{x \in X^T \\ \text{isolé}}} e_{x,X}[Y \to X][x \to x] + \sum_{F \subseteq X^T} e_F[F' \to F] \right)$$

où $e_F \in S(T)_{\mathbb{Q}}[M^{-1}]$ et $[F' \to F] \in \Omega_T^*(F)_{\mathbb{Q}}$ qui est toujours satisfaite dans l'anneau $\Omega_T^*(X)_{\mathbb{Q}}[M^{-1}]$.

En utilisant cette définition, nous pouvons énoncer le dernier résultat principal (cf. Proposition 6.7) qui est utilisé afin de calculer les classes dans $\Omega_T^*(X)_{\mathbb{Q}}$ pour les schémas lisses T-filtrables X.

Proposition. *Soit X un schéma lisse T-filtrable avec une action de T. Soit $x \in X$ un point fixe isolé et $[f : Y \to X]$ une classe dans la $S(T)_{\mathbb{Q}}$-algèbre $\Omega_T^*(X)_{\mathbb{Q}}$. Supposons en outre que tous les points fixes de la fibre $f^{-1}(x)$ sont isolés. Nous avons alors*

$$e_{x,X}[Y \to X] = \sum_{\substack{y \in Y^T \\ f(y) = x}} e_{y,Y}[Y \to Y].$$

La deuxième généralisation évidente de la première Proposition consiste à considérer un anneau de coefficients raffiné, ce qui a pour conséquence de prouver principalement le théorème de localisation (cf. [31, Theorem 7.6]) avec un anneau de coefficients raffiné. Le résultat suivant (cf. Theorem 3.59) prouve le résultat de localisation pour un anneau de coefficients raffiné $\mathbb{Z}[S_X^{-1}]$ où S_X désigne un ensemble multiplicatif qui sera discuté dans la Définition 3.58. Ce théorème généralise également le résultat de Brion [7, Theorem 3.3] pour les groupes de Chow car il considère un anneau de coefficients plus fin.

Théorème. *Soit X une variété projective lisse où un tore T de rang n agit avec un nombre fini de points fixes isolés $x_0, ..., x_p$. Soit $i : X^T \to X$ l'inclusion du lieu des points fixes. Alors l'image de $i^* : \Omega_T^*(X)_{\mathbb{Z}[S_X^{-1}]} \to \Omega_T^*(X^T)_{\mathbb{Z}[S_X^{-1}]}$ est l'intersection des images des cartes de restriction*

$$i_{T'}^* : \Omega_T^*(X^{T'})_{\mathbb{Z}[S_X^{-1}]} \to \Omega_T^*(X^T)_{\mathbb{Z}[S_X^{-1}]}$$

où T' parcours tous les sous-tores de codimension un de T.

Ce théorème nous permet de faire presque tous nos calculs également pour cet anneau de coefficients raffiné. Dans certains cas, nous devons inverser $p = 2$ en plus, s'il n'est pas déjà inversé dans $\mathbb{Z}[S_X^{-1}]$, car la différence entre SL_2 et PSL_2, qui est un groupe d'ordre deux, joue un rôle majeur dans la preuve de notre Théorème principal. Ce résultat généralise bien sûr la description des groupes de Chow rationnels T-équivariants des G-variétés sphériques projectives lisses à l'anneau de coefficients plus fin $\mathbb{Z}[S_X^{-1}]$ et de plus, cela donne un nouveau résultat même pour les groupes de Chow.

1.2.3 Organisation de la thèse

Dans cette section, nous décrivons la structure de la thèse, y compris l'organisation des cinq chapitres.

Dans le **Chapitre 2**, nous présentons principalement des notions et des constructions connues qui seront utilisées dans la suite de la thèse. Tout d'abord, nous rappelons la construction du *cobordisme algébrique* Ω^* de Levine et Morel [37] dans la Section 2.2 et discutons ses principales propriétés fournies à l'origine dans [37]. Dans ce qui suit, nous définissons le *cobordisme algébrique équivariant* (cf. Définition 2.40) inventé indépendamment par Krishna [32] et Heller, Malagón-López [20]. Nous analysons ensuite ses propriétés les plus importantes démontrées dans [20, 32].

Dans la dernière section du Chapitre 2, nous discutons des variétés *sphériques* et *horosphériques* (cf. Définition 2.47) et étudions des exemples qui seront essentiels tout au long de la thèse.

Dans le **Chapitre 3**, nous fournissons les notions nécessaires pour pouvoir énoncer les principaux résultats connus pour les calculs en cobordisme algébrique équivariant (cf. [20, 28, 31, 34]). Dans ce chapitre, nous nous intéressons principalement aux actions des tores sur les schémas.

Nous commençons ce chapitre en introduisant la notion de schémas *T-filtrable* (cf. Section 3.1). Ensuite, nous prouvons que l'anneau de cobordisme rationnel T-équivariant est isomorphe à l'anneau de cobordisme rationnel T/F-équivariant si F est un sous-groupe fini de T (cf. Lemme 3.36). En effet, cette affirmation est également valable si l'on ne considère que les coefficients dans $\mathbb{Z}[\frac{1}{p_1}, ..., \frac{1}{p_\ell}]$ pour les nombres premiers appropriés $p_1, ..., p_\ell$. Dans ce qui suit, nous prouvons que le cobordisme T-équivariant satisfait une certaine relation (cf. Théorème 3.39) qui se réduit à la relation de Brion (cf. [7, Théorème 2.1]) dans les groupes de Chow décrite précédemment dans ce chapitre.

Après avoir défini les notions nécessaires, nous présentons la description de l'anneau de cobordisme rationnel T-équivariant des variétés projectives lisses avec un nombre fini de points fixes de T et un nombre fini de courbes T-stables qui a été prouvé par Krishna dans [31, Théorème 7.8]. L'un des principaux objectifs de ce chapitre est de généraliser l'énoncé précédent à des anneaux de coefficients plus fins (cf. Proposition 3.60) en utilisant le théorème de localisation raffiné (cf. Proposition 3.59).

Pour une utilisation ultérieure, nous démontrons une formule de Künneth pour le cobordisme T-équivariant des variétés projectives lisses avec un nombre fini de points fixes de T (cf. Proposition 3.63).

Dans le **Chapitre 4** nous décrivons les anneaux de cobordisme rationnels T-équivariants des G-variétés sphériques projectives lisses X (cf. Théorème 4.13). La preuve de ce dernier résultat nécessite spécifiquement le fait que l'on travaille sur un corps algébriquement clos car il faut prendre les n-ièmes racines de l'unité pour décrire les composantes connexes de $X^{T'}$ pour un sous-tore $T' \subseteq T$ de codimension un ce qui se voit en étudiant la preuve de [7, Theorem 7.1]. Nous montrons que le Théorème 4.13 peut se réduire au calcul du cobordisme rationnel T-équivariant des plans projectifs et des surfaces de Hirzebruch (cf. Proposition 4.16). Les résultats précédents peuvent également être prouvés sur un anneau de coefficients raffiné $\mathbb{Z}[\frac{1}{p_1}, \dots \frac{1}{p_\ell}]$ (cf. Remarque 4.23).

On termine le chapitre en calculant les anneaux de cobordisme rationnels T-équivariants des grassmanniennes symplectiques impaires $\mathrm{IG}(k, 2n+1)$ pour $n \geq 2$ et $k \in [2, n]$ (cf . Exemple 4.31).

Le but principal du **Chapitre 5** est le calcul des anneaux de cobordisme rationnels T-équivariants pour les variétés horosphériques projectives lisses en utilisant le Théorème 4.13. Ceci est fait en décrivant les courbes T-stables et les points fixes de T des variétés correspondantes.

Dans la suite, nous nous intéressons principalement aux G-variétés horosphériques projectives lisses X avec $\mathrm{Pic}(X) \cong \mathbb{Z}$. Ces variétés ont été classées dans [43] et la géométrie a été décrite dans [16]. Nous étendons la compréhension de la géométrie des variétés données et en utilisant cela, nous décrivons leurs anneaux de cobordisme rationnels T-équivariants.

Pour terminer ce chapitre, nous calculons le cobordisme rationnel T-équivariant de certaines G-variétés horosphériques projectives lisses spécifiques X avec $\mathrm{Pic}(X) \cong \mathbb{Z}^2$.

Dans le **Chapitre 6** de clôture, nous étudions enfin les classes dans les anneaux de cobordisme rationnels T-équivariants qui n'ont été décrites que de manière combinatoire jusqu'à maintenant. Dans un premier temps, nous définissons la notion de *multiplicité équivariante* (cf. Définition 6.5) basée sur l'idée de multiplicités équivariantes pour des groupes de Chow équivariants (cf. [7]). En utilisant cette définition, nous étudions les multiplicités équivariantes pour certaines classes $[f : Y \to X]$ dans l'anneau de cobordisme rationnel T-équivariant des schémas lisses T-filtrables X aux points fixes isolés $x \in X$ (cf. Proposition 6.7). Ce résultat peut être prouvé de manière similaire pour l'anneau de coefficients raffiné. Un ingrédient essentiel de la preuve est la formule d'auto-intersection démontrée par Krishna [31, Proposition 3.1]. Le reste de ce chapitre est principalement consacré à des exemples qui montrent l'importance des résultats de cette thèse. Dans tous les calculs, on peut déduire qu'il suffit d'inverser un nombre fini de nombres premiers, et dans la plupart des cas il suffit même d'inverser $p = 2$.

De plus, nous discutons les différences entre le cobordisme équivariant et d'autres théories cohomologiques orientées équivariantes telles que la K-théorie équivariante ou les groupes de Chow équivariants.

De nombreux résultats de cette thèse se trouvent dans [26].

2 Preliminaries

The main goal of this chapter is to present the established notions and constructions of algebraic cobordism and equivariant algebraic cobordism. Therefore, we mainly refer to [20, 32, 37, 38].

Notations 2.1. Let k be a field of characteristic zero. Firstly, we require all k-schemes to be separated and of finite type over k. Since we will be concerned with the study of schemes with group actions of linear algebraic groups and the corresponding quotient schemes which often require the underlying scheme to be quasi-projective, we assume throughout this thesis that all schemes over k are quasi-projective if nothing else is explicitly mentioned. Next, we establish the well known notations which are mainly used in the literature. Therefore, the category of quasi-projective k-schemes will be denoted by $\mathbf{Sch}_k$. A scheme is meant to be an object of this category $\mathbf{Sch}_k$. Varieties will be irreducible and reduced schemes. Furthermore, the full subcategory of $\mathbf{Sch}_k$ consisting of smooth and quasi-projective k-schemes will be denoted by $\mathbf{Sm}_k$. Similarly, if G is a linear algebraic group over k, we denote the category of quasi-projective k-schemes with a group action of G and G-equivariant maps by $G - \mathbf{Sch}_k$. Frequently these schemes will be called G-schemes. The corresponding category of smooth quasi-projective G-schemes will be denoted by $G - \mathbf{Sm}_k$. We assume all group actions to be linear, i.e. for any G-action on a scheme X there exists a representation $G \to \mathrm{GL}(V)$ on a finite-dimensional vector space V and a G-equivariant immersion $X \to \mathbb{P}(V)$.

We will pass freely between vector bundles over X and the corresponding locally free sheaves of $\mathcal{O}_X$-modules where $\mathcal{O}_X$ denotes the structure sheaf of a scheme X. For a locally free coherent sheaf $\mathcal{E}$ on a scheme X, we let $q : \mathbb{P}(\mathcal{E}) \to X$ denote the projective bundle associated to $\mathcal{E}$. For a vector bundle $E \to X$, we write $\mathbb{P}(E)$ instead of $\mathbb{P}(\mathcal{O}_X(E))$, where $\mathcal{O}_X(E)$ denotes the sheaf of sections of E. We might omit the subscripts if there is no confusion possible.

2.1 Oriented cohomology theories

In this section, we recall the main notions of oriented cohomology theories (cf. [37]). Here we take the notations and conventions from [37].

Definition 2.2. *Let $\mathcal{V}$ be a full subcategory of $\mathbf{Sch}_k$. This subcategory $\mathcal{V}$ is said to be **admissible** if it satisfies the following conditions.*

(i) $\operatorname{Spec} k$ *and the empty scheme* $\emptyset$ *are in* $\mathcal{V}$.

(ii) If $Y \to X$ is a smooth and quasi-projective morphism in $\mathbf{Sch}_k$ such that $X \in \mathcal{V}$, then $Y \in \mathcal{V}$.

(iii) If X and Y are in $\mathcal{V}$, then so is the product $X \times_k Y$.

(iv) If X and Y are in $\mathcal{V}$, then so is $X \coprod Y$.

It immediately follows that any such $\mathcal{V}$ contains $\mathbf{Sm}_k$. In this thesis we will be mainly interested in the case where $\mathcal{V}$ is the admissible subcategory $\mathbf{Sm}_k$.

Notations 2.3. For $Z \in \mathbf{Sm}_k$ we denote by $\dim_k(Z, z)$ the dimension over k of the connected component of Z containing $z \in Z$. Let $d \in \mathbb{Z}$ be an integer. A morphism $f : Y \to X$ in $\mathbf{Sm}_k$ has **relative dimension d** if we have $\dim_k(Y, y) - \dim_k(X, f(y)) = d$ for each $y \in Y$. We also say that f has **relative codimension $-d$**.

Definition 2.4. *Let $f : X \to Z$ and $g : Y \to Z$ be two morphisms in an admissible subcategory $\mathcal{V}$ of $\mathbf{Sch}_k$. We say that f and g are* ***transverse*** *in $\mathcal{V}$ if*

(i) $\mathrm{Tor}_q^{\mathcal{O}_Z}(\mathcal{O}_Y, \mathcal{O}_X) = 0$ *holds for all $q > 0$ and*

(ii) the fibre product $X \times_Z Y$ is in $\mathcal{V}$.

If $\mathcal{V} = \mathbf{Sm}_k$, then we just say that f and g are transverse.

For what follows $\boldsymbol{R}^*$ denotes the category of commutative, graded rings with unit. We recall that a functor $A^* : (\mathbf{Sm}_k)^{\mathrm{op}} \to \boldsymbol{R}^*$ is **additive** if $A^*(\emptyset) = 0$ and for any pair $(X, Y) \in (\mathbf{Sm}_k)^2$ the canonical ring map $A^*(X \coprod Y) \to A^*(X) \times A^*(Y)$ is an isomorphism.

Definition 2.5. *Let $\mathcal{V}$ be an admissible subcategory of $\mathbf{Sch}_k$. An* ***oriented cohomology theory*** *on $\mathcal{V}$ is given by*

(D1) An additive functor $A^* : \mathcal{V}^{\mathrm{op}} \to \boldsymbol{R}^*$.

(D2) For each projective morphism $f : Y \to X$ in $\mathcal{V}$ of relative codimension d, a homomorphism

$$f_* : A^*(Y) \to A^{*+d}(X)$$

of graded $A^*(X)$-modules. Furthermore, we observe that the ring homomorphism $f^* : A^*(X) \to A^*(Y)$ gives $A^*(Y)$ the structure of an $A^*(X)$-module.

These satisfy the following conditions.

(A1) One has $(\mathrm{Id}_X)_* = \mathrm{Id}_{A^*(X)}$ for any $X \in \mathcal{V}$. Moreover, given projective morphisms $f : Y \to X$ and $g : Z \to Y$ in $\mathcal{V}$ of relative codimensions d and e, respectively, one has

$$(f \circ g)_* = f_* \circ g_* : A^*(Z) \to A^{*+d+e}(X).$$

(A2) Let $f : X \to Z$ and $g : Y \to Z$ be transverse morphisms in $\mathcal{V}$, giving the cartesian square

$$\begin{array}{ccc} W & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z \end{array}$$

in which we assume f to be projective of relative dimension d (thus so is f'). This implies $g^* \circ f_* = f'_* \circ g'^*$.

(PB) Let $E \to X$ be a rank n vector bundle over some $X \in \mathcal{V}$ and $O(1) \to \mathbb{P}(E)$ the canonical quotient line bundle with zero section $s : \mathbb{P}(E) \to O(1)$. Let $1 \in A^0(\mathbb{P}(E))$ denote the multiplicative unit element. Define $\psi \in A^1(\mathbb{P}(E))$ by

$$\psi := s^*(s_*(1)).$$

Then $A^*(\mathbb{P}(E))$ is a free $A^*(X)$-module with basis

$$(1, \psi, ..., \psi^{n-1}).$$

(EH) Let $E \to X$ be a vector bundle over some $X \in \mathcal{V}$ and let $p : V \to X$ be an E-torsor. Then $p^* : A^*(X) \to A^*(V)$ is an isomorphism.

A morphism of oriented cohomology theories on $\mathcal{V}$ is a natural transformation of functors $\mathcal{V}^{\mathrm{op}} \to \boldsymbol{R}^*$ which commutes with the maps f_*.

Remark 2.6. The morphisms of the form f^* are called **pullbacks** and the morphisms of the form f_* are called **pushforwards**. Furthermore, the axiom (PB) will be referred to as the **projective bundle formula** and the axiom (EH) as the **extended homotopy property**.

Given any oriented cohomology theory A^*, one may use Grothendieck's method (cf. [18]) to define Chern classes $c_i(E) \in A^i(X)$ of a vector bundle $E \to X$ of rank n over X which will be very important in the study of oriented cohomology theories. Axiom (PB) from Definition 2.5 implies that there exist unique elements $c_i(E) \in A^*(X)$ for $0 \le i \le n$ such that $c_0(E) = 1$ and

$$\sum_{i=0}^{n} (-1)^i c_i(E) \psi^{n-i} = 0.$$

These Chern classes are characterised by the following properties where the properties can be checked using the axioms listed above (cf. [37]).

(i) For any line bundle L over $X \in \mathbf{Sm}_k$, $c_1(L)$ equals $s^* s_*(1) \in A^1(X)$, where the morphism $s : X \to L$ denotes the zero section.

(ii) For any morphism $f : Y \to X$ in $\mathbf{Sm}_k$ and any vector bundle $E \to X$, one has

$$c_i(f^*E) = f^*(c_i(E))$$

for each $i \ge 0$.

(iii) If

$$0 \to E' \to E \to E'' \to 0$$

is a short exact sequence of vector bundles over $X \in \mathbf{Sm}_k$, then one has

$$c_n(E) = \sum_{i=0}^{n} c_i(E') c_{n-i}(E'')$$

for each integer $n \ge 0$. This formula is referred to as **Whitney sum formula**.

Following Quillen [48], the main difference with Grothendieck's axioms [18] is that in general it is not true that

$$c_1(L \otimes M) = c_1(L) + c_1(M)$$

holds for line bundles L and M over the same base scheme. This leads to the theory of formal group laws.

Definition 2.7. *A **commutative formal group law of rank one** with coefficients in R is a pair (R, F_R) consisting of a commutative ring R and a formal power series $F_R(u, v) = \sum a_{i,j} u^i v^j \in R[[u, v]]$ satisfying the following conditions.*

(i) $F(u, 0) = F(0, u) = u \in R[[u]]$.

(ii) $F(u, v) = F(v, u) \in R[[u, v]]$.

(iii) $F(u, F(v, w)) = F(F(u, v), w) \in R[[u, v, w]]$.

Construction 2.8. Now, we will analyse the construction of the so-called **Lazard Ring**. Therefore, we set $\widetilde{\mathbb{L}} := \mathbb{Z}[\{A_{i,j} | (i, j) \in \mathbb{N}^2\}]$ and $\widetilde{F}(u, v) = \sum_{i,j} A_{i,j} u^i v^j \in \widetilde{\mathbb{L}}[[u, v]]$. We define the Lazard Ring $\mathbb{L}$ to be the quotient of $\widetilde{\mathbb{L}}$ by the relations obtained by imposing the conditions of a commutative formal group law to $\widetilde{F}$. Then we denote by

$$F_{\mathbb{L}} = \sum_{i,j} a_{i,j} u^i v^j \in \mathbb{L}[[u, v]]$$

the image of $\widetilde{F}$ under the projection map $p : \widetilde{\mathbb{L}} \to \mathbb{L}$ where we define $a_{i,j}$ to be the image $p(A_{i,j})$. We would like the Lazard Ring to be a commutative graded ring and therefore, we assign the degree $1 - i - j$ to the corresponding coefficient $a_{i,j}$. We denote this commutative graded ring by $\mathbb{L}^*$. We could have also assigned the degree $i + j - 1$ to the corresponding coefficient $a_{i,j}$. The resulting commutative graded ring will be denoted by $\mathbb{L}_*$.

Lemma 2.9. *[37, Lemma 1.1.3] Let A^* be an oriented cohomology theory on $\mathbf{Sm}_k$. Then there is a unique power series*

$$F_A(u, v) = \sum_{i,j} a_{i,j} u^i v^j \in A^*(k)[[u, v]]$$

with $a_{i,j} \in A^{1-i-j}(k)$ such that for any $X \in \mathbf{Sm}_k$ and any pair of line bundles L and M on X, we have

$$F_A(c_1(L), c_1(M)) = c_1(L \otimes M).$$

Furthermore, the pair $(A^*(k), F_A)$ is a commutative formal group law of rank one.

Lazard proved in [36] that a universal commutative formal group law of rank one exists and also showed that it is a polynomial ring with integer coefficients in countably many variables. The above construction leads to the following proposition.

Proposition 2.10. *The pair $(\mathbb{L}, F_{\mathbb{L}})$ is the universal commutative formal group law of rank one. Furthermore, for every formal group law of rank one (F, A), there exists a unique ring homomorphism $\Phi_F : \mathbb{L} \to A$ such that $\Phi(F_{\mathbb{L}}) = F$ holds.*

Example 2.11. Fulton shows in [13] that the Chow ring $\mathrm{CH}^*(X)$ is an example of an oriented cohomology theory on $\mathbf{Sm}_k$. Additionally, he shows the additivity of the Chern classes which leads to the **additive** formal group law $F_{\mathrm{CH}}(u, v) = u + v$ on $\mathbb{Z} = \mathrm{CH}^*(k)$ by Lemma 2.9.

Remark 2.12. In this thesis we will be mainly concerned with Chow rings and algebraic cobordism, their equivariant theories and the comparison between them. Therefore, the preceding example is essential for this thesis and the underlying theory.

2.2 Algebraic cobordism

In this section, we describe the construction and the main properties of algebraic cobordism following Levine and Morel (cf. [37]). The main goal of this section is to give a reminder of the most important characteristics of algebraic cobordism for the convenience of the reader.

Definition 2.13. *Let X be a k-scheme.*

(i) A ***cobordism cycle*** *over X is a family $(f : Y \to X, L_1, ..., L_r)$ consisting of*

   (i') a projective morphism $f : Y \to X$ where $Y \in \mathbf{Sm}_k$ is integral and

   (ii') a finite sequence $(L_1, ..., L_r)$ of r line bundles over Y (this might be empty).

   The ***dimension*** *of this cobordism cycle is $\dim_k(Y) - r \in \mathbb{Z}$.*

(ii) An ***isomorphism Φ of cobordism cycles***

$$(Y \to X, L_1, ..., L_r) \cong (Y' \to X, L'_1, ..., L'_{r'})$$

   is a triple

$$\Phi = (\phi : Y \to Y', \sigma, (\psi_1, ..., \psi_r))$$

   consisting of

   (i") an isomorphism $\phi : Y \to Y'$ of X-schemes,

   (ii") a bijection $\sigma : \{1, ..., r\} \cong \{1, ..., r'\}$ and

   (iii") an isomorphism of line bundles over Y, i.e. $\psi_i : L_i \cong \phi^*(L'_{\sigma(i)})$ for each $i \in \{1, ..., r\}$.

(iii) Let $\mathcal{Z}(X)$ be the free abelian group generated by the isomorphism classes of cobordism cycles over X. The dimension of the cobordism cycles makes $\mathcal{Z}(X)$ into a graded abelian group $\mathcal{Z}_*(X)$. The image of a cobordism cycle $(f : Y \to X, L_1, ..., L_r)$ in this group is denoted by $[f : Y \to X, L_1, ..., L_r]$.

Remark 2.14. Given $Y = \coprod_j Y_j$ in $\mathbf{Sm}_k$ with line bundles $L_1, ..., L_r$ on Y and a projective morphism $f : Y \to X$ in $\mathbf{Sch}_k$, we write $[f : Y \to X, L_1, ..., L_r]$ for the sum $\sum_j [f_j : Y_j \to X, L_{j1}, ..., L_{jr}]$ in $\mathcal{Z}_*(X)$, where f_j and L_{ji} are the restrictions of f and L_i to Y_j.

Now, we want to describe pullbacks, pushforwards and first Chern class homomorphisms for the graded abelian group $\mathcal{Z}_*(X)$.

Definition 2.15. *Let X and X' be smooth k-schemes.*

(i) Let $g : X \to X'$ be a projective morphism. The ***pushforward along g*** *is defined as the map*

$$\begin{aligned} g_* : \mathcal{Z}_*(X) &\to \mathcal{Z}_*(X') \\ [f : Y \to X, L_1, ..., L_r] &\mapsto [g \circ f : Y \to X', L_1, ..., L_r] \end{aligned}$$

   of graded abelian groups.

*(ii) Let $g : X \to X'$ be a smooth equi-dimensional morphism of relative dimension d. The **pullback along g** is defined as the map*

$$g^* : \mathcal{Z}_*(X') \to \mathcal{Z}_{*+d}(X)$$
$$[f : Y \to X', L_1, ..., L_r] \mapsto [p_2 : (Y \times_{X'} X) \to X, p_1^*(L_1), ..., p_1^*(L_r)]$$

of graded abelian groups where p_i denote the corresponding projections.

*(iii) Let L be a line bundle on X. Then the **first Chern class homomorphism of L** is defined as the map*

$$\widetilde{c}_1(L) : \mathcal{Z}_*(X) \to \mathcal{Z}_{*-1}(X)$$
$$[f : Y \to X, L_1, ..., L_r] \mapsto [f : Y \to X, L_1, ..., L_r, f^*(L)]$$

of graded abelian groups.

Lastly, we define the external product on the graded abelian group $\mathcal{Z}_*(X)$ before we are able to begin the construction of algebraic cobordism explicitly.

Definition 2.16. *Let $\alpha = [f : X' \to X, L_1, ..., L_r]$ and $\beta = [g : Y' \to Y, M_1, ..., M_s]$ denote two cobordism cycles. Then the **external product** is defined as*

$$\times : \mathcal{Z}_*(X) \times \mathcal{Z}_*(Y) \to \mathcal{Z}_*(X \times_k Y)$$
$$(\alpha, \beta) \mapsto [f \times g : X' \times Y' \to X \times Y, p_1^*(L_1), ..., p_1^*(L_r), p_2^*(M_1), ..., p_2^*(M_s)].$$

In addition, $\times$ is associative and commutative.

Remark 2.17. This definition implies that $\mathcal{Z}_*(k)$ has the structure of a commutative graded ring with unit $1 := [\mathrm{Id}_k] \in \mathcal{Z}_0(k)$. Therefore, for each $X \in \mathbf{Sm}_k$, the group $\mathcal{Z}_*(X)$ has the structure of a graded $\mathcal{Z}_*(k)$-module.

Now, we want to construct algebraic cobordism $\Omega_*(X)$ by successively imposing three relations on $\mathcal{Z}_*(X)$. As a graded abelian group, we build the appropriate quotient by the corresponding subgroups. In fact, these relations will not affect any of the above mentioned structures of the functor $\mathcal{Z}_*$. This can be verified using [37, Section 2.1.5] where this procedure is described in detail.

Construction 2.18. The construction begins by imposing that every composition of n Chern class homomorphisms vanishes if n exceeds the dimension of the base scheme. Precisely, this means that algebraic cobordism satisfies the following axiom

(Dim) For any $X \in \mathbf{Sm}_k$ and any family $(L_1, ..., L_n)$ of line bundles on X with $n > \dim_k(X)$, one has

$$\widetilde{c}_1(L_1) \circ \cdots \circ \widetilde{c}_1(L_n)(1_X) = 0 \in \Omega_*(X),$$

where $1_X = [\mathrm{Id}_X : X \to X]$ which is also known as the **fundamental class** of X.

For any irreducible $X \in \mathbf{Sm}_k$ this is imposed by denoting $\mathcal{R}_*^{\mathrm{Dim}}(X) \subseteq \mathcal{Z}_*(X)$ the subset consisting of all elements of the form

$$[Y \to X, L_1, ..., L_r],$$

where $\dim_k(Y) < r$ is satisfied. We denote by $\underline{\mathcal{Z}}_*(X)$ the corresponding quotient $\mathcal{Z}_*(X)/\mathcal{R}_*^{\mathrm{Dim}}(X)$.

After having imposed axiom (Dim), we want to impose the second axiom which gives a relation between the first Chern class homomorphism associated to a line bundle and the fundamental class of the zero-subscheme of its sections. Expressed differently, this means

(Sect) For any $X \in \mathbf{Sm}_k$, any line bundle L on X and any section s which is transverse to the zero section of L, one has

$$\widetilde{c}_1(L)(1_X) = i_*(1_Z),$$

where $i : Z \to X$ is the closed immersion of the zero-subscheme of s.

Now, one repeats the procedure of imposing relations which, for each irreducible $X \in \mathbf{Sm}_k$, leads to the subset $\mathcal{R}_*^{\mathrm{Sect}}(X) \subseteq \underline{\mathcal{Z}}_*(X)$ consisting of all elements of the form

$$\widetilde{c}_1(L)(1_X) - [Z \to X],$$

where L is a line bundle over X, $s : X \to L$ a section transverse to the zero section and $Z \to X$ the closed zero-subscheme of s which is smooth by the assumption that s is transverse to the zero section. We denote by $\underline{\Omega}_*(X)$ the corresponding quotient $\underline{\mathcal{Z}}_*(X)/\mathcal{R}_*^{\mathrm{Sect}}(X)$ which we refer to as **algebraic pre-cobordism**.

Lastly, we want to obain algebraic cobordism $\Omega_*(X)$ as a quotient of $\mathbb{L}_* \otimes \underline{\Omega}_*(X)$ which implies the existence of a homomorphism $\Phi : \mathbb{L}_* \to \Omega_*(k)$, being the given composite $\mathbb{L}_* \to \mathbb{L}_* \otimes \underline{\Omega}_*(k) \to \Omega_*(k)$. This will be in fact the ring homomorphism from Proposition 2.10 for which it can be checked that it is a graded ring homomorphism. After we will have defined $\Omega_*(X)$, we want it to satisfy the axiom (FGL) given by

(FGL) Let $\Phi : \mathbb{L}_* \to \Omega_*(k)$ be the ring homomorphism giving the $\mathbb{L}_*$-structure and let $F_\Omega \in \Omega_*(k)[[u, v]]$ be the image of the universal formal group law $F_\mathbb{L} \in \mathbb{L}_*[[u, v]]$ under Φ. Then for any $X \in \mathbf{Sm}_k$ and any pair (L, M) of line bundles on X, one has

$$F_\Omega(\widetilde{c}_1(L), \widetilde{c}_1(M))(1_X) = \widetilde{c}_1(L \otimes M)(1_X) \in \Omega_*(X).$$

Here we remark that the order of imposing these relations matters as the left-hand side of the axiom (FGL) only makes sense if the axiom (Dim) has already been imposed.

The axiom (FGL) is imposed by taking for any irreducible $X \in \mathbf{Sm}_k$ the subset $\mathcal{R}_*^{\mathrm{FGL}}(X) \subseteq \mathbb{L}_* \otimes \underline{\Omega}_*(X)$ consisting of elements of the form

$$F_\mathbb{L}(\widetilde{c}_1(L), \widetilde{c}_1(M))(1_X) - \widetilde{c}_1(L \otimes M)(1_X),$$

where L and M are line bundles over X and $F_\mathbb{L}$ is the universal formal group law. Unfortunately, this extension of scalars is necessary as we need a ring homomorphism from $\mathbb{L}_*$ to the corresponding coefficient ring and since we want to keep the universal property which we will point out later. Furthermore, we have to consider the subset $\mathbb{L}_*\mathcal{R}_*^{\mathrm{FGL}}(X) \subseteq \mathbb{L}_* \otimes \underline{\Omega}_*(X)$ of elements of the form $a \otimes \rho$ for $a \in \mathbb{L}_*$ and $\rho \in \mathcal{R}_*^{\mathrm{FGL}}(X)$. This is necessary since the procedure of taking the quotient of these functors only works in this case if the corresponding $\mathbb{L}_*$-module structure is given. Finally, building the quotient yields **algebraic cobordism**

$$\Omega_*(X) := \mathbb{L}_* \otimes \underline{\Omega}_*(X)/\mathbb{L}_*\mathcal{R}_*^{\mathrm{FGL}}(X).$$

Remark 2.19. For the formal group law $(\mathbb{L}, F_\Omega)$ there exists a unique power series $\chi(u) \in \mathbb{L}[[u]]$ with leading term $-u$ which satisfies the equality $F_\Omega(u, \chi(u)) = 0$ (cf. [37]). We introduce the following notations:

$$\begin{aligned} u +_{F_\Omega} v &:= F_\Omega(u, v) \in \mathbb{L}[[u, v]], \\ [-1]_{F_\Omega} u &:= \chi(u) \in \mathbb{L}[[u]], \\ u -_{F_\Omega} v &:= F_\Omega(u, \chi(v)) \in \mathbb{L}[[u, v]]. \end{aligned}$$

We will use the same notations when we will be considering equivariant algebraic cobordism in the sequel.

Remark 2.20. In fact, Ω_* is an oriented Borel-Moore homology theory on $\mathbf{Sm}_k$ and using [37, Proposition 5.2.1] it is possible to construct a functor $\Omega^* : \mathbf{Sm}_k{}^{\mathrm{op}} \to \boldsymbol{R}^*$ which is an oriented cohomology theory. Furthermore, for an equi-dimensional $X \in \mathbf{Sm}_k$ of dimension d, one has $\Omega^i(X) \cong \Omega_{d-i}(X)$ for all $i \in \mathbb{Z}$.

The following results describe the main properties of algebraic cobordism and its relevance.

Proposition 2.21. *[37, Theorem 1.2.6] The functor $X \mapsto \Omega^*(X)$ is the universal oriented cohomology theory on $\mathbf{Sm}_k$. Thus, given an oriented cohomology theory A^* on $\mathbf{Sm}_k$, there is a unique morphism*

$$\theta : \Omega^* \to A^*$$

of oriented cohomlogy theories.

Proposition 2.22. *[37, Theorem 1.2.7] The canonical homomorphism $\Psi : \mathbb{L}^* \to \Omega^*(k)$ is an isomorphism.*

Proposition 2.23. *[37, Theorem 3.2.7] Let $i : Z \to X$ be a closed immersion in $\mathbf{Sch}_k$ and $j : U \to X$ the open immersion of the complement of Z. Then the localisation sequence*

$$\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j^*} \Omega_*(U) \to 0$$

is exact.

Now, we want to give another result which relates Chow rings with algebraic cobordism. This gives rise to the motivation of generalising known results about Chow rings to algebraic cobordism as these two theories behave similarly in several situations.

Proposition 2.24. *[37, Theorem 1.2.19] The canonical morphism $\Omega^* \to \mathrm{CH}^*$ induces an isomorphism $\Omega^* \otimes_{\mathbb{L}^*} \mathbb{Z} \xrightarrow{\cong} \mathrm{CH}^*$.*

2.3 Some computations for toric varieties

In this section, we want to illustrate and motivate the study of wider classes of varieties in order to obtain new computations of algebraic cobordism. First, we have a look at toric varieties for which Chow groups and algebraic cobordism are known (cf. [14, 34]). Nevertheless, this class of varieties already gives us a lot of information about the behaviour of Chow rings and algebraic cobordism. Therefore, we have a look at some examples. The theory of toric varieties is assumed to be known throughout the whole thesis. For further details, we refer to [14] where the theory is explained in detail. Nevertheless, we recap some essential results which will be relevant for our examples.

Proposition 2.25. *[14, Proposition, p. 29] A toric variety X is smooth if and only if the corresponding fan Δ consists of cones which are each generated by a part of a basis for the lattice N.*

Definition 2.26. *A cone is called **simplicial** if it is generated by linearly independent generators.*

Remark 2.27. The most known results of computations of Chow rings concern the theory of smooth toric varieties. In fact, in the simplicial case one can obtain similar results with rational coefficients ([14]).

Now, we want to establish the key assumption which is used by Fulton (cf. [14]) in order to describe the Chow ring of toric varieties.

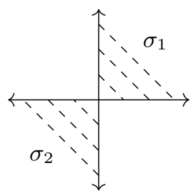
Definition 2.28. *For any ordering $\sigma_1, ..., \sigma_m$ of the top-dimensional cones in a fan corresponding to a complete, smooth (or simplicial) toric variety, we define a subsequence of subcones $\tau_i \subseteq \sigma_i$, $1 \le i \le m$, by letting τ_i be the intersection of σ_i with all those σ_j that come after σ_i, i.e. $j > i$, and that meet σ_i in a cone of dimension $n-1$. In particular, $\tau_1 = \{0\}$ and $\tau_m = \sigma_m$. This sequence has to satisfy the following condition.*

$$(*) \quad \textit{If } \tau_i \textit{ is contained in } \sigma_j, \textit{ then } i \le j.$$

Lemma 2.29. *[14, Lemma, p. 101] If X is projective and the corresponding fan Δ simplicial, then the top-dimensional cones of Δ can be ordered so that condition $(*)$ holds.*

Remark 2.30. Fulton computes the Chow ring for smooth (or simplicial) projective toric varieties X in [14, Proposition, p. 106]. It is remarked that this computation can be generalised to complete smooth (or simplicial) toric varieties. Our motivating example is not complete and therefore, we cannot use Fulton's result. Thus, we will compute the Chow ring using localisation sequences for Chow rings.

Example 2.31. Let Δ be the following two-dimensional fan in $N = \mathbb{Z}^2$.



The corresponding toric variety is $X_\Delta = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \{(0,\infty),(\infty,0)\}$. Fulton describes the Chow ring of toric varieties under certain conditions. He considers complete toric varieties that are non-singular or at least simplicial and furthermore satisfy the ordering condition $(*)$. Our example is not complete and therefore, condition $(*)$ is not defined. Thus, we will compute the Chow ring of this variety using localisation sequences for Chow rings. This leads to the exact sequences

$$\mathrm{CH}^0(\{(0,\infty),(\infty,0)\}) \xrightarrow{i_*} \mathrm{CH}^2(\mathbb{P}^1 \times \mathbb{P}^1) \xrightarrow{j^*} \mathrm{CH}^2(X_\Delta) \to 0,$$

$$\mathrm{CH}^{-1}(\{(0,\infty),(\infty,0)\}) \xrightarrow{i_*} \mathrm{CH}^1(\mathbb{P}^1\times\mathbb{P}^1) \xrightarrow{j^*} \mathrm{CH}^1(X_\Delta) \to 0,$$
$$\mathrm{CH}^{-2}(\{(0,\infty),(\infty,0)\}) \xrightarrow{i_*} \mathrm{CH}^0(\mathbb{P}^1\times\mathbb{P}^1) \xrightarrow{j^*} \mathrm{CH}^0(X_\Delta) \to 0.$$

The reader recognises immediately that the only difference between $\mathrm{CH}^*(\mathbb{P}^1\times\mathbb{P}^1)$ and $\mathrm{CH}^*(X_\Delta)$ is described by the first sequence. We know by the Künneth formula that $\mathrm{CH}^*(\mathbb{P}^1\times\mathbb{P}^1) \cong \mathbb{Z}[c,d]/(c^2,d^2)$ holds, where c and d are the classes of the corresponding one-dimensional subspaces. Furthermore, it is known that $\mathrm{CH}^0(\{(0,\infty),(\infty,0)\}) \cong \mathbb{Z}^2$ holds. The remaining question is to which element the class of a point is being mapped under the pushforward i_*. Because of the equality $\mathrm{CH}^2(\mathbb{P}^1\times\mathbb{P}^1) = \mathrm{CH}_0(\mathbb{P}^1\times\mathbb{P}^1)$, the only candidate for the image of the class of the point under i_* is $c\cdot d$ by degree reasons. This can be seen, as the image has to be a class of a zero-dimensional subspace which corresponds to the element $c\cdot d$ in $\mathrm{CH}^*(\mathbb{P}^1\times\mathbb{P}^1)\cong\mathbb{Z}[c,d]/(c^2,d^2)$. Finally, we see by the exactness of the first sequence that cd has to vanish inside $\mathrm{CH}^*(X_\Delta)$ which leads to the result $\mathrm{CH}^*(X_\Delta) = \mathbb{Z}[c,d]/(c^2,cd,d^2)$.

Next, we want to compare this result with the algebraic cobordism of the same toric variety. Therefore, we state the main result of [34]. We remark that the problematic localisation sequence for equivariant algebraic cobordism (cf. discussion before Proposition 2.43) was used in order to obtain this result and it is not clear whether it can be proved without using the equivariant localisation sequence. We will nevertheless assume for now that the main result of [34] holds. This assumption will only affect the remaining part of this section and none of the main results of this thesis.

First, we have to introduce the main notions. For a smooth toric variety associated to a fan Δ in the lattice $M_\mathbb{R}$, we denote by Δ_r the set of r-dimensional cones in Δ. For any $\rho\in\Delta_1$, let v_ρ denote the generator of the monoid $\rho\cap M$. Furthermore, if S is a subset of Δ_1 which is not contained in any maximal cone of Δ, then we denote the union of such subsets of Δ_1 by Δ_1^0. Let $\mathbb{L}^*[t_\rho] = \mathbb{L}^*[t_\rho, \rho\in\Delta_1]$ be the graded polynomial ring over $\mathbb{L}^*$ with each t_ρ homogeneous of degree one. Let $\overline{I}_\Delta$ denote the graded ideal generated by the set of monomials $\{\prod_{\rho\in S} t_\rho | S\in\Delta_1^0\}\bigcup\{t_\rho^{n+1}|\rho\in\Delta_1\}$.

Proposition 2.32. *[34, Theorem 8.2] Let X_Δ be a smooth toric variety associated to a fan Δ in the lattice $M_\mathbb{R}$. Then there is an $\mathbb{L}^*$-algebra isomorphism*

$$\overline{\Psi}_X : \frac{\mathbb{L}^*[t_\rho,\rho\in\Delta_1]}{\left(\overline{I}_\Delta, \sum_{\rho\in\Delta_1}[<\chi, v_\rho>]_F t_\rho\right)} \to \Omega^*(X_\Delta),$$

where χ runs over all elements in the character group $M^\vee$.

Remark 2.33. We want to remark here that this result holds for all smooth toric varieties. Compared to the result of Fulton which is discussed above, we do not need the assumption of completeness or projectivity. Since algebraic cobordism is the universal oriented cohomology theory, this implies in particular the more general result for Chow rings.

Now, we are able to compute the algebraic cobordism of the toric variety X_Δ which has already been discussed above.

Example 2.34. Firstly, we mention that $v_{\rho_i} = p_i$ for $p_1 = (1,0)$, $p_2 = (0,1)$, $p_3 = (-1,0)$ and $p_4 = (0,-1)$. The ideal $\overline{I}_\Delta$ is generated by

$$\{t_{\rho_1}t_{\rho_3}, t_{\rho_1}t_{\rho_4}, t_{\rho_2}t_{\rho_3}, t_{\rho_2}t_{\rho_4}\}\bigcup\{t_{\rho_1}^3, t_{\rho_2}^3, t_{\rho_3}^3, t_{\rho_4}^3\}$$

and the sums in the quotient are given by

$$\begin{aligned}&[<(1,0),(1,0)>]_F t_{\rho_1} + [<(1,0),(0,1)>]_F t_{\rho_2}\\&+[<(1,0),(-1,0)>]_F t_{\rho_3} + [<(1,0),(0,-1)>]_F t_{\rho_4}\\&= t_{\rho_1} - u_1 t_{\rho_3}\end{aligned}$$

for an invertible element u_1 of degree zero. Similarly, we obtain $t_{\rho_2} - u_2 t_{\rho_4}$. Therefore, we conclude that $t_{\rho_1} = t_{\rho_3}$ and $t_{\rho_2} = t_{\rho_4}$ hold in the quotient. This gives rise to

$$\Omega^*(X_\Delta) \cong \frac{\mathbb{L}^*[t_{\rho_1}, t_{\rho_2}, t_{\rho_3}, t_{\rho_4}]}{\left(\overline{I}_\Delta, t_{\rho_1} - t_{\rho_3}, t_{\rho_2} - t_{\rho_4}\right)} \cong \frac{\mathbb{L}^*[t_{\rho_1}, t_{\rho_2}]}{\overline{I}_\Delta}.$$

Now, we see that $\overline{I}_\Delta$ is generated by $\{t_{\rho_1}^2, t_{\rho_1}t_{\rho_2}, t_{\rho_2}^2\}$ which leads to the result

$$\Omega^*(X) \cong \frac{\mathbb{L}^*[t_{\rho_1}, t_{\rho_2}]}{\left(t_{\rho_1}^2, t_{\rho_1}t_{\rho_2}, t_{\rho_2}^2\right)}.$$

Remark 2.35. The preceding examples illustrate the relation between Chow rings and algebraic cobordism and confirm Proposition 2.24.

2.4 Equivariant algebraic cobordism

The current method of computing algebraic cobordism is introducing the notion of equivariant algebraic cobordism and then deducing the results for algebraic cobordism. Therefore, equivariant Chow groups will also be important for our studies as well as their comparison with equivariant algebraic cobordism. In order to be able to define equivariant algebraic cobordism, we have to define the niveau filtration on algebraic cobordism which will lead to a refined version of the localisation sequence. The latter will be crucial in the further applications of equivariant algebraic cobordism.

Following the conventions in [32], let X be k-scheme of dimension d. For $j \in \mathbb{Z}$, let Z_j be the set of all closed subschemes $Z \subseteq X$ such that $\dim_k(Z) \leq j$ holds where the empty scheme is assumed to have infinite negative dimension. Then Z_j is ordered by inclusion and we define

$$\Omega_i(Z_j) = \varinjlim_{Z \in Z_j} \Omega_i(Z) \text{ and } \Omega_*(Z_j) = \bigoplus_{i \geq 0} \Omega_i(Z_j).$$

We see immediately that the latter is a graded $\mathbb{L}_*$-module and that there is a graded $\mathbb{L}_*$-linear map $\Omega_*(Z_j) \to \Omega_*(X)$.

Definition 2.36. *We define $F_j\Omega_*(X)$ to be the image of the natural $\mathbb{L}_*$-linear map $\Omega_*(Z_j) \to \Omega_*(X)$. Expressed differently, $F_j\Omega_*(X)$ is the image of all $\Omega_*(W) \to \Omega_*(X)$, where $W \to X$ is a projective map such that* $\dim(\mathrm{Im}(W)) \leq j$. *This leads to the canonical* ***niveau filtration***

$$0 = F_{-1}\Omega_*(X) \subseteq F_0\Omega_*(X) \subseteq \cdots \subseteq F_{d-1}\Omega_*(X) \subseteq F_d\Omega_*(X) = \Omega_*(X).$$

Proposition 2.37. *[32, Theorem 3.5] (Refined localisation sequence) Let X be a scheme over k and let Z be a closed subscheme of X with complement U. Then for every $j \in \mathbb{Z}$, there is an exact sequence*

$$\frac{\Omega_*(Z)}{F_j\Omega_*(Z)} \to \frac{\Omega_*(X)}{F_j\Omega_*(X)} \to \frac{\Omega_*(U)}{F_j\Omega_*(U)} \to 0.$$

For this section, let G be a linear algebraic group of dimension g over k. Furthermore, all representations of G will be finite dimensional. The definition of equivariant algebraic cobordism considers a certain kind of mixed spaces which may not be a scheme even if the original space is a scheme. From [32, Lemma 4.1] we know that this problem does not occur in any of the situations described in this work.

Now, we consider for any integer $j \geq 0$ a corresponding pair (V_j, U_j) where V_j is an l_j-dimensional representation of G and U_j is a G-stable open subset of V_j such that the codimension of the complement $(V_j \setminus U_j)$ in V_j is at least j. Furthermore, G acts freely on U_j such that the quotient U_j/G is a quasi-projective scheme. Such a pair will be called a **good pair** for the G-action corresponding to j. It is well known that such a good pair always exists (cf. [12, Lemma 9]).

Let $X \times^G U_j$ denote the mixed quotient of the product $X \times U_j$ by the free diagonal action of G which exists as a scheme because the G-action on X is linear. For a k-scheme X of dimension d with a G-action and an integer $j \geq 0$, let (V_j, U_j) be an l_j-dimensional good pair corresponding to j. For all $i \in \mathbb{Z}$, we set

$$\Omega_i^G(X)_j = \frac{\Omega_{i+l_j-g}\left(X \times^G U_j\right)}{F_{d+l_j-g-j}\Omega_{i+l_j-g}\left(X \times^G U_j\right)}. \tag{2.1}$$

Lemma 2.38. *([32, Lemma 4.2]) For a fixed $j \geq 0$, the group $\Omega_i^G(X)_j$ is independent of the choice of the good pair (V_j, U_j).*

Corollary 2.39. *([32, Lemma 4.3]) For $j' \geq j \geq 0$ there is a natural surjective map $\Omega_i^G(X)_{j'} \twoheadrightarrow \Omega_i^G(X)_j$.*

Definition 2.40. *Let X be a k-scheme of dimension d with a G-action. For any $i \in \mathbb{Z}$, we define the **equivariant algebraic cobordism** of X to be*

$$\Omega_i^G(X) := \varprojlim_j \Omega_i^G(X)_j.$$

Remark 2.41. One should note that the equivariant algebraic cobordism can be non-zero for any $i \in \mathbb{Z}$ unlike the ordinary algebraic cobordism Ω^*. Furthermore, we set

$$\Omega_*^G(X) := \bigoplus_{i \in \mathbb{Z}} \Omega_i^G(X).$$

If in addition X is an equi-dimensional k-scheme with G-action, we let $\Omega_G^i(X) = \Omega_{d-i}^G(X)$ and analogously $\Omega_G^*(X) := \bigoplus_{i \in \mathbb{Z}} \Omega_G^i(X)$. We denote the equivariant cobordism $\Omega_G^*(k)$ of the underlying base field by $S(G)$. This is also called the algebraic cobordism of the classifying space of G which is denoted by $\Omega^*(BG)$. Furthermore, if G is the trivial group, equivariant algebraic cobordism reduces to ordinary algebraic cobordism. Besides that, equivariant algebraic cobordism with rational coefficients is defined by the graded topological tensor product $\Omega^*_G(X)_\mathbb{Q} := \Omega^*_G(X)\widehat{\otimes}_\mathbb{Z}\mathbb{Q}$ which will be discussed in more detail in the sequel (cf. Section 3.5).

Secondly, for any $X \in G - \mathbf{Sch}_k$ and a projective morphism $f : Y \to X$ in $G - \mathbf{Sch}_k$ where Y is smooth of dimension d we obtain for any $j \geq 0$ and any l_j-dimensional good pair (V_j, U_j) an ordinary cobordism cycle $[Y \times^G U_j \to X \times^G U_j]$ of dimension $d + l_j - g$ by [32, Lemma 5.1]. Hence, this defines an element $\alpha_j \in \Omega^G_d(X)_j$. Since the image of $\alpha_{j'}$ is α_j for all $j' \geq j$ we obtain a unique element $\alpha \in \Omega^G_d(X)$ which we call the ***G*-equivariant fundamental class** of the cobordism cycle $[f : Y \to X]$.

Remark 2.42. For a closed subgroup $H \leq G$ of dimension h, any l_j-dimensional good pair (V_j, U_j) for the G-action is also a good pair for the induced H-action. With some further arguments this induces the natural restriction map

$$r^G_{H,X} : \Omega^G_*(X) \to \Omega^H_*(X)$$

for any $X \in G - \mathbf{Sch}_k$ via taking the inverse limit of the corresponding pullback maps of $X \times^H U_j \to X \times^G U_j$. Taking $H = \{1\}$ we obtain the forgetful map $r^G_X : \Omega^G_*(X) \to \Omega_*(X)$ from equivariant algebraic cobordism to ordinary algebraic cobordism.

In the following, we want to establish the main properties of equivariant algebraic cobordism which will be useful and necessary for our further purposes. Firstly, we remark that there is a gap in the proof of the localisation theorem given by Heller and Malagón-López (cf. [20, Theorem 20]) which is described in more detail in the work of Khan and Ravi (cf. [27, Example 12.18]). Nevertheless, we obtain a slightly weaker result. Later on, we will present a case where the localisation in equivariant cobordism still works under milder assumptions (cf. Corollary 3.48).

Proposition 2.43. *[32, Theorem 5.3] Let X be a G-scheme over k of dimension d and $f : U \to X$ a G-stable open subscheme. Then the restriction map $f^* : \Omega^G_*(X) \to \Omega^G_*(U)$ is surjective.*

Proposition 2.44. *[32, Theorem 5.2] The equivariant algebraic cobordism satisfies the following properties.*

(i) Functoriality: The assignment $X \mapsto \Omega^G_*(X)$ is covariant for projective maps and contravariant for smooth (and even l.c.i.) morphisms in $G - \mathbf{Sch}_k$. Moreover, for the diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

in $G - \mathbf{Sch}_k$ with f projective and g smooth, one has $g^* \circ f_* = f'_* \circ g'^*$.

(ii) Homotopy invariance: If $f : E \to X$ is a G-equivariant vector bundle over some $X \in G - \mathbf{Sch}_k$, then f^* is an isomorphism.

(iii) Chern classes: For any G-equivariant vector bundle $f : E \to X$ of rank r, there are equivariant Chern class operators $c^G_m(E) : \Omega^G_*(X) \to \Omega^G_{*-m}(X)$ for $0 \leq m \leq r$ with $c^G_0(E) = 1$. These Chern classes have the same functorial properties as in the non-equivariant case. Moreover, they satisfy the Whitney sum formula.

(iv) *Free action: If G acts freely on X with quotient Y, then* $\Omega_*^G(X) \xrightarrow{\cong} \Omega_*(Y)$.

(v) *Exterior Product: There is a natural product map*

$$\Omega_i^G(X) \otimes_{\mathbb{Z}} \Omega_{i'}^G(X') \to \Omega_{i+i'}^G(X \times X').$$

In particular, $\Omega_*^G(k)$ *is a graded algebra and* $\Omega_*^G(X)$ *is a graded* $\Omega_*^G(k)$*-module for every* $X \in G - \mathbf{Sch}_k$. *Furthermore, for X smooth, the pullback of the diagonal morphism* $X \to X \times X$ *turns* $\Omega_G^*(X)$ *into an* $S(G)$*-algebra.*

(vi) *Projection formula: For a projective map* $f : X' \to X$ *in* $G - \mathbf{Sm}_k$, *one has* $f_*(x' \cdot f^*(x)) = f_*(x') \cdot x$ *for any* $x \in \Omega_*(X)$ *and* $x' \in \Omega_*^G(X')$.

Remark 2.45. [31, Section 2.5] If X is smooth, we identify the commutative subalgebra of $\mathrm{End}_{\mathbb{L}}(\Omega_G^*(X))$ generated by the Chern classes of vector bundles with a subalgebra of the equivariant cobordism ring $\Omega_G^*(X)$ via $c_i^G(E) \mapsto c_i^G(E)\,([\mathrm{Id} : X \to X])$. Therefore, we will also denote this image by $c_i^G(E)$. Since we pass freely between vector bundles E and their corresponding locally free coherent sheaves we will also write $c_1^G(\mathcal{E})$ for a locally free coherent sheaf $\mathcal{E}$.

To finish this section, we present one of the main results concerning actual computations of equivariant algebraic cobordism. Since taking the quotient by the niveau filtration is very hard to be computed in general, one can make use of the following result by Krishna [32].

Proposition 2.46. *[32, Theorem 6.1] Let* $\{(V_j, U_j)\}_{j\geq 0}$ *be a sequence of* l_j*-dimensional good pairs such that*

(i) $V_{j+1} = V_j \oplus W_j$ *as representations of G with* $\dim(W_j) > 0$.

(ii) $U_j \oplus W_j \subseteq U_{j+1}$ *as G-stable open subsets.*

(iii) $\mathrm{codim}_{V_{j+1}}(V_{j+1} \setminus U_{j+1}) > \mathrm{codim}_{V_j}(V_j \setminus U_j)$.

Then for any scheme $X \in G - \mathbf{Sch}_k$ *of dimension d and any* $i \in \mathbb{Z}$, *one has*

$$\Omega_i^G(X) \xrightarrow{\cong} \varprojlim_j \Omega_{i+l_j-g}\left(X \times^G U_j\right).$$

Moreover, such a sequence of good pairs always exists.

2.5 Spherical and horospherical varieties

In this section, we want to summarise the main properties of spherical varieties for convenience of the reader which are mainly taken from [41, 42, 47].

For the remaining part of this chapter, let G be a connected reductive algebraic group over an algebraically closed field k of characteristic zero, $B \subseteq G$ a Borel subgroup, $T \subseteq B$ a maximal torus, W the Weyl group of G corresponding to the maximal torus of B and U the unipotent radical of B.

Definition 2.47. *Let X be a normal G-variety.*

(i) *We call a closed subgroup $H \subseteq G$ containing U* ***horospherical****. In this case, the homogeneous space G/H is said to be* ***horospherical****.*

(ii) *A **G/H-embedding** for a homogeneous space G/H is a couple (X, x) for $x \in X$ such that the orbit $G \cdot x$ is open and isomorphic to G/H.*

(iii) *We call X **horospherical** if it contains an open orbit isomorphic to a horospherical homogeneous space.*

(iv) *A homogeneous space G/H is said to be **spherical** if it contains an open orbit under the action of the Borel subgroup of G.*

(v) *We call X **spherical** if it contains an open orbit isomorphic to a spherical homogeneous space.*

Remark 2.48. A horospherical homogenous space G/H can be equivalently described as a torus bundle over a flag variety G/P with fibre P/H. In this case we have $P = N_G(H)$ by [42, Proposition 2.2]. Furthermore, using this description one also has $P = TH = BH$ for all Borel subgroups B of G contained in P and for all maximal tori T of B.

Remark 2.49. Two G/H-embeddings (X, x) and (X', x') are isomorphic if there exists a G-equivariant isomorphim between X and X' mapping x to x'.

Definition 2.50. *For a horospherical homogeneous space G/H, we call the dimension of the fibre P/H the **rank of G/H**. Furthermore, for a horospherical variety X, the **rank of X** is defined as the rank of its open G-orbit.*

Proposition 2.51. *[4, Thereom 14.12] The **Bruhat decomposition** of G is the decomposition*

$$G = \coprod_{w \in W} BwB$$

of G as a disjoint union of double cosets of B.

Remark 2.52. In fact, by [4, Remark 14.16] we can also write G as the disjoint union of double cosets

$$G = \coprod_{w \in W/W_I} BwP_I$$

where $I \subseteq S$ is a subset of the simple roots, W_I the subgroup of W generated by the simple reflections s_α for $\alpha \in I$ and P_I the parabolic subgroup given by $P_I = \coprod_{w \in W_I} BwB$.

Remark 2.53. There are several equivalent definitions of spherical varieties which will also be used here. Namely, the following conditions for a normal G-variety X are equivalent ([47, Theorem 2.1.2]).

(i) X contains an open B-orbit.

(ii) X has finitely many B-orbits.

(iii) X is spherical.

Example 2.54. We want to point out some easy examples of spherical and horospherical varieties.

(i) Normal toric varieties are spherical.

(ii) Flag varieties of the form G/P for a parabolic subgroup P are spherical by the Bruhat decomposition which gives us the Schubert cells of the flag G/P.

(iii) Normal toric varieties and flag varieties are horospherical.

(iv) Horospherical varieties are spherical. This is true because if we have a horospherical variety X, it contains an open orbit isomorphic to a homogeneous horospherical space G/H, i.e. H contains U. We consider the open dense orbit $BwB \subseteq G$ coming from the Bruhat decomposition. Since $B = TU$ in G, we have that $BwTU = BwU$ is dense in G. Consequently, we know that also BwH is dense in G since H contains U by definition and thus, we conclude that BwH/H is dense in G/H. Therefore, G/H has a dense B-orbit, namely $B(wH)$, which implies the sphericity.

(v) There are more classes of spherical varieties which are not necessarily horospherical. One example is the class of wonderful varieties for which it was proved by Luna [39] that they are spherical, but in the sequel we will be mainly interested in horospherical varieties and therefore, we do not introduce further details.

Definition 2.55. *Let G/H be a spherical homogeneous space. We denote by $\mathcal{D}$ the set of the irreducible divisors of G/H which are B-stable, but not stable under the G-action. Furthermore, we call the elements of $\mathcal{D}$ the* ***colors****.*

Remark 2.56. Let X be a G/H-embedding for a spherical homogeneous space G/H. We denote by $X_1, ..., X_m$ the irreducible G-stable divisors of X. We can furthermore identify $\mathcal{D}$ with the set of the irreducible divisors of X which are B-stable, but again not stable under the G-action. In fact, the latter divisors are the closures of the colors in X. Therefore, we have $\mathcal{D} \cup \{X_1, ..., X_m\}$ as the union of the irreducible B-stable divisors of X.

Definition 2.57. *We call a color containing a closed G-orbit a* ***color of X*** *in the setting of the previous remark.*

Example 2.58. Next, we present the colors of two common classes of horospherical varieties.

(i) We know about toric varieties that they do not have any colors since $G = B = T$ holds and therefore, any B-stable divisor will be also G-stable.

(ii) Secondly, flag varieties do not have G-stable divisors since they only have one G-orbit and furthermore, the closures of the B-orbits in G/P coming from the Bruhat-decomposition are the Schubert varieties. Therefore, the colors of G/P are the codimension one Schubert varieties.

Remark 2.59. Following Pasquier [42], the union of the B-orbits of codimension one of a horospherical homogeneous space G/H are the inverse images of the codimension one Schubert varieties of G/P where G/H is given by the torus fibration $G/H \to G/P$ which is the natural map coming from the fact that $P = N_G(H)$, as described in Remark 2.48. Since we have a well known 1-to-1 correspondence of subsets $I \subseteq S$ of the simple roots S and the parabolic subgroups P_I containing B from above (cf. [49, Theorem 8.4.3]), the horospherical homogeneous space is in fact given by the torus fibration $G/H \to G/P_I$. Therefore, the B-orbits of codimension one in G/H are given by $Bw_0s_\alpha P_I/H$ for some $\alpha \in S \setminus I$, its associated simple reflection s_α and the longest element in $w_0 \in W/W_I$.

The colors of G/H are thus the closures D_α of the B-orbits $Bw_0 s_\alpha P_I/H$ which leads to a bijection of $\mathcal{D}$ and $S \setminus I$.

Definition 2.60. *A spherical variety X is **simple** if it contains only one closed orbit.*

Remark 2.61. [47, Section 3.1] If G/H is a spherical homogeneous space, then all G/H-embeddings X are recovered by the simple G/H-embeddings which are contained in X. Indeed, let Y be a G-orbit in X for a G/H-embedding X. Then the subset $X_{Y,G} = \{x \in X \mid Y \subseteq \overline{Gx}\}$ is G-stable and open in X. Let Gx for $x \in X_{Y,G}$ be a closed G-orbit in $X_{Y,G}$, then $Gx = X_{Y,G} \cap \overline{Gx}$ is the closure of Gx in $X_{Y,G}$. Since Y is contained in $X_{Y,G}$ and $\overline{Gx}$, we conclude $Y = Gx$ and therefore, the unique closed G-orbit in $X_{Y,G}$ is Y. In particular, this implies that X can be covered by finitely many G-stable open subsets with a unique closed G-orbit.

Definition 2.62. *A spherical variety X is called **toroidal** if it does not have any color.*

In the following, we want to recap the main results of the Luna-Vust theory for horospherical varieties and consider some easy examples at the end of this section. Therefore, we introduce some further notations.

From now, we fix a horospherical homogeneous space G/H and keep the notations as above. Additionally, we denote by M the characters of $P = P_I$ whose restrictions to H are trivial. This determines a sublattice of the characters of T or B since $P = TH = BH$ for B, T contained in P. Further, we define N to be the dual of M, $M_\mathbb{R} := M \otimes_\mathbb{Z} \mathbb{R}$, and $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$. Lastly, we denote by $k(G/H)^{(B)}$ the set of rational functions f on G/H such that there exists a character χ_f of B such that $f(bgH) = \chi_f(b)f(gH)$ holds for all $b \in B$ and $g \in G$. It is known that the map $k(G/H)^{(B)}/k^* \to M, f \mapsto \chi_f$ is an isomorphism (cf. [42]).

Now, let X be a G/H-embedding. We define a map

$$\sigma : \mathcal{D} \cup \{X_1, ..., X_m\} \to N$$

by the following: Let D be a B-stable divisor of X which leads to the natural B-stable valuation v_D, i.e. $v_D(b \cdot f) = v_D(f)$ for all $b \in B$, on the function field $k(G/H) = k(X)$. Therefore, we obtain a homomorphism $k(G/H)^{(B)}/k^* \to \mathbb{Z}$ which is in fact the restriction of v_D to $k(G/H)^{(B)}/k^*$. Using the above isomorphism, we get an element in N which we denote by $\sigma(D)$.

Lastly, when we restrict σ to $\mathcal{D}$ we sometimes denote the image of the color D_α associated to $\alpha \in S \setminus I$ by $\alpha_M^\vee$ instead of $\sigma(D_\alpha)$ since it is the restriction of $\alpha^\vee$ to M by [42]. We remark that σ may not be injective, e.g. in the case $H = P$ we have $N = 0$.

Definition 2.63. *Let G/H be a horospherical homogeneous space of rank n with the associated data containing the parabolic subgroup $P = P_I$, the subset $I \subseteq S$ of simple roots and the lattice N of rank n with the associated map $\sigma : \mathcal{D} \to N$.*

*(i) Then we define a **colored cone** to be a pair $(\mathcal{C}, \mathcal{F})$ with $\mathcal{C} \subseteq N_\mathbb{R}$ and $\mathcal{F} \subseteq \mathcal{D}$ with the following properties.*

(i') $\mathcal{C}$ is a convex cone generated by finitely many elements of N containing $\sigma(\mathcal{F})$.

(ii') $\mathcal{C}$ does not contain lines and $\sigma(F) \neq 0$ for all $F \in \mathcal{F}$.

*(ii) A **colored face** of a colored cone $(\mathcal{C}, \mathcal{F})$ is a pair $(\mathcal{C}', \mathcal{F}')$ such that $\mathcal{C}'$ is a face of $\mathcal{C}$ and $\mathcal{F}'$ is a subset of $\mathcal{F}$ such that $\sigma(F') \in \mathcal{C}'$ for all $F' \in \mathcal{F}'$.*

*(iii) A **colored fan** is a finite set $\mathbb{F}$ with the following properties.*

(i') Every colored face of a colored cone of $\mathbb{F}$ is also in $\mathbb{F}$.

(ii') For each element $u \in N_\mathbb{R}$, there exists at most one colored cone $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}$ such that u is contained in the relative interior of $\mathcal{C}$.

*(iv) The **support of a colored fan** $\mathbb{F}$ is the set of elements of N contained in the cone of any colored cone of $\mathbb{F}$.*

*(v) We say that the fan $\mathbb{F}$ is **complete** if for any element $x \in N_\mathbb{R}$ there exists a colored cone $(\mathcal{C}, \mathcal{F})$ in $\mathbb{F}$ such that x is contained in $\mathcal{C}$.*

*(vi) A **color of a colored cone** of $\mathbb{F}$ is an element of $D \in \mathcal{D}$ such that there exists a colored cone $(\mathcal{C}, \mathcal{F}) \in \mathbb{F}$ such that $D \in \mathcal{F}$.*

Remark 2.64. For $G = T$ there are no colors and therefore, we obtain the definition of a fan which is used to describe the geometry of toric varieties.

Let X be a simple G/H-embedding and Y be the unique closed G-orbit in X. Then we set $\mathcal{F}$ to be the union of the colors containing Y. Furthermore, let $\mathcal{C}$ be the cone generated by $\sigma(\mathcal{F})$ and $\sigma(D)$ for any irreducible G-stable divisor D of X. The described cone $(\mathcal{C}, \mathcal{F})$ is the colored cone associated to the simple G/H-embedding X. For any G/H-embedding X, the colored fan is described by the colored cones associated to the simple G/H-embeddings which are contained in X and will be denoted by $\mathbb{F}_X$. Furthermore, the colors of X coincide with the colors of the associated colored fan $\mathbb{F}_X$.

Now, we want to recap an important result in the theory of horospherical varieties which classifies G/H-embeddings using their associated colored fans. The following proposition is a special case of [29, Theorem 4.3] in which the statement is formulated with G/H spherical.

Proposition 2.65. *[42, Theorem 2.5] Let G/H be a horospherical homogeneous space. The above construction of the colored fan $\mathbb{F}_X$ defines a bijection $X \mapsto \mathbb{F}_X$ between the isomorphism classes of G/H-embeddings and the set of colored fans in $N_\mathbb{R}$. Furthermore, complete G/H-embeddings correspond to complete colored fans.*

Remark 2.66. For G/H a torus, we obtain the classification of toric varieties associated to fans which is described in [14].

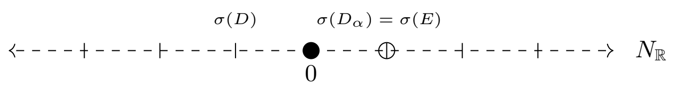
Example 2.67. [41, Example 1.13] Let $G = \mathrm{SL}_2$ and $H = U$ be the unipotent radical of SL_2. Then SL_2/U is clearly a horospherical homogeneous space and thus a torus bundle over SL_2/P. In this example we have $P = B$ where we choose B to be the upper triangular matrices. Therefore, the rank of SL_2/U is equal to one where the dimension of SL_2/U is two. Using the notation as above we have $S = \{\alpha\}$ and $I = \emptyset$ and furthermore, we observe that SL_2/U is isomorphic to $k^2 \setminus \{0\}$ by sending gU to (g_{11}, g_{21}) where g is given by $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$. One may also observe that SL_2/B is isomorphic to $\mathbb{P}^1$ and therefore, we can identify $\mathrm{SL}_2/U \to \mathrm{SL}_2/B$ by the projection from $k^2 \setminus \{0\}$ to $\mathbb{P}^1$.

Next, we consider the natural SL_2-action on k^2 which induces an action on the projective plane $\mathbb{P}^2 \cong \mathbb{P}(k \oplus k^2)$. Further, we denote the homogeneous coordinates of $\mathbb{P}^2$ by x_0, x_1, x_2 and we see that $\mathrm{SL}_2 \cdot [1:1:0]$ is isomorphic to $k^2 \setminus \{0\}$ which implies that $\mathbb{P}^2$ is an SL_2/U-embedding. Additionally, it may be observed that SL_2/U corresponds to the open subset $\{[1 : x_1 : x_2] \mid (x_1, x_2) \in k^2 \setminus \{0\}\} \subseteq \mathbb{P}^2$. We denote by 0 the SL_2-fixed point $[1 : 0 : 0] \in \mathbb{P}^2$, by D the line $\{[x_0 : x_1 : x_2] \in \mathbb{P}^2 \mid x_0 = 0\}$ which implies $\mathbb{P}^2 \setminus D = k^2$ and by E the exceptional divisor of the blow up of 0 in $\mathbb{P}^2$. Then we have the following non-trivial SL_2/U-embeddings.

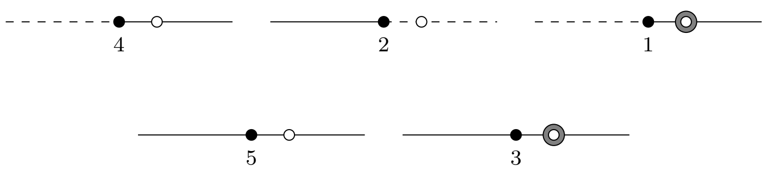
	SL_2/U-embedding X	SL_2-stable divisors	closed SL_2-orbits	color of X
1	k^2	$\emptyset$	$\{0\}$	D_α
2	$\mathbb{P}^2 \setminus \{0\}$	D	D	$\emptyset$
3	$\mathbb{P}^2$	D	D and $\{0\}$	D_α
4	$\mathrm{Bl}_{\{0\}} k^2$	E	E	$\emptyset$
5	$\mathrm{Bl}_{\{0\}} \mathbb{P}^2$	D and E	D and E	$\emptyset$

For the homogeneous horospherical space SL_2/U the only color D_α is given by the set $\{[1 : x_1 : 0] \mid x_1 \in k^*\}$. Since the embeddings $1, 2$ and 4 contain only one closed SL_2-orbit, they are simple horospherical varieties. We see that the embeddings $2, 4$ and 5 do not have any color since D_α does not contain any closed SL_2-orbit. Therefore, by Definition 2.62 these varieties are toroidal. We remark that $\mathrm{Bl}_{\{0\}} \mathbb{P}^2$ is covered by $\mathbb{P}^2 \setminus \{0\}$ and $\mathrm{Bl}_{\{0\}} k^2$ whereas $\mathbb{P}^2$ is covered by $\mathbb{P}^2 \setminus \{0\}$ and k^2. Furthermore, in our particular example we have that $M = N = \mathbb{Z}$ since the characters of $P = TU$ which become trivial after the restriction to U, correspond to the characters of T.

Now, we draw the line $N_{\mathbb{R}}$ with the images under σ of the B-stable divisors of $\mathrm{Bl}_{\{0\}} \mathbb{P}^2$ which will be used later on to construct the corresponding colored fans. The images can be obtained by considering the subset of rational functions $k(\mathrm{SL}_2/U)^{(B)}/k^*$ which consists of rational functions of the form $f(x_1/x_0, x_2/x_0) = (x_2/x_0)^n$ for $n \in \mathbb{Z}$ with the corresponding coordinates on SL_2/U. The valuations which are induced by the divisors lead to the images of σ.



Using the procedure described above to obtain colored fans from their corresponding SL_2/U-embeddings, we obtain the following five non-trivial colored fans. Hereby, the black points represent the origin, the white points the colors and the gray annuli around the white points the colors of a fan. Furthermore, the colored cones of the colored fan are given by the black half-lines starting from the origin.



Using Proposition 2.65, we can also verify that there are only five non-trivial SL_2/U-embeddings since there are no more non-trivial colored fans in the given example. Lastly, we remark that the trivial colored fan $\{(\{0\}, \emptyset)\}$ corresponds to the trivial SL_2/U-embedding SL_2/U.

Remark 2.68. We have seen that for our example of rank 1 there were finitely many non-trivial colored fans, but as soon as the rank of G/H is larger or equal to two, there will be infinitely many colored fans. This behaviour is equal to the one of the fans corresponding to toric varieties even though there are more colored fans than fans coming from toric varieties.

Now, we want to recall the smoothness criterion for horospherical varieties which was first described by Pasquier (cf. [41]). In order to be able to state it, we need some more notation which we take from [41].

Proposition 2.69. *[41, Proposition 2.2] Let X be a G/H-embedding for a horospherical homogeneous space G/H defined by the colored fan $\mathbb{F}_X$. Then X is locally factorial if and only if for every colored cone $(\mathcal{C}, \mathcal{F})$ of $\mathbb{F}_X$*

(i) the elements $F \in \mathcal{F}$ have pairwise distinct images under σ and

(ii) $\mathcal{C}$ is generated by a part of a basis of N containing $\sigma(\mathcal{F})$.

Definition 2.70. *Let I and J be two disjoint subsets of the set of simple roots S. We denote by Γ_S the corresponding Dynkin diagram of G and by $\Gamma_{I\cup J}$ the subgraph of Γ_S containing the nodes of I and J and the edges of Γ_S joining two elements of $I \cup J$. We say (I, J) is smooth if every connected component Γ of $\Gamma_{I\cup J}$ satisfies one of the following conditions.*

(i) Γ is a Dynkin diagram of type A_n for which every knot is in I except the first or the last one which is in J.

(ii) Γ is a Dynkin diagram of type C_n for which every knot is in I except the first one (connected with a simple edge) which is in J.

(iii) Γ is any Dynkin diagram for which every knot is in I.

Example 2.71. Let Γ_S be a Dynkin diagram of type C_6 where we denote the nodes by α_i for $1 \leq i \leq n$.

(i) Furthermore, let $I_1 = \{\alpha_2, \alpha_5, \alpha_6\}$ and $J_1 = \{\alpha_1, \alpha_4\}$. In this case (I_1, J_1) is smooth since the connected component containing $\{\alpha_1, \alpha_2\}$ satisfies condition (i) and the connected component $\{\alpha_4, \alpha_5, \alpha_6\}$ satisfies condition (ii).

(ii) Now, let $I_2 = \{\alpha_1, \alpha_3\}$ and $J_2 = \{\alpha_2\}$. In this case we see that there is only one connected component which does not satisfy any of the conditions since the first and the last knot are contained in I. Therefore, (I_2, J_2) is not smooth.

Now, we can state the smoothness criterion for horospherical varieties.

Proposition 2.72. *[41, Theorem 2.6] Let G/H be a horospherical homogeneous space. As above, $I \subseteq S$ denotes the subset of the simple roots associated to H (cf. Remark 2.59). A G/H-embedding X defined by a colored fan $\mathbb{F}_X$ is smooth if and only if the following two conditions are satisfied.*

(i) X is locally factorial, i.e. the colored cones satisfy the conditions of Proposition 2.69.

(ii) For each maximal colored cone $(\mathcal{C}, \mathcal{F})$ in $\mathbb{F}_X$, we denote by $J_{\mathcal{F}}$ the union of the $\alpha \in S \setminus I$ such that D_α is in $\mathcal{F}$. Then $(I, J_{\mathcal{F}})$ is smooth.

Next, we want to have a look at another horospherical homogeneous space of rank one which we will need in the sequel of this work and for which we want to apply the previous smoothness criterion.

Example 2.73. We consider the horospherical homogeneous space $\mathrm{SL}_2 / \mu_n U$ where μ_n is a finite group of order $n \geq 1$ which is given by diagonal matrices with entries (ξ, ξ^{-1}) where ξ is an n-th root of unity. In this case we have again $P = N_G(\mu_n U) = B$ and $I = \emptyset$ as in Example 2.67. The rank of $\mathrm{SL}_2 / \mu_n U$ is again one and $S = \{\alpha\}$. Now, we want to have a look at M which are the characters of P, i.e. the characters of T in this case, whose restrictions to $\mu_n U$ are trivial. We see that a character

$$\phi : T \to \mathbb{C}^*, \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t^m$$

restricts trivially to $\mu_n U$ if and only if $m = a \cdot n$ for $a \in \mathbb{Z}$. Therefore, $M = n\mathbb{Z}$ and $N = \mathrm{Hom}_{\mathbb{Z}}(n\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$ is the dual lattice. Now, we want to determine the possible fans of the $\mathrm{SL}_2 / \mu_n U$-embeddings. Therefore, we do not need to know the varieties, but only the image of the color D_α under the map σ which was already used frequently above. As mentioned above, the image of the color D_α under σ is given by the restriction of the coroot $\alpha^\vee$ to the sublattice M. In this case the coroot is given by

$$\alpha^\vee : \mathrm{Hom}(T, \mathbb{C}^*) \to \mathbb{Z}, \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t^m \right) \mapsto m$$

since $(\alpha, \alpha^\vee) = 2$ has to hold for the corresponding pairing by the definition of the coroot. If we restrict this map to M, we see that the generator $n \in n\mathbb{Z}$ is mapped to n which corresponds to the vector $n \in N$ via the identification $\mathrm{Hom}_{\mathbb{Z}}(n\mathbb{Z}, \mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}, (n \mapsto k) \mapsto k$ for $k \in \mathbb{Z}$. Therefore, we obtain the image of the color $\sigma(D_\alpha) = \alpha^\vee_M = n$.

Later on, we will be interested in the complete $\mathrm{SL}_2 / \mu_n U$-embeddings and therefore, we consider the complete colored fans in this example. It is clear that there are only two complete colored fans in this case, one containing the color D_α and the other one not containing the color as in Example 2.67. Now, we want to know which ones are smooth and therefore, we use the smoothness criterion described above. We have already seen in Example 2.67 that for $n = 1$ both complete $\mathrm{SL}_2 / \mu_n U$-embeddings are smooth. For $n > 1$ we need to verify the criterion. First, we consider the colored fan not containing the color. It contains the colored cones $(-1, \emptyset)$ and $(1, \emptyset)$ for both of which the pair $(I, J_{\mathcal{F}}) = (\emptyset, \emptyset)$ is trivial and therefore smooth. Furthermore, this $\mathrm{SL}_2 / \mu_n U$-embedding is locally factorial because the conditions of Proposition 2.69 are satisfied.

Next, we take the complete fan containing the color which changes only the colored cone $(1, \emptyset)$ to $(1, D_\alpha)$. This leads to the pair $(\emptyset, \alpha)$ which is smooth by Definition 2.70 (i). Lastly, we need to verify the local factoriality, but here we see that $\sigma(D_\alpha) = n$ and therefore, the generators of the cone $\mathcal{C}$ do not contain $\sigma(D_\alpha)$. This implies that the complete $\mathrm{SL}_2 / \mu_n U$-embedding containing the color is not smooth. The upshot is that there exists only one smooth complete $\mathrm{SL}_2 / \mu_n U$-embedding and therefore, only one smooth equivariant completion of $\mathrm{SL}_2 / \mu_n U$.

3 Computations of equivariant algebraic cobordism

3.1 Filtrable schemes

We recap some basic notation from Krishna [33] which was introduced following Brion's work [7] on Chow groups of torus actions. Therefore, let G be a linear algebraic group over k and let $X \in G - \mathbf{Sch}_k$. We say that X is ***G*-filtrable** if the fixed point subscheme X^G is smooth and projective, there is an ordering $X^G = \coprod_{m=0}^{n} Z_m$ of the connected components Z_m of the fixed point subscheme such that there is a filtration of X by G-stable closed subschemes

$$\emptyset = X_{-1} \subsetneq X_0 \subsetneq ... \subsetneq X_n = X \tag{3.1}$$

with $Z_m \subseteq W_m := X_m \setminus X_{m-1}$ and maps $\phi_m : W_m \to Z_m$ for all $0 \leq m \leq n$ which are all G-equivariant vector bundles such that the inclusions $Z_m \hookrightarrow W_m$ are the 0-section embeddings. One should note that if X is G-filtrable then so is every closed subscheme X_m.

We say furthermore that a k-scheme X is filtrable if there are closed, connected, smooth and projective subschemes $Z_0, ..., Z_m$ of X and a filtration of X by closed subschemes

$$\emptyset = X_{-1} \subsetneq X_0 \subsetneq ... \subsetneq X_n = X$$

with $Z_m \subseteq W_m := X_m \setminus X_{m-1}$ and maps $\phi_m : W_m \to Z_m$ for all $0 \leq m \leq n$ which are vector bundles such that the inclusions $Z_m \hookrightarrow W_m$ are the corresponding 0-section embeddings. Clearly each G-filtrable scheme is also filtrable. Furthermore, one observes that any smooth and projective k-scheme X is filtrable by choosing $Z_i = W_i$ to be the connected components of X. Lastly, we remark that any smooth projective k-scheme X is also $\{e\}$-fitrable by the same choice as above.

3.2 *T*-filtrable schemes

In this section, we will discuss the specific case $G = T$ where T is a torus which acts on a scheme X. For the rest of this chapter we will only consider algebraically closed fields k of characteristic zero. First, we recap the Bialynicki-Birula decomposition (cf. [7, Section 3]) in order to be able to discuss known results for Chow groups ([7]) which we would like to generalise to algebraic cobordism.

Definition 3.1. *For $X \in T - \mathbf{Sch}_k$, we denote by X^T its fixed point subscheme. For any one-parameter subgroup λ of T, we have $X^\lambda \supseteq X^T$ where $X^\lambda = X^{\lambda(k)}$ is being used as an abbreviation. We call λ **generic** if $X^T = X^\lambda$ holds.*

Remark 3.2. It is known [7, Section 3] that such a generic one-parameter group always exists due to the linearity of the actions.

Construction 3.3. For a subvariety $Y \subseteq X^\lambda$, we define

$$X_+(Y, \lambda) = \{x \in X \mid \lim_{t \to 0} \lambda(t)x \text{ exists and is in } Y\} \text{ and}$$
$$X_-(Y, \lambda) = \{x \in X \mid \lim_{t \to 0} \lambda(t^{-1})x \text{ exists and is in } Y\}.$$

Furthermore, we define the maps $p_\pm : X_\pm(Y,\lambda) \to Y,\ x \mapsto \lim_{t\to 0} \lambda(t^\pm)x$. Then following [7] $X_+(Y,\lambda)$ and $X_-(Y,\lambda)$ are locally closed T-stable subvarieties of X and p_+, p_- are T-equivariant morphisms. It is a fact that any complete T-variety X is the disjoint union of locally closed subvarieties $X_+(Y,\lambda)$ where λ is a fixed generic one-parameter subgroup and where Y runs over all connected components of X^T.

Proposition 3.4. *[2, 3] Let X be a complete non-singular T-variety with finitely many T-fixed points and let λ be a generic one-parameter subgroup. Then for any fixed point x_i of X^T, the maps $p_\pm : X_\pm(x_i,\lambda) \to x_i$ make $X_\pm(x_i,\lambda)$ into an equivariant vector bundle over x_i.*

Definition 3.5. *A scheme $X \in T-\mathbf{Sch}_k$ is called **T-filtrable** if it satisfies the following conditions.*

(i) X is the disjoint union of its plus strata $X_+(Y,\lambda)$ for some generic one-parameter subgroup λ of T.

(ii) There is an indexing $\Sigma_0, ..., \Sigma_n$ of the set of strata such that the closure $\overline{\Sigma_i}$ is contained in the union $\bigcup_{j\geq i} \Sigma_j$.

Remark 3.6. The preceding definition directly implies that there exists a closed stratum $\Sigma_n = X_+(Y,\lambda)$ for some connected component Y of X^T.

The following proposition was formulated in [31, Theorem 4.5] without the assumption of finitely many T-fixed points and a similar result was stated in [7, Theorem 3.1]. Without the assumption on the T-fixed points, the term "vector bundle" in the definition of T-filtrable has to be replaced by the more general notion of an affine space bundle. In fact, there is a counterexample to the assertion that $X_+(Y,\lambda) \to Y$ is a vector bundle which is described in [25].

Proposition 3.7. *[2, 3] Let X be a smooth projective T-variety with finitely many T-fixed points. Then X is T-filtrable.*

Remark 3.8. In the case of T-filtrable schemes we will be mostly interested in the examples given by the preceding proposition, i.e. smooth and projective T-varieties with finitely many T-fixed points. Therefore, let X be a smooth and projective T-variety with finitely many T-fixed points.

Definition 3.5 following [7] differs from our definition of G-filtrable schemes in Section 3.1 at first sight, but if we choose $X_0 := \Sigma_n$ where Σ_n is the closed stratum we mentioned above and $X_i := \bigcup_{j\geq n-i} \overline{\Sigma_j} = \bigcup_{j\geq n-i} \Sigma_j$ coming from condition (ii) of Definition 3.5, this leads to the inclusion $X_i \subsetneq X_{i+1}$. One is left with the condition that $X_{i+1} \setminus X_i$ needs to be an equivariant vector bundle. Therefore, one verifies $X_{i+1} \setminus X_i = \Sigma_{n-i-1}$ which is a T-equivariant vector bundle over the corresponding T-fixed point by Proposition 3.4.

Conversely, starting with the filtration of Section 3.1, we obtain the Σ_i similarly and furthermore, it is clear that an appropriate generic one-parameter subgroup λ has to exist because of the fact that any complete T-variety is the disjoint union of the $X_+(x_i,\lambda)$ for any generic one-parameter subgroup λ and the fact that those $X_+(x_i,\lambda)$ are T-equivariant vector bundles over x_i by Proposition 3.4.

We conclude that these two definitions coincide for the class of smooth and projective T-varieties with finitely many T-fixed points.

Example 3.9. Now, we want to discuss some very elementary examples of T-filtrable varieties.

(i) We consider $X = \mathbb{P}^1_{\mathbb{C}}$ with maximal torus $T = \mathbb{G}_m$ and trivial group action. In this case we can choose $\lambda : \mathbb{G}_m \to \mathbb{G}_m, t \mapsto t$ to be the identity on $\mathbb{G}_m$. This is a generic one-parameter subgroup. Therefore, we have $X^T = X^\lambda = \mathbb{A}^1 \cup \{\infty\}$. Now, we compute

$$X_+(\mathbb{P}^1, \lambda) = \{x \in \mathbb{P}^1 \mid \lim_{t\to 0} t \cdot x \in \mathbb{P}^1\} = \mathbb{P}^1.$$

We see that the Bialynicki-Birula cells do not necessarily coincide with the cellular decomposition of $\mathbb{P}^1$. We remark that this is an example of a smooth projective variety with infinitely many T-fixed points.

(ii) Now, we want to have a look at the same variety $X = \mathbb{P}^1$ with a different group action. Therefore, let $\mathbb{P}^1_{\mathbb{C}} = \{[x : 1] \mid x \in \mathbb{C}\} \cup [1 : 0]$ and the torus $T = \mathbb{G}_m$ acts on X by $t \cdot [x : 1] := [tx : 1]$ and $t \cdot [1 : 0] = [t1 : 0] = [1 : 0]$. Again we can choose λ to be the identity on $\mathbb{G}_m$. This time we have $X^T = \{0\} \cup \{\infty\}$ and for these we compute the Bialynicki-Birula cells

$$X_+(\{0\}, \lambda) = \{x \in \mathbb{P}^1 \mid \lim_{t\to 0} t \cdot x = 0\} = \mathbb{A}^1 \text{ and}$$
$$X_+(\{\infty\}, \lambda) = \{x \in \mathbb{P}^1 \mid \lim_{t\to 0} t \cdot x = \infty\} = \{\infty\}.$$

Contrary to the previous example, we see that we obtain the cellular decomposition of the projective line with the natural group action. Therefore, the decomposition into Bialynicki-Birula cells depends on the group action.

(iii) Next, we consider an example whose decomposition into Bialynicki-Birula cells will be needed later on. Let $V = \mathrm{Sym}^0(k^2) \oplus \mathrm{Sym}^1(k^2) = k \oplus k^2$ and $X = \mathbb{P}(V) = \mathbb{P}^2$ be the projectivisation of V where Sym^n denotes the space of symmetric tensors of order n. Furthermore, let $G = \mathrm{SL}_2(k)$ be the special linear group with torus T being the diagonal matrices in $\mathrm{SL}_2(k)$. By reordering of the basis elements we have the following group action

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot [x' : y' : z'] = [tx' : y' : t^{-1}z']$$

for all $t \in \mathbb{G}_m$ and $[x' : y' : z'] \in \mathbb{P}^2$. Now, we are interested in the fixed point subscheme which is given by $X^T = \{[1 : 0 : 0]\} \cup \{[0 : 1 : 0]\} \cup \{[0 : 0 : 1]\}$. Before we are able to compute the Bialynicki-Birula cells we need a generic one-parameter subgroup of T. Therefore, we choose

$$\lambda : \mathbb{G}_m \to T,\ t \mapsto \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$$

to be the given map. First, we compute the limits

$$\lim_{t\to 0} \lambda(t) \cdot [1 : y' : z'] = \lim_{t\to 0}\, [t^{-1} : y' : tz'] = \lim_{t\to 0}\, [1 : ty' : t^2 z'] = [1 : 0 : 0],$$
$$\lim_{t\to 0} \lambda(t) \cdot [0 : 1 : z'] = \lim_{t\to 0}\, [0 : 1 : tz'] = [0 : 1 : 0],$$

$$\lim_{t\to 0} \lambda(t) \cdot [0:0:z'] = \lim_{t\to 0} [0:0:tz'] = [0:0:1].$$

Thus, we have

$$X_+(\{[1:0:0]\}, \lambda) = \{[1:y':z'] \in \mathbb{P}^2 \mid y', z' \in k\} = \mathbb{A}^2,$$
$$X_+(\{[0:1:0]\}, \lambda) = \{[0:1:z'] \in \mathbb{P}^2 \mid z' \in k\} = \mathbb{A}^1,$$
$$X_+(\{[0:0:1]\}, \lambda) = \{[0:0:1] \in \mathbb{P}^2\} = \mathbb{A}^0.$$

This leads to the Bialynicki-Birula decomposition $\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{A}^1 \cup \mathbb{A}^0$. For later use we remark that the closures of the Bialynicki-Birula cells are the T-fixed point $z := [0:0:1]$, the line connecting $y = [0:1:0]$ and z which we denote by (yz), and the whole space $X = \mathbb{P}(V)$. In this case, the indexing of the strata in Definition 3.5 is obviously given by $\Sigma_2 := \mathbb{A}^0, \Sigma_1 := \mathbb{A}^1$ and $\Sigma_0 := \mathbb{A}^2$.

(iv) In the following, we consider an example which is similar to the previous example. Therefore, let $V = \mathfrak{sl}_2$ be the Lie algebra of SL_2 which is a non-trivial SL_2-module of dimension three and let $X = \mathbb{P}(V)$ as above. First, we have a look at the action of SL_2 on $\mathfrak{sl}_2$:

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & -a \end{pmatrix} := \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} a & t^2 b \\ t^{-2} c & -a \end{pmatrix}.$$

We deduce again by reordering of the basis elements the torus action on X which is then given by

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot [x:y:z] = [t^2 x : y : t^{-2} z].$$

With the same generic one-parameter subgroup and the same calculations as in the previous example, we obtain the Bialynicki-Birula decomposition of X. Later on, we will come back to this example and show that the different induced actions on $\mathbb{P}^2$ will lead to different results in computations we are going to need in the sequel.

(v) Lastly, for any positive integer n we will analyse the example of the rational ruled surfaces $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ which are also known as Hirzebruch surfaces. We know by [7] that $\mathbb{F}_n$ can be obtained as the smooth equivariant completion of $\mathrm{SL}_2 / \mu_n U$. The fans for the corresponding $\mathrm{SL}_2 / \mu_n U$-embeddings have been described in Example 2.73. Alternatively, we want to realise the surface $\mathbb{F}_n$ as a closed subvariety of the projectivisation of an SL_2-module. Let $V_{n+1} := \mathrm{Sym}^{n+1}(k^2)$ be the space of symmetric tensors of order $n+1$. We consider the SL_2-module $V := V_1 \oplus V_{n+1}$. At this point, we remark that each element $P \in V_{n+1}$ is a homogeneous polynomial of degree $n+1$ in two variables $z_1 := (1,0)^t$ and $z_2 := (0,1)^t$, i.e.

$$P = a_0 z_1^{n+1} + a_1 z_1^n z_2 + \ldots + a_n z_1 z_2^n + a_{n+1} z_2^{n+1}.$$

Next, we consider the SL_2-action on V_{n+1} for $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which can be described by the componentwise action of A on the space of symmetric tensors.

We want to show that the Hirzebruch surface $\mathbb{F}_n$, $n \geq 1$, is the closure of the SL_2-orbit $\mathrm{SL}_2 \cdot [1:0:1:0:...:0]$ where the given vector

$$v = v_1 + v_{n+1} = [1:0:1:0:...:0] \in \mathbb{P}(V)$$

is the sum of the two highest weight vectors in $v_1 \in V_1$ and $v_{n+1} \in V_{n+1}$, respectively. The torus action of SL_2 on $\mathbb{P}(V)$ is given by

$$t \cdot [x' : y' : a_0 : ... : a_{n+1}] = [tx' : t^{-1}y' : t^{n+1}a_0 : t^{n-1}a_1 : ... : t^{-n-1}a_{n+1}]$$

for $t = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$. In order to be able to determine the T-fixed points of $\mathbb{F}_n$ we consider the above mentioned SL_2-orbit

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [1:0:1:0:...:0] = [a : c : a^{n+1} : a^n c : ... : ac^n : c^{n+1}].$$

We remark that the stabiliser of $v = [1:0:1:0:...:0]$ is given by $\mu_n U$ which shows that $\mathrm{SL}_2[v] \cong \mathrm{SL}_2 / \mu_n U$ holds. Now, we consider the closure $X := \overline{\mathrm{SL}_2[v]}$ of $\mathrm{SL}_2[v]$ in $\mathbb{P}(V)$ for which we want to know the T-fixed points. If we choose $a = t$ and $c = ta$, we obtain

$$[t : ta : t^{n+1} : t^n ta : ... : t(ta)^n : (ta)^{n+1}] = [1 : a : t^n : ... : t^n a^{n+1}].$$

Then we let t and a go to zero and we see that the point $[v_1]$ is in X. Similar choices for a and c lead to four T-fixed points

$$[1:0:...:0]\ ,\ [0:1:0:...:0]\ ,\ [0:0:1:0:...:0] \text{ and } [0:...:0:1].$$

These four points are the only T-fixed points in the closure of this orbit and therefore, we are left to show that $X \subseteq \mathbb{P}(V)$ coincides with the smooth equivariant completion of $\mathrm{SL}_2 / \mu_n U$.

By the construction as an $\mathrm{SL}_2 / \mu_n U$-embedding we know that there are at most two G-stable divisors in X. This leads to the SL_2-stable divisor $\mathrm{SL}_2[v_1]$ which is given by elements of the form $[x' : y' : 0 : ... : 0]$ as it is closed and SL_2-stable. Similarly, the point $[v_{n+1}]$ leads to the second SL_2-stable divisor $\mathrm{SL}_2[v_{n+1}]$. Recall, that a color of X is a B-stable divisor which is not G-stable and contains a closed G-orbit. Therefore, in our example a color would have to contain an SL_2-stable divisor which implies that there is no color in X. We conclude that X is the smooth equivariant completion of $\mathrm{SL}_2 / \mu_n U$ whose fan has been described in Example 2.73. Consequently, X describes the n-th Hirzebruch surface $\mathbb{F}_n$.

Now, we want to compute the Bialynicki-Birula decompostion for $\mathbb{F}_n$. Therefore, we choose again the identity as generic one-parameter subgroup and for later use we name the T-fixed points by

$$w := [0:0:1:0:...:0],$$
$$x := [1:0:0:...:0:0],$$

$$y := [0 : 1 : 0 : ... : 0 : 0],$$
$$z := [0 : 0 : 0 : ... : 0 : 1].$$

This leads to

$$\begin{aligned}\lim_{t\to 0}\lambda(t)\left[\frac{a}{c^{n+1}} : \frac{1}{c^n} : \left(\frac{a}{c}\right)^{n+1} : ... : 1\right] &= \lim_{t\to 0} t\cdot\left[\frac{a}{c^{n+1}} : \frac{1}{c^n} : \left(\frac{a}{c}\right)^{n+1} : ... : 1\right]\\ &= \lim_{t\to 0}\left[\frac{ta}{c^{n+1}} : \frac{t^{-1}}{c^n} : \left(\frac{ta}{c}\right)^{n+1} : ... : t^{-n-1}\right]\\ &= z,\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to 0}\lambda(t)\left[0 : 0 : \left(\frac{a}{c}\right)^{n+1} : \left(\frac{a}{c}\right)^{n} : ... : 1\right] &= \lim_{t\to 0}\ t\cdot\left[0 : 0 : \left(\frac{a}{c}\right)^{n+1} : \left(\frac{a}{c}\right)^{n} : ... : 1\right]\\ &= \lim_{t\to 0}\ \left[0 : 0 : \left(\frac{ta}{c}\right)^{n+1} : ... : t^{-n-1}\right]\\ &= z,\end{aligned}$$

as well as

$$\begin{aligned}\lim_{t\to 0}\lambda(t)[1 : 0 : a^n : 0 : ... : 0] &= \lim_{t\to 0}\ [t : 0 : t^{n+1}a^n : ... : 0] &= x,\\ \lim_{t\to 0}\lambda(t)[a : 1 : 0 : 0 : ... : 0] &= \lim_{t\to 0}\ [ta : t^{-1} : 0 : 0 : ... : 0] &= y,\\ \lim_{t\to 0}\lambda(t)[0 : 0 : 1 : 0 : ... : 0] &= \lim_{t\to 0}\ [0 : 0 : t^{n+1} : 0 : ... : 0] &= w.\end{aligned}$$

We set

$$\begin{aligned}M &:= \left\{\left[a : c : a^{n+1} : ... : c^{n+1}\right]\ \middle|\ a\in k, c\neq 0\in k\right\} = \mathbb{A}^2\setminus\mathbb{A}^1,\\ N &:= \left\{\left[0 : 0 : \left(\frac{a}{c}\right)^{n+1} : ... : 1\right]\ \middle|\ \frac{a}{c}\in k\right\} = \mathbb{A}^1.\end{aligned}$$

By the geometry of the Hirzebruch surface $\mathbb{F}_n$, we obtain

$$X_+(z,\lambda) = M\cup N = \mathbb{A}^2.$$

Further, we get

$$\begin{aligned}X_+(x,\lambda) &= \{[1 : 0 : a^n : 0... : 0]\mid a\in k\} = \mathbb{A}^1,\\ X_+(y,\lambda) &= \{[a : 1 : 0 : ... : 0]\mid a\in k\} = \mathbb{A}^1,\\ X_+(w,\lambda) &= \{[0 : 0 : 1 : 0 : ... : 0]\} = \mathbb{A}^0.\end{aligned}$$

Therefore, the Bialynicki-Birula decomposition of the n-th Hirzebruch surface $\mathbb{F}_n$ is given by $\mathbb{F}_n = \mathbb{A}^2\cup\mathbb{A}^1\cup\mathbb{A}^1\cup\mathbb{A}^0$. The closures of the Bialynicki-Birula cells are the point w, the lines (xy) and (wx), and the whole surface $\mathbb{F}_n$.

(vi) The case $n = 0$ is an exception since $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and therefore, the four T-fixed points are $w := ([1 : 0], [1 : 0]), x := ([1 : 0], [0 : 1]), y := ([0 : 1], [1 : 0])$ and $z := ([0 : 1], [0 : 1])$ for the natural SL_2-action on $\mathbb{P}^1$. We compute the Bialynicki-Birula decomposition in this case now. As above we need to choose a generic one-parameter subgroup which in this case will be the map

$$\lambda : \mathbb{G}_m \to T,\ t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

We compute the limits

$$\begin{aligned}
\lim_{t\to 0} \lambda(t) \cdot ([a : 0], [c : 0]) &= \lim_{t\to 0} ([ta : 0], [tc : 0]) = ([1 : 0], [1 : 0]),\\
\lim_{t\to 0} \lambda(t) \cdot ([a : 0], [c : 1]) &= \lim_{t\to 0} ([ta : 0], [tc : t^{-1}]) = ([1 : 0], [0 : 1]),\\
\lim_{t\to 0} \lambda(t) \cdot ([a : 1], [c : 0]) &= \lim_{t\to 0} ([ta : t^{-1}], [tc : 0]) = ([0 : 1], [1 : 0]),\\
\lim_{t\to 0} \lambda(t) \cdot ([a : 1], [c : 1]) &= \lim_{t\to 0} ([ta : t^{-1}], [tc : t^{-1}]) = ([0 : 1], [0 : 1]).
\end{aligned}$$

This leads to the Bialynicki-Birula decomposition $\mathbb{F}_0 = \mathbb{A}^2 \cup \mathbb{A}^1 \cup \mathbb{A}^1 \cup \mathbb{A}^0$. In this example, the indexing of the strata according to Definition 3.5 is given by $\Sigma_3 := \mathbb{A}^0, \Sigma_2 := \mathbb{A}^1, \Sigma_1 := \mathbb{A}^1$ and $\Sigma_0 := \mathbb{A}^2$ where the two different affine lines come from the above computation. For later use we remark that the closures of the Bialynicki-Birula cells are the point $w := ([1 : 0], [1 : 0])$, the lines (wx) and (wy), and the whole space $\mathbb{P}^1 \times \mathbb{P}^1$.

(vii) Finally, we will again compute the Bialynicki-Birula decomposition with a different description of the Hirzebruch surfaces $\mathbb{F}_n$, $n \geq 1$. We will not show that these descriptions are T-isomorphic, but we will nevertheless compute the Bialynicki-Birula decomposition by using its corresponding torus action.

Therefore, we consider first the case $n = 1$ and the fact that the above description of $\mathbb{F}_1$ is isomorphic to the blow up of $\mathbb{P}^2$ at a point. Therefore, it is well known (cf. [24, p. 97]) that the first Hirzebruch surface $\mathbb{F}_1$ can be described as

$$\mathbb{F}_1 = \{([x' : y' : z'], [u : v]) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid uy' - vz' = 0\}.$$

Now, we consider the action

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot ([x' : y' : z'], [u : v]) = ([x' : t^{-1}y' : tz'], [tu : t^{-1}v])$$

of the torus of SL_2 on $\mathbb{F}_1$ which clearly preserves the given relation for $\mathbb{F}_1$. This action has six T-fixed points in $\mathbb{P}^2 \times \mathbb{P}^1$, but the points $([0 : 0 : 1], [0 : 1])$ and $([0 : 1 : 0], [1 : 0])$ are not in $\mathbb{F}_1$ and therefore, $\mathbb{F}_1$ has exactly four T-fixed points

$$\begin{aligned}
w &:= ([0 : 0 : 1], [1 : 0]),\\
x &:= ([1 : 0 : 0], [1 : 0]),\\
y &:= ([1 : 0 : 0], [0 : 1]),\\
z &:= ([0 : 1 : 0], [0 : 1]).
\end{aligned}$$

In order to compute the Bialynicki-Birula decomposition of $\mathbb{F}_1$ we take the same generic one-parameter subgroup as above and obtain

$$\begin{aligned}
&\lim_{t\to 0}\lambda(t)\cdot([x':1:z'],[u:1]) = \lim_{t\to 0}([x':t^{-1}:tz'],[tu:t^{-1}]) = z,\\
&\lim_{t\to 0}\lambda(t)\cdot([1:0:z'],[1:0]) = \lim_{t\to 0}([1:0:tz'],[t:0]) = x,\\
&\lim_{t\to 0}\lambda(t)\cdot([1:0:z'],[u:1]) = \lim_{t\to 0}([1:0:tz'],[tu:t^{-1}]) = y,\\
&\lim_{t\to 0}\lambda(t)\cdot([0:0:1],[1:0]) = \lim_{t\to 0}([0:0:t],[t:0]) = w.
\end{aligned}$$

where in the third computation the variable z' has to vanish in $\mathbb{F}_1$. Thus, we have

$$\begin{aligned}
X_+(z,\lambda) &= \{([x':1:z'],[u:1]) \in \mathbb{F}_1 \mid x',z',u \in k\} = \mathbb{A}^2,\\
X_+(x,\lambda) &= \{([1:0:z'],[1:0]) \in \mathbb{F}_1 \mid z' \in k\} = \mathbb{A}^1,\\
X_+(y,\lambda) &= \{([1:0:0],[u:1]) \in \mathbb{F}_1 \mid u \in k\} = \mathbb{A}^1,\\
X_+(w,\lambda) &= \{([0:0:1],[1:0]) \in \mathbb{F}_1\} = \mathbb{A}^0.
\end{aligned}$$

These components again cover the Hirzebruch surface $\mathbb{F}_1$. The other components of $\mathbb{P}^2 \times \mathbb{P}^1$ do not satisfy the relation $uy' - vz' = 0$ and therefore, we obtain the exact same Bialynicki-Birula decomposition $\mathbb{F}_1 = \mathbb{A}^2 \cup \mathbb{A}^1 \cup \mathbb{A}^1 \cup \mathbb{A}^0$. Again, we remark that the closures of the Bialynicki-Birula cells are the point w, the lines (xy) and (wx) and the whole space $\mathbb{F}_1$.

Finally, we will tackle the general case $\mathbb{F}_n$, $n \geq 1$, for which we choose the same generic one-parameter subgroup as above. Besides that, we again use the isomorphic description

$$\mathbb{F}_n = \{([x':y':z'],[u:v]) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid u^n y' - v^n z' = 0\}.$$

In this case the T-action on $\mathbb{F}_n$ is given by

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot ([x':y':z'],[u:v]) = ([x':t^{-n}y':t^n z'],[tu:t^{-1}v])$$

which preserves the given relation for $\mathbb{F}_n$. The T-fixed points have already been described and therefore, we compute again the Bialynicki-Birula cells as for $\mathbb{F}_1$ and thus, we obtain the Bialynicki-Birula decomposition $\mathbb{F}_n = \mathbb{A}^2 \cup \mathbb{A}^1 \cup \mathbb{A}^1 \cup \mathbb{A}^0$.

3.3 Equivariant Chow groups for torus actions

In this section, we want to recap the most important results of equivariant Chow groups for torus actions following [7]. Later on, we want to compare these results to the known results of equivariant cobordism for torus actions (cf .[31]) whereas Brion proves his results for algebraically closed fields of arbitrary characteristic. First, we recall the definition of equivariant Chow groups.

Definition 3.10. *A **rational G-module** is a vector space V equipped with a linear action of G such that every $v \in V$ is contained in a finite-dimensional G-stable subspace on which G acts algebraically.*

Definition 3.11. *[7, Section 2.1] Let X be a scheme with an action of a linear algebraic group G. Let V be a finite-dimensional rational G-module and $U \subseteq V$ a G-stable open subset such that the quotient $U \to U/G$ exists and is a principal G-bundle. Then the quotient for the diagonal action on $X \times U$ exists and is a principal G-bundle. For $n := \dim(X), l := \dim(V)$ and $d := \dim(G)$ we define the* ***i-th equivariant Chow group*** *$\mathrm{CH}_i^G(X) := \mathrm{CH}_{i+l-d}((X \times U)/G)$ if $\mathrm{codim}(V \setminus U) > n - i$. As above for the equivariant algebraic cobordism, the pair (V, U) will be called* ***good pair*** *for the G-action corresponding to i. Following [12], such a good pair (V, U) always exists and $\mathrm{CH}_i^G(X)$ is independent of the choice of the good pair. Further, we set $\mathrm{CH}_*^G(X) := \bigoplus_{i \geq 0} \mathrm{CH}_i^G(X)$. Lastly, each closed G-stable subvariety $Y \subseteq X$ defines a class $[Y]$ in $\mathrm{CH}_*^G(X)$ by setting $[Y] := [(Y \times U)/G]$.*

Remark 3.12. It is known that there is an intersection product on equivariant Chow groups which makes $\mathrm{CH}_*^G(X)$ into a graded ring for a smooth scheme X which we denote by $\mathrm{CH}_G^*(X)$. If X is a smooth equi-dimensional scheme with $n := \dim X$, then $\mathrm{CH}_G^i(X) \cong \mathrm{CH}_{n-i}^G(X)$ holds for all $i \in \mathbb{Z}$. Furthermore, it is worth a remark that $\mathrm{CH}_n^G(X) = 0$ for all $n > \dim X$, but CH_n^G may be also non-trivial for $n < 0$ by definition as opposed to ordinary Chow groups which vanish in negative degrees. Lastly, we will often consider rational equivariant Chow groups by which we mean the tensor product $\mathrm{CH}_*^G(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Construction 3.13. Now, we present a general construction which is going to be used to describe the equivariant Chow groups for torus actions. Therefore, let $G = T$ be a torus, X a scheme with a T-action and M be the character group of T. Let the further notations be as above. In this case, the graded abelian group $\mathrm{CH}_*^T(X)$ has the structure of an S-module where S denotes the character ring of T. This was firstly introduced by Edidin and Graham [12] who also give an explicit good pair (V, U) for the torus case [12, Section 3.1]. To define the S-action it suffices to understand the action of a character $\chi \in M$ on $\mathrm{CH}_*^T(X)$. First of all, we define (V, U) to be the good pair described in [12] and let L_χ be the bundle which is defined by the one-dimensional representation where T acts via weight $-\chi$. This defines another line bundle $U \times L_\chi \to U$ which descends to the line bundle $(U \times L_\chi)/T \to U/T$. Finally, for any closed T-stable subvariety $Y \subseteq X$ we define multiplication by χ as the first equivariant Chern class of the pullback of the line bundle L_χ to X, i.e. $\chi \cdot [Y] = c_1^T(p^* L_\chi)[Y]$ for the map $p : X \to \operatorname{Spec} k$. As shown by [7, Theorem 2.1] the S-module $\mathrm{CH}_*^T(X)$ is generated by the elements $[Y]$ where $Y \subseteq X$ is a closed stable subvariety. This implies that the action above determines the action on the whole S-module $\mathrm{CH}_*^T(X)$. For a G-equivariant vector bundle E on X we will furthermore denote the mixed space $(U \times E)/G$ by E_G as in [12].

Lemma 3.14. *Let X be a T-variety, Y a closed T-stable subvariety and $f \in k(Y)$ a T-eigenfunction with weight χ. Then the relation*

$$[\mathrm{div}_Y(f)] = \chi \cdot [Y]$$

holds in $\mathrm{CH}_*^T(X)$ where $\mathrm{div}_Y(f) := \sum_Z \mathrm{ord}_Z(f) Z$ with $\mathrm{ord}_Z(f)$ describing the order of vanishing along the prime divisor Z for the rational eigenfunction f on Y for weight χ and the sum running over all prime divisors contained in Y.

Proof. Let f be a rational function on Y which is an eigenfunction of T of some weight χ, i.e. $(t \cdot f)(y) = \chi(t) \cdot f(y)$ for all $t \in T$ and some weight $\chi \in M$. We observe that the support of $\mathrm{div}_Y(f)$ is T-stable because we have either $f(z) = 0$ or $f(z) = \infty$ for $z \in \mathrm{Supp}(\mathrm{div}_Y(f))$. In both cases one can see that

$$f(t \cdot z) = (t^{-1} \cdot f)(z) = \chi(t^{-1}) \cdot f(z) = f(z)$$

holds since f is an eigenfunction of T of weight χ and because of the linear T-action on rational functions. Therefore, $[\mathrm{div}_Y(f)]$ defines a class in $\mathrm{CH}_*^T(X)$. Now, we want to show that the relation $[\mathrm{div}_Y(f)] = \chi \cdot [Y]$ holds in the S-module $\mathrm{CH}_*^T(X)$. In order to show this, we consider f as a rational section of $(p^*L_\chi)_T = (X \times U \times L_\chi)/T$ restricted to $(Y \times U)/T$ given by $s : (Y \times U)/T \to (Y \times U \times L_\chi)/T$, $(y, u) \mapsto (y, u, f(y))$. Furthermore,

$$s(t(y, u)) = (ty, tu, f(ty)) = (ty, tu, (t^{-1}f)(y)) = (ty, tu, \chi^{-1}(t)f(y)) = t(y, u, f(y))$$

holds because L_χ is defined by the one-dimensional T-module of weight $-\chi$. Thus, s is a rational section of the line bundle $(p^*L_\chi)_T$ restricted to $(Y \times U)/T$. Let the divisor C be given by $C = \mathrm{div}_{((Y \times U)/T)}(s)$. Therefore, the restriction of the line bundle $(p^*L_\chi)_T$ to $(Y \times U)/T$ is isomorphic to $\mathcal{O}_{(Y \times U)/T}(C)$. By definition (cf. [13, Section 2.5]), this leads to $c_1^T(p^*L_\chi)[Y] = c_1((p^*L_\chi)_T)[(Y \times U)/T] = [C]$. Lastly, C is the same as $\mathrm{div}_Y(f)$ by the definition of the rational section s. This implies the desired relation

$$[\mathrm{div}_Y(f)] = \chi \cdot [Y].$$

□

For equivariant Chow groups the above relations generate all relations as explained in the following result of Brion.

Theorem 3.15. *[7, Theorem 2.1] Let X be a variety with an action of a torus T. The $\mathrm{CH}_*^T(k)$-module $\mathrm{CH}_*^T(X)$ is defined by generators $[Y]$, where $Y \subseteq X$ is a closed T-stable subvariety, and by relations $[\mathrm{div}_Y(f)] = \chi \cdot [Y]$, where f is a non-constant rational function on Y which is an eigenvector of T of weight χ.*

In the following, we will be mainly interested in equivariant Chow groups and equivariant algebraic cobordism groups with rational coefficients which we will denote by an appropriate subscript. The main reason for this is that we want to cover a wider range of examples for which we can compute the equivariant Chow groups and by taking this extension of scalars we can also consider examples where the Chow groups of the connected components of X^T have torsion elements. For example [7, Proposition 3.2] holds for $\mathbb{Q}$-coefficients, but also for integral coefficients if $\mathrm{CH}_*(Y)$ is torsion-free for one specifically determined connected component $Y \subseteq X^T$. It is known that we can have torsion in the Chow groups (see e.g. [14, p.65]) and therefore, we would have to exclude many cases, if we did not take rational coefficients in order to compute the (equivariant) Chow groups.

Proposition 3.16. *[7, Corollary 3.2] Let X be a smooth T-filtrable variety with a group action of a torus T. Then the inclusion of the fixed point scheme $i : X^T \hookrightarrow X$ induces an injective S-algebra homomorphism*

$$i^* : \mathrm{CH}_T^*(X)_\mathbb{Q} \to \mathrm{CH}_T^*(X^T)_\mathbb{Q}$$

which is surjective over the quotient field of S where S denotes the character ring of T.

Proposition 3.17. *[7, Theorem 3.3] Let X be a smooth T-filtrable variety with an action of a torus T. Then the image of the pullback i^* of the inclusion $i : X^T \hookrightarrow X$ is the intersection of the images of*

$$i^*_{T'} : \mathrm{CH}^*_T(X^{T'})_{\mathbb{Q}} \to \mathrm{CH}^*_T(X^T)_{\mathbb{Q}}$$

where T' runs over all subtori of codimension one in T.

Remark 3.18. Following [12, Section 3.2], we know that $\mathrm{CH}^*_T(\{*\}) \cong \mathbb{Z}[t_1, ..., t_n]$ holds, where n denotes the dimension of the torus T. In this case, the character ring S of T is isomorphic to the polynomial ring $\mathbb{Z}[t_1, ..., t_n]$. Therefore, we have the isomorphism $\mathrm{CH}^*_T(\{*\}) \cong S$.

Remark 3.19. The T-action on the fixed point subscheme $X^{T'}$, where T' is a subtorus of codimension one in T, is the induced T-action. We remark here that $X^{T'}$ is T-stable since $tx = tt'x = t'tx$ holds for all $t \in T, t' \in T'$ and $x \in X^{T'}$.

Now, we present the last important statement which is proved using Propositions 3.16 and 3.17. The following proposition is being used in order to compute equivariant Chow groups for torus actions. But before we state the proposition we recall one important definition which will be frequently used in the sequel.

Definition 3.20. *Let G be a group. An element $g \in G$ is called* ***primitive*** *if, whenever $h \in G$ satisfies $h^k = g$ for some $k > 0$, it follows that $k = 1$ and $h = g$.*

Proposition 3.21. *[7, Theorem 3.4] Let X be a smooth T-filtrable variety with finitely many fixed points $x_1, ..., x_s$ and finitely many stable curves under the action of the torus T. Let $i : X^T \hookrightarrow X$ be the inclusion of the fixed point scheme. Then the image of the pullback i^* is the set of all $(f_1, ..., f_s) \in S^s_{\mathbb{Q}}$ such that $f_i \equiv f_j \mod \chi$ whenever x_i and x_j are connected by a stable curve where T acts through the weight χ. If moreover all such characters χ are primitive in the character group, then the statement holds over the integers.*

Remark 3.22. In the preceding proposition we can identify $\mathrm{CH}^*_T(X^T)_{\mathbb{Q}}$ with $S^s_{\mathbb{Q}}$ by Remark 3.18 because of the assumption that the fixed point scheme only consists of the fixed points $x_1, ..., x_s$.

Next, we want to come to a more general example as in Example 3.9 (iii) for which we will compute the equivariant Chow groups in the sequel.

Example 3.23. [12, Section 3.3] We consider one of the T-filtrable varieties above and compute the equivariant Chow groups in a greater generality. Let $X = \mathbb{P}^n_k$ where $T = \mathbb{G}_m$ acts diagonally with weights $a_0, ..., a_n$ via $t \cdot [x_0 : ... : x_n] = [t^{a_0}x_0 : ... : t^{a_n}x_n]$. From this example we can deduce some of the equivariant Chow groups for the examples discussed in Example 3.9 (iii). Firstly, we need to explicitly choose a good pair which will be given by $(V, U) = (\mathbb{A}^l, \mathbb{A}^l \setminus \{0\})$ as described in [12, Section 3.1] where V has all weights equal to -1. Then we have $\mathrm{codim}(V \setminus U) = l$ and $U/T = \mathbb{P}^{l-1}$. Therefore, $(\mathbb{P}^n \times U)/T \to \mathbb{P}^{l-1}$ defines a $\mathbb{P}^n$-bundle, namely $\mathbb{P}(\mathcal{O}(a_0) \oplus ... \oplus \mathcal{O}(a_n)) \to \mathbb{P}^{l-1}$. Using the projective bundle formula for oriented cohomology theories [37, Definition 1.1.2], we obtain

$$\begin{aligned}\mathrm{CH}^*((\mathbb{P}^n \times U)/T) &\cong \mathrm{CH}^*\left(\mathbb{P}(\mathcal{O}(a_0) \oplus ... \oplus \mathcal{O}(a_n))\right) = \mathrm{CH}^*(\mathbb{P}(\mathcal{E})) \\ &\cong \frac{\mathrm{CH}^*(\mathbb{P}^{l-1})[\xi]}{(\xi^{n+1} - c_1(\mathcal{E})\xi^n + ... + (-1)^{n+1}c_{n+1}(\mathcal{E}))}\end{aligned}$$

where $\mathcal{E} := \mathcal{O}(a_0)\oplus ... \oplus\mathcal{O}(a_n)$, $\xi := c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ and $c_i(\mathcal{E})$ denotes the i-th Chern class of $\mathcal{E}$. One can compute $c_i(\mathcal{E})$ using the Whitney sum formula (cf. Section 2) whereas one has the additive formal group law in Chow groups. By defining $t := c_1(\mathcal{O}(1)) \in \mathrm{CH}^1(\mathbb{P}^{l-1})$ we may reduce the last expression to

$$\mathrm{CH}^*(\mathbb{P}(\mathcal{E})) \cong \mathrm{CH}^*(\mathbb{P}^{l-1})[\xi]/(P(\xi,t))$$

with

$$P(\xi,t) = \sum_{i=0}^{n+1} (-1)^i \xi^{n+1-i} e_i(a_0 t, ..., a_n t),$$

where e_i denotes the i-th elementary symmetric polynomial in the variables $a_0t, ..., a_nt$. This fact can be used since $\mathcal{E}$ is a direct sum of line bundles. Letting the dimension of the representation V go to infinity we obtain

$$\mathrm{CH}^*_T(\mathbb{P}^n) = \mathrm{CH}^*((\mathbb{P}^n \times U)/T) = \mathrm{CH}^*(\mathbb{P}(\mathcal{E})) \cong \mathbb{Z}[\xi,t]/(P(\xi,t)).$$

We remark here that this direct computation is completely done with integral coefficients and does not use the above results which will be used later on for different computations.

Remark 3.24. The reader may have seen a slightly different result for this computation where the polynomial $P(\xi,t)$ does not contain the alternating signs for the theory of Chow groups. This comes from the different definitions of a projective bundle in the literature since for example in [13] a projective bundle is defined as $\mathbb{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}\,\mathcal{E}^\vee)$ via the dual sheaf of $\mathcal{E}$. Using this definition, we would end up without having the alternating signs because of the identity $c_i(\mathcal{E}^\vee) = (-1)^i c_i(\mathcal{E})$ which holds in Chow groups due to the fact that we have the additive formal group law.

3.4 Cobordism ring of classifying spaces

In this section, we discuss the cobordism ring of classifying spaces, i.e. the equivariant cobordism ring of a point. The main constructions given here mainly refer to Krishna's paper [32] where the whole equivariant cobordism theory for arbitrary k-schemes with a G-action was introduced for a linear algebraic group G.

Construction 3.25. Following [32], let R be a noetherian ring and $A = \bigoplus_{j\in\mathbb{Z}} A_j$ be a $\mathbb{Z}$-graded R-algebra with $R \subseteq A_0$. In this case, we consider the **graded power series ring** $S^{(n)} = \bigoplus_{i\in\mathbb{Z}} S_i$ which is a graded ring where S_i is the set of formal power series $f(\boldsymbol{t}) = \sum_{m(\boldsymbol{t})\in\mathcal{C}} a_{m(\boldsymbol{t})} m(\boldsymbol{t})$ such that $a_{m(\boldsymbol{t})}$ is a homogeneous element of A of degree $|a_{m(\boldsymbol{t})}|$ with $|a_{m(\boldsymbol{t})}| + |m(\boldsymbol{t})| = i$. Moreover, $\mathcal{C}$ denotes the set of all monomials in $\boldsymbol{t} = (t_1, ..., t_n)$ and $|m(\boldsymbol{t})| = i_1 + ... + i_n$ the degree of the monomial $m(\boldsymbol{t}) = t_1^{i_1} \cdots t_n^{i_n}$. We denote this graded power series ring by $A[[\boldsymbol{t}]]_{\mathrm{gr}}$ to make it easier to distinguish between the graded formal power series ring and the usual formal power series ring $A[[\boldsymbol{t}]]$. From this follows immediately that if A is only non-negatively graded, then $S^{(n)}$ is the standard polynomial ring $A[t_1, ..., t_n]$ over A. We will later see an example where this particular case happens. We remark here as well that $S^{(n)}$ is a graded ring which is a subring of the formal power series ring $A[[t_1, ..., t_n]]$.

Now, we want to summarise the most important properties of the graded power series ring.

Lemma 3.26. *[32, Lemma 6.5] Let the notations be as above.*

(i) There are inclusions of rings $A[t_1, ..., t_n] \subseteq S^{(n)} \subseteq A[[t_1, ..., t_n]]$*, where the first inclusion is an inclusion of graded rings.*

(ii) $S^{(n-1)}[[t_n]]_{\text{gr}} \xrightarrow{\cong} S^{(n)}$.

(iii) $\frac{S^{(n)}}{(t_{i_1}, ..., t_{i_r})} \xrightarrow{\cong} S^{(n-r)}$ *for any* $n \geq r \geq 1$*, where* $S^{(0)} = A$.

(iv) The sequence $\{t_1, ..., t_n\}$ *is a regular sequence in* $S^{(n)}$.

(v) If $A = R[x_1, x_2, ...]$ *is a polynomial ring with negative degree of the variables* x_i *and* $\lim_{i\to\infty} |x_i| = -\infty$*, then*

$$S^{(n)} \xrightarrow{\cong} \varprojlim_i R[x_1, ..., x_i][[\boldsymbol{t}]]_{\text{gr}}$$

holds.

In the following, we will give some examples of the cobordism ring of classifying spaces which have been already computed.

Proposition 3.27. *[32, Proposition 6.7] Let* $\{\chi_1, ..., \chi_n\}$ *be a basis of the character group of a torus* T *of rank* n*. Then the assignment* $t_i \mapsto c_1^T(L_{\chi_i})$ *yields a graded* $\mathbb{L}$*-algebra isomorphism*

$$\mathbb{L}[[t_1, ..., t_n]]_{\text{gr}} \xrightarrow{\cong} \Omega^*(\text{BT}) = \Omega_T^*(k),$$

where L_{χ_i} *is the one-dimensional representation of* T *on which* T *acts via the weight* $-\chi_i$ *and* $c_1^T(L_{\chi_i})$ *are the equivariant Chern classes defined in Remark 2.45.*

In the sequel, let $S(T)$ denote the cobordism ring $\Omega_T^*(k)$ of the classifying space of T.

Remark 3.28. In [32] the previous proposition is stated with the one-dimensional representation L_{χ_i} of T on which T acts via weight χ_i. For our purposes, we will use the convention described above because it will not change the results in [32], but it will be useful later on in this work. The explicit construction in the proof of the previous proposition involves a specific choice of good pairs (V_j, U_j) such that the quotient of U_j/T is a product of projective spaces. Then one uses the projective bundle formula for the ordinary algebraic cobordism to conclude the result. This construction can be seen in [32, Section 6.3].

Remark 3.29. For the case $G = \text{GL}_n$ we can take the good pairs (V_j, U_j) described in [50, Remark 1.4] in order to obtain Grassmannians as the corresponding mixed quotients. Then Krishna [32] argues that one can compute the cobordism ring of this mixed quotient using the projective bundle formula again which leads to the isomorphism

$$\Omega^*(\text{BGL}_n) \xrightarrow{\cong} \mathbb{L}[[\gamma_1, ..., \gamma_n]]_{\text{gr}}$$

of graded $\mathbb{L}$-algebras where the γ_i are the elementary symmetric polynomials in the variables $t_1, ..., t_n$.

3.5 Equivariant algebraic cobordism for torus actions

In this section, we want to compare Section 3.3 with the results of Krishna [31] which we will later use to compute some equivariant algebraic cobordism groups, but firstly we introduce the notion of the topological tensor product which we will need in the sequel. Furthermore, this contruction will be compared to the regular tensor product. The main parts of this section are taken from [32] and [31].

Construction 3.30. Following the notations in [32], let A be a commutative ring with unit and let $\{L_n\}$ and $\{M_n\}$ be two inverse systems of A-modules with inverse limits L and M respectively. We define the **topological tensor product** of L and M by

$$L\widehat{\otimes}_A M := \varprojlim_n (L_n \otimes_A M_n).$$

Following [32], if D is an integral domain with quotient field F and if $\{A_n\}$ is an inverse system of D-modules with inverse limit A, one has

$$A\widehat{\otimes}_D F = \varprojlim_n (A_n \otimes_D F).$$

In order to simplify the notations we denote $A\widehat{\otimes}_D F$ in the sequel by A_F.

If R is a $\mathbb{Z}$-graded ring and M and N are graded R-modules, then $M \otimes_R N$ is a graded R-module given as the quotient of $M \otimes_{R_0} N$ modulo the graded submodule generated by the homogeneous elements of type $ax \otimes y - x \otimes ay$ where a, x and y are homogeneous elements of R, M and N, respectively, and R_0 denotes the degree zero part of the ring R. If furthermore the graded parts M_i and N_i are limits of inverse systems $\{M_i^\lambda\}$ and $\{N_i^\lambda\}$ of R_0-modules, then we define the **graded topological tensor product** as $M\widehat{\otimes}_R N = \bigoplus_{i\in\mathbb{Z}} (M\widehat{\otimes}_R N)_i$, where

$$(M\widehat{\otimes}_R N)_i = \varprojlim_\lambda \left(\bigoplus_{j+j'=i} \frac{M_j^\lambda \otimes_{R_0} N_{j'}^\lambda}{(ax \otimes y - x \otimes ay)} \right).$$

The reader may observe that this graded topological tensor product reduces to the ordinary graded tensor product if the underlying inverse systems are trivial.

Similarly to Remark 2.19 we recall the existence of a unique formal graded power series $\chi(u_i) \in \mathbb{L}[[u_1, ..., u_n]]_{\text{gr}}$ which satisfies $F_\Omega(u_i, \chi(u_i)) = 0$. For any positive integer $b \in \mathbb{Z}_{\geq 1}$ we establish the following notations.

$$\begin{aligned}
u_i +_{F_\Omega} u_j &:= F_\Omega(u_i, u_j) \in \mathbb{L}[[u_i, u_j]]_{\text{gr}}, \\
[-1]_{F_\Omega} u_i &:= \chi(u_i) \in \mathbb{L}[[u_i]]_{\text{gr}}, \\
u_i -_{F_\Omega} u_j &:= F_\Omega(u_i, \chi(u_j)) \in \mathbb{L}[[u_i, u_j]]_{\text{gr}}, \\
[0]_{F_\Omega} u_i &:= 0, \\
[b]_{F_\Omega} u_i &:= F_\Omega(u_i, [b-1]_{F_\Omega} u_i) \in \mathbb{L}[[u_i]]_{\text{gr}}.
\end{aligned}$$

We remark that the final relation is an inductive definition of $[b]_{F_\Omega} u_i$. Further, it is clear that $[b]_{F_\Omega} u$ is divisible by u for any $u \in \mathbb{L}[[u_1, ..., u_n]]_{\text{gr}}$ of degree 1.

Lemma 3.31. *Let $u \in \mathbb{L}[[u_1, ..., u_n]]_{\text{gr}}$ be a homogeneous element of degree 1. Then there exists an element $g \in \mathbb{L}_{\mathbb{Q}}[[x]]$ such that $u = g([b]_{F_\Omega} u)$ for any $b \in \mathbb{Z}_{\geq 1}$.*

Proof. Fix $b \in \mathbb{Z}_{\geq 1}$ and write

$$[b]_{F_\Omega} u = b_1 u + b_2 a_{11} u^2 + b_3 a_{21} u^3 + b_4 a_{12} u^3 + b_5 a_{11}^2 u^3 + ...$$

for $b_i \in \mathbb{Z}_{\geq 0}$ for all $i \geq 1$. Now, we construct an element ρ of degree 0 such that $\rho \cdot [b]_{F_\Omega} u = u$ holds. By comparison of coefficients, we observe that ρ is given by

$$\rho = \frac{1}{b_1} - b_2 \frac{a_{11}}{b_1^2} u + \left(-\frac{b_3}{b_1^2} a_{21} - \frac{b_4}{b_1^2} a_{12} + \left(-\frac{b_5}{b_1^2} + \frac{b_2^2}{b_1^3} \right) a_{11}^2 \right) u^2 + ...$$

Successively replacing u with $\rho \cdot [b]_{F_\Omega} u$ implies the claim. ☐

Definition 3.32. *Let $u \in \mathbb{L}[[u_1, ..., u_n]]_{\mathrm{gr}}$ be a homogeneous element of degree 1. Then for $n \in \mathbb{Z}_{\geq 1}$ we define*

$$[-n]_{F_\Omega} u := [-1]_{F_\Omega} \left([n]_{F_\Omega} u \right).$$

Furthermore, if there exists a homogeneous element $u' \in (\mathbb{L}[[u_1, ..., u_n]]_{\mathrm{gr}})_{\mathbb{Q}}$ of degree 1 such that $[m]_{F_\Omega} u' = u$ holds for $m \in \mathbb{Z}_{\geq 1}$, then we define

$$\left[\frac{1}{m}\right]_{F_\Omega} u := u'.$$

Definition 3.33. *In the setting of the above definition we define the operator $\rho_{n/m}$ by*

$$\rho_{n/m} u := \frac{[n]_{F_\Omega} \left(\left[\frac{1}{m}\right]_{F_\Omega} u \right)}{u}$$

in $(\mathbb{L}[[u_1, ..., u_n]]_{\mathrm{gr}})_{\mathbb{Q}}$ for any $n \in \mathbb{Z} \setminus \{0\}$ and $m \in \mathbb{Z}_{\geq 1}$.

Remark 3.34. The quotient $\rho_{n/m} u$ is indeed in $(\mathbb{L}[[u_1, ..., u_n]]_{\mathrm{gr}})_{\mathbb{Q}}$ for any $n \in \mathbb{Z} \setminus \{0\}$ and $m \in \mathbb{Z}_{\geq 1}$ because $\left[\frac{1}{m}\right]_{F_\Omega} u \in (\mathbb{L}[[u_1, ..., u_n]]_{\mathrm{gr}})_{\mathbb{Q}}$ is homogeneous of degree 1 and therefore, $\left[\frac{1}{m}\right]_{F_\Omega} u = g(u)$ holds for some $g \in \mathbb{L}_{\mathbb{Q}}[[x]]$ by Lemma 3.31. Further, $g(u)$ is divisible by u by construction and thus, $[n]_{F_\Omega} \left(\left[\frac{1}{m}\right]_{F_\Omega} u \right)$ is divisible by u.

Remark 3.35. Let M be the character group of a torus T of finite rank. Using Definition 3.33 one observes that

$$\rho_{n/m} c_1^T(L_\chi) = \frac{c_1^T(L_{n\chi/m})}{c_1^T(L_\chi)}$$

holds in $S(T)_{\mathbb{Q}}$ for any character $\chi \in M$, $n \in \mathbb{Z} \setminus \{0\}$ and $m \in \mathbb{Z}_{\geq 1}$ if $\frac{n\chi}{m}$ is also a character in M.

Now, we will prove a lemma which will be of great importance in the sequel for comparing the equivariant algebraic cobordism with respect to a torus T and its quotient T/F by a finite subgroup F.

Lemma 3.36. *Let T be a torus of rank n and F be a finite subgroup. Then we have a graded $\mathbb{L}$-algebra isomorphism*

$$\Omega_T^*(k)_{\mathbb{Q}} \cong \Omega_{T/F}^*(k)_{\mathbb{Q}}.$$

Proof. Let $\{\chi_1, ..., \chi_n\}$ be a basis of the character group of T such that the basis of the character group of T/F is then given by $\{a_1\chi_1, ..., a_n\chi_n\}$ for some positive integers $a_1|a_2|\cdots|a_n$. Using Proposition 3.27 we know that there is an $\mathbb{L}$-algebra isomorphism $\Omega_T^*(k) \cong \mathbb{L}[[t_1, ..., t_n]]_{\text{gr}}$ mapping $c_1^T(L_{\chi_i}) \mapsto t_i$ where L_{χ_i} is the one-dimensional representation of weight $-\chi_i$. Furthermore, we have $\Omega_{T/F}^*(k) \cong \mathbb{L}[[t_1', ..., t_n']]_{\text{gr}}$ for $c_1^{T/F}(L_{a_i\chi_i}) \mapsto t_i'$. Since we consider the L_{χ_i} as the one-dimensional representations of T and similarly those of T/F, we know that

$$c_1^T(L_{a_i\chi_i}) = c_1^T(L_{\chi_i+...+\chi_i}) = c_1^T(L_{\chi_i} \otimes ... \otimes L_{\chi_i}) = [a_i]_{F_\Omega} c_1^T(L_{\chi_i})$$

holds in $\Omega_T^*(k)$ where F_Ω denotes again the universal formal group law in cobordism. On the other hand, we know that we can take $c_1^T(L_{a_i\chi_i})$ as generators of $\Omega_T^*(k)_{\mathbb{Q}}$ instead of $c_1^T(L_{\chi_i})$ as soon as we consider rational coefficients by Lemma 3.31. This leads to the desired isomorphism. □

Remark 3.37. The preceding lemma implies the same statement for equivariant Chow groups and furthermore we remark that the finite subgroup F has order $a_1 \cdots a_n$. Lastly, the statement also holds if we only take coefficients in $\mathbb{Z}[\frac{1}{p_1}, ..., \frac{1}{p_\ell}]$ where $p_1, .., p_\ell$ are the primes occuring in the prime factorisation of a_n. Therefore, we only have to invert a finite number of primes in order to obtain the isomorphism of Lemma 3.36.

The next step for the computations in this article is to describe a result in equivariant cobordism which is similar to the following one in Chow groups. Recall that for any T-scheme X, any closed T-stable subvariety $Y \subseteq X$ and any rational function f on Y which is an eigenvector of T for weight χ, we have $\chi \cdot [Y] = \text{div}_Y(f)$ in the $\text{CH}_T^*(k)$-module $\text{CH}_T^*(X)$ (cf. [7, Theorem 2.1]). We would like to have such a relation for smooth schemes X in equivariant cobordism and therefore, we need to understand properly the $S(T)$-action on $\Omega_*^T(X)$ for $X \in \mathbf{Sm}_k$.

Construction 3.38. Now, we present a similar construction to the one introduced to prove the above relation in Chow groups in [7, Theorem 2.1]. By Proposition 3.27 we know that for any basis $\{\chi_1, ..., \chi_n\}$ of the character group of T we have the isomorphism $\mathbb{L}[[t_1, ..., t_n]]_{\text{gr}} \cong S(T), t_i \mapsto c_1^T(L_{\chi_i})$, where in this case we set L_{χ_i} to be the one-dimensional representation of T on which T acts via weight $-\chi_i$. Hereby, $c_1^T(L_{\chi_i})$ means $c_1^T(L_{\chi_i})[\text{Spec}\, k \to \text{Spec}\, k]$ where $[\text{Spec}\, k \to \text{Spec}\, k]$ is by abuse of notation the equivariant fundamental class of the ordinary cobordism cycle $[\text{Spec}\, k \to \text{Spec}\, k]$. For any character χ and an l_j-dimensional good pair (V_j, U_j) we have the line bundle $(L_\chi \times U_j)/T \to U_j/T$ which we denote by $(L_\chi)_T$. Since equivariant cobordism is defined via an inverse limit construction we consider the elements

$$c_1^T(L_\chi)[\text{Spec}\, k \to \text{Spec}\, k] = \varprojlim_j \widetilde{c}_1((L_\chi)_T)[U_j/T \to U_j/T].$$

We take one of the ordinary cobordism cycles $[h : Y \to X]$ and consider the ordinary cobordism cycle $[(Y \times U_j)/T \to (X \times U_j)/T]$ in the j-th component of the equivariant fundamental class which we denote by $[Y \to X]_j$ for some good pair (V_j, U_j). For the morphism $g : (X \times U_j \times U_j)/T \to U_j/T$, that is induced by the second projection $p_2 : U_j \times U_j \to U_j$, we use the exterior product on equivariant cobordism which was described in the proof of [32, Theorem 5.2] and thus, we obtain

$$\begin{aligned} c_1^T(L_\chi) \cdot [Y \to X] &= \varprojlim_j (\widetilde{c}_1((L_\chi)_T)[U_j/T \to U_j/T] \cdot [(Y \times U_j)/T \to (X \times U_j)/T]) \\ &= \varprojlim_j \widetilde{c}_1(g^*(L_\chi)_T)[(Y \times U_j \times U_j)/T \to (X \times U_j \times U_j)/T] \end{aligned}$$

in $\Omega_*^T(X)$. We observe that in this case the line bundle $g^*(L_\chi)_T$ is obtained by the good pair $(V_j \times V_j, U_j \times U_j)$ of dimension $2l_j$ for $j \geq 0$.

Theorem 3.39. *Let X be a smooth T-variety, $[h : Y \to X]$ the equivariant fundamental class of a T-stable cobordism cycle and $f \in k(Y)$ a rational T-eigenfunction with weight χ. Denote by Z_0 and Z_∞ the zeros and poles of f, respectively, and assume that they are smooth. Then the relation*

$$c_1^T(L_\chi) \cdot [Y \to X] = h_* F_\Omega ([Z_0 \to Y], [-1]_{F_\Omega}[Z_\infty \to Y])$$

holds in $\Omega_*^T(X)$ where F_Ω denotes the universal formal group law and $[-1]_{F_\Omega}$ is the inverse in the universal formal group law.

Proof. We consider the rational function f on Y. One may observe that

$$s : (Y \times U_j \times U_j)/T \to (Y \times U_j \times U_j \times L_\chi)/T, (y, u_1, u_2) \mapsto (y, u_1, u_2, f(y))$$

is a rational section of the line bundle $h^*g^*(L_\chi)_T$. For this line bundle with the given rational section, we can also write

$$h^*g^*(L_\chi)_T = \mathcal{O}_{(Y \times U_j \times U_j)/T}(Z_0 - Z_\infty) \cong \mathcal{O}_{(Y \times U_j \times U_j)/T}(Z_0) \otimes \mathcal{O}_{(Y \times U_j \times U_j)/T}(Z_\infty)^\vee$$

by the known correspondence between Cartier divisors and pairs (L, s) consisting of a line bundle and a rational section. We simplify by setting $L_0 = \mathcal{O}_{(Y \times U_j \times U_j)/T}(Z_0)$ and similarly $L_\infty = \mathcal{O}_{(Y \times U_j \times U_j)/T}(Z_\infty)$. By the smoothness assumption we know that the corresponding sections of L_0 and L_∞ coming from the rational section s are transverse to the zero sections of L_0 and L_∞, respectively. Furthermore, the zero-subschemes of these sections are T-stable and hence, they define cobordism cycles whose equivariant fundamental classes are in $\Omega_*^T(Y)$. In the following computation we will use [37, Definition 2.1.2] axiom (A3) and [37, Definition 2.2.1] axiom (Sect). By [37, Proposition 5.2.4], we know further that the Chern class operator $\widetilde{c}_1(L)$ on a smooth scheme X is given by $\widetilde{c}_1(L)(\eta) = c_1(L) \cdot \eta$ for $\eta \in \Omega^*(X)$. Lastly, we have the embeddings of the zero-subschemes $i_0 : (Z_0 \times U_j \times U_j)/T \to (Y \times U_j \times U_j)/T$ and similarly $i_\infty : (Z_\infty \times U_j \times U_j)/T \to (Y \times U_j \times U_j)/T$. Using all those properties, we obtain

$$\begin{aligned} &\widetilde{c}_1(g^*(L_\chi)_T)[(Y \times U_j \times U_j)/T \to (X \times U_j \times U_j)/T] \\ =&\widetilde{c}_1(g^*(L_\chi)_T)h_*[1_{(Y \times U_j \times U_j)/T}] \\ =&h_*\widetilde{c}_1(h^*g^*(L_\chi)_T)[1_{(Y \times U_j \times U_j)/T}] \\ =&h_*\widetilde{c}_1(L_0 \otimes L_\infty^\vee)[1_{(Y \times U_j \times U_j)/T}] \\ =&h_*F_\Omega(\widetilde{c}_1(L_0), \widetilde{c}_1(L_\infty^\vee))[1_{(Y \times U_j \times U_j)/T}] \end{aligned}$$

$$
\begin{aligned}
=&h_*\left(\widetilde{c}_1(L_0)[1_{(Y\times U_j\times U_j)/T}]+[-1]_{F_\Omega}\widetilde{c}_1(L_\infty)[1_{(Y\times U_j\times U_j)/T}]\right)\\
&+h_*\left(\sum_{i,k\geq 1}a_{ik}\widetilde{c}_1(L_0)^i\circ\widetilde{c}_1(L_\infty^\vee)^k[1_{(Y\times U_j\times U_j)/T}]\right)\\
=&h_*\left(\widetilde{c}_1(L_0)[1_{(Y\times U_j\times U_j)/T}]+[-1]_{F_\Omega}\widetilde{c}_1(L_\infty)[1_{(Y\times U_j\times U_j)/T}]\right)\\
&+h_*\left(\sum_{i,k\geq 1}a_{ik}c_1(L_0)^i\cdot c_1(L_\infty^\vee)^k\right)\\
=&h_*\left(i_{0*}(1_{(Z_0\times U_j\times U_j)/T})+[-1]_{F_\Omega}i_{\infty*}(1_{(Z_\infty\times U_j\times U_j)/T})\right)\\
&+h_*\left(\sum_{i,k\geq 1}a_{ik}i_{0*}\left(1_{(Z_0\times U_j\times U_j)/T}\right)^i\cdot\left([-1]_{F_\Omega}i_{\infty*}(1_{(Z_\infty\times U_j\times U_j)/T})\right)^k\right)\\
=&h_*\left([Z_0\to Y]_j+[-1]_{F_\Omega}[Z_\infty\to Y]_j+\sum_{i,k\geq 1}a_{ik}[Z_0\to Y]_j^i\cdot([-1]_{F_\Omega}[Z_\infty\to Y]_j)^k\right)
\end{aligned}
$$

and furthermore, the sum is finite since the ordinary first Chern classes are nilpotent. We conclude the claim because taking the limit on these elements commutes with the pushforward h_* by the definition of the equivariant pushforward maps. □

We recall the morphism $g : (X \times U_j \times U_j)/T \to U_j/T$ and the given line bundle $(L_\chi \times U_j)/T \to U_j/T$ which we denote by $(L_\chi)_T$ (cf. Construction 3.38). In the sequel, we will only consider the cases in which there exists a global section of the line bundle $h^*g^*(L_\chi)_T$ which is transverse to the zero section. In this particular case, the terms containing Z_∞ disappear and one obtains the following statement.

Corollary 3.40. *Assume there exists a global section s of the line bundle $h^*g^*(L_\chi)_T$ which is transverse to the zero section. In this case, the relation*

$$c_1^T(L_\chi)\cdot[Y\to X]=[Z_0\to X]$$

holds in $\Omega_*^T(X)$ where Z_0 is the zero-subscheme of s on Y.

Remark 3.41. Similarly to Theorem 3.15, we know that $\Omega_*^T(X)$ is generated by the equivariant fundamental classes of the T-stable cobordism cycles in $\Omega_*(X)$ by [31, Corollary 4.8] for a smooth projective variety X with an action of a torus and finitely many T-fixed points. At present the author does not know whether the equivariant cobordism modules $\Omega_*^T(X)$ are given by the equivariant fundamental classes of T-stable cobordism cycles in $\Omega_*(X)$ modulo the previously described relations from Proposition 3.39, but it might be enough for smooth projective varieties X with an action of a torus and finitely many T-fixed points.

Remark 3.42. Now, we want to restrict the previously described relation in equivariant algebraic cobordism to equivariant Chow groups by using the natural transformation $\vartheta : \Omega^*(X) \to \mathrm{CH}^*(X)$. Therefore, we consider the relation from Theorem 3.39. Firstly, we define

$$
\begin{aligned}
n &:= [K(Y) : K(h(Y))],\\
n_0 &:= [K(Z_0) : K(h(Z_0))],
\end{aligned}
$$

$$n_\infty := [K(Z_\infty) : K(h(Z_\infty))],$$

where $n_0 = n_\infty$ by construction since $[Z_0] = [Z_\infty] \in \mathrm{CH}^*(Y)$ and therefore, their images under the pushforward must coincide. We compute the left-hand side of the relation which leads to

$$\begin{aligned}\vartheta(c_1^T(L_\chi) \cdot [Y \to X]) &= c_1^T(L_\chi) h_* \vartheta([1_Y]) \\ &= c_1^T(L_\chi) h_*[Y] \\ &= c_1^T(L_\chi) n[h(Y)].\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\vartheta(h_* F_\Omega ([Z_0 \to Y], [-1]_{F_\Omega}[Z_\infty \to Y]) \\ &= h_*\vartheta\left([Z_0 \to Y] + [-1]_{F_\Omega}[Z_\infty \to Y] + \sum_{i,k\geq 1} a_{ik}[Z_0 \to Y]^i \cdot ([-1]_{F_\Omega}[Z_\infty \to Y])^k\right) \\ &= h_* i_{0*}\vartheta([1_{Z_0}]) - h_* i_{\infty *}\vartheta([1_{Z_\infty}]) \\ &= h_*[Z_0] - h_*[Z_\infty] \\ &= n_0[h(Z_0)] - n_0[h(Z_\infty)]\end{aligned}$$

since the a_{ik} and all the summands of $[-1]_{F_\Omega}[Z_\infty \to Y]$ except the first one vanish in $\mathrm{CH}_T^*(X)$ because they are in $\mathbb{L}^{<0}$. Furthermore, i_0 and i_∞ are the inclusions of the corresponding zero-subschemes and therefore, the degree of the extension of the corresponding function fields is one. This leads to

$$\chi \cdot n \cdot [h(Y)] = n_0([h(Z_0)] - [h(Z_\infty)])$$

which describes the relation in Lemma 3.14 for equivariant Chow groups. Let X be a smooth projective T-variety X with finitely many T-fixed points. By [31, Corollary 4.8] we know that the $S(T)$-module $\Omega_*^T(X)$ is generated by the fundamental classes of the T-stable cobordism cycles $[\widetilde{X}_m \to X]$ where $\widetilde{X}_m$ denotes a T-equivariant resolution of singularities of X_m, the latter coming from the filtration of X. Therefore, the map h is a closed immersion up to the exceptional divisor which implies $n = n_0 = n_\infty = 1$ and hence

$$\chi_i \cdot [h(Y)] = [h(Z_0)] - [h(Z_\infty)].$$

Next, we want to state analogous versions of the results for equivariant Chow groups in equivariant algebraic cobordism, but in order to be able to formulate them, we have to introduce some further notation which we recall from [31, Section 6].

Construction 3.43. Let T be a torus of rank n and let $\{\chi_1, ..., \chi_n\}$ be a basis for the character group M. We recall the isomorphism $S(T) = \Omega_T^*(k) \cong \mathbb{L}[[t_1, ..., t_n]]_{\mathrm{gr}}$ by Proposition 3.27 where $t_j = c_1^T(L_{\chi_j})$ for all $1 \leq j \leq n$. From the formal group law for equivariant cobordism (cf. [32, Section 6.2]), i.e. $c_1^T(L_1 \otimes L_2) = c_1^T(L_1) +_{F_\Omega} c_1^T(L_2)$, we obtain

$$c_1^T(L_{m\chi_j}) = mt_j + t_j^2 \sum_{i\geq 0} a_i t_j^i.$$

Furthermore, for a character $\chi = \sum_{j=1}^{n} m_j\chi_j$ we have

$$c_1^T(L_\chi) = \sum_{j=1}^{n} [m_j]_{F_\Omega} t_j = \sum_{j=1}^{n} m_j t_j + \sum_{|m(\boldsymbol{t})|\geq 2} a_{m(\boldsymbol{t})} m(\boldsymbol{t})$$

where we use the notation from Construction 3.25. Let $S(T)[M^{-1}]$ denote the ring obtained by inverting all non-zero linear forms $\sum_{j=1}^{n} m_j t_j$ in $S(T)$. Then $S(T)[M^{-1}]$ is a graded ring and for $f = \sum_{j=1}^{n} m_j t_j$ we can write

$$c_1^T(L_\chi) = f\left(1 + f^{-1} \sum_{|m(\boldsymbol{t})|\geq 2} a_{m(\boldsymbol{t})} m(\boldsymbol{t})\right)$$

in $S(T)[M^{-1}]$. By construction, the element inside the parenthesis is homogeneous of degree zero in $S(T)[M^{-1}]$ and also invertible because the constant term is 1. This implies that $c_1^T(L_\chi)$ is invertible in $S(T)[M^{-1}]$. For a smooth k-scheme X with a torus action, we denote $\Omega_T^*(X) \otimes_{S(T)} S(T)[M^{-1}]$ by $\Omega_T^*(X)[M^{-1}]$.

We recall that in general we only know that the restriction map is surjective (cf. 2.43), but now we present some results which suffice to prove [31, Theorem 7.1].

Lemma 3.44. *[31, Lemma 4.6] Let X be a T-filtrable variety with a T-action and a filtration (3.1). Then for every $0 \leq m \leq n$, there is a canonical split exact sequence*

$$0 \to \Omega_*^T(X_{m-1}) \xrightarrow{i_{(m-1)*}} \Omega_*^T(X_m) \xrightarrow{j_m^*} \Omega_*^T(U_m) \to 0.$$

Remark 3.45. The proof of the previous lemma does not rely on the localisation sequence claimed by Heller and Malagón-López, but instead uses arguments via the Motivic Borel-Moore cobordism theory MGL and the refined localisation sequence (cf. Proposition 2.37).

Corollary 3.46. *[31, Corollary 4.8] Let T be a torus of rank r acting on a T-filtrable variety X such that X^T is the finite set of smooth closed points $\{x_0, ..., x_n\}$. For any $0 \leq m \leq n$, let $f_m : \widetilde{X}_m \to X_m$ be a T-equivariant resolution of singularities and let $\widetilde{x}_m$ be the fundamental class of the T-stable cobordism cycle $[\widetilde{X}_m \to X]$ in $\Omega_*^T(X)$. Then $\Omega_*^T(X)$ is a free $S(T)$-module with basis $\{\widetilde{x}_0, ..., \widetilde{x}_n\}$.*

It is not clear to the author whether [31, Theorem 3.4] still holds in its generality because the proof uses the unknown localisation result [20, Theorem 20]. Therefore, we formulate a weaker version of [31, Theorem 3.4] which is relevant for our setting.

Proposition 3.47. *Let X be a T-filtrable variety with a T-action and finitely many T-fixed points. Then the forgetful map $r_X^T : \Omega_*^T(X) \to \Omega_*(X)$ induces an isomorphism of $\mathbb{L}$-modules*

$$\overline{r}_X^T : \Omega_*^T(X) \otimes_{S(T)} \mathbb{L} \xrightarrow{\cong} \Omega_*(X).$$

If X is smooth, this is an $\mathbb{L}$-algebra isomorphism.

Proof. This statement can be obtained immediately from Corollary 3.46 and [21, Theorem 2.5] once one observes that T-filtrable varieties with finitely many T-fixed points are indeed cellular (cf. [21, Definition 2.4]). □

Using the previous two results, we can deduce the following corollary.

Corollary 3.48. *Let X be a T-filtrable variety with a T-action and finitely many T-fixed points. For any $0 \leq m \leq n$, the inclusions $i : X_m \to X_n$ and $j : X_n \setminus X_m \to X_n$ induce the short exact sequence*

$$0 \to \Omega_*^T(X_m) \xrightarrow{i_*} \Omega_*^T(X_n) \xrightarrow{j^*} \Omega_*^T(X_n \setminus X_m) \to 0.$$

Proof. Using Proposition 2.43 and Lemma 3.44 repetitively, we know that j^* is surjective and that i_* is injective. Thus, we only need to show exactness at the middle term in order to obtain the statement. Since X_m is T-filtrable, we may use Corollary 3.46 which implies that a general element $a \in \Omega_*^T(X_m)$ is of the form

$$a = a_0[\widetilde{X}_0 \to X_m] + ... + a_m[\widetilde{X}_m \to X_m]$$

for $a_i \in S(T)$ for all $0 \leq i \leq m$. Hence, we have

$$j^* i_* a = a_0 j^*[\widetilde{X}_0 \to X_n] + ... + a_m j^*[\widetilde{X}_m \to X_n] = 0$$

where $j^*[\widetilde{X}_i \to X_n]$ vanish for all $0 \leq i \leq m$. This implies $\mathrm{Im}(i_*) \subseteq \mathrm{Ker}(j^*)$. Now, let $b \in \mathrm{Ker}(j^*)$. Again by Corollary 3.46, we have

$$b = b_0[\widetilde{X}_0 \to X_n] + ... + b_n[\widetilde{X}_n \to X_n]$$

for $b_i \in S(T)$ for all $0 \leq i \leq n$. As above we know that $j^*[\widetilde{X}_i \to X_n] = 0$ if and only if $0 \leq i \leq m$. Since $j^* b = 0$ we have

$$0 = b_{m+1} j^*[\widetilde{X}_{m+1} \to X_n] + ... + b_n j^*[\widetilde{X}_n \to X_n]$$

and therefore, we conclude $b_i = 0$ for all $m+1 \leq i \leq n$ because the pullbacks $j^*[\widetilde{X}_i \to X_n]$ do not vanish for $m + 1 \leq i \leq n$. This implies $\mathrm{Ker}(j^*) \subseteq \mathrm{Im}(i_*)$ which finishes the proof. ☐

Now, we will state the analogous version of Proposition 3.16 for equivariant algebraic cobordism. The upcoming statement was originally formulated for smooth T-filtrable schemes using the definition in Section 3.1, but in this thesis we will be mainly interested in smooth projective schemes with finitely many T-fixed points and thus, we state it only for this specific subclass of T-schemes.

The proof of the following proposition (cf. [31, Theorem 7.1]) relies heavily on the possibly incorrect localisation sequence (cf. discussion before Proposition 2.43). Nevertheless, we are able to use the following three propositions since one can simply replace [31, Proposition 4.1 (i)] with Corollary 3.48 in the proofs of the given statements in our setting with finitely many T-fixed points. The above mentioned fact will become even clearer in the proof of the refined localisation theorem (cf. Proposition 3.59).

Proposition 3.49. *[31, Theorem 7.1] Let X be a smooth projective T-scheme with finitely many T-fixed points. For the inclusion $i : X^T \hookrightarrow X$ of the fixed point subscheme the $S(T)_{\mathbb{Q}}$-algebra map*

$$i^* : \Omega_T^*(X)_{\mathbb{Q}} \to \Omega_T^*(X^T)_{\mathbb{Q}}$$

is injective. Furthermore, i^* is an isomorphism over $S(T)_{\mathbb{Q}}[M^{-1}]$.

Remark 3.50. The preceding proposition slightly differs from Proposition 3.16 because inverting all non-zero linear forms as described in Construction 3.43 does not lead to the quotient field of $S(T)_{\mathbb{Q}}$.

The following proposition will be crucial for the upcoming computations and results. This is the analogue of Proposition 3.17 in equivariant algebraic cobordism, but in this case Krishna needed the assumption that the fixed point scheme consists only of finitely many isolated points which was not needed in the original result from Brion for equivariant Chow groups.

Proposition 3.51. *[31, Theorem 7.6] Let X be a smooth projective scheme with an action of a torus T. Further, let X^T consist of finitely many fixed points $x_1, ..., x_p$ and let $i : X^T \hookrightarrow X$ denote the inclusion of the fixed point subscheme. Then the image of $i^* : \Omega^*_T(X)_{\mathbb{Q}} \to \Omega^*_T(X^T)_{\mathbb{Q}}$ is the intersection of the images of*

$$i^*_{T'} : \Omega^*_T(X^{T'})_{\mathbb{Q}} \to \Omega^*_T(X^T)_{\mathbb{Q}}$$

where T' runs over all subtori of codimension one in T.

Lastly, we present the analogous result of Proposition 3.21.

Proposition 3.52. *[31, Theorem 7.8] Let X be a smooth projective scheme where a torus T acts with finitely many fixed points $x_1, ..., x_p$ and finitely many stable curves. Then the image of*

$$i^* : \Omega^*_T(X)_{\mathbb{Q}} \to \Omega^*_T(X^T)_{\mathbb{Q}}$$

is the set of $(f_1, ..., f_p) \in S(T)^p_{\mathbb{Q}}$ such that $f_i \equiv f_j \mod \chi$ whenever x_i and x_j are connected by a stable irreducible curve where T acts through the weight χ.

Remark 3.53. In the preceding two propositions we can identify $\Omega^*_T(X^T)_{\mathbb{Q}}$ with the ring $S(T)^p_{\mathbb{Q}} \cong (\mathbb{L}[[t_1, ..., t_n]]_{\text{gr}})^p_{\mathbb{Q}}$ because of Proposition 3.27 and the assumption that the fixed point subscheme only consists of finitely many fixed points $x_1, ..., x_p$.

Before we discuss the above introduced theories for spherical varieties, we come back to the intensively described example of the projective space.

Example 3.54. [20, Example 3.2.2] As in the Chow group case, we will compute the equivariant algebraic cobordism for the projective space $\mathbb{P}^n$ with the weighted $\mathbb{G}_m$-action, i.e. the action described in Example 3.23. We choose $\{(V_j, U_j)\} = \{(\mathbb{A}^j, \mathbb{A}^j \setminus \{0\})\}_{j \geq 0}$ to be the sequence of j-dimensional good pairs which satisfies the conditions of Proposition 2.46 for the given $\{(V_j, U_j)\}_{j \geq 0}$ and $W_j = \mathbb{A}^1$. Therefore, we do not have to take the quotient by the niveau filtration in order to compute the equivariant algebraic cobordism for the given $\mathbb{G}_m$-action. First of all, we have again as in Example 3.23 that $(\mathbb{P}^n \times U_j)/T$ is the $\mathbb{P}^n$-bundle $\mathbb{P}(\mathcal{O}(a_0) \oplus ... \oplus \mathcal{O}(a_n))$ over $\mathbb{P}^{j-1}$ and therefore, we obtain

$$\begin{aligned} \Omega^*\left((\mathbb{P}^n \times U_j)/T\right) &\cong \Omega^*\left(\mathbb{P}(\mathcal{O}(a_0) \oplus ... \oplus \mathcal{O}(a_n))\right) \\ &= \Omega^*(\mathbb{P}(\mathcal{E})) \\ &\cong \frac{\Omega^*(\mathbb{P}^{j-1})[\xi]}{(\xi^{n+1} - c_1(\mathcal{E})\xi^n + ... + (-1)^{n+1}c_{n+1}(\mathcal{E})} \end{aligned}$$

where $\mathcal{E} := \mathcal{O}(a_0) \oplus ... \oplus \mathcal{O}(a_n)$, $\xi := c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ and $c_i(\mathcal{E})$ denotes the i-th Chern class of $\mathcal{E}$. Furthermore, we let $t := c_1(\mathcal{O}(1)) \in \Omega^1(\mathbb{P}^{j-1})$ which implies $c_1(\mathcal{O}(a_k)) = [a_k]_\Omega t$ where $[a_k]_\Omega t$ is defined inductively by $F_\Omega(t, [a_k - 1]_\Omega t)$. By using the Whitney sum formula we can simplify this to

$$\Omega^*(\mathbb{P}(\mathcal{E})) \cong \Omega^*(\mathbb{P}^{j-1})[\xi]/(P_\Omega(\xi, t))$$

with

$$P_\Omega(\xi, t) = \sum_{i=0}^{n+1} (-1)^i \xi^{n+1-i} e_i([a_0]_\Omega t, ..., [a_n]_\Omega t)$$

where e_i denotes the i-th elementary symmetric polynomial which is given in the variables $[a_0]_\Omega t, ..., [a_n]_\Omega t$. As above we obtain this because of the structure of the vector bundle $\mathcal{E}$. By using Proposition 2.46 we conclude

$$\begin{aligned} \Omega_T^*(\mathbb{P}^n) &\cong \varprojlim_j \Omega^*((\mathbb{P}^n \times U_j)/T) \\ &= \varprojlim_j \Omega^*(\mathbb{P}^{j-1})[\xi]/(P_\Omega(\xi, t)) \\ &\cong \varprojlim_j \frac{\Omega^*(k)[t][\xi]}{(P_\Omega(\xi, t), t^j)} \\ &\cong \frac{\mathbb{L}[[t]]_{\mathrm{gr}}[\xi]}{(P_\Omega(\xi, t))}. \end{aligned}$$

Hereby we remark that here one has to take the colimit in the category of graded rings. Furthermore, in this example we computed the equivariant algebraic cobordism with integral coefficients as in the Chow group case.

3.6 Refinement of coefficients in the localisation theorem

In this section, we want to generalise Proposition 3.51 such that we do not need rational coefficients and instead only need to invert finitely many primes in order to obtain the result. Throughout the thesis, we will often consider the question whether one can refine the statements which are proved with rational coefficients.

Remark 3.55. Let T be a torus of rank n acting trivially on a smooth variety X and E be a T-equivariant vector bundle of rank d on X which is a direct sum of non-equivariant line bundles. Further, assume that in the eigenspace decomposition of E with respect to T, the submodule corresponding to the trivial character is zero. For the structure morphism $p : X \to \operatorname{Spec} k$ and following the proof of [31, Lemma 6.3] we have a unique direct sum decomposition

$$E = \bigoplus_{i=1}^{m} E_i \otimes p^*(L_{\chi_i}) \tag{3.2}$$

where each E_i is an ordinary vector bundle on X. Further, we can rearrange the terms by writing

$$E = \bigoplus_{j=1}^{s} \bigoplus_{q\in\mathbb{Z}} E_{jq} \otimes p^*(L_{\psi_j^q}) \tag{3.3}$$

where the ψ_j are primitive characters for all $1 \leq j \leq s$.

Definition 3.56. *Let T be a torus of rank n acting trivially on a smooth variety X and E be a T-equivariant vector bundle of rank d on X which is a direct sum of non-equivariant line bundles. Further, assume that in the eigenspace decomposition of E with respect to T, the submodule corresponding to the trivial character is zero. We define $\boldsymbol{S_E} \subseteq \mathbb{Z}$ to be the smallest multiplicative set associated to the T-equivariant vector bundle E such that*

(i) each χ_i occurring in the decomposition (3.2), $1 \leq i \leq m$, can be extended to a basis of $M[S_E^{-1}] = M \otimes_{\mathbb{Z}} \mathbb{Z}[S_E^{-1}]$ where M is the character group of T and $\mathbb{Z}[S_E^{-1}]$ the localisation of $\mathbb{Z}$ by S_E.

(ii) each pair of primitive characters $\{\psi_j, \psi_{j'}\}$ in the decomposition (3.3) is a part of a basis of $M[S_E^{-1}]$ for all $1 \leq j \neq j' \leq s$.

Lemma 3.57. *Let T be a torus of rank n acting trivially on a smooth variety X and let E be a T-equivariant vector bundle of rank d on X which is a direct sum of non-equivariant line bundles. Assume that in the eigenspace decomposition of E with respect to T, the submodule corresponding to the trivial character is zero. Then $c_d^T(E)$ is a non-zero divisor in $\Omega_T^*(X)_{\mathbb{Z}[S_E^{-1}]}$.*

Proof. The proof can be simply taken from [31, Lemma 6.3] with the adaption of taking the ring $\mathbb{Z}[S_E^{-1}]$ as coefficients instead of $\mathbb{Q}$. □

Definition 3.58. *Let X be a smooth projective variety where a torus T of rank n acts with finitely many isolated fixed points $x_0, ..., x_p$. Let $N_i := N_{x_i/X}$ be the normal bundle of x_i in X for $0 \leq i \leq p$. We define the multiplicative set $\boldsymbol{S_X} := \bigcup_{i=0}^p S_{N_i}$ in $\mathbb{Z}$. In this case, $\mathbb{Z}[S_X^{-1}]$ is the localisation of $\mathbb{Z}$ by S_X.*

The following statement is a refinement of Proposition 3.51. In order to prove it, we mainly use the technique from [31, Theorem 7.6] (cf. Proposition 3.51) which was already used in the proof of a similar result for equivariant Chow groups in [7, Theorem 3.3] (cf. Proposition 3.17).

Theorem 3.59. *Let X be a smooth projective variety where a torus T of rank n acts with finitely many isolated fixed points $x_0, ..., x_p$. Let $i : X^T \to X$ be the inclusion of the fixed point locus. Then the image of $i^* : \Omega_T^*(X)_{\mathbb{Z}[S_X^{-1}]} \to \Omega_T^*(X^T)_{\mathbb{Z}[S_X^{-1}]}$ is the intersection of the images of the restriction maps*

$$i_{T'}^* : \Omega_T^*(X^{T'})_{\mathbb{Z}[S_X^{-1}]} \to \Omega_T^*(X^T)_{\mathbb{Z}[S_X^{-1}]}$$

where T' runs over all subtori of codimension one in T.

Proof. We prove the statement by induction over the length of the filtration (3.1). If $n = 0$ in the filtration (3.1), then X is a T-equivariant vector bundle over X^T. In that case, the map i^* is surjective by homotopy invariance and therefore also $i_{T'}^*$. In the general case let $X_0 \subseteq X$ be given as in the filtration (3.1) and let U_0 be its complement. Then U_0 is a smooth T-filtrable variety with a shorter filtration where T acts with finitely many isolated fixed points. Further, let x_0 be the T-fixed point such that $X_0 \to x_0$ is the vector bundle coming from the definition of T-filtrable varieties. We recall that the normal bundle of x_0 in X is given by $N_{x_0/X} = T_{x_0}X/(T_{x_0}x_0)$. We know (cf. [7, Theorem 3.1]) that $T_{x_0}x_0$ is the weight zero subspace $(T_{x_0}X)_0$ of the tangent space $T_{x_0}X$. It follows that in the eigenspace decomposition of the T-equivariant vector bundle $N_{x_0/X}$, the submodule corresponding to the trivial character is zero. Since $N_{x_0/X} \cong T_{x_0}X$ is a direct sum of line bundles we can apply Lemma 3.57 and thus, the top Chern class of the normal bundle of x_0 in X is a non-zero divisor in $\Omega_T^*(x_0)_{\mathbb{Z}[S_{N_0}^{-1}]}$. By homotopy invariance for the vector bundle $X_0 \to x_0$ we know that $c_0 := c_{\mathrm{top}}^T(N_{X_0/X})$ is also a non-zero divisor in $\Omega_T^*(X_0)_{\mathbb{Z}[S_X^{-1}]}$. Using Corollary 3.48, we have the short exact sequence

$$0 \to \Omega_T^*(X_0)_{\mathbb{Z}[S_X^{-1}]} \xrightarrow{i_*} \Omega_T^*(X)_{\mathbb{Z}[S_X^{-1}]} \xrightarrow{j^*} \Omega_T^*(U_0)_{\mathbb{Z}[S_X^{-1}]} \to 0$$

and using the proof of [31, Proposition 4.1 (ii)], we obtain another short exact sequence

$$0 \to \Omega_T^*(X)_{\mathbb{Z}[S_X^{-1}]} \xrightarrow{(i^*,j^*)} \Omega_T^*(X_0)_{\mathbb{Z}[S_X^{-1}]} \times \Omega_T^*(U_0)_{\mathbb{Z}[S_X^{-1}]} \to \frac{\Omega_T^*(X_0)_{\mathbb{Z}[S_X^{-1}]}}{(c_0)} \to 0.$$

We identify $\Omega_T^*(X_0)_{\mathbb{Z}[S_X^{-1}]}$ with $\Omega_T^*(x_0)_{\mathbb{Z}[S_X^{-1}]}$ by homotopy invariance which in turn is identified with $\Omega^*(x_0)_{\mathbb{Z}[S_X^{-1}]}[[t_1, ..., t_n]]_{\mathrm{gr}}$. In particular, we can pull back c_0 to an element in $\Omega_T^*(x_0)_{\mathbb{Z}[S_X^{-1}]}$ along the isomorphism given by the zero-section. Using this pullback, we obtain

$$N_{x_0/X} \cong T_{x_0}X = \bigoplus_{k=1}^{m} L_{\chi_k}$$

because $T_{x_i}X$ splits into equivariant line bundles for every $0 \leq i \leq p$. We form the direct sum of L_{χ_k} and $L_{\chi'_k}$ whenever the characters χ_k and χ'_k are multiples of a common primitive character of T. This leads to

$$N_{x_0/X} = \bigoplus_{j=1}^{s} E_j$$

where each E_j is of the form

$$E_j = \bigoplus_q L_{\psi_j^q}$$

for a primitive character ψ_j. Additionally, we remark that the sum runs over a finite set of potentially repetitive integers. Using the Whitney sum formula, we obtain

$$c_0 = \prod_{j=1}^{s} c_{\mathrm{top}}^T(E_j) =: \prod_{j=1}^{s} c_{\psi_j}.$$

Following the proof of [7, Theorem 3.3], the identity component of the kernel of χ is a subtorus of codimension one in T for any primitive character χ. Furthermore, c_χ is the top equivariant Chern class of the normal bundle of $X_0^{\mathrm{Ker}(\chi)^0}$ in $X^{\mathrm{Ker}(\chi)^0}$. Conversely, recall from the proof of [7, Theorem 3.3] that any subtorus of codimension one in T can be written as $\mathrm{Ker}(\chi)^0$ for some primitive character χ of T, uniquely determined up to sign.

Now, let $\gamma \in \Omega_T^*(X^T)_{\mathbb{Z}[S_X^{-1}]}$ be in the image of all $i_{T'}^*$ and $j_{U_0^T} : U_0^T \to X^T$ the inclusion. By the induction hypothesis and a small diagram chase the class $j_{U_0^T}^*\gamma$ is in the image of

$$\Omega_T^*(U_0)_{\mathbb{Z}[S_X^{-1}]} \to \Omega_T^*(U_0^T)_{\mathbb{Z}[S_X^{-1}]}.$$

Because $j_{U_0}^* : \Omega_T^*(X)_{\mathbb{Z}[S_X^{-1}]} \to \Omega_T^*(U_0)_{\mathbb{Z}[S_X^{-1}]}$ is surjective and since $j_{U_0^T}^*$ is just the projection onto all but one factor, we may conclude that there exist $\alpha \in \Omega_T^*(x_0)_{\mathbb{Z}[S_X^{-1}]}$ and $\beta \in \Omega_T^*(X)_{\mathbb{Z}[S_X^{-1}]}$ such that $\gamma = \alpha + i^*\beta$ holds. Let ψ_j be a primitive character of T. Then $\alpha = \gamma - i^*\beta$ is in the image of $i^*_{\mathrm{Ker}(\psi_j)^0}$ because γ and $i^*\beta$ are. Now, we want to use Corollary 3.48 and the proof of [31, Proposition 4.1 (ii)] for $X^{\mathrm{Ker}(\psi_j)^0}$ which we can do due to the fact that c_{ψ_j} is a non-zero divisor in $\Omega_T^*\left(X_0^{\mathrm{Ker}(\psi_j)^0}\right)_{\mathbb{Z}[S_X^{-1}]}$ which comes from the fact that $N_{x_0/X^{\mathrm{Ker}(\psi_j)^0}}$ is a subbundle of $N_{x_0/X}$. This leads to the short exact sequence

$$0 \to \Omega_T^*\left(X^{\mathrm{Ker}(\psi_j)^0}\right)_{\mathbb{Z}[S_X^{-1}]} \xrightarrow{(i_{T0}^*, j_{T0}^*)} \Omega_T^*\left(X_0^{\mathrm{Ker}(\psi_j)^0}\right)_{\mathbb{Z}[S_X^{-1}]} \times \Omega_T^*\left(U_0^{\mathrm{Ker}(\psi_j)^0}\right)_{\mathbb{Z}[S_X^{-1}]}$$
$$\longrightarrow \frac{\Omega_T^*\left(X_0^{\mathrm{Ker}(\psi_j)^0}\right)_{\mathbb{Z}[S_X^{-1}]}}{(c_{\psi_j})} \to 0.$$

We remark that $X_0^{\mathrm{Ker}(\psi_j)^0}$ is a vector bundle over x_0. Next, we want to show that α is divisible by c_{ψ_j} in $\Omega_T^*\left(X_0^{\mathrm{Ker}(\psi_j)^0}\right)_{\mathbb{Z}[S_X^{-1}]} = \Omega_T^*(x_0)_{\mathbb{Z}[S_X^{-1}]}$ and therefore, it suffices to show that α is in the image of i_{T0}^*. But this is clear because $\alpha \in \Omega_T^*(x_0)_{\mathbb{Z}[S_X^{-1}]}$ is in the image of $i^*_{\mathrm{Ker}(\psi_j)^0}$ and since i_{T0}^* is just the map in the first component of $i^*_{\mathrm{Ker}(\psi_j)^0}$. We conclude that α is divisible by c_{ψ_j} in $\Omega_T^*\left(X_0^{\mathrm{Ker}(\psi_j)^0}\right)_{\mathbb{Z}[S_X^{-1}]}$. We now use the equality

$$(c_0) = \bigcap_{j=1}^{s} (c_{\psi_j}) \tag{3.4}$$

as ideals in $\Omega^*(x_0)_{\mathbb{Z}[S_X^{-1}]}[[t_1, ..., t_n]]_{\mathrm{gr}}$ which will be shown further below. Thus, we know that α is also divisible by c_0. For $i_{x_0} : x_0 \to X_0$ and using the self-intersection formula [31, Proposition 3.1], we have

$$\alpha \in c_0 \Omega_T^*(x_0)_{\mathbb{Z}[S_X^{-1}]} = i_{x_0}^* i_{x_0 *} \Omega_T^*(x_0)_{\mathbb{Z}[S_X^{-1}]}.$$

We can identify $i_{x_0 *}\Omega_T^*(x_0)_{\mathbb{Z}[S_X^{-1}]}$ as a subset of $\Omega_T^*(X)_{\mathbb{Z}[S_X^{-1}]}$ because i_* is injective. Furthermore, we remark that $i_{x_0}^*$ and i^* are the same map for elements of the form $i_{x_0 *}\Omega_T^*(x_0)_{\mathbb{Z}[S_X^{-1}]}$ and the identified elements in $\Omega_T^*(X)_{\mathbb{Z}[S_X^{-1}]}$, respectively. Thus, the elements α and $i^*\beta$ both are in $i^*\Omega_T^*(X)_{\mathbb{Z}[S_X^{-1}]}$ and therefore, one can conclude that $\gamma \in i^*\Omega_T^*(X)_{\mathbb{Z}[S_X^{-1}]}$.

Lastly, we need to prove the equality (3.4) in order to finish the proof. Let $d_j \geq 1$ be the rank of E_j for all $1 \leq j \leq s$. The statement [31, Lemma 6.4] also holds over $\mathbb{Z}[S_X^{-1}]$ using the same proof even though it was only proved rationally and thus, we have $c_{d_j}^T(E_j) = u_j(c_1^T(L_{\psi_j}))^{d_j}$ where u_j is invertible in $\Omega_T^*(x_0)_{\mathbb{Z}[S_X^{-1}]}$ for all $1 \leq j \leq s$ because all vector bundles on x_0 are trivial. By assumption, the $\{\psi_1, ..., \psi_s\}$ are pairwise a part of a basis of the character group $M[S_X^{-1}]$ because we consider coefficients in $\mathbb{Z}[S_X^{-1}]$. Setting $\gamma_i = (c_1^T(L_{\psi_j}))^{d_j}$ leads to

$$(\gamma_1 \cdots \gamma_s) = \bigcap_{j=1}^{s} (\gamma_j)$$

by [31, Lemma 5.4]. Since u_j, $1 \leq j \leq s$, are units in $\Omega_T^*(x_0)_{\mathbb{Z}[S_X^{-1}]}$, the equality (3.4) follows. This completes the proof of the theorem. □

Using the same proof as [31, Theorem 7.8] with the refined localisation theorem (cf. Theorem 3.59) one may deduce the following proposition.

Proposition 3.60. *Let X be a smooth projective scheme where a torus T acts with finitely many fixed points $x_1, ..., x_p$ and finitely many T-stable curves. Then the image of*

$$i^* : \Omega_T^*(X)_{\mathbb{Z}[S_X^{-1}]} \to \Omega_T^*(X^T)_{\mathbb{Z}[S_X^{-1}]}$$

is the set of $(f_1, ..., f_p) \in S(T)^p_{\mathbb{Z}[S_X^{-1}]}$ such that $f_i \equiv f_j \mod \chi$ whenever x_i and x_j are connected by a stable irreducible curve where T acts through the weight χ.

Remark 3.61. In Chapter 6, we will see examples where we actually compute the coefficient ring $\mathbb{Z}[S_X^{-1}]$.

Remark 3.62. The refined localisation theorem (cf. Theorem 3.59) of course also generalises the localisation theorem with rational coefficients for Chow groups (cf. [7, Theorem 3.3]) in the setting of finitely many fixed points in some smooth projective variety.

3.7 Künneth formula for T-equivariant cobordism

In this section, we will denote by G a connected reductive algebraic group over an algebraically closed field k of characteristic zero. Further, denote by T a maximal torus of G.

Proposition 3.63. *(Künneth formula) Let X, Y be smooth projective G-varieties such that $X \times Y$ has finitely many T-fixed points with respect to the diagonal action. Then there exists an isomorphism*

$$\Omega_T^*(X) \otimes_{\Omega_T^*(k)} \Omega_T^*(Y) \cong \Omega_T^*(X \times Y).$$

Proof. By assumption, we know that X and Y must have finitely many T-fixed points. Hence, X and Y are T-filtrable and cellular (cf. [21, Definition 2.4]). Let

$$\emptyset = X_{-1} \subsetneq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = X$$

and

$$\emptyset = Y_{-1} \subsetneq Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_\ell = Y$$

be filtrations of X and Y, respectively, such that $X_m \setminus X_{m-1}$ and $Y_i \setminus Y_{i-1}$ are affine spaces for all $0 \leq m \leq n$ and $0 \leq i \leq \ell$. We show by induction on m that the morphism

$$\Omega^*_T(X_m) \otimes_{\Omega^*_T(k)} \Omega^*_T(Y) \to \Omega^*_T(X_m \times Y)$$

induced by the external product is an isomorphism. We remark that X_0 is closed in X and an affine space so that X_0 must be a point. Thus, the isomorphism holds for $X_0 = \operatorname{Spec} k$. Let $U_{m+1} := X_{m+1} \setminus X_m$ be the corresponding affine space. We remark that the variety $X \times Y$ is T-filtrable. One filtration of $X \times Y$ is given by

$$\begin{aligned}\emptyset = X_{-1} &\subsetneq X_0 \times Y_0 \subsetneq X_0 \times Y_1 \subsetneq \cdots \subsetneq X_0 \times Y_\ell \subsetneq X_0 \times Y_\ell \cup X_1 \times Y_0 \\ &\subsetneq X_0 \times Y_\ell \cup X_1 \times Y_1 \subsetneq \cdots \subsetneq X_1 \times Y_\ell \subsetneq \cdots \subsetneq X_m \times Y_\ell \\ &\subsetneq \cdots \subsetneq X_{m+1} \times Y_\ell \subsetneq \cdots \subsetneq X_n \times Y_\ell = X \times Y.\end{aligned}$$

We verify that the complement

$$\begin{aligned}X_m \times Y_\ell \setminus ((X_{m-1} \times Y_\ell) \cup (X_m \times Y_{\ell-1})) &= ((X_m \setminus X_{m-1}) \times Y_\ell) \cap (X_m \times (Y_\ell \setminus Y_{\ell-1})) \\ &= (X_m \setminus X_{m-1}) \times (Y_\ell \setminus Y_{\ell-1})\end{aligned}$$

is an affine space for all $1 \leq m \leq n$ and so is the complement

$$\begin{aligned}&((X_m \times Y) \cup (X_{m+1} \times Y_{i+1})) \setminus ((X_m \times Y) \cup (X_{m+1} \times Y_i)) \\ &= \emptyset \cup (X_{m+1} \times Y_{i+1}) \setminus ((X_m \times Y) \cup (X_{m+1} \times Y_i)) \\ &= ((X_{m+1} \times Y_{i+1}) \setminus (X_m \times Y)) \cap ((X_{m+1} \times Y_{i+1}) \setminus (X_{m+1} \times Y_i)) \\ &= ((X_{m+1} \setminus X_m) \times Y_{i+1}) \cap (X_{m+1} \times (Y_{i+1} \setminus Y_i)) \\ &= (X_{m+1} \setminus X_m) \times (Y_{i+1} \setminus Y_i)\end{aligned}$$

for all $0 \leq m \leq n-1$ and $0 \leq i \leq \ell - 1$.

Using Corollary 3.48, we obtain the following diagram of short exact sequences where all tensor products are over $\Omega^*_T(k)$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^*_T(X_m) \otimes \Omega^*_T(Y) & \longrightarrow & \Omega^*_T(X_{m+1}) \otimes \Omega^*_T(Y) & \longrightarrow & \Omega^*_T(U_{m+1}) \otimes \Omega^*_T(Y) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^*_T(X_m \times Y) & \longrightarrow & \Omega^*_T(X_{m+1} \times Y) & \longrightarrow & \Omega^*_T(U_{m+1} \times Y) & \longrightarrow & 0. \end{array}$$

We remark that tensoring with $\Omega^*_T(Y)$ is exact because the latter is a free $\Omega^*_T(k)$-module by Corollary 3.46. The right vertical arrow is an isomorphism by homotopy invariance and the left vertical arrow is an isomorphism by the induction hypothesis. By five lemma we conclude the statement. □

3.8 Comparison of equivariant cohomology theories

In the following, we want to relate the theory of algebraic cobordism with the one of Chow groups and therefore, we state the known results in the ordinary and equivariant case.

Proposition 3.64. *[37, Theorem 1.2.19] The canonical morphism $\Omega_* \to \mathrm{CH}_*$ coming from the universality of ordinary algebraic cobordism induces an isomorphism*

$$\Omega_*(X) \otimes_{\mathbb{L}} \mathbb{Z} \overset{\cong}{\to} \mathrm{CH}_*(X)$$

of abelian groups. Furthermore, this is a graded $\mathbb{L}$-algebra isomorphism if X is smooth.

Remark 3.65. By the constructions of equivariant Chow groups and equivariant algebraic cobordism, one obtains a map of graded $\mathbb{L}$-modules

$$\Phi_X : \Omega_*^G(X) \to \mathrm{CH}_*^G(X)$$

for a k-scheme X of with a G-action.

Now, we will consider the example we care most about in the sequel of this chapter.

Example 3.66. Let T be a torus of rank n over k. From Proposition 3.27 we know that $\Omega^*(\mathrm{BT}) \cong \mathbb{L}[[t_1, ..., t_n]]_{\mathrm{gr}}$. Furthermore, as discuseed in Remark 3.18 we know that $\mathrm{CH}^*(\mathrm{BT})$ is isomorphic to the polynomial ring $\mathbb{Z}[t_1, ..., t_n]$. In this case, we obtain the map

$$\Phi_k : \Omega^*(\mathrm{BT}) \to \mathrm{CH}^*(\mathrm{BT}), \text{ i.e. } \Phi_k : \mathbb{L}[[t_1, ..., t_n]]_{\mathrm{gr}} \to \mathbb{Z}[t_1, ..., t_n]$$

which is given by killing the ideal $\mathbb{L}^{<0}$.

One of the main results obtained by Krishna [32] concerning the relation of equivariant Chow groups and equivariant algebraic cobordism is the following proposition.

Proposition 3.67. *[32, Proposition 7.2] The map Φ_X induces an isomorphism of graded $\mathbb{L}$-modules*

$$\Phi_X : \Omega_*^G(X) \widehat{\otimes}_{\mathbb{L}} \mathbb{Z} \overset{\cong}{\to} \mathrm{CH}_*^G(X).$$

Remark 3.68. The proof of the above result shows that the map $\Omega_*^G(X) \to \mathrm{CH}_*^G(X)$ is surjective. This is the map originally coming from the universality of ordinary algebraic cobordism and below, we will illustrate the difference between equivariant algebraic cobordism and equivariant Chow groups in some more detail.

Corollary 3.69. *[32, Corollary 7.3] For a k-scheme X with a G-action, the map*

$$\Omega_*^G(X) \otimes_{S(G)} \mathrm{CH}_*^G(k) \to \mathrm{CH}_*^G(X)$$

is an isomorphism of $\mathrm{CH}_*^G(k)$-modules. If X is smooth, then this map is a ring isomorphism.

Example 3.70. In the case

$$\mathrm{CH}_*^T(k) \cong \Omega_*^T(k) \widehat{\otimes}_{\mathbb{L}} \mathbb{Z} \cong \mathbb{L}[[t_1, ..., t_n]]_{\mathrm{gr}} \widehat{\otimes}_{\mathbb{L}} \mathbb{Z} \cong \mathbb{Z}[t_1, ..., t_n]$$

we see that the graded topological tensor product reduces to the ordinary graded tensor product. This shows that one can also deduce the Chow group of the classifying space BT from Proposition 3.67.

Example 3.71. Finally, we come back to Example 3.54 and Example 3.23 which we shortly recall. Above we computed $\mathrm{CH}^*_T(\mathbb{P}^n)$ and $\Omega^*_T(\mathbb{P}^n)$ for $T = \mathbb{G}_m$ and the weighted $\mathbb{G}_m$-action on $\mathbb{P}^n$. We made the remark that one has to take the colimit in the category of graded rings which implies

$$\mathrm{CH}^*_T(\mathbb{P}^n) \cong \Omega^*_T(\mathbb{P}^n)\widehat{\otimes}_{\mathbb{L}}\mathbb{Z} \cong \frac{\mathbb{L}[[t]]_{\mathrm{gr}}[\xi]}{(P_\Omega(\xi,t))}\widehat{\otimes}_{\mathbb{L}}\mathbb{Z} \cong \mathbb{Z}[\xi,t]/(P(\xi,t)).$$

This example well illustrates the powerful statement of Proposition 3.67.

4 Equivariant cobordism of spherical varieties

4.1 Equivariant Chow groups of spherical varieties

For this section, let G be a connected reductive algebraic group over an algebraically closed field k of characteristic zero, $B \subseteq G$ a Borel subgroup and $T \subseteq B$ a maximal torus. Furthermore, let X be a smooth projective G-variety. This section is based on [7].

Definition 4.1. *A subtorus $T' \subseteq T$ is **regular** if its centraliser*

$$C_G(T') = \{g \in G \mid gt' = t'g \text{ for all } t' \in T'\}$$

*is equal to the torus T. If this is not the case, we call the subtorus T' **singular**.*

Remark 4.2. Following [22, Corollary B, Section 26.2], a subtorus T' of codimension one is singular if and only if it is the identity component of the kernel of some positive root α of (G, T). In this case we will write $T' = \mathrm{Ker}(\alpha)^0$. Then α is unique and the group $C_G(T')$ is the product of T' with a subgroup $S(\alpha) \subseteq G$ isomorphic to SL_2 or PSL_2. Moreover, the fixed point locus $X^{T'}$ is equipped with an action of

$$C_G(T')/T' = T'S(\alpha)/T' = S(\alpha)/(S(\alpha) \cap T') = \mathrm{SL}_2 \text{ or } \mathrm{PSL}_2$$

since $S(\alpha) \cap T'$ is either of order one or two. Furthermore, we have $T = T'S_m(\alpha)$ for a maximal subtorus $S_m(\alpha)$ of $S(\alpha)$, the image of the coroot of α. As above, $T' \cap S_m(\alpha)$ is a finite group $F(\alpha)$ of order one or two. Clearly, $F(\alpha)$ acts trivially on $X^{T'}$ and hence, the T-action on $X^{T'}$ factors through an action of the corresponding quotient $T/F(\alpha) \cong (T' \times S_m(\alpha))/(F(\alpha) \times F(\alpha))$.

In the following proposition, we will analyse the components of the fixed point subschemes $X^{T'}$ for regular and singular codimension one subtori $T' \subseteq T$. Recall that the surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ is called the n-th Hirzebruch surface.

Proposition 4.3. *[7, Proposition 7.1] Let X be a spherical G-variety and let $T' \subseteq T$ be a subtorus of codimension one.*

(i) Each irreducible component of $X^{T'}$ is a spherical $C_G(T')$-variety.

(ii) If T' is regular, then $X^{T'}$ is at most one-dimensional.

(iii) If T' is singular, then $X^{T'}$ is at most two-dimensional. Furthermore, any two-dimensional connected component of $X^{T'}$ is either a rational ruled surface

$$\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$$

where $C_G(T')$ acts through the natural action of SL_2, or the projective plane where $C_G(T')$ acts through the projectivisation of a non-trivial SL_2-module of dimension three.

Now, let D be the torus of the diagonal matrices in SL_2 and α be the character of D given by

$$\alpha \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = t^2.$$

Further, let β be the square root of α and therefore, we could identify the rational character ring of D with $\mathbb{Q}[\beta]$ where we again denote the former by $S_{\mathbb{Q}}$. For later use and because of the rational coefficients, we rather identify the rational character ring of D with $\mathbb{Q}[\alpha]$ using Lemma 3.36 for Chow groups. Nevertheless, we will keep the square root β in mind since it will be needed in the sequel when we compare the equivariant Chow groups for the tori of SL_2 and PSL_2.

Definition 4.4. *[49, Definition 7.1.1] Let T be a torus and $p : T \to \mathrm{GL}(V)$ a rational representation of T. Then V is a direct sum of one-dimensional subspaces in each of which an element $t \in T$ acts as multiplication by $\chi(t)$ where χ is a character of T. The characters obtained in this manner are called* ***weights*** *of T in V.*

Remark 4.5. Proposition 4.3 is a very strong result which gives us the opportunity to compute the images in Proposition 3.17 using the action given by Proposition 4.3 since $T \subseteq C_G(T')$ holds. Therefore, we only have to consider the cases from Proposition 4.3 in order to get a result for the image of the pullback of $i : X^T \hookrightarrow X$.

Now, we want to compute the equivariant Chow rings for projective planes and Hirzebruch surfaces. Therefore, we describe the irreducible components of $X^{T'}$ for singular codimension one subtori T' coming from Proposition 4.3 in some more detail.

First, we want to consider the two cases of $\mathbb{P}(V)$ for a non-trivial SL_2-module V of dimension three arising from the previous theorem. Set $V_{n+1} := \mathrm{Sym}^{n+1}(k^2)$. Let $V = V_0 \oplus V_1$ be one of the non-trivial SL_2-modules of dimension three. The weights of D in V are $\alpha/2, 0$ and $-\alpha/2$ by Definition 4.4 and the given group action of D on V from Example 3.9 (iii). We denote by x, y and z the corresponding fixed points of D in $\mathbb{P}(V)$ which we have already seen in Example 3.9 (iii). To be more explicit, the corresponding fixed points to the weights $\alpha/2, 0, -\alpha/2$ are $x = [1 : 0 : 0], y = [0 : 1 : 0]$ and $z = [0 : 0 : 1]$, respectively. Therefore, we identify $\mathrm{CH}^*_D(\mathbb{P}(V)^D)_{\mathbb{Q}} \cong S^3_{\mathbb{Q}}$ with $\mathbb{Q}[\alpha]^3$.

Similarly, for the other non-trivial SL_2-module $V = V_2 = \mathfrak{sl}_2$ of dimension three, the corresponding weights are $\alpha, 0$ and $-\alpha$ whereas the corresponding fixed points are again $x = [1 : 0 : 0], y = [0 : 1 : 0]$ and $z = [0 : 0 : 1]$, respectively.

Next, we consider the case $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ with D-action given by

$$d \cdot ([a : b], [u : v]) = ([da : d^{-1}b], [du : d^{-1}v]).$$

We denote by w and z the D-fixed points $([1 : 0], [1 : 0])$ and $([0 : 1], [0 : 1])$, respectively. Further, we denote the remaining two D-fixed points $([1 : 0], [0 : 1])$ and $([0 : 1], [1 : 0])$ by x and y, respectively.

Lastly, we have a look at the rational ruled surface $\mathbb{F}_n$, $n \geq 1$, which has been described in large detail in Example 3.9 (v). The open SL_2-orbit cannot contain any D-fixed points. Since the two closed SL_2-orbits are given by $\mathrm{SL}_2 \cdot [v_1]$ and $\mathrm{SL}_2 \cdot [v_{n+1}]$, which are both projective lines, we observe that $\mathbb{F}_n$ has four D-fixed points w, x, y and z with corresponding weights $(n + 1)\alpha/2, \alpha/2, -\alpha/2$ and $-(n + 1)\alpha/2$, respectively, by the induced D-action on $\mathbb{F}_n$. Therefore, we can identify $\mathrm{CH}^*_D(\mathbb{F}_n^D)_{\mathbb{Q}} \cong S^4_{\mathbb{Q}}$ with $\mathbb{Q}[\alpha]^4$.

Proposition 4.6. *[7, Proposition 7.2] Notations being as above, the image of*

$$i^* : \mathrm{CH}^*_D(\mathbb{F}_n)_{\mathbb{Q}} \to S^4_{\mathbb{Q}}$$

consists of all $(f_w, f_x, f_y, f_z) \in S^4_{\mathbb{Q}}$ *such that*

$$\begin{aligned} f_w \equiv f_x \equiv f_y \equiv f_z \quad &\mathrm{mod}\ \alpha \text{ and} \\ f_w - f_x - f_y + f_z \equiv 0 \quad &\mathrm{mod}\ \alpha^2 \end{aligned}$$

hold. Moreover, the image of

$$i^* : \mathrm{CH}^*_D(\mathbb{P}(V))_{\mathbb{Q}} \to S^3_{\mathbb{Q}}$$

consists of all (f_x, f_y, f_z) *such that*

$$\begin{aligned} f_x \equiv f_y \equiv f_z \quad &\mathrm{mod}\ \alpha \text{ and} \\ f_x - 2f_y + f_z \equiv 0 \quad &\mathrm{mod}\ \alpha^2 \end{aligned}$$

hold.

Remark 4.7. We state one more equation in each of the cases in order to obtain the statement symmetric in its arguments although one could prove the proposition with one equation less in each of the cases. Furthermore, the last equation follows from the other ones and therefore, we will keep the statement as it was formulated by Brion.

Proof. First, we consider the case of $\mathbb{P}(V)$ for $V = V_0 \oplus V_1$. By Example 3.9 (iii), the closures of the Bialynicki-Birula cells are the point z, the line connecting y and z, and the whole $\mathbb{P}(V)$. We obtain

$$\mathrm{CH}^*_D(\mathbb{P}(V)^D)_{\mathbb{Q}} \cong \mathbb{Q}[t]^3 \cong \mathbb{Q}[\alpha]^3 \cong S^3_{\mathbb{Q}}$$

where $c_1(L_\alpha) \mapsto t$ and $t \mapsto \alpha \in S$ are the corresponding isomorphisms. Since $[\mathbb{P}(V)]$ is the identity in $\mathrm{CH}^*_D(\mathbb{P}(V))_{\mathbb{Q}}$ and i^* is a ring homomorphism, this class maps onto $(1,1,1)$. Now, we consider the images of the closures of the remaining two Bialynicki-Birula cells because these images generate the equivariant Chow ring of $\mathbb{P}(V)$ as a subalgebra of $S^3_{\mathbb{Q}}$ (cf. [7, Corollary 3.2 (iii)]). Therefore, we will have a look at the pullbacks $i^*[Y] = (i^*_x[Y], i^*_y[Y], i^*_z[Y])$ separately where $i^*_x[Y]$ denotes the pullback of any class $[Y]$ under the inclusion i_x of the corresponding fixed point in $\mathbb{P}(V)$. First, we can consider the composition $z \overset{j}{\hookrightarrow} U_z \overset{k}{\hookrightarrow} \mathbb{P}(V)$ where U_z is any open D-stable neighbourhood of z. This is possible since $z \hookrightarrow \mathbb{P}(V)$ factors through the above composition and $(k \circ j)^* = j^* \circ k^* = i^*_z$ holds. This implies that it is enough to replace $\mathbb{P}(V)$ by U_z.

In our particular case, we choose U_z to be the affine chart of $\mathbb{P}(V)$ in which the coordinate associated with z does not vanish. Therefore, we first introduce the coordinates a, b and c for V in order to be able to describe the coordinates on the affine space $U_z = \mathbb{A}^2$. By our choice for U_z, we get a/c and b/c as coordinates for U_z. The induced action implies now $t \cdot [x : y : z] = [tx : y : t^{-1}z]$ which maps to $(tx/t^{-1}z, y/t^{-1}z) = (t^2x/z, ty/z)$ in U_z. Thus, D acts linearly on U_z with weights α and $\alpha/2$.

Now, we come to the closure (yz) of one of the other Bialynicki-Birula cells. Every line in the projective plane can be defined by an equation $a_1a + a_2b + a_3c = 0$. Here we want y and z to be on the line. Since $y = [0 : 1 : 0]$ and $z = [0 : 0 : 1]$ we have

$a_2 = 0$ and $a_3 = 0$ and we are left with $a_1 a = 0$ but in projective space this is the same as $a = 0$. We choose f to be an eigenfunction of D with respect to weight $-\alpha$, i.e. $(t \cdot f)(a/c, b/c) = -\alpha(t) f(a/c, b/c)$ must hold. This leads to the following equation

$$(t \cdot f)(a/c, b/c) = f(t^{-1} \cdot (a/c, b/c)) = f(t^{-2}a/c, t^{-1}b/c) = t^{-2} f(a/c, b/c).$$

Now, we choose $f(a/c, b/c) = (a/c)$ explicitly which leads to $f(t^{-2}a/c, tb/c) = t^{-2}a/c$. This implies that f is an eigenfunction of D with respect to weight $-\alpha$. Furthermore, one has to compute $\operatorname{div}(f)$ in order to be able to apply [7, Theorem 2.1]. For the line $(yz) \cap U_z$ which is described by $a = 0$ it is clear that the order of vanishing of the line described by $a = 0$ is one and the other prime divisors have order of vanishing zero since c does not vanish on U_z. Therefore, we conclude $\operatorname{div}(f) = (yz) \cap U_z$. By [7, Theorem 2.1] we know that $[(yz) \cap U_z] = [\operatorname{div}(f)] = -\alpha[U_z]$ holds in $CH^*_D(U_z)_{\mathbb{Q}}$. Pulling back this relation to $\mathrm{CH}^*_D(z)_{\mathbb{Q}}$ yields $i_z^*[(yz) \cap U_z] = -\alpha$. We can apply the same argument for the pullback $i_y^*[(yz)]$ by choosing U_y to be the open affine neighbourhood of y such that the coordinate associated to y does not vanish. This leads to weights $\alpha/2$ and $-\alpha/2$. Then we take the same eigenfunction as above for the weight $-\alpha/2$. Therefore, $i_y^*[(yz) \cap U_y] = -\alpha/2$ by the same argument as given above.

Now, we consider the last closure of the Bialynicki-Birula cells, i.e. the point z. Clearly, this point is the complete intersection of the two lines (xz) and (yz). Therefore, we want to compute the pullback of the class $[(xz) \cap (yz) \cap U_z] = [z]$. In this case, we choose $g(a/c, b/c) = b/c$ to be the eigenfunction of D with respect to weight $-\alpha/2$, i.e. $(t \cdot g)(a/c, b/c) = g(t^{-2}a/c, t^{-1}b/c) = t^{-1}b/c = -\alpha/2(t) g(a/c, b/c)$ holds for all $t \in D$. Since $\operatorname{div}_{(yz)\cap U_z}(g) = (xz) \cap (yz) \cap U_z = z$ by the same argument as above, we obtain again by [7, Theorem 2.1] that $[z] = [(xz) \cap (yz) \cap U_z] = -\alpha/2[(yz) \cap U_z]$ holds in $\mathrm{CH}^*_D(U_z)_{\mathbb{Q}}$. Inserting the identity $[(yz) \cap U_z] = -\alpha[U_z]$, we obtain

$$[z] = -\alpha/2[(yz) \cap U_z] = -\alpha/2 \cdot (-\alpha)[U_z]$$

in $\mathrm{CH}^*_D(U_z)_{\mathbb{Q}}$. Pulling back this relation yields $i_z^*[z] = -\alpha/2 \cdot (-\alpha) = \alpha^2/2$.

Since the remaining pullbacks $i_x^*[z] = i_y^*[z] = i_x^*[(yz)]$ vanish we conclude that

$$\begin{aligned} i^* : \mathrm{CH}^*_D(\mathbb{P}(V))_{\mathbb{Q}} &\to S^3_{\mathbb{Q}} \text{ maps} \\ [\mathbb{P}(V)] &\mapsto (1, 1, 1) \\ [(yz)] &\mapsto (0, -\alpha/2, -\alpha) \\ [z] &\mapsto (0, 0, \alpha^2/2). \end{aligned}$$

We observe that these images satisfy the given equations and that the closures of the Bialynicki-Birula cells generate the equivariant Chow group by [7, Corollary 3.1]. Conversely, there cannot be more elements $(f_x, f_y, f_z) \in S^3_{\mathbb{Q}}$ satisfying the equations than linear combinations of the given images of the Bialynicki-Birula cells. The latter can be seen by taking $(f_x, f_y, f_z) \in S^3_{\mathbb{Q}}$ satisfying the equations which can be written as

$$\begin{aligned} (f_x, f_y, f_z) &= f_x(1, 1, 1) + (0, f_y - f_x, f_z - f_x) \\ &= f_x(1, 1, 1) + \frac{2(f_y - f_x)}{-\alpha}(0, -\alpha/2, -\alpha) + (0, 0, f_x - 2f_y + f_z) \\ &= f_x(1, 1, 1) + \frac{2(f_y - f_x)}{-\alpha}(0, -\alpha/2, -\alpha) + \frac{2(f_x - 2f_y + f_z)}{\alpha^2}(0, 0, \alpha^2/2) \end{aligned}$$

where $\frac{2(f_y-f_x)}{-\alpha}$ and $\frac{2(f_x-2f_y+f_z)}{\alpha^2}$ are well-defined elements in $S^3_{\mathbb{Q}}$ because of the given equations. This completes the proof in the case $V = V_0 \oplus V_1$.

Now, we consider the case of $V = V_2 = \mathfrak{sl}_2$ in which we choose the same U_z as above with the same coordinates. The computation will be almost the same as in the case above, but nevertheless we will write it down in order to have complete computations for all the cases. The induced action results in the element $(t^4x/z, t^2y/z)$ in U_z implying that D acts linearly on U_z with weights α and 2α where we denote the weights again additively. Since the class of $[\mathbb{P}(V)] \in \mathrm{CH}^*_D(\mathbb{P}(V))_{\mathbb{Q}}$ is mapped to $(1,1,1)$ we need to compute the remaining two images of the closures of the Bialynicki-Birula cells. Luckily, we have computed in Example 3.9 (iv) that the Bialynicki-Birula cells are the same as in the previous case.

Therefore, we take the closure (yz) of one of the remaining Bialynicki-Birula cells. This time, we choose f to be an eigenfunction of D with respect to weight -2α which leads again to $f(a/c, b/c) = a/c$. As above we know that $\mathrm{div}(f) = (yz) \cap U_z$ holds. Thus, again by [7, Theorem 2.1] we obtain $[(yz) \cap U_z] = [\mathrm{div}(f)] = -2\alpha[U_z]$ in $\mathrm{CH}^D_*(U_z)_{\mathbb{Q}}$. Pulling back this relation to $\mathrm{CH}^*_D(z)_{\mathbb{Q}}$ yields $i^*_z[(yz) \cap U_z] = -2\alpha$. By the same argument as above, we conclude $i^*_y[(yz) \cap U_y] = -\alpha$.

Lastly, we take the point z which leads to the computation of the pullback of the class of the intersection $[(xz) \cap (yz) \cap U_z] = [z]$. Therefore, we apply the same argument as in the previous case and obtain $\mathrm{div}_{(yz)\cap U_z}(g) = (xz) \cap (yz) \cap U_z = z$. Now, we apply again [7, Theorem 2.1] and get $[z] = [\mathrm{div}_{(yz)\cap U_z}(g)] = -\alpha[(yz) \cap U_z]$ in $\mathrm{CH}^*_D(U_z)_{\mathbb{Q}}$. Inserting $[(yz) \cap U_z] = -2\alpha[U_z]$, we obtain

$$[z] = -\alpha[(yz) \cap U_z] = \alpha \cdot 2\alpha[U_z]$$

in $\mathrm{CH}^*_D(U_z)_{\mathbb{Q}}$. Pulling back this relation yields $i^*_z[z] = \alpha \cdot 2\alpha$. The remaining pullbacks vanish again and thus, we summarise

$$\begin{aligned} i^* : \mathrm{CH}^*_D(\mathbb{P}(V))_{\mathbb{Q}} &\to S^3_{\mathbb{Q}} \text{ maps} \\ [\mathbb{P}(V)] &\mapsto (1,1,1) \\ [(yz)] &\mapsto (0,-\alpha,-2\alpha) \\ [z] &\mapsto (0,0,2\alpha^2). \end{aligned}$$

A similar computation to the one above completes the proof in the case $V = \mathfrak{sl}_2$.

Next, we take care of the case $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ from Example 3.9 (vi) in which we choose U_w to be an open D-stable neighbourhood of $w = ([1:0],[1:0])$. Therefore, we obtain $(t^{-2}b/a, t^{-2}v/u)$ for coordinates $([a:b],[u:v])$ which implies that D acts linearly on U_w with weight $-\alpha$. The class $[\mathbb{F}_0] \in \mathrm{CH}^*_D(\mathbb{F}_0)_{\mathbb{Q}}$ is again mapped to $(1,1,1,1)$ and we need to compute the remaining images of the closures of the Bialynicki-Birula cells.

Therefore, we take the closure (wx) of one of the remaining Bialynicki-Birula cells. We choose $f(b/a, v/u) = b/a$ to be an eigenfunction of D with respect to weight α. By the same argument as above, we obtain $\mathrm{div}(f) = (wx) \cap U_w$ and by [7, Theorem 2.1] we get $[(wx) \cap U_w] = [\mathrm{div}(f)] = \alpha[U_w]$ in $\mathrm{CH}^*_D(U_w)_{\mathbb{Q}}$. Pulling back this relation to $\mathrm{CH}^*_D(w)_{\mathbb{Q}}$ yields $i^*_w[(wx) \cap U_w] = \alpha$. With the eigenfunction $f(b/a, u/v) = b/a$ and a neighbourhood U_x of the fixed point x we obtain $i^*_x[(wx) \cap U_x] = \alpha$.

For the computations of the pullbacks of (wy) let U_w be as above and consider the eigenfunction $f(b/a, v/u) = v/u$ of D with respect to weight α, but here we have $\mathrm{div}(f) = (wy) \cap U_w$ and therefore, $[(wy) \cap U_w] = [\mathrm{div}(f)] = \alpha \in \mathrm{CH}^*_D(U_w)_{\mathbb{Q}}$. Pulling back this relation yields $i_w^*[(wy)\cap U_w]=\alpha$. Similarly, we obtain $i_y^*[(wy)\cap U_y]=\alpha$.

Lastly, we consider the pullback of the point w which is the complete intersection of the lines (wy) and (wx) for which we have already computed the corresponding pullbacks. Therefore, we obtain

$$i_w^*[w]=i_w^*[(wy)\cap U_w]\cdot i_w^*[(wx)\cap U_w]=\alpha\cdot\alpha=\alpha^2$$

whereas the other pullbacks of the class of the point w vanish. Thus, we summarise

$$\begin{aligned} i^*:\mathrm{CH}_D^*(\mathbb{F}_0)_{\mathbb{Q}}&\to S_{\mathbb{Q}}^4 \text{ maps}\\ [\mathbb{F}_0]&\mapsto(1,1,1,1)\\ [(wx)]&\mapsto(\alpha,\alpha,0,0)\\ [(wy)]&\mapsto(\alpha,0,\alpha,0)\\ [w]&\mapsto(\alpha^2,0,0,0). \end{aligned}$$

As above, we see immediately that these images satisfy the given equations. Conversely, let $(f_w,f_x,f_y,f_z)\in S_{\mathbb{Q}}^4$ satisfy the equations. This leads to

$$\begin{aligned} (f_w,f_x,f_y,f_z)=&f_z(1,1,1,1)+(f_w-f_z,f_x-f_z,f_y-f_z,0)\\ =&f_z(1,1,1,1)+\frac{f_y-f_z}{\alpha}(\alpha,0,\alpha,0)+(f_w-f_y,f_x-f_z,0,0)\\ =&f_z(1,1,1,1)+\frac{f_y-f_z}{\alpha}(\alpha,0,\alpha,0)+\frac{f_x-f_z}{\alpha}(\alpha,\alpha,0,0)\\ &+(f_w-f_x-f_y+f_z,0,0,0)\\ =&f_z(1,1,1,1)+\frac{f_y-f_z}{\alpha}(\alpha,0,\alpha,0)+\frac{f_x-f_z}{\alpha}(\alpha,\alpha,0,0)\\ &+\frac{f_w-f_x-f_y+f_z}{\alpha^2}(\alpha^2,0,0,0) \end{aligned}$$

which completes the case of $\mathbb{F}_0$.

We continue with the case $\mathbb{F}_n$ from Example 3.9 (v). The class $[\mathbb{F}_n]\in\mathrm{CH}_D^*(\mathbb{F}_n)_{\mathbb{Q}}$ is again mapped to $(1,1,1,1)$.

Next, we compute the remaining pullbacks of the closures of the Bialynicki-Birula cells. For chosen coordinates $[x_0:x_1:y_0:\ldots:y_{n+1}]$ we choose the open affine D-stable neighbourhood $U_w=\{y_0\neq 0\}$ of the point

$$w=v_{n+1}=[0:0:1:0:\ldots:0]\in\mathbb{P}(V_1\oplus V_{n+1})$$

in the surface $\mathbb{F}_n$. The induced D-action on U_w is given by $(t^{-n}x_0/y_0,t^{-2}y_1/y_0)$ and therefore, D acts linearly on U_w with weights $-n\alpha/2$ and $-\alpha$. Hence, we choose $f(x_0/y_0,y_1/y_0)=y_1/y_0$ to be an eigenfunction of D with respect to weight α. By the relations of the coordinates in $\mathbb{F}_n$ we obtain $\mathrm{div}(f)=(wx)\cap U_w$ with the notations of the D-fixed points from Example 3.9 (v). As above, we get $i_w^*[(wx)\cap U_w]=\alpha$. We remark here that one could have chosen different coordinates for the D-stable affine neighbourhood U_w and would obtain the same pullback since the $\mathrm{div}(f)$ would contain multiplicities due to the relations in the coordinates of $\mathbb{F}_n$. For the point $x=[1:0:\ldots:0]$ and a D-stable neighbourhood U_x we choose the eigenfunction $f(x_1/x_0,y_0/x_0)=x_1/x_0$ of D with respect to weight α which leads to $i_x^*[(wx)\cap U_x]=\alpha$.

For the pullback of (xy) let U_x be given by coordinates $(y_0/x_0, x_1/x_0)$ and take the eigenfunction $f(y_0/x_0, x_1/x_0) = y_0/x_0$ of weight $-n\alpha/2$ which leads to $\mathrm{div}(f) = (xy)\cap U_x$ and therefore to $i_x^*[(xy) \cap U_x] = -n\alpha/2$. For the coordinates $(x_0/x_1, y_{n+1}/x_1)$ for U_y and the eigenfunction $f(x_0/x_1, y_{n+1}/x_1) = y_{n+1}/x_1$ we get $\mathrm{div}(f) = (xy)\cap U_y$ and hence $i_y^*[(xy) \cap U_y] = n\alpha/2$.

Now, we consider the pullback of the point w again by introducing an eigenfunction on $(wx)\cap U_w$. Therefore, we choose $g(x_0/y_0, y_1/y_0) = x_0/y_0$ which is an eigenfunction of weight $n\alpha/2$. This leads to $\mathrm{div}_{(wx)\cap U_w}(g) = (wz) \cap (wx) \cap U_w = w$ and thus, we obtain $[w] = n\alpha/2[(wx) \cap U_w]$ in $\mathrm{CH}_D^*(U_w)_\mathbb{Q}$ again by [7, Theorem 2.1]. As above, we conclude $[w] = n\alpha/2 \cdot \alpha[U_w]$ and therefore also $i_w^*[w] = n\alpha/2 \cdot \alpha = n\alpha^2/2$. We obtain the images

$$\begin{aligned} i^* : \mathrm{CH}_D^*(\mathbb{F}_n)_\mathbb{Q} &\to S_\mathbb{Q}^4 \text{ maps} \\ [\mathbb{F}_n] &\mapsto (1,1,1,1) \\ [(wx)] &\mapsto (\alpha,\alpha,0,0) \\ [(xy)] &\mapsto (0,-n\alpha/2, n\alpha/2, 0) \\ [w] &\mapsto (n\alpha^2/2, 0,0,0). \end{aligned}$$

which satisfy the given equations and generate the whole image of i^*. Conversely, let $(f_w, f_x, f_y, f_z) \in S_\mathbb{Q}^4$ be an element fulfulling the conditions. This leads to

$$\begin{aligned} (f_w, f_x, f_y, f_z) =& f_z(1,1,1,1) + \frac{(f_y - f_z)}{n\alpha/2}(0, -n\alpha/2, n\alpha/2, 0) \\ &+ \frac{f_y - 2f_z + f_x}{\alpha}(\alpha,\alpha,0,0) \\ &+ \frac{f_w - f_x - f_y + f_z}{n\alpha^2/2}(n\alpha^2/2, 0, 0, 0) \end{aligned}$$

which completes the proof. □

Now, we present a well known fact for which a reference could not be found.

Lemma 4.8. *Let X be a spherical G-variety where T is a maximal torus. Then X has only finitely many T-fixed points.*

Proof. It is known that any spherical G-variety has only finitely many G-orbits. Therefore, $X = \bigcup G \cdot x$ for finitely many $x \in X$. If there is a T-fixed point $x_0 \in X$, then $G \cdot x_0$ has finitely many T-fixed points by [10, Lemma 2.2]. So if there is a T-fixed point in any of the G-orbits $G \cdot x$, this implies that there are finitely many T-fixed points in this given orbit. Otherwise, the G-orbit $G \cdot x$ does not have any T-fixed point. Therefore, any G-orbit $G \cdot x$ can have at most finitely many T-fixed points which implies the statement. □

Remark 4.9. By the Borel fixed point theorem [4, Theorem 10.4] we know that the fixed point subscheme X^T must be non-empty in case X is complete.

Proposition 4.10. *[7, Theorem 7.3] For any smooth projective spherical G-variety X, the pullback map*

$$i^* : \mathrm{CH}^*_T(X)_{\mathbb{Q}} \to \mathrm{CH}^*_T(X^T)_{\mathbb{Q}}$$

is injective. Moreover, the image of i^* consists of all families $(f_x)_{x \in X^T}$ such that

(i) $f_x \equiv f_y \mod \chi$ *whenever x and y are connected by a T-stable curve where T acts through the weight χ.*

(ii) $f_x - 2f_y + f_z \equiv 0 \mod \alpha^2$ *whenever α is a positive root of G relative to T, x, y and z lie in a connected component of $X^{\mathrm{Ker}(\alpha)^0}$ isomorphic to $\mathbb{P}^2$ and $x \geq y \geq z$ are ordered by their corresponding weights.*

(iii) $f_w - f_x - f_y + f_z \equiv 0 \mod \alpha^2$ *whenever α is a positive root of G relative to T, w, x, y and z lie in a connected component of $X^{\mathrm{Ker}(\alpha)^0}$ isomorphic to a rational ruled surface and $w \geq x \geq y \geq z$ are ordered by their corresponding weights.*

Proof. This result is an immediate consequence of Propositions 3.17, 3.16, 4.3 and 4.6. Concretely, Proposition 3.16 ensures injectivity whereas Proposition 4.3 classifies the components of fixed point subschemes $X^{T'}$ for subtori T' of codimension one in T. Furthermore, the results of Proposition 3.21 and Proposition 4.6 describe how these components behave under the pullback map i^* which is needed in order to apply Proposition 3.17. Since we compute the pullbacks with rational coefficients the proof of the latter result can be adapted to any positive root α of G relative to T which leads to a singular codimension one subtorus $T' = \mathrm{Ker}(\alpha)^0$. For a regular subtorus we know from Proposition 4.3 that $\dim X^{T'} \leq 1$ holds and each irreducible component of $X^{T'}$ is a spherical T-variety and therefore, each irreducible component of $X^{T'}$ contains only finitely many one-dimensional T-orbits and consequently, only finitely many T-stable curves. The fixed point subscheme $X^{T'}$ is noetherian as a finite type scheme over a noetherian base scheme which implies that it has only finitely many irreducible components. We can now conclude that $X^{T'}$ contains only finitely many T-stable curves. Furthermore, $X^{T'}$ has only finitely many T-fixed points by Lemma 4.8. Thus, since $X^{T'}$ is smooth by [7, Theorem 3.1], we can apply Proposition 3.21 to $X^{T'}$ in order to obtain the image of $i^*_{T'} : \mathrm{CH}^*_T(X^{T'})_{\mathbb{Q}} \to \mathrm{CH}^*_T(X^T)_{\mathbb{Q}}$ which coincides with condition (i) of the proposition.

For the cases (ii) and (iii) which cover the singular subtori T' of codimension one in T we use Remark 4.2 in order to compute $\mathrm{CH}^*_T(X^{T'})_{\mathbb{Q}}$ which we need by Proposition 3.17. More precisely, we consider the module $\mathrm{CH}^*_T(X^{T'})_{\mathbb{Q}}$ which is isomorphic to $\mathrm{CH}^*_{T/F(\alpha)}(X^{T'})_{\mathbb{Q}}$ by [7, Theorem 2.1] which gives us the corresponding generators and relations with the notations from Remark 4.2. In this case, one has to take into consideration that we consider rational coefficients and therefore, both modules are in fact modules over the same polynomial ring $\mathbb{Q}[\alpha_1, ..., \alpha_n] \cong \mathbb{Q}[\beta_1, ..., \beta_n]$ by Lemma 3.36 where

$$\alpha_i \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} = t_i^2$$

and $2\beta_i = \alpha_i$ are given as above. Thus, we conclude $\mathrm{CH}^*_T(X^{T'})_{\mathbb{Q}} \cong \mathrm{CH}^*_{T/F(\alpha)}(X^{T'})_{\mathbb{Q}}$.

This leads to

$$\begin{aligned}
\mathrm{CH}^*_T(X^{T'})_{\mathbb{Q}} &\cong \mathrm{CH}^*_{T/F(\alpha)}(X^{T'})_{\mathbb{Q}} \\
&\cong \mathrm{CH}^*_{(T'\times S_m(\alpha))/(F(\alpha)\times F(\alpha))}(X^{T'})_{\mathbb{Q}} \\
&\cong \mathrm{CH}^*_{T'\times S_m(\alpha)}(X^{T'})_{\mathbb{Q}} \\
&\cong \mathrm{CH}^*((\operatorname{Spec} k \times U_2 \times X^{T'} \times U_1)/(T' \times S_m(\alpha)))_{\mathbb{Q}} \\
&\cong \mathrm{CH}^*((\operatorname{Spec} k \times U_2)/T' \times (X^{T'} \times U_1)/(S_m(\alpha)))_{\mathbb{Q}} \\
&\cong \mathrm{CH}^*_{T'}(\operatorname{Spec} k)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathrm{CH}^*_{S_m(\alpha)}(X^{T'})_{\mathbb{Q}}
\end{aligned}$$

by a special case of the Künneth formula for Chow groups where U_1 and U_2 are the corresponding parts of the good pairs (V_1, U_1) and (V_2, U_2) for $S_m(\alpha)$ and T' respectively. This reduces the computations to the setting of Proposition 4.6 by the construction of the subtorus T' and the corresponding $S_m(\alpha)$ in Remark 4.2. Lastly, it is enough to reduce to the setting of SL_2 as in Proposition 4.6 because the rational equivariant Chow group of X for the torus of PSL_2 is the same as the one for the torus of SL_2 by [7, Theorem 2.1] since the generators and relations are the same in both cases. □

Remark 4.11. As already used in the above proof, [7, Theorem 2.1] implies that whenever a torus T acts on a variety X and the action factors through a quotient T/F where F acts trivially on X we obtain the same rational equivariant Chow groups. The same holds of course if we extend the action from T/F to T with a trivial action of F. The latter occurs for the group PSL_2 as we can extend the action to SL_2 where $\{\mathrm{Id}, -\mathrm{Id}\}$ act trivially on X. Using [12, Proposition 6], this implies for example that for those actions the rational equivariant Chow groups $\mathrm{CH}^*_{\mathrm{SL}_2}(X)_{\mathbb{Q}} \cong \mathrm{CH}^*_{\mathrm{PSL}_2}(X)_{\mathbb{Q}}$ are isomorphic since SL_2 and PSL_2 have the same Weyl group with respect to their maximal tori.

Remark 4.12. If one computes the rational Chow ring of the smooth projective horospherical G_2-variety X of Picard number one (cf. Proposition 5.15) using Proposition 4.10 and the forgetful map to ordinary rational Chow rings, there is no obvious isomorphism to the description of the rational Chow ring of X which was recently presented in [45, Section 3]. The latter gives an explicit description of $\mathrm{CH}^*(X)_{\mathbb{Q}}$ in terms of two generators and two relations.

4.2 Equivariant cobordism of spherical varieties

In this section, we let G be a connected reductive algebraic group, T a maximal torus and X a smooth projective spherical G-variety. The goal of this section is to generalise the presentations of the rational equivariant Chow rings of smooth projective spherical G-varieties (cf. [7, Theorem 7.3]) to rational equivariant algebraic cobordism. In order to be able to generalise those, we need to compute the equivariant algebraic cobordism of the projective plane and the Hirzebruch surfaces as required by Proposition 4.3 and Proposition 3.51. Using notation as in Proposition 3.52, we can now formulate the main result of this section which is the analogue of [7, Theorem 7.3] and which will be proved later on in this section. We remark that the ordering of the T-fixed points in $\mathbb{P}^2$ and $\mathbb{F}_n$, $n \geq 0$, which is used in the upcoming theorem, is discussed in Section 4.1.

Theorem 4.13. *For any smooth projective and spherical G-variety X, the pullback map*

$$i^* : \Omega^*_T(X)_{\mathbb{Q}} \to \Omega^*_T(X^T)_{\mathbb{Q}}$$

is injective. Moreover, the image of i^* consists of all families $(f_x)_{x \in X^T}$ such that

(i) $f_x \equiv f_y \mod c_1^T(L_\chi)$ *whenever x and y are connected by a T-stable curve where T acts through the weight χ.*

(ii) $(f_x - f_y) + \rho_{1/2} c_1^T(L_\alpha)(f_z - f_x) \equiv 0 \mod c_1^T(L_\alpha)^2$ *whenever α is a positive root of G relative to T, x, y and z lie in a connected component of $X^{\mathrm{Ker}(\alpha)^0}$ isomorphic to a projective plane $\mathbb{P}^2$ and $x \geq y \geq z$ are ordered by their corresponding weights.*

(iii) $f_w - f_x - f_y + f_z \equiv 0 \mod c_1^T(L_\alpha)^2$ *whenever α is a positive root of G relative to T, w, x, y and z lie in a connected component of $X^{\mathrm{Ker}(\alpha)^0}$ isomorphic to $\mathbb{F}_0$ and $w \geq x, y \geq z$ are ordered by their corresponding weights.*

(iv) $\rho_{-n/2} c_1^T(L_\alpha)(f_y - f_z) + \rho_{n/2} c_1^T(L_\alpha)(f_w - f_x) \equiv 0 \mod c_1^T(L_\alpha)^2$ *whenever α is a positive root of G relative to T, w, x, y and z lie in a connected component of $X^{\mathrm{Ker}(\alpha)^0}$ isomorphic to a rational ruled surface $\mathbb{F}_n$, $n \geq 1$, and $w \geq x \geq y \geq z$ are ordered by their corresponding weights.*

Remark 4.14. We will see later in the proof that condition (i) in the preceding theorem comes from Proposition 3.52. Further, the formulation of the equations in the conditions (ii) and (iv) slightly differs from the one in Brion's description. We need to introduce the terms $\rho_{n/2}$, $n \in \mathbb{Z}\backslash\{0\}$, from Definition 3.35 because of the universal formal group law in cobordism. Moreover, we have to distinguish between the cases $\mathbb{F}_0$ and $\mathbb{F}_n$, $n \geq 1$, again due to the universal formal group law. In the case of a smooth projective spherical G-variety X, the theorem is a generalisation of Proposition 3.52 because the cases (ii)-(iv) do not occur if the variety has only finitely many T-stable curves.

In the sequel, we will show that information concerning the formal group law gets lost if one tries to generalise the known results regarding rational T-equivariant Chow groups from Section 4.1 to a finer theory in a primitive way.

Construction 4.15. In the preceding section, we described the relation between equivariant Chow groups and equivariant algebraic cobordism and we have seen that the map $\Omega^*_T(X) \to \mathrm{CH}^*_T(X)$ coming from the universality of ordinary algebraic cobordism is surjective for any k-scheme X with a T-action. With the results of Section 3.5 and Section 4.1 and the notations from these sections we are left with computing the image of the two pullbacks

$$i^* : \Omega^*_T(\mathbb{F}_n)_{\mathbb{Q}} \to S(T)^4_{\mathbb{Q}}$$

and

$$i^* : \Omega^*_T(\mathbb{P}(V))_{\mathbb{Q}} \to S(T)^3_{\mathbb{Q}}.$$

This observation mainly relies on the Proposition 4.3 and 3.51. Here we remark that the assumption in Proposition 3.51 of having finitely many T-fixed points will not affect our result since we only consider spherical varieties which always have finitely many T-fixed points by Lemma 4.8.

First, one could think of the following commutative diagram for the case $\mathbb{P}(V)$ for $V = k^3$ and $i : \mathbb{P}(V)^T \to \mathbb{P}(V)$

$$\begin{array}{ccc} \Omega^*_T(\mathbb{P}(V))_{\mathbb{Q}} & \xrightarrow{i^*} & \Omega^*_T(\mathbb{P}(V)^T)_{\mathbb{Q}} \cong (\mathbb{L}[[t_1,...,t_n]]_{\mathrm{gr}})^3_{\mathbb{Q}} \\ \downarrow \Phi_{\mathbb{P}(V)} & & \downarrow \Phi_{\mathbb{P}(V)^T} \\ \mathrm{CH}^*_T(\mathbb{P}(V))_{\mathbb{Q}} & \xrightarrow{i^*} & \mathrm{CH}^*_T(\mathbb{P}(V)^T)_{\mathbb{Q}} \cong \mathbb{Z}[t_1,...,t_n]^3_{\mathbb{Q}} \end{array}$$

where Φ_X are the induced maps coming from the morphism of oriented cohomology theories $\theta : \Omega^* \to \mathrm{CH}^*$ as in Proposition 3.67. As we know that $\Omega^*_T(\mathbb{P}(V))$ is generated by the T-stable cobordism cycles in $\Omega^*(\mathbb{P}(V))$ by [31, Corollary 4.8], we need to compute the images of the fundamental classes of the cobordism cycles $[\mathbb{P}(V) \to \mathbb{P}(V)]$, $[(yz) \to \mathbb{P}(V)]$, and $[z \to \mathbb{P}(V)]$ where we keep the same notation as in Section 4.1. Furthermore, we know that the map $\Phi_{\mathbb{P}(V)^T}$ is given by killing the ideal $\mathbb{L}^{<0}$. Therefore, the class $\alpha := i^*[(yz)] \in \mathrm{CH}^*_T(\mathbb{P}(V)^T)_{\mathbb{Q}}$ comes from an element $\alpha + l$ where $l \in \mathbb{L}^{<0}$. Thus, the fundamental class of the cobordism cycle $[(yz) \to \mathbb{P}(V)]$ in $\Omega^*_T(\mathbb{P}(V))_{\mathbb{Q}}$ must be mapped to $\alpha + l$ for some $l \in \mathbb{L}^{<0}$. Now, we see that we can determine the image of i^* up to an element $l \in \mathbb{L}^{<0}$, but unfortunately this element contains all the information about the formal group law and therefore, we lose a lot of information if we can only determine the image up to this element l. In order to be able to determine the image of i^* properly, we need a different approach, but nevertheless this construction gives us an intuitive way of thinking about the difference between the two given equivariant cohomology theories.

Now, we want to prove Proposition 4.6 for equivariant algebraic cobordism. Therefore, we use the notations from Section 4.1. Recall that we consider the two cases $\mathbb{P}(V)$ for a non-trivial SL_2-module V of dimension three which have three D-fixed points where D denotes the maximal torus of SL_2 with positive root α. Furthermore, we consider the Hirzebruch surfaces $\mathbb{F}_n$ containing four D-fixed points which have been described precisely in Example 3.9 (v).

Proposition 4.16. *Let X be a Hirzebruch surface $\mathbb{F}_n$ or a projective plane $\mathbb{P}(V)$.*

(i) The image of the pullback

$$i^* : \Omega^*_D(\mathbb{F}_n)_{\mathbb{Q}} \to S(D)^4_{\mathbb{Q}}$$

consists of all $(f_w, f_x, f_y, f_z) \in S(D)^4_{\mathbb{Q}}$ such that

$$f_w \equiv f_x \equiv f_y \equiv f_z \mod c_1^D(L_\alpha) \text{ and} \tag{4.1}$$
$$f_w - f_x - f_y + f_z \equiv 0 \mod c_1^D(L_\alpha)^2 \tag{4.2}$$

hold for $n = 0$ and of all $(f_w, f_x, f_y, f_z) \in S(D)^4_{\mathbb{Q}}$ such that

$$f_w \equiv f_x \equiv f_y \equiv f_z \mod c_1^D(L_\alpha) \text{ and} \tag{4.3}$$
$$\rho_{-n/2}c_1^D(L_\alpha)(f_y - f_z) + \rho_{n/2}c_1^D(L_\alpha)(f_w - f_x) \equiv 0 \mod c_1^D(L_\alpha)^2 \tag{4.4}$$

hold for $n \geq 1$.

(ii) *The image of the pullback*

$$i^* : \Omega^*_D(\mathbb{P}(V))_{\mathbb{Q}} \to S(D)^3_{\mathbb{Q}}$$

consists of all (f_x, f_y, f_z) *such that*

$$f_x \equiv f_y \equiv f_z \mod c_1^D(L_\alpha) \text{ and} \tag{4.5}$$
$$(f_x - f_y) + \rho_{1/2} c_1^D(L_\alpha)(f_z - f_x) \equiv 0 \mod c_1^D(L_\alpha)^2 \tag{4.6}$$

hold.

Remark 4.17. Following the Chow group case, we state one more equation than we will need for the proof in each of the cases because of the symmetry in the arguments which makes the proposition comparable to the Chow group case. For example, in the $\mathbb{F}_n$ case we could remove the equation $f_w \equiv f_x \mod c_1^D(L_\alpha)$. As above, the last equation follows from the other ones in each of the cases.

Proof. We first consider the case of $\mathbb{P}(V)$ for $V = V_0 \oplus V_1$. Since i^* is a ring homomorphism, the class $[\mathbb{P}(V) \to \mathbb{P}(V)]$ maps to $(1, 1, 1)$. We remark that the closures of the Bialynicki-Birula cells are smooth in the case of the projective plane and the Hirzebruch surfaces. Now, we compute the images of the closures of the Bialynicki-Birula cells, i.e. the images of the equivariant fundamental classes of the D-stable cobordism cycles $[(yz) \to \mathbb{P}(V)]$ and $[z \to \mathbb{P}(V)]$ because these images generate the equivariant cobordism ring of $\mathbb{P}(V)$ as a subalgebra of $S(D)^3_{\mathbb{Q}}$ (cf. [31, Corollary 4.8]). We have a look at the pullback

$$i^*[Y \to \mathbb{P}(V)] = (i_x^*[Y \to \mathbb{P}(V)], i_y^*[Y \to \mathbb{P}(V)], i_z^*[Y \to \mathbb{P}(V)])$$

where $i_x^*[Y \to \mathbb{P}(V)]$ denotes the pullback of any class $[Y \to \mathbb{P}(V)]$ under the inclusion i_x of the corresponding fixed point in $\mathbb{P}(V)$. To compute $i_z^*[Y \to \mathbb{P}(V)]$ we can replace $\mathbb{P}(V)$ by any open D-stable neighbourhood U_z of z. In this case we choose U_z to be the affine chart of $\mathbb{P}(V)$ in which the coordinate associated to z does not vanish. We introduce the coordinates a, b and c for V such that our coordinates for U_z become a/c and b/c. Therefore, D acts linearly on U_z with weights α and $\alpha/2$. We choose $f(a/c, b/c) = a/c$ which is an eigenfunction of D with respect to weight $-\alpha$. In this situation, we can apply Corollary 3.40 because f gives rise to the global section

$$\begin{aligned} s : (U_z \times U_j \times U_j)/T &\to (U_z \times U_j \times U_j \times L_{-\alpha})/T, \\ (a/c, b/c, u_1, u_2) &\mapsto (a/c, b/c, u_1, u_2, f(a/c, b/c)) \end{aligned}$$

which is transverse to the zero section with zero-subscheme $Z_0 = (yz) \cap U_z$. Thus, we know that

$$[(yz) \cap U_z \to U_z] = c_1^D(L_{-\alpha})[\operatorname{Spec} k \to \operatorname{Spec} k] \cdot [U_z \to U_z]$$

holds in $\Omega^*_D(U_z)_{\mathbb{Q}}$. Pulling back to $\Omega^*_D(z)_{\mathbb{Q}}$ yields $i_z^*[(yz) \cap U_z \to U_z] = c_1^D(L_{-\alpha})$. We can apply the same argument for the pullback $i_y^*[(yz) \to \mathbb{P}(V)]$ by choosing U_y to be the open affine neighbourhood of y such that the coordinate associated to y does not vanish. Thus, D acts linearly on U_y with weights $\alpha/2$ and $-\alpha/2$. We take $f(a/b, c/b) = a/b$ which is an eigenfunction of D with respect to weight $-\alpha/2$. Therefore, we conclude $i_y^*[(yz) \cap U_y] = c_1^D(L_{-\alpha/2})$ by the same argument as above.

Finally, we consider the last closure of the Bialynicki-Birula cells, i.e. the point z. Clearly, z is the complete intersection of the two lines (yz) and (xz). Therefore, we want to compute the pullback of $[(xz) \cap (yz) \cap U_z \to U_z] = [z \to U_z]$. We want to apply the same argument again using the relation

$$c_1^D(L_{-\alpha/2}) \cdot [(yz) \cap U_z \to U_z] = [(xz) \cap (yz) \cap U_z \to U_z] = [z \to U_z]$$

from Corollary 3.40 where $z = (xz) \cap (yz) \cap U_z$ is the zero-subscheme of the section defined by the eigenfunction $g(a/c, b/c) = b/c$ of D with respect to weight $-\alpha/2$ on $(yz) \cap U_z$. Using the equality $i_z^*[(yz) \cap U_z \to U_z] = c_1^D(L_{-\alpha})$, we obtain the pullback

$$i_z^*[z \to U_z] = c_1^D(L_{-\alpha/2}) \cdot i_z^*[(yz) \cap U_z \to U_z] = c_1^D(L_{-\alpha/2}) \cdot c_1^D(L_{-\alpha})$$

in $\Omega_D^*(z)_{\mathbb{Q}}$. The images of the D-stable cobordism cycles coming from the closures of the Bialynicki-Birula decomposition generate the equivariant cobordism ring by [31, Corollary 4.8]. Therefore, the image of the pullback $i^* : \Omega_D^*(\mathbb{P}(V))_{\mathbb{Q}} \to S(D)_{\mathbb{Q}}^3$ is generated by the images

$$\begin{aligned}
[\mathbb{P}(V) \to \mathbb{P}(V)] &\mapsto (1,1,1) \\
[(yz) \to \mathbb{P}(V)] &\mapsto (0, c_1^D(L_{-\alpha/2}), c_1^D(L_{-\alpha})) \\
[z \to \mathbb{P}(V)] &\mapsto (0,0, c_1^D(L_{-\alpha/2})c_1^D(L_{-\alpha})).
\end{aligned}$$

These images satisfy the equations (4.5) and (4.6) which can be seen by expressing $c_1^D(L_{\alpha/2})$ as a formal power series in the variable $c_1^D(L_\alpha)$ with rational coefficients. For the following computation and similar ones upcoming in the sequel of this proof, we remark that any element which is divisible by $c_1^D(L_\alpha)$ will also be divisible by $c_1^D(L_{n\alpha/m})$ for $n \in \mathbb{Z} \setminus \{0\}$ and $m \in \mathbb{Z}_{\geq 1}$ because we can again express the former Chern class in terms of the latter one and factor out. Therefore, for an element $(f_x, f_y, f_z) \in S(T)_{\mathbb{Q}}^3$ satisfying the given equations we have

$$\begin{aligned}
(f_x, f_y, f_z) =& f_x(1,1,1) + (0, f_y - f_x, f_z - f_x) \\
=& f_x(1,1,1) + \frac{f_y - f_x}{c_1^D(L_{-\alpha/2})}\Big(0, c_1^D(L_{-\alpha/2}), c_1^D(L_{-\alpha})\Big) \\
&+ \left(0,0,(f_x - f_y)\frac{c_1^D(L_{-\alpha})}{c_1^D(L_{-\alpha/2})} + f_z - f_x\right) \\
=& f_x(1,1,1) + \frac{f_y - f_x}{c_1^D(L_{-\alpha/2})}\Big(0, c_1^D(L_{-\alpha/2}), c_1^D(L_{-\alpha})\Big) \\
&+ \left(0,0,c_1^D(L_{-\alpha/2})c_1^D(L_{-\alpha})\left(\frac{(f_x - f_y)c_1^D(L_{-\alpha}) + c_1^D(L_{-\alpha/2})(f_z - f_x)}{c_1^D(L_{-\alpha/2})^2 c_1^D(L_{-\alpha})}\right)\right) \\
=& f_x(1,1,1) + \frac{f_y - f_x}{c_1^D(L_{-\alpha/2})}\Big(0, c_1^D(L_{-\alpha/2}), c_1^D(L_{-\alpha})\Big) \\
&+ \frac{(f_x - f_y)c_1^D(L_{-\alpha}) + c_1^D(L_{-\alpha/2})(f_z - f_x)}{c_1^D(L_{-\alpha/2})^2 c_1^D(L_{-\alpha})}\Big(0,0,c_1^D(L_{-\alpha/2})c_1^D(L_{-\alpha})\Big)
\end{aligned}$$

which completes the proof in the case $V = V_0 \oplus V_1$.

The computation for $V = V_2$ can be done similarly. We get

$$\begin{aligned}
i^* : \Omega^*_D(\mathbb{P}(V))_{\mathbb{Q}} &\to S(D)^3_{\mathbb{Q}} \\
[\mathbb{P}(V) \to \mathbb{P}(V)] &\mapsto (1,1,1) \\
[(yz) \to \mathbb{P}(V)] &\mapsto \Big(0, c_1^D(L_{-\alpha}), c_1^D(L_{-2\alpha})\Big) \\
[z \to \mathbb{P}(V)] &\mapsto \Big(0, 0, c_1^D(L_{-\alpha})c_1^D(L_{-2\alpha})\Big)
\end{aligned}$$

which satisfy the given equations using the properties of the formal group law. Again, we obtain

$$\begin{aligned}
(f_x, f_y, f_z) =& f_x(1,1,1) + (0, f_y - f_x, f_z - f_x) \\
=& f_x(1,1,1) + \frac{f_y - f_x}{c_1^D(L_{-\alpha})}\Big(0, c_1^D(L_{-\alpha}), c_1^D(L_{-2\alpha})\Big) \\
&+ \left(0, 0, (f_x - f_y)\frac{c_1^D(L_{-2\alpha})}{c_1^D(L_{-\alpha})} + f_z - f_x\right) \\
=& f_x(1,1,1) + \frac{f_y - f_x}{c_1^D(L_{-\alpha})}\Big(0, c_1^D(L_{-\alpha}), c_1^D(L_{-2\alpha})\Big) \\
&+ \left(0, 0, c_1^D(L_{-\alpha})c_1^D(L_{-2\alpha})\left(\frac{(f_x - f_y)c_1^D(L_{-2\alpha}) + c_1^D(L_{-\alpha})(f_z - f_x)}{c_1^D(L_{-\alpha})^2 c_1^D(L_{-2\alpha})}\right)\right) \\
=& f_x(1,1,1) + \frac{f_y - f_x}{c_1^D(L_{-\alpha})}\Big(0, c_1^D(L_{-\alpha}), c_1^D(L_{-2\alpha})\Big) \\
&+ \frac{(f_x - f_y)c_1^D(L_{-2\alpha}) + c_1^D(L_{-\alpha})(f_z - f_x)}{c_1^D(L_{-\alpha})^2 c_1^D(L_{-2\alpha})}\Big(0, 0, c_1^D(L_{-\alpha})c_1^D(L_{-2\alpha})\Big)
\end{aligned}$$

which completes the proof for $V = V_2$ since the last coefficient is well-defined using the properties of the formal group law and the equations (4.5) and (4.6). More precisely, the quotient $c_1^D(L_{-\alpha})/c_1^D(L_{-2\alpha})$ has the same coefficients as $\rho_{1/2}c_1^D(L_\alpha)$ and the only difference will be the variable $c_1^D(L_{-2\alpha})$ in the first quotient as opposed to $c_1^D(L_\alpha)$ in the second one. As we consider the reduction modulo $c_1^D(L_\alpha)^2$, we only need to take the first two summands of $c_1^D(L_{-\alpha})/c_1^D(L_{-2\alpha})$ into account. Therefore, $c_1^D(L_{-\alpha})/c_1^D(L_{-2\alpha})$ differs from $\rho_{1/2}c_1^D(L_\alpha)$ only by a factor of -2 in the second summand. We consider the difference

$$\frac{c_1^D(L_{-\alpha})}{c_1^D(L_{-2\alpha})}(f_z - f_x) - \rho_{1/2}c_1^D(L_\alpha)(f_z - f_x)$$

which is a product containing a factor $c_1^D(L_\alpha)(f_z - f_x)$. This implies that the above difference vanishes modulo $c_1^D(L_\alpha)^2$ because of the equation (4.5) and thus, we reduced the coefficient to the known equation $(f_x - f_y) + \rho_{1/2}c_1^D(L_\alpha)(f_z - f_x)$ which finishes the argument.

Next, we consider the case $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ for which we choose U_w to be an open D-stable neighbourhood of $w = ([1:0],[1:0])$. We get $(t^{-2}b/a, t^{-2}v/u)$ for coordinates $([a:b],[u:v])$ which implies that D acts linearly on U_w with weight $-\alpha$. The class $[\mathbb{F}_0] \in \Omega^*_D(\mathbb{F}_0)_{\mathbb{Q}}$ again maps to $(1,1,1,1)$ and we want to compute the remaining images of the closures of the Bialynicki-Birula cells.

Therefore, we take the closure (wx) of one of the remaining Bialynicki-Birula cells.

We choose $f(b/a, v/u) = b/a$ to be an eigenfunction of D with respect to weight α. By Corollary 3.40 we obtain

$$[(wx) \cap U_w \to U_w] = c_1^D(L_\alpha)[\operatorname{Spec} k \to \operatorname{Spec} k] \cdot [U_w \to U_w]$$

in $\Omega_D^*(U_w)_\mathbb{Q}$. Pulling this relation back yields $i_w^*[(wx) \cap U_w \to U_w] = c_1^D(L_\alpha)$. With the eigenfunction $f(b/a, u/v) = b/a$ and an open D-stable neighbourhood U_x of the fixed point x we obtain $i_x^*[(wx) \cap U_x \to U_x] = c_1^D(L_\alpha)$. For the pullbacks of (wy) we take the eigenfunction $f(b/a, v/u) = v/u$ of D with respect to weight α on the open D-stable U_w from above, but in this case we have $V(f) = (wy) \cap U_w$ and therefore, $i_w^*[(wy) \cap U_w \to U_w] = c_1^D(L_\alpha)$. Similarly, we obtain $i_y^*[(wy) \cap U_y \to U_y] = c_1^D(L_\alpha)$.

Lastly, we consider the pullback of the point w which is again the complete intersection of (wy) and (wx). By the same argument as in the above cases, we get

$$\begin{aligned} i_w^*[w \to U_w] &= i_w^*[(wx) \cap (wy) \cap U_w \to U_w] \\ &= c_1^D(L_\alpha) \cdot i_w^*[(wy) \cap U_w \to U_w] \\ &= c_1^D(L_\alpha)^2 \end{aligned}$$

whereas the other pullbacks of the class of the point w vanish. We summarise that the image of i^* is determined by the image of the basis, displayed below

$$\begin{aligned} i^* : \Omega_D^*(\mathbb{F}_0)_\mathbb{Q} &\to S(D)_\mathbb{Q}^4 \\ [\mathbb{F}_0 \to \mathbb{F}_0] &\mapsto (1, 1, 1, 1) \\ [(wx) \to \mathbb{F}_0] &\mapsto (c_1^D(L_\alpha), c_1^D(L_\alpha), 0, 0) \\ [(wy) \to \mathbb{F}_0] &\mapsto (c_1^D(L_\alpha), 0, c_1^D(L_\alpha), 0) \\ [w \to \mathbb{F}_0] &\mapsto (c_1^D(L_\alpha)^2, 0, 0, 0). \end{aligned}$$

which satisfies the equations (4.1) and (4.2).

Conversely, for an element $(f_w, f_x, f_y, f_z) \in S(T)_\mathbb{Q}^4$ fulfilling the conditions, we have

$$\begin{aligned} (f_w, f_x, f_y, f_z) =& f_z(1, 1, 1, 1) + (f_w - f_z, f_x - f_z, f_y - f_z, 0) \\ =& f_z(1, 1, 1, 1) + \frac{f_y - f_z}{c_1^D(L_\alpha)} \left(c_1^D(L_\alpha), 0, c_1^D(L_\alpha), 0\right) \\ &+ (f_w - f_y, f_x - f_z, 0, 0) \\ =& f_z(1, 1, 1, 1) + \frac{f_y - f_z}{c_1^D(L_\alpha)} \left(c_1^D(L_\alpha), 0, c_1^D(L_\alpha), 0\right) \\ &+ \frac{f_x - f_z}{c_1^D(L_\alpha)} \left(c_1^D(L_\alpha), c_1^D(L_\alpha), 0, 0\right) + (f_w - f_x - f_y + f_z, 0, 0, 0) \\ =& f_z(1, 1, 1, 1) + \frac{f_y - f_z}{c_1^D(L_\alpha)} \left(c_1^D(L_\alpha), 0, c_1^D(L_\alpha), 0\right) \\ &+ \frac{f_x - f_z}{c_1^D(L_\alpha)} \left(c_1^D(L_\alpha), c_1^D(L_\alpha), 0, 0\right) \\ &+ \frac{f_w - f_x - f_y + f_z}{c_1^D(L_\alpha)^2} \left(c_1^D(L_\alpha)^2, 0, 0, 0\right) \end{aligned}$$

which completes the proof for the case of $\mathbb{F}_0$.

In the following, we consider the case $\mathbb{F}_n$ for $n \geq 1$. The class $[\mathbb{F}_n] \in \Omega^*_D(\mathbb{F}_n)_\mathbb{Q}$ is again mapped to $(1,1,1,1)$.

Now, we compute the remaining pullbacks of the closures of the Bialynicki-Birula cells. Following the strategy from above, we choose again an open D-stable neighbourhood U_w of the D-fixed point $w = [0 : 0 : 1 : 0 : ... : 0]$. The induced D-action on U_w is given by $(t^{-n}x_0/y_0, t^{-2}y_1/y_0)$ for coordinates $[x_0 : x_1 : y_0 : y_1 : ... : y_{n+1}]$ and therefore, D acts linearly on U_w with weights $-n\alpha/2$ and $-\alpha$. We choose $f(x_0/y_0, y_1/y_0) = y_1/y_0$ to be an eigenfunction of D with respect to weight α. By the relations on the coordinates in $\mathbb{F}_n$ we obtain $V(f) = (wx) \cap U_w$ with the given notations of the D-fixed points. As above, we get $i_w^*[(wx) \cap U_w \to U_w] = c_1^D(L_\alpha)$. One may observe that the pullback does not depend on the choice of coordinates for U_w. For the point $x = [1 : 0 : ... : 0]$ and a D-stable neighbourhood U_x we choose the eigenfunction $f(x_1/x_0, y_1/x_0) = x_1/x_0$ of D with respect to weight α which leads to $i_x^*[(wx) \cap U_x \to U_x] = c_1^D(L_\alpha)$.

For the pullback of (xy) let U_x be given by coordinates $(y_0/x_0, x_1/x_0)$ and take the eigenfunction $f(y_0/x_0, x_1/x_0) = y_0/x_0$ of weight $-n\alpha/2$ which leads to $V(f) = (xy) \cap U_x$ and therefore to $i_x^*[(xy) \cap U_x \to U_x] = c_1^D(L_{-n\alpha/2})$. For the coordinates $(y_0/x_1, y_{n+1}/x_1)$ for U_y and the eigenfunction $f(y_0/x_1, y_{n+1}/x_1) = y_{n+1}/x_1$ we get $V(f) = (xy) \cap U_y$ and hence $i_y^*[(xy) \cap U_y \to U_y] = c_1^D(L_{n\alpha/2})$.

Finally, we consider the pullback of the point w by introducing an eigenfunction on $(wx) \cap U_w$. We choose $g(x_0/y_0, y_1/y_0) = x_0/y_0$ which is an eigenfunction of weight $n\alpha/2$. This leads to $V(g) = (wz) \cap (wx) \cap U_w = w$ and thus, we obtain

$$[w \to U_w] = c_1^D(L_{n\alpha/2})[(wx) \cap U_w \to U_w]$$

in $\Omega^*_D(U_w)_\mathbb{Q}$, again by Corollary 3.40.

We conclude $i_w^*[w \to U_w] = c_1^D(L_{n\alpha/2})c_1^D(L_\alpha)$ and obtain the image

$$\begin{aligned} i^* : \Omega^*_D(\mathbb{F}_n)_\mathbb{Q} &\to S(D)^4_\mathbb{Q} \\ [\mathbb{F}_n \to \mathbb{F}_n] &\mapsto (1,1,1,1) \\ [(wx) \to \mathbb{F}_n] &\mapsto \left(c_1^D(L_\alpha), c_1^D(L_\alpha), 0, 0\right) \\ [(xy) \to \mathbb{F}_n] &\mapsto \left(0, c_1^D(L_{-n\alpha/2}), c_1^D(L_{n\alpha/2}), 0\right) \\ [w \to \mathbb{F}_n] &\mapsto \left(c_1^D(L_\alpha)c_1^D(L_{n\alpha/2}), 0, 0, 0\right) \end{aligned}$$

which satisfies the equations (4.3) and (4.4).

Conversely, let $(f_w, f_x, f_y, f_z) \in S(T)^4_\mathbb{Q}$ be an element fulfilling these equations. This leads to

$$\begin{aligned} &(f_w, f_x, f_y, f_z) = f_z(1,1,1,1) + (f_w - f_z, f_x - f_z, f_y - f_z, 0) \\ =&f_z(1,1,1,1) + \frac{f_y - f_z}{c_1^D(L_{n\alpha/2})}\left(0, c_1^D(L_{-n\alpha/2}), c_1^D(L_{n\alpha/2}), 0\right) \\ &+ \left(f_w - f_z, \frac{(f_z - f_y)c_1^D(L_{-n\alpha/2})}{c_1^D(L_{n\alpha/2})} + f_x - f_z, 0, 0\right) \\ =&f_z(1,1,1,1) + \frac{f_y - f_z}{c_1^D(L_{n\alpha/2})}\left(0, c_1^D(L_{-n\alpha/2}), c_1^D(L_{n\alpha/2}), 0\right) \end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{(f_z-f_y)c_1^D(L_{-n\alpha/2})+(f_x-f_z)c_1^D(L_{n\alpha/2})}{c_1^D(L_{n\alpha/2})c_1^D(L_\alpha)}\right)\left(c_1^D(L_\alpha),c_1^D(L_\alpha),0,0\right)\\
&+\left(\frac{(f_y-f_z)c_1^D(L_{-n\alpha/2})+(f_w-f_x)c_1^D(L_{n\alpha/2})}{c_1^D(L_{n\alpha/2})},0,0,0\right)\\
=&f_z(1,1,1,1)+\frac{f_y-f_z}{c_1^D(L_{n\alpha/2})}\left(0,c_1^D(L_{-n\alpha/2}),c_1^D(L_{n\alpha/2}),0\right)\\
&+\left(\frac{(f_z-f_y)c_1^D(L_{-n\alpha/2})+(f_x-f_z)c_1^D(L_{n\alpha/2})}{c_1^D(L_{n\alpha/2})c_1^D(L_\alpha)}\right)\left(c_1^D(L_\alpha),c_1^D(L_\alpha),0,0\right)\\
&+\left(\frac{(f_y-f_z)c_1^D(L_{-n\alpha/2})+c_1^D(L_{n\alpha/2})(f_w-f_x)}{c_1^D(L_{n\alpha/2})^2c_1^D(L_\alpha)}\right)\left(c_1^D(L_\alpha)c_1^D(L_{n\alpha/2}),0,0,0\right)
\end{aligned}
$$

which completes the proof in the case $\mathbb{F}_n$ because of the equations (4.3) and (4.4), and the above mentioned fact that an element which is divisible by $c_1^D(L_\alpha)$ will also be divisible by $c_1^D(L_{n\alpha/m})$ for $n \in \mathbb{Z} \setminus \{0\}$ and $m \in \mathbb{Z}_{\geq 1}$ since we consider rational coefficients. □

Remark 4.18. The equations given in Proposition 4.16 reduce to the equations given in Proposition 4.6 for rational equivariant Chow rings. In order to be able to compute rational equivariant cobordism rings one has to consider the universal formal group law and not the additive formal group law which simplifies the computations in the Chow group case.

Lemma 4.19. *The product of projective spaces is cellular.*

Proof. The product of projective spaces is projective by the Segre embedding and smooth since the product of smooth schemes is smooth. By Proposition 3.7 any smooth projective scheme is T-filtrable. By Lemma 4.8 the product of projective spaces has only finitely many T-fixed points. Therefore, by Definition 3.5, we know that it is the union of its plus strata which in this case are affine spaces. □

Next, we want to prove Theorem 4.13 from the beginning of this section which is a refinement of [7, Theorem 7.3]. One way to prove it would be to use an analogue of [7, Theorem 2.1] for equivariant algebraic cobordism which is not known at present. Luckily, we do not need such a deep result in order to be able to prove Theorem 4.13. In our situation it will be enough to use some known results on T-filtrable varieties and their equivariant algebraic cobordism rings.

Proof of Theorem 4.13. In order to compute the ring structure of $\Omega_T^*(X)_\mathbb{Q}$ we need to apply Proposition 3.51 to the given variety X. Due to Proposition 4.3 we know which fixed point subschemes $X^{T'}$ can occur and therefore, we distinguish between codimension one subtori T' with $\dim X^{T'} \leq 1$ and those with $\dim X^{T'} = 2$.

Recall that for a subtorus T' with $\dim X^{T'} \leq 1$ there are only finitely many T-stable curves in $X^{T'}$ and furthermore, in the setting of a smooth projective spherical G-variety X, we have only finitely many T-fixed points by [10, Lemma 2.2]. This implies that the assumptions of Proposition 3.52 are fulfilled and thus, we can apply Proposition 3.52 to $X^{T'}$ which leads to case (i).

Now, we consider the case where $\dim X^{T'} = 2$ for which we know that $X^{T'}$ is either a projective plane or a Hirzebruch surface $\mathbb{F}_n$. The T-orbits in $X^{T'}$ are always at most one-dimensional and thus, the surfaces occurring in (ii)-(iv) must consist of infinitely many T-stable curves. For these cases we need some different results. We claim that there is an isomorphism $\Omega_*^T(X^{T'})_{\mathbb{Q}} \cong \Omega_*^{T/F(\alpha)}(X^{T'})_{\mathbb{Q}}$ where $F(\alpha)$ is given as in Remark 4.2. We will use [31, Theorem 4.7] in order to prove our claim. This theorem states that we have an isomorphism of $S(T)$-modules $\Omega_*^T(X^{T'}) \cong \Omega_*(X^{T'})[[t_1, ..., t_r]]_{\mathrm{gr}}$ where r is the rank of T and t_i corresponds to $c_1^T(L_{\chi_i})$ for a chosen basis of the character group of T. We remark that $X^{T'}$ is also a $T/F(\alpha)$-filtrable variety as $F(\alpha)$ acts trivially on $X^{T'}$ and therefore, the T-action factors through a $T/F(\alpha)$-action. As above, we obtain the isomorphism $\Omega_*^{T/F(\alpha)}(X^{T'}) \cong \Omega_*(X^{T'})[[t_1, ..., t_r]]_{\mathrm{gr}}$ of $S(T/F(\alpha))$-modules where t_i here corresponds to $c_1^{T/F(\alpha)}(L_{a_i\chi_i})$, but as we are considering rational coefficients we have $S(T)_{\mathbb{Q}} \cong S(T/F(\alpha))_{\mathbb{Q}}$ by Lemma 3.36. This implies the claim and using the same argument for the torus $T' \times S_m(\alpha)$, we obtain

$$
\begin{aligned}
\Omega_*^T(X^{T'})_{\mathbb{Q}} &\cong \Omega_*^{(T'\times S_m(\alpha))/(F(\alpha)\times F(\alpha))}(X^{T'})_{\mathbb{Q}} \\
&\cong \bigoplus_{i\in\mathbb{Z}} \Omega_i^{T'\times S_m(\alpha)}(X^{T'})_{\mathbb{Q}} \\
&\cong \bigoplus_{i\in\mathbb{Z}} \varprojlim_j \Omega_i((\operatorname{Spec} k \times U_j^2 \times X^{T'} \times U_j^1)/(T' \times S_m(\alpha)))_{\mathbb{Q}} \\
&\cong \bigoplus_{i\in\mathbb{Z}} \varprojlim_j \Omega_i((\operatorname{Spec} k \times U_j^2)/T' \times (X^{T'} \times U_j^1)/S_m(\alpha))_{\mathbb{Q}} \\
&\cong \bigoplus_{i\in\mathbb{Z}} \varprojlim_j \bigoplus_{i_1+i_2=i} \Omega_{i_1}((\operatorname{Spec} k \times U_j^2)/T')_{\mathbb{Q}} \otimes_{\mathbb{L}_{\mathbb{Q}}} \Omega_{i_2}((X^{T'} \times U_j^1)/S_m(\alpha))_{\mathbb{Q}} \\
&\cong \bigoplus_{i\in\mathbb{Z}} \bigoplus_{i_1+i_2=i} \varprojlim_j \Omega_{i_1}((\operatorname{Spec} k \times U_j^2)/T')_{\mathbb{Q}} \otimes_{\mathbb{L}_{\mathbb{Q}}} \Omega_{i_2}((X^{T'} \times U_j^1)/S_m(\alpha))_{\mathbb{Q}} \\
&\cong \bigoplus_{i\in\mathbb{Z}} \bigoplus_{i_1+i_2=i} \Omega_{i_1}^{T'}(\operatorname{Spec} k)_{\mathbb{Q}} \otimes_{\mathbb{L}_{\mathbb{Q}}} \Omega_{i_2}^{S_m(\alpha)}(X^{T'})_{\mathbb{Q}} \\
&\cong \Omega_*^{T'}(\operatorname{Spec} k)_{\mathbb{Q}} \otimes_{\mathbb{L}_{\mathbb{Q}}} \Omega_*^{S_m(\alpha)}(X^{T'})_{\mathbb{Q}}
\end{aligned}
$$

where U_j^1 and U_j^2 are the parts of the sequences of good pairs $\{(V_j^1, U_j^1)\}_{j\geq 0}$ and $\{(V_j^2, U_j^2)\}_{j\geq 0}$ for $S_m(\alpha)$ and T', respectively. In this case we know that U_j^2/T' are products of projective spaces by the choice of good pairs in the proof of [31, Lemma 6.1]. As a product of projective spaces, the U_j^2/T' are cellular which means that we can use a special version of a Künneth formula (cf. [20, Proposition 7]) from line 4 to 5. The ordinary cobordism vanishes for negative degrees and therefore, the inverse limit and the finite sum commute in our setting. We conclude the proof by Proposition 4.16. □

Remark 4.20. Proposition 3.47 leads to an abstract description of the rational ordinary algebraic cobordism ring of any smooth projective and spherical G-variety X. Using this result, we would be able to describe $\Omega^*(X)_{\mathbb{Q}}$ explicitly if we could compute all the classes in $\Omega_T^*(X)_{\mathbb{Q}}$. We will come back to this problem in Section 6.

In the remark below, we discuss a result which illustrates the relation between the rational Chow module and the rational cobordism module. Therefore, we first state a well known fact about Chow groups.

Lemma 4.21. *[7, Corollary 2.3] Let T be a torus acting on a k-variety X. Then the forgetful map for Chow groups r_X^T induces an isomorphism*

$$\overline{r}_X^T : \mathrm{CH}_*^T(X) \otimes_{\mathrm{CH}_*^T(k)} \mathbb{Z} \xrightarrow{\cong} \mathrm{CH}_*(X).$$

If X is smooth, this is an $\mathbb{Z}$-algebra isomorphism.

Remark 4.22. Using the notations from chapter four in [37], we know that there is an isomorphism $(\Omega_*)_{\mathbb{Q}} \cong \mathrm{CH}_* \otimes_{\mathbb{Z}} \mathbb{Q}[t]^{(t)}$ of oriented Borel-Moore weak homology theories by [37, Theorem 4.1.28 and Theorem 4.5.1]. We remark that we have $A_*(X) = A_*^{(t)}(X)$ by [37, Section 4.1.9] for any Borel-Moore weak homology theory A_*. Therefore, we consider the following diagram for any T-filtrable variety with finitely many T-fixed points

$$\begin{array}{ccc} \mathrm{CH}_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[t] & \xrightarrow{\cong} & \Omega_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{CH}_*^T(X) \otimes_{\mathrm{CH}_*^T(k)} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}[t] & \xrightarrow{\cong} & \Omega_*^T(X) \otimes_{S(T)} \mathbb{L}_{\mathbb{Q}} \end{array}$$

which leads to $\mathrm{CH}_*^T(X) \otimes_{\mathrm{CH}_*^T(k)} \mathbb{Q}[t] \cong \Omega_*^T(X) \otimes_{S(T)} \mathbb{L}_{\mathbb{Q}}$ by Corollary 4.21 and Proposition 3.47.

On the other hand, one has

$$\begin{aligned} \Omega_n^T(X)_{\mathbb{Q}} &= \varprojlim_j (\Omega_n((X \times U_j)/T)_{\mathbb{Q}}) \\ &\cong \varprojlim_j (\mathrm{CH}_n((X \times U_j)/T) \otimes_{\mathbb{Z}} \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[t]) \\ &\cong \varprojlim_j (\mathrm{CH}_n((X \times U_j)/T)[t] \otimes_{\mathbb{Z}} \mathbb{Q}) \\ &\cong \varprojlim_j (\mathrm{CH}_n((X \times U_j)/T)[t]) \otimes_{\mathbb{Z}} \mathbb{Q} \\ &\cong \mathrm{CH}_n^T(X)[t] \otimes_{\mathbb{Z}} \mathbb{Q} \end{aligned}$$

using the above isomorphism and the property that the above limit stabilises for some j. Therefore, we can omit the limit in the last step. The above shows that one can obtain the rational T-equivariant cobordism module using some information of the rational T-equivariant Chow groups via extension of scalars by $\mathbb{Z}[t]$. More details regarding the precise constructions can be found in [37, Chapter 4].

Remark 4.23. By now we have computed (equivariant) algebraic cobordism with rational coefficients whereas the group structure is already known by [37, Theorem 4.1.28], but the latter result does not imply the description of the (equivariant) cobordism rings with rational coefficients using the one of the (equivariant) Chow rings. We remark that the results in this section could be proved with the coefficient ring $\mathbb{Z}[S_X^{-1}]$ using the refined localisation result (cf. Theorem 3.59). This is not quite true because we would need to possibly invert $p = 2$ additionally, due to the fact that we use Lemma 3.36 for a subgroup F which could be of order 2.

4.3 Equivariant cobordism of odd symplectic Grassmannians

In this section, we will compute several smaller examples in order to be able to generalise the results for the most general case of the odd symplectic Grassmannians. We will start with the usual Grassmannian $\mathrm{Gr}(2,4)$.

Example 4.24. Let $X = \mathrm{Gr}(2,4)$ be the Grassmannian of 2-dimensional planes in k^4 for an algebraically closed field k of characteristic zero. This is certainly smooth, projective and spherical and therefore, Theorem 4.13 can be used to describe the structure of the rational equivariant cobordism ring $\Omega_T^*(\mathrm{Gr}(2,4))_{\mathbb{Q}}$. As we will see, there are only finitely many T-stable curves contained in $\mathrm{Gr}(2,4)$ which implies that we will only need the description given by Proposition 3.52, but nevertheless it will give an intuitive understanding of how computations can be done using the given result. First, we consider the natural GL_4-action on X induced by the one on k^4 which induces the following torus action

$$\begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_3 & 0 \\ 0 & 0 & 0 & t_4 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} = \begin{pmatrix} t_1 a_{11} & t_1 a_{12} \\ t_2 a_{21} & t_2 a_{22} \\ t_3 a_{31} & t_3 a_{32} \\ t_4 a_{41} & t_4 a_{42} \end{pmatrix}.$$

We see that this action has six T-fixed points, namely

$$\begin{aligned} x_{12} = (e_1, e_2),\ x_{13} = (e_1, e_3),\ x_{14} = (e_1, e_4), \\ x_{23} = (e_2, e_3),\ x_{24} = (e_2, e_4),\ x_{34} = (e_3, e_4) \end{aligned}$$

where e_i denotes the i-th basis vector of k^4. In order to obtain the singular subtori of codimension one we need to consider the positive roots of (GL_n, T) which are given by

$$(\varepsilon_i - \varepsilon_j) \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_3 & 0 \\ 0 & 0 & 0 & t_4 \end{pmatrix} = t_i t_j^{-1} \text{ for } i < j.$$

The identity components of the kernels of the $\varepsilon_i - \varepsilon_j$ lead to the singular subtori of codimension one (cf. Remark 4.2). Next, we consider the fixed point subschemes for $T'_{ij} := \mathrm{Ker}(\varepsilon_i - \varepsilon_j)^0$. Exemplarily, we analyse it for $X^{T'_{12}} = X^{\mathrm{Ker}(\varepsilon_1 - \varepsilon_2)^0}$. We obtain

$$\begin{pmatrix} t_2 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_3 & 0 \\ 0 & 0 & 0 & t_4 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} = \begin{pmatrix} t_2 a_{11} & t_2 a_{12} \\ t_2 a_{21} & t_2 a_{22} \\ t_3 a_{31} & t_3 a_{32} \\ t_4 a_{41} & t_4 a_{42} \end{pmatrix}$$

as the induced T'_{12}-action. Clearly, the six T-fixed points are contained in $X^{T'_{12}}$ and furthermore, the points

$$\begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

in Gr(2, 4). We see that these are 2 T-stable curves and therefore, summarise that $X^{T'_{12}}$ consists of these two T-stable curves and the two remaining isolated fixed points. One can verify that there are 12 T-stable curves of the form as above and two different of those are each contained in some $X^{\mathrm{Ker}(\varepsilon_i-\varepsilon_j)^0}$ such that every T-stable curve is contained in some $X^{\mathrm{Ker}(\varepsilon_i-\varepsilon_j)^0}$.

We remark that the regular subtori T' are obtained as $T' = \mathrm{Ker}(\chi)^0$ for some primitive character χ which is not a root (see e.g. [7]). In this particular case one can verify that we have $X^{T'} = X^T$ (see also [8, Section 2]) whereas in the general case it might happen that $X^{T'} \neq X^T$, but only for finitely many subtori $T' = \ker(\chi)^0$ which has been proved by Brion [9, Theorem 1.2]. One also observes that for the singular subtori and their fixed point subschemes, the kernel of the T-action on the T-stable curves contained in $X^{\mathrm{Ker}(\varepsilon_i-\varepsilon_j)^0}$ coincides with the kernel of the positive root $\varepsilon_i - \varepsilon_j$.

Now, we can apply the localisation theorem and compute the images of the pullback maps

$$i^*_{T'_{ij}} : \Omega^*_T(X^{\mathrm{Ker}(\varepsilon_i-\varepsilon_j)^0})_{\mathbb{Q}} \to \Omega^*_T(X^T)_{\mathbb{Q}}$$

and take their intersection. As example we consider

$$\begin{aligned}
i^*_{T'_{12}} : \Omega^*_T(X^{\mathrm{Ker}(\varepsilon_1-\varepsilon_2)^0})_{\mathbb{Q}} &\to \Omega^*_T(X^T)_{\mathbb{Q}} \\
[x_{12} \to X^{\mathrm{Ker}(\varepsilon_1-\varepsilon_2)^0}] &\mapsto (1, 0, 0, 0, 0, 0) \\
[x_{13} \to X^{\mathrm{Ker}(\varepsilon_1-\varepsilon_2)^0}] &\mapsto (0, c_1^T(L_{\varepsilon_1-\varepsilon_2}), 0, 0, 0, 0) \\
[x_{14} \to X^{\mathrm{Ker}(\varepsilon_1-\varepsilon_2)^0}] &\mapsto (0, 0, c_1^T(L_{\varepsilon_1-\varepsilon_2}), 0, 0, 0) \\
[x_{23} \to X^{\mathrm{Ker}(\varepsilon_1-\varepsilon_2)^0}] &\mapsto (0, 0, 0, c_1^T(L_{\varepsilon_2-\varepsilon_1}), 0, 0) \\
[x_{24} \to X^{\mathrm{Ker}(\varepsilon_1-\varepsilon_2)^0}] &\mapsto (0, 0, 0, 0, c_1^T(L_{\varepsilon_2-\varepsilon_1}), 0) \\
[x_{34} \to X^{\mathrm{Ker}(\varepsilon_1-\varepsilon_2)^0}] &\mapsto (0, 0, 0, 0, 0, 1) \\
[C_{1323} \to X^{\mathrm{Ker}(\varepsilon_1-\varepsilon_2)^0}] &\mapsto (0, 1, 0, 1, 0, 0) \\
[C_{1424} \to X^{\mathrm{Ker}(\varepsilon_1-\varepsilon_2)^0}] &\mapsto (0, 0, 1, 0, 1, 0)
\end{aligned}$$

using the same technique as in the proof of Proposition 4.16 for C_{ijkl} being the T-stable curves connecting x_{ij} and x_{kl}.

By Proposition 3.52 the intersection of these images and thus, the tuples $(f_{12}, ..., f_{34})$ contained in the image of $i^* : \Omega^*_T(X)_{\mathbb{Q}} \to \Omega^*_T(X^T)_{\mathbb{Q}}$ can be described by the equations

$$\begin{aligned}
f_{13} &\equiv f_{23} \mod c_1^T(L_{\varepsilon_1-\varepsilon_2}), & f_{14} &\equiv f_{24} \mod c_1^T(L_{\varepsilon_1-\varepsilon_2}), \\
f_{12} &\equiv f_{23} \mod c_1^T(L_{\varepsilon_1-\varepsilon_3}), & f_{14} &\equiv f_{34} \mod c_1^T(L_{\varepsilon_1-\varepsilon_3}), \\
f_{12} &\equiv f_{24} \mod c_1^T(L_{\varepsilon_1-\varepsilon_4}), & f_{13} &\equiv f_{34} \mod c_1^T(L_{\varepsilon_1-\varepsilon_4}), \\
f_{12} &\equiv f_{13} \mod c_1^T(L_{\varepsilon_2-\varepsilon_3}), & f_{24} &\equiv f_{34} \mod c_1^T(L_{\varepsilon_2-\varepsilon_3}), \\
f_{12} &\equiv f_{14} \mod c_1^T(L_{\varepsilon_2-\varepsilon_4}), & f_{23} &\equiv f_{34} \mod c_1^T(L_{\varepsilon_2-\varepsilon_4}), \\
f_{13} &\equiv f_{14} \mod c_1^T(L_{\varepsilon_3-\varepsilon_4}), & f_{23} &\equiv f_{24} \mod c_1^T(L_{\varepsilon_3-\varepsilon_4}).
\end{aligned}$$

From these equations it can easily be seen that the image of the fixed point x_{12} under

i^* must be given by

$$(a \cdot c_1^T(L_{\varepsilon_1-\varepsilon_3})c_1^T(L_{\varepsilon_1-\varepsilon_4})c_1^T(L_{\varepsilon_2-\varepsilon_3})c_1^T(L_{\varepsilon_2-\varepsilon_4}), 0, ..., 0)$$

for some $a \in S(T)_{\mathbb{Q}}$ of degree zero and similarly for the other T-fixed points. Furthermore, the image under i^* of the T-stable curve C_{1213} connecting x_{12} and x_{13} must be given by

$$(ac_1^T(L_{\varepsilon_1-\varepsilon_3})c_1^T(L_{\varepsilon_1-\varepsilon_4})c_1^T(L_{\varepsilon_2-\varepsilon_4}), bc_1^T(L_{\varepsilon_1-\varepsilon_2})c_1^T(L_{\varepsilon_1-\varepsilon_4})c_1^T(L_{\varepsilon_3-\varepsilon_4}), 0, ..., 0)$$

for some $a, b \in S(T)_{\mathbb{Q}}$ of degree zero. This method gives the equivariant algebraic cobordism ring, but we cannot determine in general which class corresponds to which element in $\Omega_T^*(X^T)_{\mathbb{Q}}$. This gives some constraint on the coefficients, but as mentioned above, they cannot be determined uniquely by the localisation method. Therefore, we can obtain certain elements in the subring given by the above equations corresponding to the geometric elements in $\mathrm{Gr}(2,4)$ up to coefficients. In Section 6, we will come back to the problem of determining these coefficients precisely.

For the next example we want to recall some definitions which will be necessary in this specific computation. Therefore, let V be a k-vector space of dimension $2n+1$ for $n \geq 2$ and ω an antisymmetric form of maximal rank on V.

Definition 4.25. *Let $\omega : V \times V \to k$ be an antisymmetric form. We call a subspace $W \subseteq V$ isotropic if $\omega(w_1, w_2) = 0$ holds for all $w_1, w_2 \in W$.*

Remark 4.26. Every antisymmetric form is represented by an antisymmetric matrix A via $\omega(v_1, v_2) = v_1^T A v_2$. Furthermore, the rank of an antisymmetric matrix of odd dimension $2n+1$ is at most $2n$.

Definition 4.27. *With the notations above, let $2 \leq m \leq n$ be an integer. Then the odd symplectic Grassmannian with respect to ω is defined by*

$$\mathrm{IG}_\omega(m, V) := \{\Sigma \in \mathrm{Gr}(m, V) \mid \Sigma \textit{ is isotropic for } \omega\}.$$

Remark 4.28. It is well known by [16] that it has an action of the symplectic group

$$\mathrm{Sp}(W) := \{g \in \mathrm{GL}(W) \mid \omega(gu, gv) = \omega(u, v)\ \forall u, v \in W\}$$

where W denotes the complement of the kernel K of the form ω in V. Therefore, $\omega|_W$ is a symplectic form. Furthermore, up to isomorphism the odd symplectic Grassmannian $\mathrm{IG}_\omega(m, V)$ does not depend on the $(2n+1)$-dimensional vector space V nor the antisymmetric form ω by [46]. Therefore, we denote the odd symplectic Grassmannian by $\mathrm{IG}(m, 2n+1)$ and similarly, the symplectic group by Sp_{2n}. We know that $\mathrm{IG}(m, V)$ is horospherical by [43, Proposition 1.12] for the Sp_{2n}-action and corresponds to the triple $(C_n, \omega_m, \omega_{m-1})$, i.e. the case (3) of Pasquier's classification of horospherical varieties which is described in [43].

Now, we want to have a look at the specific example of $\mathrm{IG}(2, 5)$ for which we can use Theorem 4.13 in order to compute its rational equivariant cobordism ring.

Example 4.29. Let $V = \mathbb{C}^5$ and $X = \mathrm{IG}(2,5)$ be the odd symplectic Grassmannian. Since X does not depend on the antisymmetric form we consider the one given by

$$\omega : V \times V \to \mathbb{C}, \left(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} \right) \mapsto \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix}.$$

Therefore, we have

$$\omega \left(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} \right) = a_5 b_1 + a_4 b_2 - a_2 b_4 - a_1 b_5.$$

Using the condition

$$a_5 b_1 + a_4 b_2 - a_2 b_4 - a_1 b_5 = 0,$$

we can easily verify whether an element in $\mathrm{Gr}(2,5)$ also lies in $\mathrm{IG}(2,5)$.

Next, we will consider the group action of Sp_4 on $\mathrm{IG}(2,5)$. First, we determine the torus of Sp_4. In order to do so, we consider the torus action induced by the GL_5-action which is given by

$$\begin{pmatrix} t_1 & 0 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & t_4 & 0 \\ 0 & 0 & 0 & 0 & t_5 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \\ a_5 & b_5 \end{pmatrix} = \begin{pmatrix} t_1 a_1 & t_1 b_1 \\ t_2 a_2 & t_2 b_2 \\ a_3 & b_3 \\ t_4 a_4 & t_4 b_4 \\ t_5 a_5 & t_5 b_5 \end{pmatrix}.$$

In order to be an action on $\mathrm{IG}(2,5)$, we need

$$t_5 t_1 a_5 b_1 + t_4 t_2 a_4 b_2 - t_2 t_4 a_2 b_4 - t_1 t_5 a_1 b_5 = 0$$

to hold. This implies $t_5 t_1 = t_2 t_4 = 1$ in Sp_4 and therefore, elements of the torus of Sp_4 are of the form

$$\begin{pmatrix} t_1 & 0 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & t_1^{-1} \end{pmatrix}.$$

This leads to the torus action

$$\begin{pmatrix} t_1 & 0 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & t_1^{-1} \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \\ a_5 & b_5 \end{pmatrix} = \begin{pmatrix} t_1a_1 & t_1b_1 \\ t_2a_2 & t_2b_2 \\ a_3 & b_3 \\ t_2^{-1}a_4 & t_2^{-1}b_4 \\ t_1^{-1}a_5 & t_1^{-1}b_5 \end{pmatrix}$$

of Sp_4 for which we denote the given torus by T. Let e_i denote again the i-th basis vector of $\mathbb{C}^5$. The T-fixed points in $\mathrm{IG}(2,5)$ are then given by

$$\begin{aligned} x_{12} &= (e_1, e_2),\ x_{13} = (e_1, e_3),\ x_{14} = (e_1, e_4),\ x_{23} = (e_2, e_3) \\ x_{25} &= (e_2, e_5),\ x_{34} = (e_3, e_4),\ x_{35} = (e_3, e_5),\ x_{45} = (e_4, e_5) \end{aligned}$$

whereas the points $x_{15} = (e_1, e_5)$ and $x_{24} = (e_2, e_4)$ are not in $\mathrm{IG}(2,5)$.

The positive roots of (Sp_4, T) are given by $\varepsilon_1 - \varepsilon_2, \varepsilon_1 + \varepsilon_2, 2\varepsilon_1$ and $2\varepsilon_2$ for the characters

$$\varepsilon_i \begin{pmatrix} t_1 & 0 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & t_1^{-1} \end{pmatrix} = t_i$$

which we will denote by $\varepsilon_{12}, \varepsilon_{1(-2)}, \varepsilon_{1(-1)}$ and $\varepsilon_{2(-2)}$, respectively, for later use.

For $T'_{12} = \mathrm{Ker}(\varepsilon_{12})^0$, the subtorus action is given by

$$\begin{pmatrix} t_1 & 0 & 0 & 0 & 0 \\ 0 & t_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & t_1^{-1} & 0 \\ 0 & 0 & 0 & 0 & t_1^{-1} \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \\ a_5 & b_5 \end{pmatrix} = \begin{pmatrix} t_1a_1 & t_1b_1 \\ t_1a_2 & t_1b_2 \\ a_3 & b_3 \\ t_1^{-1}a_4 & t_1^{-1}b_4 \\ t_1^{-1}a_5 & t_1^{-1}b_5 \end{pmatrix}.$$

Therefore, $X^{T'_{12}}$ consists of the points

$$\begin{pmatrix} a_1 & 0 \\ a_2 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & b_4 \\ 0 & b_5 \end{pmatrix}, \begin{pmatrix} a_1 & 0 \\ a_2 & 0 \\ 0 & 0 \\ 0 & b_4 \\ 0 & b_5 \end{pmatrix}$$

where the equation $-a_1b_5 - a_2b_4 = 0$ has to hold for the third kind of points. These are three T-stable curves which are contained in $X^{T'_{12}}$ in addition to two isolated fixed points x_{12} and x_{45}.

For the subtorus $T'_{1(-2)} = \mathrm{Ker}(\varepsilon_{1(-2)})^0$ we obtain the action

$$\begin{pmatrix} t_1 & 0 & 0 & 0 & 0 \\ 0 & t_1^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & t_1 & 0 \\ 0 & 0 & 0 & 0 & t_1^{-1} \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \\ a_5 & b_5 \end{pmatrix} = \begin{pmatrix} t_1a_1 & t_1b_1 \\ t_1^{-1}a_2 & t_1^{-1}b_2 \\ a_3 & b_3 \\ t_1a_4 & t_1b_4 \\ t_1^{-1}a_5 & t_1^{-1}b_5 \end{pmatrix}.$$

The fixed point subscheme $X^{T'_{1(-2)}}$ consists of the points of the form

$$\begin{pmatrix} a_1 & 0 \\ 0 & 0 \\ 0 & 1 \\ a_4 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & b_2 \\ 1 & 0 \\ 0 & 0 \\ 0 & b_5 \end{pmatrix}, \begin{pmatrix} a_1 & 0 \\ 0 & b_2 \\ 0 & 0 \\ a_4 & 0 \\ 0 & b_5 \end{pmatrix}$$

where the equation $-a_1b_5 + a_4b_2 = 0$ has to hold for the third kind of points. These are again three T-stable curves which are contained in $X^{T'_{1(-2)}}$ in addition to the two other isolated fixed points x_{14} and x_{25}.

Next, we consider the subtorus $T'_{1(-1)} = \mathrm{Ker}(\varepsilon_{1(-1)})^0$ and its fixed point subscheme. The subtorus action is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & t_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \\ a_5 & b_5 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ t_2a_2 & t_2b_2 \\ a_3 & b_3 \\ t_2^{-1}a_4 & t_2^{-1}b_4 \\ a_5 & b_5 \end{pmatrix}$$

since $t_1 = 1$ as $T'_{1(-1)}$ is by definition the identity component of the kernel of $\varepsilon_{1(-1)}$. This leads to the fixed point subscheme $X^{T'_{1(-1)}}$ which consists of points of the form

$$\begin{pmatrix} a_1 & 0 \\ 0 & 1 \\ a_3 & 0 \\ 0 & 0 \\ a_5 & 0 \end{pmatrix}, \begin{pmatrix} a_1 & 0 \\ 0 & 0 \\ a_3 & 0 \\ 0 & 1 \\ a_5 & 0 \end{pmatrix}, \begin{pmatrix} a_1 & b_1 \\ 0 & 0 \\ a_3 & b_3 \\ 0 & 0 \\ a_5 & b_5 \end{pmatrix}$$

where the first two components are a projective plane $\mathbb{P}^2$ and the last one is the set of two dimensional subspaces in a three dimensional subspace of $\mathbb{C}^5$. Therefore, the last component is isomorphic to the Grassmannian $\mathrm{Gr}(2,3)$ with the condition $a_5b_1 - a_1b_5 = 0$ which leads to a T-stable curve connecting x_{13} and x_{35}.

The computations for the positive root $\varepsilon_{2(-2)}$ are similar. They lead to the points

$$\begin{pmatrix} 0 & 1 \\ a_2 & 0 \\ a_3 & 0 \\ a_4 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ a_2 & 0 \\ a_3 & 0 \\ a_4 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \\ 0 & 0 \end{pmatrix}$$

in the fixed point subscheme $X^{T'_{2(-2)}}$. Again, we obtain two projective planes and one T-stable curve connecting x_{23} and x_{34}.

The other codimension one subtori are regular. Those are given by $T' = \mathrm{Ker}(\chi)^0$ for some primitive character χ of T which is not a multiple of a root and further, they will not contribute to the computations of cobordism since one may verify that $X^{T'} = X^T$ holds for those T'.

These precise descriptions of the fixed point subschemes lead to the equations describing the image of $i^* : \Omega_T^*(\mathrm{IG}(2,5))_{\mathbb{Q}} \to \Omega_T^*((\mathrm{IG}(2,5))^T)_{\mathbb{Q}}$.

Using Theorem 4.13, the equations are given by

$$\begin{aligned}
&f_{13} \equiv f_{23} \mod c_1^T(L_{\varepsilon_{12}}), \quad f_{34} \equiv f_{35} \mod c_1^T(L_{\varepsilon_{12}}),\\
&f_{14} \equiv f_{25} \mod c_1^T(L_{\varepsilon_{12}}), \quad f_{13} \equiv f_{34} \mod c_1^T(L_{\varepsilon_{1(-2)}}),\\
&f_{23} \equiv f_{35} \mod c_1^T(L_{\varepsilon_{1(-2)}}), \quad f_{12} \equiv f_{45} \mod c_1^T(L_{\varepsilon_{1(-2)}}),\\
&f_{13} \equiv f_{35} \mod c_1^T(L_{\varepsilon_{1(-1)}}), \quad f_{23} \equiv f_{34} \mod c_1^T(L_{\varepsilon_{2(-2)}}),\\
&f_{12} \equiv f_{23} \equiv f_{25} \mod c_1^T(L_{\varepsilon_{1(-1)}}), \quad f_{14} \equiv f_{34} \equiv f_{45} \mod c_1^T(L_{\varepsilon_{1(-1)}}),\\
&f_{25} \equiv f_{35} \equiv f_{45} \mod c_1^T(L_{\varepsilon_{2(-2)}}), \quad f_{12} \equiv f_{13} \equiv f_{14} \mod c_1^T(L_{\varepsilon_{2(-2)}}),\\
&(f_{12} - f_{23}) + \rho_{1/2} c_1^T(L_{\varepsilon_{1(-1)}})(f_{25} - f_{12}) \equiv 0 \mod c_1^T(L_{\varepsilon_{1(-1)}})^2,\\
&(f_{14} - f_{34}) + \rho_{1/2} c_1^T(L_{\varepsilon_{1(-1)}})(f_{45} - f_{14}) \equiv 0 \mod c_1^T(L_{\varepsilon_{1(-1)}})^2,\\
&(f_{25} - f_{35}) + \rho_{1/2} c_1^T(L_{\varepsilon_{2(-2)}})(f_{45} - f_{25}) \equiv 0 \mod c_1^T(L_{\varepsilon_{2(-2)}})^2,\\
&(f_{12} - f_{13}) + \rho_{1/2} c_1^T(L_{\varepsilon_{2(-2)}})(f_{14} - f_{12}) \equiv 0 \mod c_1^T(L_{\varepsilon_{2(-2)}})^2.
\end{aligned}$$

These equations give a complete description of the rational equivariant algebraic cobordism ring of $\mathrm{IG}(2,5)$. As above, we want to know which elements correspond to the pullback of which T-stable classes. Therefore, we remark that the procedure for points and T-stable curves is similar to the one described in the previous example. Thus, we consider the pullback of the T-stable class given by the points

$$\begin{pmatrix} a_1 & 0 \\ 0 & 0 \\ a_3 & 0 \\ 0 & 1 \\ a_5 & 0 \end{pmatrix}$$

which is some projective plane $\mathbb{P}^2_{14,34,45}$ in $\mathrm{IG}(2,5)$. It contains the fixed points x_{14}, x_{34} and x_{45} and therefore, the corresponding entries in the tuple will be non-zero. First we determine the first Chern classes which have to be in each of the entries coming from the equations above. This leads to

$$\begin{aligned}
i_{x_{14}}^*[\mathbb{P}^2_{14,34,45} \to \mathrm{IG}(2,5)] &= n_1 c_1^T(L_{\varepsilon_{12}}) c_1^T(L_{\varepsilon_{2(-2)}})^2,\\
i_{x_{34}}^*[\mathbb{P}^2_{14,34,45} \to \mathrm{IG}(2,5)] &= n_2 c_1^T(L_{\varepsilon_{12}}) c_1^T(L_{\varepsilon_{1(-2)}}) c_1^T(L_{\varepsilon_{2(-2)}}),\\
i_{x_{45}}^*[\mathbb{P}^2_{14,34,45} \to \mathrm{IG}(2,5)] &= n_3 c_1^T(L_{\varepsilon_{1(-2)}}) c_1^T(L_{\varepsilon_{2(-2)}})^2
\end{aligned}$$

for some $n_1, n_2, n_3 \in S(T)_{\mathbb{Q}}$ of degree zero. As mentioned above, we do not know at present which particular choice of the n_i determines the pullback of $[\mathbb{P}^2_{14,34,45} \to \mathrm{IG}(2,5)]$, but one of those tuples is certainly its image under the pullback map i^*. We will come back to this problem in Section 6.

Example 4.30. Now, we consider a more general example for which we want to describe the rational equivariant cobordism ring. Therefore, we let $V = \mathbb{C}^{2n+1}$ and $X = \mathrm{IG}(2, 2n+1)$ be again the odd symplectic Grassmannian for $n \geq 2$.

As above, we consider the antisymmetric form $\omega : V \times V \to \mathbb{C}$ given by

$$\left(\begin{pmatrix} a_1 \\ \vdots \\ a_{2n+1} \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_{2n+1} \end{pmatrix} \right) \mapsto \begin{pmatrix} a_1 \\ \vdots \\ a_{2n+1} \end{pmatrix}^T \begin{pmatrix} 0 & 0 & -J \\ 0 & 0 & 0 \\ J & 0 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_{2n+1} \end{pmatrix}$$

for the $(n \times n)$-matrix

$$J := \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & \iddots & 0 \\ 0 & \iddots & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Similar to the above, we obtain the equation

$$a_{2n+1}b_1 + ... + a_{n+2}b_n - a_n b_{n+2} - ... - a_1 b_{2n+1} = 0.$$

The torus action of Sp_{2n} on $\mathrm{IG}(2, 2n+1)$ is given by

$$\begin{pmatrix} t_1 & 0 & \cdots & & & \cdots & 0 \\ 0 & \ddots & \ddots & & & & \vdots \\ \vdots & \ddots & t_n & & & & \\ & & & 1 & & & \\ & & & & t_n^{-1} & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ 0 & & & & & 0 & t_1^{-1} \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_n & b_n \\ a_{n+1} & b_{n+1} \\ a_{n+2} & b_{n+2} \\ \vdots & \vdots \\ a_{2n+1} & b_{2n+1} \end{pmatrix} = \begin{pmatrix} t_1 a_1 & t_1 b_1 \\ \vdots & \vdots \\ t_n a_n & t_n b_n \\ a_{n+1} & b_{n+1} \\ t_n^{-1} a_{n+2} & t_n^{-1} b_{n+2} \\ \vdots & \vdots \\ t_1^{-1} a_{2n+1} & t_1^{-1} b_{2n+1} \end{pmatrix}.$$

There are

$$\binom{2n+1}{2} = (2n+1)n = 2n^2 + n$$

T-fixed points in $\mathrm{Gr}(2, 2n+1)$ since the fixed points in $\mathrm{Gr}(2, 2n+1)$ are generated by two distinct basis vectors e_i and e_j of $\mathbb{C}^{2n+1}$. We denote from now on $e_{2n+1-(i-1)}$ by e_{-i} for $1 \leq i \leq n$ since it will give us the opportunity to realise similar equations as in the previous example in order to describe the rational equivariant cobordism. Because of the above equation we need to subtract n of those $2n^2 + n$ fixed points in order to obtain the T-fixed points of $\mathrm{IG}(2, 2n+1)$. More precisely, the two-dimensional subspaces generated by e_i and e_{-i} are not in $\mathrm{IG}(2, 2n+1)$ and therefore, there are $2n^2$ T-fixed points in $\mathrm{IG}(2, 2n+1)$ which we denote by $x_{ij} = x_{ji} = (e_i, e_j)$ as above.

With the notations from the previous example, the positive roots of (Sp_{2n}, T) are given by $\varepsilon_i \pm \varepsilon_j = \varepsilon_{i(\mp j)}$ for $i < j$ and $2\varepsilon_i = \varepsilon_{i(-i)}$ for $1 \leq i \leq n$ which results in

$$2 \cdot \binom{n}{2} + n = \frac{2n!}{(n-2)!2!} + n = n(n-1) + n = n^2$$

positive roots of (Sp_{2n}, T).

For the subtori $T'_{ij} = \mathrm{Ker}(\varepsilon_{ij})^0$, one can observe that the fixed point subschemes $X^{T'_{ij}}$ contain $4n-5$ T-stable curves which are of the same form as the curves occurring in Example 4.29. The same holds for the fixed point subschemes $X^{T'_{i(-j)}}$ coming from the subtori $T'_{i(-j)} = \mathrm{Ker}(\varepsilon_{i(-j)})^0$. This describes the n^2-n fixed point subschemes corresponding to the positive roots $\varepsilon_{i(\mp j)}$ for $i<j$.

Lastly, we consider the positive roots $\varepsilon_{i(-i)}$, $1 \leq i \leq n$, for which we obtain the subtori $T'_{i(-i)} = \mathrm{Ker}(\varepsilon_{i(-i)})^0$ and the fixed point subschemes $X^{T'_{i(-i)}}$ each containing $2n-2$ projective planes $\mathbb{P}^2$ and one T-stable curve which can easily be seen by generalising the computations from Example 4.29. As above, $X^{T'} = X^T$ holds for the other codimension subtori $T' = \mathrm{Ker}(\chi)^0$ for a primitive character χ which is not a root.

One obtains the equations coming from Theorem 4.13 describing the image of the injective pullback map

$$i^* : \Omega^*_T(\mathrm{IG}(2,2n+1))_{\mathbb{Q}} \to \Omega^*_T((\mathrm{IG}(2,2n+1))^T)_{\mathbb{Q}} = S(T)^{2n^2}_{\mathbb{Q}}$$

and therefore the rational equivariant cobordism ring $\Omega^*_T(\mathrm{IG}(2,2n+1))_{\mathbb{Q}}$.

More precisely, for any positive short root ε_{st} with $-n \leq t \leq n$, $1 \leq s \leq n$ and $t \neq 0, \pm s$ we obtain the equations

$$\begin{aligned} f_{sk} &\equiv f_{tk} && \mod c_1^T(L_{\varepsilon_{st}}) \\ f_{-sk} &\equiv f_{-tk} && \mod c_1^T(L_{\varepsilon_{st}}) \\ f_{s(-t)} &\equiv f_{t(-s)} && \mod c_1^T(L_{\varepsilon_{st}}) \end{aligned}$$

for $-n \leq k \leq n+1$ and $k \neq 0, \pm s, \pm t$. These $4n-5$ equations describe the T-stable curves in $X^{T'_{st}}$.

Lastly, the equations coming from the positive long roots $\varepsilon_{i(-i)}$ with $1 \leq i \leq n$ are given by

$$\begin{aligned} f_{ik} \equiv f_{(n+1)k} &\equiv f_{-ik} && \mod c_1^T(L_{\varepsilon_{i(-i)}}) \\ (f_{ik} - f_{(n+1)k}) + \rho_{1/2} c_1^T(L_{\varepsilon_{i(-i)}})(f_{-ik} - f_{ik}) &\equiv 0 && \mod c_1^T(L_{\varepsilon_{i(-i)}})^2 \\ f_{i(n+1)} &\equiv f_{-i(n+1)} && \mod c_1^T(L_{\varepsilon_{i(-i)}}) \end{aligned}$$

for $-n \leq k \leq n$ and $k \neq 0, \pm i$ coming from the projective planes and the T-stable curve in $X^{T'_{i(-i)}}$.

These equations together determine precisely the image of i^* and therefore the rational equivariant cobordism ring $\Omega^*_T(\mathrm{IG}(2,2n+1))_{\mathbb{Q}}$.

To finish the examples of type (3) in Pasquier's classification of horospherical varieties [43], we consider the general case $\mathrm{IG}(k,2n+1)$ for $n \geq 2$ and $k \in [2,n]$.

Example 4.31. Let $V = \mathbb{C}^{2n+1}$ and the antisymmetric form ω be given as in the previous example. Similarly, the torus action of Sp_{2n} on $\mathrm{IG}(k,2n+1)$ will be given as above. Next, we need to count the possible T-fixed points in $\mathrm{IG}(k,2n+1)$. We reformulate this combinatorial question as follows. We need to choose k different vectors which have one non-zero entry each where we do not allow to choose two vectors within these k vectors for which the i-th and the $(2n+1-i+1)$-st entry are non-zero, respectively. Furthermore, it is enough to consider the first n entries in the given vectors because we can multiply those possibilities by two for each vector in order to obtain the T-fixed points in $\mathrm{IG}(k, 2n+1)$ which are isotropic subspaces for ω.

First, we consider the possibilities for which the $(n+1)$-st entry in each of the k vectors is always zero. Therefore, we have $\binom{n}{k}$ possibilities to choose the ones in the k vectors. This results in $2^k\binom{n}{k}$ possibilities for the first case.

Secondly, we count the possibilities where one of the k vectors has a non-zero entry at the $(n+1)$-st position. In this case, we can only choose $(k-1)$ elements within the first n entries which leads to $\binom{n}{k-1}$ possibilities which we need to multiply by 2^{k-1} this time since the vector with a non-zero entry at the middle position cannot be chosen differently. Therefore, we have $2^{k-1}\binom{n}{k-1}$ possibilities for the second case.

This leads to

$$2^k\binom{n}{k} + 2^{k-1}\binom{n}{k-1}$$

T-fixed points in $\mathrm{IG}(k, 2n+1)$ which we denote by $x_{i_1 \dots i_k} = (e_{i_1}, \dots, e_{i_k})$ where we define e_{-i} to be $e_{2n+1-(i-1)}$ as above for all $1 \leq i \leq n$.

With the notations from the previous example, we consider $X^{T'_{ij}}$ which contains

$$2\left(\binom{n-2}{k-1}2^{k-1} + \binom{n-2}{k-2}2^{k-2}\right) + \binom{n-2}{k-2}2^{k-2} + \binom{n-2}{k-3}2^{k-3}$$

T-stable curves by applying the same counting method as described above. Similarly, we obtain the fixed point subschemes $X^{T'_{i(-j)}}$.

Lastly, we consider the positive long roots $\varepsilon_{i(-i)}$, $1 \leq i \leq n$, for which we obtain $X^{T'_{i(-i)}}$ consisting of

$$\binom{n-1}{k-1}2^{k-1}$$

projective planes $\mathbb{P}^2$ and

$$\binom{n-1}{k-2}2^{k-2}$$

T-stable curves which can be seen by using the same combinatorial arguments as above. Again, the equality $X^{\mathrm{Ker}(\chi)^0} = X^T$ holds for every other codimension one subtorus being the identity component of the kernel of a primitive character χ which is not a root.

One then obtains the equations coming from Theorem 4.13 describing the image of the injective pullback map

$$i^* : \Omega_T^*(\mathrm{IG}(k, 2n+1))_{\mathbb{Q}} \to \Omega_T^*((\mathrm{IG}(k, 2n+1))^T)_{\mathbb{Q}} = S(T)_{\mathbb{Q}}^{2^k\binom{n}{k}+2^{k-1}\binom{n}{k-1}}$$

and therefore the rational equivariant cobordism ring $\Omega_T^*(\mathrm{IG}(k, 2n+1))_{\mathbb{Q}}$.

More precisely, for any positive short root ε_{st} with $-n \leq t \leq n$, $1 \leq s \leq n$ and $t \neq 0, \pm s$, we obtain the equations

$$f_{si_2 \dots i_k} \equiv f_{ti_2 \dots i_k} \qquad \mod c_1^T(L_{\varepsilon_{st}})$$

$$\begin{aligned} f_{-si_2\ldots i_k} &\equiv f_{-ti_2\ldots i_k} && \mod c_1^T(L_{\varepsilon_{st}}) \\ f_{s(-t)i_3\ldots i_k} &\equiv f_{t(-s)i_3\ldots i_k} && \mod c_1^T(L_{\varepsilon_{st}}) \end{aligned}$$

with $-n \le i_m \le n+1$, $i_m \ne 0, \pm s, \pm t$ and $i_m \ne \pm i_{m'}$ for all $m \ne m' \in \{2, ..., k\}$. These equations describe the T-stable curves in $X^{T'_{st}}$.

Lastly, the equations coming from the positive long roots $\varepsilon_{i(-i)}$ for $1 \le i \le n$ are given by

$$\begin{aligned} f_{ii_2\ldots i_k} \equiv f_{(n+1)i_2\ldots i_k} &\equiv f_{-ii_2\ldots i_k} && \mod c_1^T(L_{\varepsilon_{i(-i)}}) \\ (f_{ii_2\ldots i_k} - f_{(n+1)i_2\ldots i_k}) + \rho_{1/2} c_1^T(L_{\varepsilon_{i(-i)}})(f_{-ii_2\ldots i_k} - f_{ii_2\ldots i_k}) &\equiv 0 && \mod c_1^T(L_{\varepsilon_{i(-i)}})^2 \\ f_{i(n+1)i_3\ldots i_k} &\equiv f_{-i(n+1)i_3\ldots i_k} && \mod c_1^T(L_{\varepsilon_{i(-i)}}) \end{aligned}$$

with $-n \le i_m \le n$, $i_m \ne 0, \pm i$ and $i_m \ne \pm i_{m'}$ for all $m \ne m' \in \{2, ..., k\}$ coming from the projective planes and T-stable curves in $X^{T'_{i(-i)}}$.

These equations determine the image of i^* and therefore the rational equivariant cobordism ring $\Omega_T^*(\mathrm{IG}(k, 2n+1))_{\mathbb{Q}}$.

Remark 4.32. Using Proposition 3.47, one immediately obtains an expression for the ordinary rational cobordism rings in any of the examples discussed above.

5 Equivariant cobordism of horospherical varieties

In this chapter, we let G be a complex connected reductive algebraic group, B a fixed Borel subgroup with maximal torus T and $W = N(T)/T$ the Weyl group. The ground field must be restricted to the complex numbers $\mathbb{C}$ because the results in [43] were only proved in this setting. We want to compute the equivariant algebraic cobordism of smooth projective horospherical varieties of Picard number one. We begin this section by recalling the description of the T-stable curves in flag varieties which will be important in order to describe the geometry of horospherical varieties. After that, we will recall some basic properties of horospherical varieties as well as their geometry. Excellent references for the geometry of horospherical varieties are for example [16, 43]. Using these descriptions, we will be able to describe the rational equivariant cobordism ring of horospherical varieties of Picard number one in terms of Theorem 4.13.

5.1 T-stable curves in flag varieties

In this section, we will recall the main notions and results on T-stable curves in flag varieties G/P from [15, Section 3]. We denote by $R = R^+ \cup R^-$ the positive and negative roots, and by S the simple roots. Furthermore, we denote by s_α the reflections in W which are indexed by positive roots α. These are simple reflections if α is in S. For a subset $I \subseteq S$, let W_I be the subgroup of W which is generated by the reflections s_α for α in I. In addition, let $P_I = \coprod_{w \in W_I} BwB$ and $R^+_{P_I}$ be the set of positive roots that can be written as sums of roots in I. This is the well known correspondence between parabolic subgroups P_I of G containing B and subsets $I \subseteq S$. The length $\ell(w)$ of an element $w \in W$ is the minimum number of simple reflections whose product is w.

For any $u \in W/W_I$, we let $X(u) = \overline{BuP_I/P_I}$ be the corresponding Schubert variety which is of dimension $\ell(u)$ where $\ell(u)$ denotes the unique minimum length of a representative of u in W. We denote its cohomology class $[X(u)]$ by $\sigma(u) \in H^*(G/P_I)$. Further, for any $u \in W/W_I$, we denote by $x(u) := uP_I/P_I$ the corresponding T-fixed point in G/P_I. The Schubert classes of dimension one have the form $\sigma(s_\beta)$ as β varies over $S \setminus I$. We define a degree d to be a nonnegative integral combination $d = \sum d_\beta \sigma(s_\beta)$. The degrees are the classes of curves on G/P_I. For any positive root α, we write $\alpha = \sum n_{\alpha\beta}\beta$ as the positive integral combination of simple roots β. Then we define the degree $d(\alpha)$ of α by

$$d(\alpha) := \sum_{\beta \in S\setminus I} n_{\alpha\beta} \frac{(\beta,\beta)}{(\alpha,\alpha)} \sigma(s_\beta).$$

Remark 5.1. If $h_\alpha = 2\alpha/(\alpha,\alpha)$ and ω_β is the fundamental weight corresponding to β, then $h_\alpha(\omega_\beta) = n_{\alpha\beta}(\beta,\beta)/(\alpha,\alpha)$ which implies

$$d(\alpha) = \sum_{\beta \in S\setminus I} h_\alpha(\omega_\beta)\sigma(s_\beta).$$

Lemma 5.2. *[15, Lemma 3.1] If w is in W_I, then we have $d(w(\alpha)) = d(\alpha)$.*

For any positive root α which is not in $R^+_{P_I}$, there is a unique T-stable curve C_α in G/P_I that contains the points $x(1)$ and $x(s_\alpha)$. We know that $C_\alpha = S(\alpha) \cdot P_I/P_I$ where $S(\alpha)$ is the 3-dimensional subgroup of G whose Lie algebra is $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$.

Lemma 5.3. *[15, Lemma 3.4] The degree $[C_\alpha]$ of C_α is $d(\alpha)$.*

Definition 5.4. *We say that two unequal elements u and v in W/W_I are **adjacent** if there is a reflection s in W such that $v = s \cdot u$. In this case we define $d(u, v)$ to be the degree $d(\alpha)$ where s_α is a reflection relating them.*

Remark 5.5. If u and v are adjacent, then for any $w \in W$, the elements wu and wv are also adjacent and $d(wu, wv) = d(u, v)$ holds.

Lemma 5.6. *[15, Lemma 4.2] Elements u and v in W/W_I are adjacent if and only if $x(u) \neq x(v)$ and there is a T-stable curve C connecting $x(u)$ and $x(v)$. In this case, the curve C is unique, isomorphic to $\mathbb{P}^1$, and its degree is equal to $d(u, v)$.*

Remark 5.7. A general T-stable curve in G/P_I has the form $w \cdot C_\alpha$ for some $\alpha \in R^+ \backslash R^+_{P_I}$ and $w \in W$. This curve is the unique T-stable curve connecting $x(w) = w \cdot x(1)$ and $x(w \cdot s_\alpha) = w \cdot x(s_\alpha)$.

Example 5.8. We consider the flag variety G_2/P_α where α and β denote the simple roots of G_2, β being the long root. The flag variety G_2/P_α is a 5-dimensional quadric whose geometry was also studied in [16]. The positive roots are given by

$$R^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}.$$

The geometric picture of the roots of G_2 is given by the following figure.

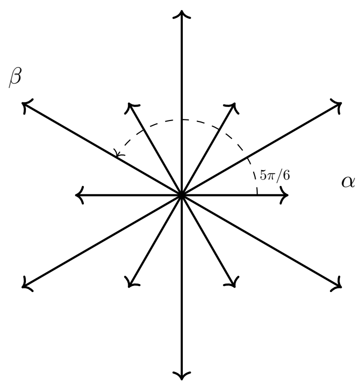


Figure 1: Root system of G_2.

In this case, we have $W(G_2) \cong D_6$ where the counterclockwise rotation r by an angle of $\pi/3$ and the reflection s at the y-axis denote the generators of the dihedral group D_6 of order 12. Furthermore, we know that the closed orbits in this horospherical variety are given by G_2/P_α and G_2/P_β. Therefore, we obtain that W_α is generated by s_α which has order 2. Thus, we have 6 T-fixed points in G_2/P_α and they are indexed by the elements of W/W_α. From the above we know that for any $\gamma \in R^+ \setminus R^+_{P_\alpha}$ there exists a unique T-stable curve connecting $x(1)$ and $x(s_\gamma)$. We identify s_α with the element $s \in D_6$ and therefore, one can check that s_β corresponds to $sr \in D_6$. We write

$$W/W_\alpha = \{\overline{r^0}, \overline{r^1}, \overline{r^2}, \overline{r^3}, \overline{r^4}, \overline{r^5}\}$$

where $\overline{r^i}$ denotes the class $r^i W_\alpha = \{r^i s, r^i\}$ as representatives in the quotient W/W_α. Using Figure 1 or the formulas for reflections, one may check that pairwise different reflections s_γ correspond to pairwise different $r^i s$ for $1 \leq i \leq 5$. Therefore, all the reflections are given by $r^i s$ for $0 \leq i \leq 5$. Now, we need to check which elements $\overline{r^k}, \overline{r^j} \in W/W_\alpha$ are adjacent for $0 \leq k \neq j \leq 5$. By Definition 5.4 we need to find a reflection $r^i s$ in W, $0 \leq i \leq 5$, such that $r^i s \cdot r^k = r^j$ holds. By the relations in D_6 we have $r^i s r^k = r^{i-k} s$ and thus, we can find a reflection $r^i s$ in W for any two fixed elements $0 \leq k \neq j \leq 5$ such that $\overline{r^j}$ and $\overline{r^k}$ are adjacent. This implies that there is a T-stable curve connecting any two of the T-fixed points in G_2/P_α which leads to a total of 15 T-stable curves in the flag variety G_2/P_α. Similar computations can be done for the flag variety G_2/P_β.

Later on, we will be interested in the weight of a T-stable curve C and also its degree. To obtain those one can use the following lemma.

Lemma 5.9. *[15, Lemma 2.1] Let a torus T act on a curve $C \cong \mathbb{P}^1$ with two different T-fixed points p and q and let L be a T-equivariant line bundle on C. Let χ_p and χ_q be the weights of T acting on the fibres L_p and L_q, respectively, and ψ_p the weight of T acting on the tangent space of C at p. Then we have*

$$\chi_p - \chi_q = n\psi_p$$

where n is the degree of L.

Remark 5.10. In our case, and more specifically in the previous Example 5.8, the degree can also be obtained by the formulas in Section 5.1. These computations (cf. Section A.1) lead to 6 T-stable curves of degree 1, 6 T-stable curves of degree 3 and 3 T-stable curves of degree 2 in the flag variety G_2/P_α in Example 5.8.

5.2 Geometry of horospherical varieties of Picard number one

In this section, we want to consider the geometry of smooth projective horospherical varieties of Picard number one. First, we will recall some basic notions of horospherical varieties. Excellent references are for example [16, 41, 42, 43].

Definition 5.11. *[19, II. Remark 6.10.3] Let X be a smooth projective variety of dimension ≥ 2. Then we define the **Néron-Severi group** $N^1(X)$ as the divisor class group modulo algebraic equivalence. Its rank is called the **Picard number** of X.*

Remark 5.12. [41, Remark 4.5ff.] Let $S \setminus I$ be the set of simple roots corresponding to the horospherical variety X as described in Remark 2.59. Then the Picard number ρ of a smooth G/H-embedding X satisfies

$$\rho_X = r_X + \#(S \setminus I) - \#(\mathcal{D}_X)$$

where r_X is the number of rays in the colored fan of X minus the rank n (cf. Definition 2.50) and $\mathcal{D}_X$ denotes the set of simple roots in $S \setminus I$ which correspond to colors in the colored fan $\mathbb{F}_X$.

Remark 5.13. [43, Section 1.2] Since we will be mainly interested in smooth projective horospherical varieties X, we remark that the colored fan associated to X is complete and therefore, $r_X \geq 1$ holds. Moreover, $\rho_X = 1$ if and only if $r_X = 1$ and $\mathcal{D}_X = S \setminus I$. In particular, the colored fan of X has exactly $n + 1$ rays.

Remark 5.14. Let X be any smooth projective spherical G-variety. Then algebraic equivalence coincides with rational equivalence (cf. [6, Section 1.3]). Since X is normal by definition, rational equivalence of codimension one cycles coincides with linear equivalence of Weil divisors (cf. [19, Appendix A Section 1]). The divisor class group coincides with the Picard group because X is assumed to be smooth and thus, the Néron-Severi group is isomorphic to the Picard group. The Néron-Severi group is known to be finitely generated (cf. [35]) and the Picard group $\mathrm{Pic}(X)$ of X is torsion-free by [47, Corollary 3.2.6]. Thus, we know that $\mathrm{Pic}(X)$ is free. We will be mainly interested in smooth projective horospherical G-varieties X of Picard number one and resulting from the previous discussion we know that this is equivalent to $\mathrm{Pic}(X) \cong \mathbb{Z}$.

We will consider smooth projective horospherical varieties of Picard number one. These varieties were classified by Pasquier [43] and the classification is given by the following theorem.

Proposition 5.15. *[43, Theorem 0.1] Let G be a connected reductive algebraic group. Let X be a smooth projective horospherical G-variety with Picard number one. Then one of the following cases can occur.*

(i) X is homogeneous.

(ii) X is horospherical of rank 1. Its automorphism group is a connected non-reductive linear algebraic group, acting with exactly two orbits.

Moreover, in the second case X is uniquely determined by its two closed G-orbits Y and Z, isomorphic to G/P_Y and G/P_Z, respectively, and (G, P_Y, P_Z) is one of the triples of the following list.

(1) $(B_n, P(\omega_{n-1}), P(\omega_n))$ *for* $n \geq 3$

(2) $(B_3, P(\omega_1), P(\omega_3))$

(3) $(C_n, P(\omega_m), P(\omega_{m-1}))$ *for* $n \geq 2$ *and* $m \in [2, n]$

(4) $(F_4, P(\omega_2), P(\omega_3))$

(5) $(G_2, P(\omega_1), P(\omega_2)$

Here we denote by $P(\omega_i)$ the maximal parabolic subgroup of G corresponding to the dominant weight ω_i using the notations from Bourbaki [5].

Remark 5.16. In our notation $P(\omega_i)$ denotes the maximal parabolic subgroup $P_{S\setminus\alpha_i}$ for the simple root α_i associated to the fundamental weight ω_i.

Lemma 5.17. *[43, Lemma 1.2] Let G/H be a horospherical homogeneous space. Up to G-equivariant isomorphism of varieties, there exists at most one smooth projective G/H-embedding with Picard number one.*

In the sequel, we are only interested in the cases which are not homogeneous because the rational T-equivariant cobordism for the homogeneous varieties can be described using [31, Theorem 7.8]. Therefore, we recall the construction from [16, Section 1.3].

Let X be a smooth projective horospherical but non homogeneous variety of Picard number one with associated triple (G, P_Y, P_Z). In this case, we denote the previous triple also by $(G, P(\omega_Y), P(\omega_Z))$ for the corresponding fundamental weights ω_Y and ω_Z. Furthermore, the dense orbit is given by $G/H = G \cdot [v_Y + v_Z] \subseteq \mathbb{P}(V_Y \oplus V_Z)$ where V_Y and V_Z are the irreducible G-representations with highest weights ω_Y and ω_Z and the corresponding highest weight vectors v_Y and v_Z. By construction, we conclude that P_Y and P_Z are the stabilisers of $[v_Y]$ and $[v_Z]$ in $\mathbb{P}(V_Y)$ and $\mathbb{P}(V_Z)$, and that Y and Z are the G-orbits of $[v_Y]$ and $[v_Z]$ in $\mathbb{P}(V_Y)$ and $\mathbb{P}(V_Z)$, respectively. Lastly, we have that $X = \overline{G \cdot [v_Y + v_Z]} \subseteq \mathbb{P}(V_Y \oplus V_Z)$ is the closure of the G-orbit $G \cdot [v_Y + v_Z]$ in $\mathbb{P}(V_Y \oplus V_Z)$.

The T-fixed points of X are given by the T-fixed points of the two closed G-orbits. Now, we analyse the T-stable curves and the fixed point subschemes $X^{T'}$ for some given X in order to be able to use Theorem 4.13 with the aim of obtaining the rational T-equivariant cobordism of X. In the previous section, we have already seen how to determine the T-stable curves in the closed orbits G/P_Y and G/P_Z which are flag varieties. Next, we investigate the T-stable curves meeting the dense open orbit G/H for any smooth projective horospherical variety X of Picard number one. We will use the diagram

$$\begin{array}{ccccc} & & G/H & & \\ & & \big\downarrow{\scriptstyle \pi} & & \\ & & G/(P_Y \cap P_Z) & & \\ & \swarrow^{p_Y} & & \searrow^{p_Z} & \\ G/P_Y & & & & G/P_Z \end{array} \tag{5.1}$$

where π is the corresponding $\mathbb{C}^*$-bundle.

Definition 5.18. *Let C be a T-stable irreducible curve in the dense open orbit G/H. Then we define $S := \pi^{-1}(\pi(C))$ to be the preimage of $\pi(C)$.*

Lemma 5.19. *Let C be a T-stable irreducible curve in the dense open orbit G/H. Then S is given by one of the following cases.*

(i) S is the curve C itself.

(ii) S is an irreducible surface containing C.

Proof. Let C be a given T-stable irreducible curve in the dense open orbit G/H. Then $\pi(C)$ is also T-stable. The following two cases can occur for $\pi(C)$.

(i) $\pi(C) = \{*\}$ is a point. Without loss of generality, we can choose this point to be the B-fixed point in $G/P := G/(P_Y \cap P_Z)$ where $P_Y \cap P_Z$ is the same as the normaliser $N_G(H)$ which was mentioned in Remark 2.48. The B-fixed point is $1 \cdot P/P$ and therefore, the closure $\overline{C} \subseteq X$ of the fibre $\pi^{-1}(1 \cdot P/P) = C$ is the line joining the B-fixed points $1 \cdot P_Y/P_Y = [v_Y] \in Y$ and $1 \cdot P_Z/P_Z = [v_Z] \in Z$ because π is a $\mathbb{C}^*$-bundle and B-fixed points are mapped to B-fixed points via the projections p_Y and p_Z. The other lines will be obtained by the Weyl group action. Those lines are T-stable by assumption.

(ii) $\pi(C)$ is a T-stable irreducible curve. Without loss of generality, we can choose $\pi(C) = S(\alpha) \cdot P/P$ for some positive root α which is not in R_P^+ where $S(\alpha)$ is the 3-dimensional subgroup of G whose Lie algebra is $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. This curve joins the B-fixed point $x(1)$ and $x(s_\alpha) = s_\alpha \cdot P/P$. Then we obtain an irreducible surface $S := \pi^{-1}(\pi(C))$ because π is a $\mathbb{C}^*$-bundle. This surface S contains C and is T-stable since π is equivariant. The other curves are obtained by the Weyl group action. □

Lemma 5.20. *Any surface in a connected component of $X^{T'}$ for a singular codimension one subtorus $T' = \mathrm{Ker}(\alpha)^0$ for some positive root α is of the form $\overline{S} \subseteq X$ for some T-stable curve C and $S = \pi^{-1}(\pi(C))$.*

Proof. Let A be a surface in $X^{T'}$. This is itself a connected component of $X^{T'}$ and A is a spherical $C_G(T')$-variety by Proposition 4.3. We know that $A \cap G/H \neq \emptyset$ because the two closed orbits $Y \cong G/P_Y$ and $Z \cong G/P_Z$ contain only finitely many T-stable curves. Now, let $a \in A$. Then we have $ta = tt'a = t'ta$ for all $t, t' \in T$ which implies that A is T-stable. We have $A \cap G/H \subseteq \pi^{-1}\pi(A \cap G/H)$ and the reversed inclusion is also true because A is T-stable and T acts transitively on the fibres of π as P/H is a quotient of T. Therefore, the whole fibre must be in $A \cap G/H$. Furthermore, A has only zero- and one-dimensional T-orbits since T/T' is one-dimensional. The image under π of those orbits is either a T-fixed point or the T-stable irreducible curve $\pi(A \cap G/H)$. We conclude that there must be a T-stable curve $C \subseteq A \cap G/H$ such that $A \cap G/H = \pi^{-1}\pi(C)$ because if the T-orbits were only the fibres then there would be infinitely many T-fixed points in G/P. □

Remark 5.21. Let a connected component of $X^{T'}$ be given for some codimension one subtorus T'. As mentioned already in the previous proof these are T-stable with only zero- and one-dimensional T-orbits since T/T' is one-dimensional. To be more precise, either an orbit is a T-fixed point or a one-dimensional T-orbit $T/T' \cdot x$ for some $x \in X^{T'}$. Therefore, the stabilisers of x in T are subtori of codimension one or zero.

Definition 5.22. *Let X be a smooth projective horospherical G-variety of Picard number one of the form $(G, P(\omega_Y), P(\omega_Z))$. Then we denote by $\chi := \omega_Y - \omega_Z$ the difference of the two fundamental weights ω_Y and ω_Z.*

In the following, we want to analyse which surfaces S are a connected component in some $X^{T'}$ for some codimension one subtorus T'. Therefore, we formulate the following lemma.

Lemma 5.23. *For any smooth projective horospherical variety X of Picard number one we have the following properties.*

(1) The only T-stable curves in X meeting the open orbit G/H occurring as a connected component of $X^{T'}$ for some codimension one subtorus T' are of the form $\overline{\pi^{-1}(z)}$ where $z \in G/(P_Y \cap P_Z)$ is a T-fixed point.

(2) The surfaces occurring in $X^{T'}$ only arise from codimension one subtori of the form $T' = \mathrm{Ker}(w\alpha)^0 = \mathrm{Ker}(w\chi)^0$ for some positive root α which is a non-zero multiple of χ and some $w \in W$.

Proof. As above, we have the B-fixed point $1 \cdot P/P$ in $G/P := G/(P_Y \cap P_Z)$. We need to consider the previously discussed case from Lemma 5.19 (ii). Therefore, we assume that there exists a T-stable curve $C \subseteq G/H$ such that a general point in the T-stable curve $\pi(C)$ has the form $w \cdot u_{-\alpha}(x) \cdot P/P$ where $u_{-\alpha}(x)$ denotes the corresponding element in the root subgroup $U_{-\alpha}$. A general point in $S = \pi^{-1}\pi(C)$ has the form

$$w \cdot u_{-\alpha}(x)tH = u_{-w\alpha}(x')wtH = u_{-w\alpha}(x')wtw^{-1}wH = u_{-w\alpha}(x')t'wH$$

for $t \in P/H = \mathbb{C}^*$.

Now, we consider the T-action on those points for $z \in T$:

$$\begin{aligned} zu_{-w\alpha}(x')t'wH &= u_{-w\alpha}((w\alpha)(z)^{-1}x')zt'wH \\ &= u_{-w\alpha}((w\alpha)(z)^{-1}x')t'zwH \\ &= u_{-w\alpha}((w\alpha)(z)^{-1}x')t'ww^{-1}zwH. \end{aligned}$$

This implies that a point z acts trivially if and only if $w^{-1}zw \in H = \mathrm{Ker}(\chi)$ and $z \in \mathrm{Ker}(w\alpha)$ hold. This implies by the Weyl group action on the character group that this is equivalent to $z \in \mathrm{Ker}(w\chi) \cap \mathrm{Ker}(w\alpha)$.

If $\mathrm{Ker}(w\chi)^0 \neq \mathrm{Ker}(w\alpha)^0$ holds, then $\mathrm{Ker}(w\chi)^0 \cap \mathrm{Ker}(w\alpha)^0$ has codimension two in T. Therefore, we obtain a T-stable surface S or a T-stable curve in $\pi^{-1}\pi(C)$. It remains to check whether those are fixed by some codimension one subtorus. If one of those was a connected component of $X^{T'}$, then the stabiliser of any point in $X^{T'}$ would have at most codimension one in T, but as we computed above, the stabiliser of a general point in S and therefore, also in every potential T-stable curve in $\pi^{-1}\pi(C)$ is precisely $\mathrm{Ker}(w\chi) \cap \mathrm{Ker}(w\alpha)$. Therefore, the stabiliser of a general point would be of codimension two in T and thus, the T-stable surface S is not a connected component of $X^{T'}$ and there exists no T-stable curve in $\pi^{-1}\pi(C)$ which is a connected component of $X^{T'}$.

If $\mathrm{Ker}(w\alpha) = \mathrm{Ker}(w\chi)$ holds, then we have $\mathrm{Ker}(w\chi)^0 = \mathrm{Ker}(w\alpha)^0 = T'$ and S is some connected component of $X^{T'}$ because z acts trivially on a general point of S. This implies property (2) because in this case α must be a non-zero multiple of χ.

Now, let $\overline{C}$ be a T-stable curve in $X^{T'}$ meeting the open orbit G/H for some codimension one subtorus $T' \subseteq T$. The curve $\overline{C}$ meets the open orbit G/H along a T-stable curve C. By Lemma 5.19, C is either a fiber of π or C is contained in the T-stable surface $S = \pi^{-1}\pi(C)$. In the latter case, we distinguish whether α is non-zero multiple of χ or not. As discussed above, there exists no T-stable curve in S whose closure is a connected component of $X^{T'}$ if α is not a non-zero multiple of χ. We also showed above that the closure of S is itself a connected component of $X^{T'}$ containing C if α is a non-zero multiple of χ. We conclude that $\overline{C}$ is never a connected component of $X^{T'}$ if C is contained in the T-stable surface S. This implies property (1). □

Algorithm: We analyse the occuring surfaces in the connected components of $X^{T'}$. As we have seen above, we need to consider roots α which are multiples of the difference χ of the two fundamental weights ω_Y and ω_Z up to the Weyl group action. After that, we look at the curves in the closed orbits Y and Z. Up to Weyl group action these are given by $S(\alpha)[v_{\omega_Y}]$ which connect $s_\alpha[v_{\omega_Y}]$ and $[v_{\omega_Y}]$ in Y and similarly in Z. Thus, we need to compute $s_\alpha(\omega_Y) = \omega_Y - (\alpha^\vee, \omega_Y)\alpha$ and similarly for ω_Z. Then we will know how many T-fixed points we have in the relevant connected component of $X^{T'}$ and in which orbits they occur. If we obtain 3 T-fixed points then we obtain a projective plane and if we obtain 4 T-fixed points, then we will have a Hirzebruch surface $\mathbb{F}_n$. We remark that $s_\alpha(\omega_Y) = \omega_Y$ holds if and only if $(\alpha^\vee, \omega_Y)$ vanishes. Now, it remains to be checked which Hirzebruch surface $\mathbb{F}_n$ we obtain in the case of 4 T-fixed points. For details concerning the geometry of Hirzebruch surfaces as ruled surfaces we refer the reader to [19, Chapter V]. So let S be some surface of type $\mathbb{F}_n$ with T-stable curves of degrees x and y with $x \leq y$ where the T-stable curves are the two closed SL_2-orbits in $\mathbb{F}_n$. Now, we want to determine the non-negative integer n and the embedding of S. Let $p : S \to \mathbb{P}^1$ be the ruling. Then p has a section C_0 such that the self-intersection is

$C_0^2 = -n$. It is furthermore well known that this is the only section of p with negative (non-positive) self-intersection for $n > 0$ ($n = 0$). Let f be a fibre of p. We know that the Picard group of S is generated by C_0 and f. Furthermore, we have $f.C_0 = 0$ and $f^2 = 0$. Geometrically, the morphism p describes the projection to one of the closed SL_2-orbits. Thus, we may choose f to be the line $[\lambda v + \mu v'] \subseteq \mathbb{P}(V \oplus V')$ where $v \in V$ and $v' \in V'$ are the highest weights vectors. So let $L = aC_0 + bf$ be the line bundle defining the embedding of S. Following [19, Chapter V, Corollary 2.18], we know that L is very ample if and only if $a > 0$ and $b > an$. In addition, we have two special sections of p which are the two closed SL_2-orbits, i.e. the curves C and C' of degrees x and y, respectively. As they are sections of p, the intersection with f is 1 and therefore, they are of the form $C = C_0 + cf$ and $C' = C_0 + c'f$. For the embedding $\phi : S \to \mathbb{P}(V \oplus V')$ we know that $L = \phi^*(\mathcal{O}(1))$. Further, for a general hyperplane H in $\mathbb{P}(V \oplus V')$ we have $H.f = 1$ because f is a line in $\mathbb{P}(V \oplus V')$. We also have $H.C = x$ and $H.C' = y$ in $\mathbb{P}(V \oplus V')$ by the assumption on the degrees of the curves C and C'. Considering now the line bundle L on S, we compute $(H \cap S).f = 1$ for the divisor $H \cap S$ whose class corresponds to the line bundle L. This can then be also written as $L.f = 1$. Thus, we have $1 = L.f = (aC_0 + bf).f = a$. Similarly, we can pull $\mathcal{O}(1)$ back to $\mathbb{P}(V)$ and $\mathbb{P}(V')$ which are the curves C and C', respectively. We obtain $(H \cap C).C = x$ and $(H \cap C').C' = y$ which can be written as $L.C = x$ and $L.C' = y$. Further, we have $C.C' = 0$ because the two closed SL_2-orbits do not meet. On the other hand, we have

$$\begin{aligned} x &= C.L = (C_0 + cf).(C_0 + bf) = -n + c + b, \\ y &= C'.L = (C_0 + c'f).(C_0 + bf) = -n + c' + b, \\ 0 &= C.C' = (C_0 + cf).(C_0 + c'f) = -n + c + c' \end{aligned}$$

which leads to

$$\begin{aligned} c &= \frac{n + x - y}{2}, \\ c' &= \frac{n - x + y}{2}. \end{aligned}$$

These observations yield $C^2 = 2c - n = x - y \leq 0$. This leads to $C = C_0$ because C_0 is the unique irreducible curve with non-positive self-intersection (or $x = y$ and $n = 0$). We conclude $c = 0$ and hence, $n = y - x$ which also holds for $n = 0$ and $y = x$. Thus, the embedding is given by $L = C_0 + yf$ and the curve C' is $C' = C_0 + (y - x)f$.

We consider some examples of the classification of Pasquier which are given by triples (G, P_Y, P_Z). We will study their geometry using the above algorithm and the classification of Bourbaki [5].

Example 5.24. In this example we will discuss all the possible cases from Proposition 5.15.

(i) We consider type (1), i.e. $(B_n, P(\omega_{n-1}), P(\omega_n))$ for $n \geq 3$. The fundamental weights are given by

$$\begin{aligned} \omega_{n-1} &= \varepsilon_1 + ... + \varepsilon_{n-1} \\ &= \alpha_1 + 2\alpha_2 + ... + (n-2)\alpha_{n-2} + (n-1)(\alpha_{n-1} + \alpha_n) \text{ and} \\ \omega_n &= \frac{1}{2}(\varepsilon_1 + ... + \varepsilon_n) \end{aligned}$$

$$= \frac{1}{2}(\alpha_1 + 2\alpha_2 + ... + n\alpha_n)$$

for $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \leq i \leq n-1$, and $\alpha_n = \varepsilon_n$. Therefore, we have

$$\begin{aligned}\chi &= \omega_{n-1} - \omega_n \\ &= \frac{1}{2}(\varepsilon_1 + ... + \varepsilon_{n-1} - \varepsilon_n) \\ &= \frac{1}{2}(\alpha_1 + 2\alpha_2 + ... + (n-1)\alpha_{n-1} + (n-2)\alpha_n).\end{aligned}$$

The positive roots are given by ε_i for $1 \leq i \leq n$ and $\varepsilon_i \pm \varepsilon_j$ for $1 \leq i < j \leq n$. This implies that they will not be any surface in $X^{T'}$ because there is no root which is a non-zero multiple of χ.

(ii) Now, we consider type (2), i.e. the triple $(B_3, P(\omega_1), P(\omega_3))$. The fundamental weights are given by

$$\begin{aligned}\omega_1 &= \varepsilon_1 \\ &= \alpha_1 + \alpha_2 + \alpha_3 \text{ and} \\ \omega_3 &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \\ &= \frac{1}{2}(\alpha_1 + 2\alpha_2 + 3\alpha_3)\end{aligned}$$

for α_i as above. Therefore, we have

$$\begin{aligned}\chi &= \omega_1 - \omega_3 \\ &= \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3) \\ &= \frac{1}{2}(\alpha_1 - \alpha_3).\end{aligned}$$

As above, there will not be any surface in $X^{T'}$ by the same argument.

(iii) Next, we consider type (3), i.e. $(C_n, P(\omega_m), P(\omega_{m-1}))$ with $n \geq 2$ and $m \in [2, n]$. The fundamental weights are given by

$$\begin{aligned}\omega_i &= \varepsilon_1 + ... + \varepsilon_i \\ &= \alpha_1 + 2\alpha_2 + ... + (i-1)\alpha_{i-1} + i\left(\alpha_i + \alpha_{i+1} + ... + \alpha_{n-1} + \frac{1}{2}\alpha_n\right)\end{aligned}$$

for $1 \leq i \leq n$ and $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \leq i \leq n-1$, and $\alpha_n = 2\varepsilon_n$. Therefore, we have

$$\begin{aligned}\chi &= \omega_m - \omega_{m-1} \\ &= \varepsilon_m \\ &= \alpha_m + ... + \alpha_{n-1} + \frac{1}{2}\alpha_n.\end{aligned}$$

The positive roots are given by $\varepsilon_i \pm \varepsilon_j$ for $1 \leq i < j \leq n$ and $2\varepsilon_i$ for $1 \leq i \leq n$. Thus, there is a positive root which is a non-zero multiple of χ, namely $\alpha := 2\varepsilon_m$.

Consequently, we have

$$\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)} = \frac{2\cdot 2\varepsilon_m}{(2\varepsilon_m, 2\varepsilon_m)} = \varepsilon_m$$

and therefore, we obtain

$$(\alpha^\vee, \omega_m) = (\varepsilon_m, \varepsilon_1 + ... + \varepsilon_m) = 1$$

and

$$(\alpha^\vee, \omega_{m-1}) = (\varepsilon_m, \varepsilon_1 + ... + \varepsilon_{m-1}) = 0.$$

This implies that we have 3 T-fixed points and thus, we obtain a projective plane in $X^{T'}$.

(iv) Next, we consider type (4), i.e. the triple $(F_4, P(\omega_2), P(\omega_3))$. The fundamental weights are given by

$$\begin{aligned}\omega_2 &= 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \\ &= 3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 4\alpha_4 \text{ and} \\ \omega_3 &= \frac{1}{2}(3\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) \\ &= 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4\end{aligned}$$

for

$$\alpha_1 = \varepsilon_2 - \varepsilon_3, \quad \alpha_2 = \varepsilon_3 - \varepsilon_4, \quad \alpha_3 = \varepsilon_4, \quad \alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4).$$

Therefore, we have

$$\begin{aligned}\chi &= \omega_2 - \omega_3 \\ &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4) \\ &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4.\end{aligned}$$

The positive roots are given by ε_i for $1 \le i \le 4$, $\varepsilon_i \pm \varepsilon_j$ for $1 \le i < j \le 4$ and $\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$. Thus, $\alpha := \chi$ is already a positive root and consequently, we have

$$\begin{aligned}\alpha^\vee &= \frac{2\alpha}{(\alpha,\alpha)} = \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4}{(\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4), \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4))} \\ &= \varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4.\end{aligned}$$

Therefore, we obtain

$$(\alpha^\vee, \omega_2) = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4, 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = 4$$

and

$$(\alpha^\vee, \omega_3) = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4, \frac{1}{2}(3\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)) = 2.$$

This implies that we have 4 T-fixed points and thus, we obtain a Hirzebruch surface. Remark 5.1 ensures that $(\alpha^\vee, \omega_2)$ and $(\alpha^\vee, \omega_3)$ give us the degrees of the curves in the two closed orbits. The only surfaces occurring in $X^{T'}$ are projective spaces $\mathbb{P}^2$ and Hirzebruch surfaces $\mathbb{F}_n$. Therefore, the given surface must be a Hirzebruch surface $\mathbb{F}_2$ using the previous algorithm.

(v) Lastly, we consider type (5), i.e. the triple $(G_2, P(\omega_1), P(\omega_2))$. The fundamental weights are given by

$$\begin{aligned}\omega_1 &= -\varepsilon_2 + \varepsilon_3 \\ &= 2\alpha_1 + \alpha_2 \text{ and} \\ \omega_2 &= -\varepsilon_1 - \varepsilon_2 + 2\varepsilon_3 \\ &= 3\alpha_1 + 2\alpha_2\end{aligned}$$

for $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$. Therefore, we have

$$\begin{aligned}\chi &= \omega_1 - \omega_2 \\ &= \varepsilon_1 - \varepsilon_3 \\ &= -\alpha_1 - \alpha_2.\end{aligned}$$

The positive roots are given by $\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2$ and $3\alpha_1 + 2\alpha_2$. Thus, $\alpha := -\chi$ is a positive root and consequently, we have

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)} = \frac{2(\varepsilon_3 - \varepsilon_1)}{(\varepsilon_3 - \varepsilon_1, \varepsilon_3 - \varepsilon_1)} = \varepsilon_3 - \varepsilon_1.$$

Therefore, we obtain

$$(\alpha^\vee, \omega_1) = (\varepsilon_3 - \varepsilon_1, -\varepsilon_2 + \varepsilon_3) = 1$$

and

$$(\alpha^\vee, \omega_2) = (\varepsilon_3 - \varepsilon_1, -\varepsilon_1 - \varepsilon_2 + 2\varepsilon_3) = 3.$$

This implies that we have 4 T-fixed points and thus, we obtain a Hirzebruch surface. Remark 5.1 ensures that $(\alpha^\vee, \omega_1)$ and $(\alpha^\vee, \omega_2)$ give us the degrees of the curves in the two closed orbits Y and Z, respectively. We conclude using the previous algorithm that $X^{\mathrm{Ker}(\alpha)^0}$ contains a Hirzebruch surface $\mathbb{F}_2$.

After having described the T-stable points and curves on all the smooth projective horospherical varieties of Picard number one, we can describe their equivariant algebraic cobordism rings. This will be done by using Theorem 4.13.

Example 5.25. Here, we will give the equivariant cobordism rings of the previous 5 cases. Therefore, we will in general consider as usual the injective map

$$i^* : \Omega_T^*(X)_{\mathbb{Q}} \to \Omega_T^*(X^T)_{\mathbb{Q}}.$$

(i) First, we consider the case $(B_n, P(\omega_{n-1}), P(\omega_n))$ for $n \geq 3$. For any element $w' \in W/W_{S\setminus\alpha_{n-1}}$ we denote by $y(w') := w'P(\omega_{n-1})/P(\omega_{n-1})$ the corresponding

T-fixed point in Y and similarly by $z(w'') := w''P(\omega_n)/P(\omega_n)$ the T-fixed point in the closed orbit Z for any $w'' \in W/W_{S\setminus\alpha_n}$.

The equations for the closed orbits Y and Z are given by

$$f_{y(w\cdot s_\alpha)} \equiv f_{y(w)} \qquad \mod c_1^T\left(L_{w\omega_{n-1}-ws_\alpha\omega_{n-1}}\right) \tag{5.2}$$

$$f_{z(w\cdot s_\beta)} \equiv f_{z(w)} \qquad \mod c_1^T\left(L_{w\omega_n-ws_\beta\omega_n}\right) \tag{5.3}$$

for $\alpha \in R^+ \setminus R^+_{P(\omega_{n-1})}, \beta \in R^+ \setminus R^+_{P(\omega_n)}$ and $w \in W$ which is true as the difference of the weights associated to the T-fixed points is a multiple of the weight of the corresponding curve and we consider rational coefficients. We have seen above that there are no surfaces in this particular case. Therefore, the last equations are given by the lines joining the two closed orbits. These are given by

$$f_{y(w)} \equiv f_{z(w)} \mod c_1^T\left(L_{w\omega_{n-1}-w\omega_n}\right) \tag{5.4}$$

for $w \in W$. This completely describes the equivariant algebraic cobordism $\Omega^*_T(X)_\mathbb{Q}$ in case (1).

(ii) Case (2) can be done similarly because there are also no surfaces in any fixed point subscheme $X^{T'}$.

(iii) Now, we consider the case $(C_n, P(\omega_m), P(\omega_{m-1}))$ for $n \geq 2$ and $m \in [2, n]$. The equations from the curves in the closed orbits can be obtained as in (5.2) and (5.3). Furthermore, the equations from the lines joining the closed orbits can be obtained as in (5.4). As we have seen in Example 5.24, we need to choose $\alpha := 2\varepsilon_m$ to be the positive root which is a non-zero multiple of $\chi = \omega_m - \omega_{m-1}$ in order to obtain a surface in $X^{T'}$ for $T' = \mathrm{Ker}(\alpha)^0$. The reflection s_α acts trivially on the T-fixed point $z(1)$ and therefore, we obtain the T-fixed points $z(1), y(1)$ and $y(s_\alpha)$ in $X^{T'}$. Having a look at the weights of the resulting surface $\mathbb{P}^2$, we see that ω_{m-1} acts trivially and that ω_m acts with weight ε_m. Therefore, we can identify the T-fixed points $z(1), y(1)$ and $y(s_\alpha)$ with y, x and z respectively where we consider the T-action on $\mathbb{P}^2$ given by $t \cdot [x : y : z] = [tx : y : t^{-1}z]$. For any $w \in W$ this leads to the equation

$$\left(f_{y(w)} - f_{z(w)}\right) + \rho_{1/2}c_1^T(L_{w\alpha})\left(f_{y(w\cdot s_\alpha)} - f_{y(w)}\right) \equiv 0 \mod c_1^T(L_{w\alpha})^2.$$

This completes the description of the equivariant algebraic cobordism in case (3).

(iv) Next, we consider case (4) which is given by the triple $(F_4, P(\omega_2), P(\omega_3))$. The equations coming from the curves can be described as above for the previous cases. From Example 5.24 we know that we can choose the positive root $\alpha := \chi = \omega_2 - \omega_3$ in order to obtain surfaces in $X^{T'}$ for $T' = \mathrm{Ker}(\alpha)^0$. The T-fixed points are given by $y(1), y(s_\alpha), z(1)$ and $z(s_\alpha)$ contained in a Hirzebruch surface $\mathbb{F}_2$ which has been described in Example 5.24. By that example, we know that we have a T-stable curve of degree 4 in the closed orbit Y and one of degree 2 in the closed orbit Z. By computing

$$s_\alpha\omega_2 = -\varepsilon_2 - \varepsilon_3 + 2\varepsilon_4 \text{ and}$$

$$s_\alpha\omega_3 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + 3\varepsilon_4),$$

we obtain $\omega_2 - s_\alpha\omega_2 = 4\alpha$ and $\omega_3 - s_\alpha\omega_3 = 2\alpha$ and therefore, we can identify $y(1), y(s_\alpha), z(1)$ and $z(s_\alpha)$ with w, z, x and y, respectively, using the notations from Example 3.9 (v). For any element $w' \in W$ we define $\zeta_{w'\cdot s_\alpha} := \left(f_{z(w's_\alpha)} - f_{y(w's_\alpha)}\right)$ and $\zeta_{w'} := \left(f_{y(w')} - f_{z(w')}\right)$ which leads to the equations

$$\rho_{-2/2}c_1^T(L_{w'\alpha})\zeta_{w'\cdot s_\alpha} + \rho_{2/2}c_1^T(L_{w'\alpha})\zeta_{w'} \equiv 0 \mod c_1^T(L_{w'\alpha})^2.$$

This completes the description of $\Omega_T^*(X)_\mathbb{Q}$ in case (4).

(v) Lastly, we consider case (5) which is given by the triple $(G_2, P(\omega_1), P(\omega_2))$ for $\omega_1 = 2\alpha_1 + \alpha_2$ and $\omega_2 = 3\alpha_1 + 2\alpha_2$. The curves can be described as above for the previous cases. In order to obtain surfaces in $X^{T'}$ we need to choose $\alpha := -\chi$ by Example 5.24. Therefore, we obtain the T-fixed points $y(1), y(s_\alpha), z(1)$ and $z(s_\alpha)$ contained in a Hirzebruch surface $\mathbb{F}_2$ which has been described in Example 5.24. By that example, we know that we have a T-stable curve of degree 1 in Y and one of degree 3 in Z. By verifying the weights $s_\alpha\cdot\omega_1 = \alpha_1$ and $s_\alpha\cdot\omega_2 = -\alpha_2$ we can identify $y(1), y(s_\alpha), z(1)$ and $z(s_\alpha)$ with x, y, w and z, respectively, by using the notations from Example 3.9 (v). For any $w' \in W$ we define $\xi_{w'\cdot s_\alpha} := \left(f_{y(w'\cdot s_\alpha)} - f_{z(w'\cdot s_\alpha)}\right)$ and $\xi_{w'} := \left(f_{z(w')} - f_{y(w')}\right)$ which leads to the equations

$$\rho_{-2/2}c_1^T(L_{w'\alpha})\xi_{w'\cdot s_\alpha} + \rho_{2/2}c_1^T(L_{w'\alpha})\xi_{w'} \equiv 0 \mod c_1^T(L_{w'\alpha})^2.$$

This completes the description of $\Omega_T^*(X)_\mathbb{Q}$ in case (5).

To finish this section, we will redo the computations for the equivariant cobordism of the symplectic Grassmannian IG(2, 5) and verify that the geometric description from Example 4.29 leads to the same result as the previously described algebraic approach.

Example 5.26. We consider the case $(C_2, P(\omega_2), P(\omega_1))$. The positive roots are given by $\alpha, \beta, \alpha+\beta$ and $2\alpha+\beta$ where $\alpha = \varepsilon_1 - \varepsilon_2$ and $\beta = 2\varepsilon_2$ denote the simple roots, β being the long root. We have $W(\mathrm{Sp}_4) \cong D_4$ where D_4 denotes the dihedral group of order 8. The geometric picture of the roots of C_2 is given by the following figure.

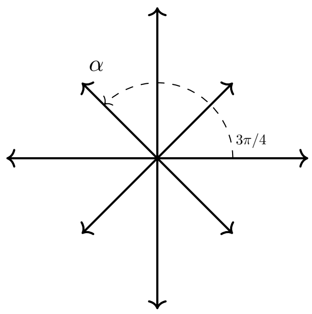


Figure 2: Root system of C_2.

With the same arguments as in Example 5.8 we obtain 4 T-fixed points in G/P_α and G/P_β. Furthermore, every two fixed points in one of the closed orbits are connected by some T-stable curve which leads to a total of 12 T-stable curves in the closed orbits. The fundamental weights are given by $\omega_1 = \varepsilon_1$ and $\omega_2 = \varepsilon_1 + \varepsilon_2 = \alpha + \beta$. The T-fixed points in G/P_α are indexed by elements of the quotient

$$W/W_\alpha = \{\{s_\alpha, \mathrm{Id}\}, \{s_\beta s_\alpha, s_\beta\}, \{s_\alpha s_\beta s_\alpha, s_\alpha s_\beta\}, \{(s_\beta s_\alpha)^2, s_\beta s_\alpha s_\beta\}\}.$$

The B-fixed point $y(1)$ in Y corresponds to the weight $\varepsilon_1 + \varepsilon_2$ which implies that the T-fixed points $y(s_\beta), y(s_\alpha s_\beta)$ and $y(s_\beta s_\alpha s_\beta)$ correspond to the weights $\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_1$ and $-\varepsilon_1 - \varepsilon_2$, respectively. Now, we compute the weights acting on the T-stable curves which are given by

$$\begin{aligned}
(y(1) \leftrightarrow y(s_\beta)) &\quad \leftrightarrow -2\varepsilon_2 \\
(y(1) \leftrightarrow y(s_\alpha s_\beta)) &\quad \leftrightarrow -2\varepsilon_1 \\
(y(1) \leftrightarrow y(s_\beta s_\alpha s_\beta)) &\quad \leftrightarrow -2(\varepsilon_1 + \varepsilon_2) \\
(y(s_\beta) \leftrightarrow y(s_\alpha s_\beta)) &\quad \leftrightarrow -2(\varepsilon_1 - \varepsilon_2) \\
(y(s_\beta) \leftrightarrow y(s_\beta s_\alpha s_\beta)) &\quad \leftrightarrow -2\varepsilon_1 \\
(y(s_\alpha s_\beta) \leftrightarrow y(s_\beta s_\alpha s_\beta)) &\quad \leftrightarrow -2\varepsilon_2.
\end{aligned}$$

We obtain 4 T-stable curves of degree 1 and 2 T-stable curves of degree 2 in the closed orbit Y.

Next, we will do the same computations for the closed orbit Z. The T-fixed points in G/P_β are indexed by elements of the quotient

$$W/W_\beta = \{\{s_\beta, \mathrm{Id}\}, \{s_\alpha s_\beta, s_\alpha\}, \{s_\beta s_\alpha s_\beta, s_\beta s_\alpha\}, \{(s_\alpha s_\beta)^2 = (s_\beta s_\alpha)^2, s_\alpha s_\beta s_\alpha\}\}.$$

The B-fixed point $z(1)$ in Z corresponds to the weight ε_1 implying that the T-fixed points $z(s_\alpha), z(s_\beta s_\alpha)$ and $z(s_\alpha s_\beta s_\alpha)$ correspond to the weights $\varepsilon_2, -\varepsilon_2$ and $-\varepsilon_1$, respectively. The weights acting on the T-stable curves are given by

$$\begin{aligned}
(z(1) \leftrightarrow z(s_\alpha)) &\quad \leftrightarrow -(\varepsilon_1 - \varepsilon_2) \\
(z(1) \leftrightarrow z(s_\beta s_\alpha)) &\quad \leftrightarrow -(\varepsilon_1 + \varepsilon_2) \\
(z(1) \leftrightarrow z(s_\alpha s_\beta s_\alpha)) &\quad \leftrightarrow -2\varepsilon_1 \\
(z(s_\alpha) \leftrightarrow z(s_\beta s_\alpha)) &\quad \leftrightarrow -2\varepsilon_2 \\
(z(s_\alpha) \leftrightarrow z(s_\alpha s_\beta s_\alpha)) &\quad \leftrightarrow -(\varepsilon_1 + \varepsilon_2) \\
(z(s_\beta s_\alpha) \leftrightarrow z(s_\alpha s_\beta s_\alpha)) &\quad \leftrightarrow -(\varepsilon_1 - \varepsilon_2).
\end{aligned}$$

These computations for the closed orbit Z lead to 6 T-stable curves of degree 1. Lastly, we obtain 8 T-stable curves of degree 1 meeting the dense orbit by letting the Weyl group act on the curve connecting the two B-fixed points $y(1)$ and $z(1)$. The corresponding weights are given by

$$\begin{aligned}
\mathrm{Id}(y(1) \leftrightarrow z(1)) = (y(1) \leftrightarrow z(1)) &\quad \leftrightarrow -\varepsilon_2 \\
s_\alpha(y(1) \leftrightarrow z(1)) = (y(1) \leftrightarrow z(s_\alpha)) &\quad \leftrightarrow -\varepsilon_1 \\
s_\beta(y(1) \leftrightarrow z(1)) = (y(s_\beta) \leftrightarrow z(1)) &\quad \leftrightarrow \varepsilon_2
\end{aligned}$$

$$
\begin{aligned}
s_\alpha s_\beta(y(1) \leftrightarrow z(1)) &= (y(s_\alpha s_\beta) \leftrightarrow z(s_\alpha)) && \leftrightarrow \varepsilon_1\\
s_\beta s_\alpha(y(1) \leftrightarrow z(1)) &= (y(s_\beta) \leftrightarrow z(s_\beta s_\alpha)) && \leftrightarrow -\varepsilon_1\\
s_\alpha s_\beta s_\alpha(y(1) \leftrightarrow z(1)) &= (y(s_\alpha s_\beta) \leftrightarrow z(s_\alpha s_\beta s_\alpha)) && \leftrightarrow -\varepsilon_2\\
s_\beta s_\alpha s_\beta(y(1) \leftrightarrow z(1)) &= (y(s_\beta s_\alpha s_\beta) \leftrightarrow z(s_\beta s_\alpha)) && \leftrightarrow \varepsilon_1\\
(s_\beta s_\alpha)^2(y(1) \leftrightarrow z(1)) &= (y(s_\beta s_\alpha s_\beta) \leftrightarrow z(s_\alpha s_\beta s_\alpha)) && \leftrightarrow \varepsilon_2.
\end{aligned}
$$

In accordance to Example 5.25 there will be projective planes $\mathbb{P}^2$ in $X^{\mathrm{Ker}(\beta)^0}$ and to be even more precise, there will be 2 projective planes in $X^{T'}$ containing $y(1), y(s_\beta), z(1)$ and $y(s_\alpha s_\beta), y(s_\beta s_\alpha s_\beta), z(s_\alpha s_\beta s_\alpha)$, respectively. Therefore, using the notation from Example 4.29, we may identify the B-fixed point $z(1)$ with the T-fixed point x_{13} by choosing the Borel subgroup to be the subgroup of upper triangular matrices in Sp_4. This implies $y(1) = x_{12}$ because of the degree 2 curve in Y containing $y(1)$ and therefore, we have $y(s_\beta) = x_{14}$. Furthermore, we can then identify $z(s_\alpha s_\beta s_\alpha) = x_{35}, y(s_\alpha s_\beta) = x_{25}$ and $y(s_\beta s_\alpha s_\beta) = x_{45}$. Lastly, we obtain $z(s_\beta s_\alpha) = x_{34}$ and $z(s_\alpha) = x_{23}$. In conclusion, we obtain the same description as in Example 4.29 which will lead to the same equations describing the equivariant cobordism ring $\Omega^*_T(\mathrm{IG}(2,5))_{\mathbb{Q}}$.

Remark 5.27. The previous example can be generalised to all the examples of type (3), i.e. to all odd symplectic Grassmannians $\mathrm{IG}(k, 2n+1)$ for $n \geq 2$ and $k \in [2, n]$ which have been described in Example 4.31. We conclude that computation of the equivariant cobordism rings using the geometric interpretation coincides with the algebraic one for all the examples occurring in case (3).

5.3 Geometry of horospherical varieties of Picard number two

In this section, we consider the geometry of smooth projective horospherical varieties of Picard number two. We will first recall some relevant notation which was used in Pasquier's classification of smooth projective horospherical varieties of Picard number two (cf. [44]).

Let G be a complex simply connected simple algebraic group, B a Borel subgroup of G and T a fixed maximal torus contained in B. This defines a root system and in particular a set of simple roots. We may associate a fundamental weight ω_α and a fundamental G-module denoted by $V(\omega_\alpha)$ to each simple root α. More generally, for any dominant weight χ we denote by $V(\chi)$ the G-module associated to χ. To be more precise, it is the unique irreducible G-module that contains a unique B-stable line where B acts via weight χ. A non-zero element of the B-stable line of $V(\chi)$ is called a highest weight vector of weight χ and the stabiliser of the B-stable line of $V(\chi)$ is denoted by $P(\chi)$ which is a parabolic subgroup of G containing B.

In this section, for $G = \mathbb{C}^*$ we call the identity automorphism of $\mathbb{C}^*$ the **simple root** of G, we denote it by α and set $\omega_\alpha = \alpha$. The natural $\mathbb{C}^*$-module $\mathbb{C}$ is denoted by $V(\omega_\alpha)$ where α is the simple root of $\mathbb{C}^*$. Further, for any $n \in \mathbb{Z}$, we denote by $V(n\omega_\alpha)$ the $\mathbb{C}^*$-module $\mathbb{C}$ where $\mathbb{C}^*$ acts with weight $n\omega_\alpha$. Moreover, if $G = \{1\}$, we call the trivial morphism from G to $\mathbb{C}^*$ the **simple root** of G which we denote by α and we set $\omega_\alpha = 0$. We remark that in both cases any non-zero vector is a highest weight vector.

Assume now that G is a product $G_0 \times \cdots \times G_t$ of simply connected simple groups, $\mathbb{C}^*$ and $\{1\}$. A simple root of G is a simple root of some G_i and it is said to be trivial if G_i is equal to $\mathbb{C}^*$ or $\{1\}$. Moreover, let $\chi_0, ..., \chi_t$ be dominant weights of $G_0, ..., G_t$, respectively, then the G-module associated to $\chi = \chi_0 + ... + \chi_t$ is the tensor product

$V(\chi_0) \otimes \cdots \otimes V(\chi_t)$ and a highest weight vector of this G-module is a decomposable tensor product of highest weight vectors.

Definition 5.28. *[44, Definition 3.9] Let $G = G_0 \times \cdots \times G_t$ be a product of simply connected simple groups, $\mathbb{C}^*$ and $\{1\}$ with $t \geq 0$.*

(1) Suppose G_0 to be a simple group. Further, let β be a non-trivial simple root of G_0, let $n \geq \max\{1, t\}$ and let $\alpha_0, ..., \alpha_n$ be distinct, possibly trivial, simple roots of G different from β and lastly, let $0 = a_0 \leq a_1 \leq \cdots \leq a_n$ be integers. Suppose also that for any $k \in \{1, ..., t\}$, $G_k = \{1\}$ if and only if $k = 1$ and α_0 is the trivial root of G_1. Denote $\underline{\alpha} := (\alpha_0, ..., \alpha_n)$ and $\underline{a} := (a_0, ..., a_n)$. Under these assumptions, we define $\mathbb{X}^1(G, \beta, \underline{\alpha}, \underline{a})$ to be the closure of the G-orbit of a sum of highest weight vectors in

$$\mathbb{P}\left(\bigoplus_{i=0}^{n} V(\omega_{\alpha_i} + (1 + a_i)\omega_\beta\right).$$

(2) Suppose now that $t \geq 1$. Let $n \geq 2$, let $0 = a_0 \leq a_1 \leq \cdots \leq a_{n-1}$ be integers, and let $\alpha_0, ..., \alpha_{n+1}$ be distinct, possibly trivial, simple roots of G. Suppose also that, for any $k \in \{0, ..., t\}$, $G_k = \{1\}$ if and only if $k = 0$ and α_0 is the trivial root of G_0, or $k = t$ and α_{n+1} is the trivial root of G_t. Denote $\underline{\alpha} := (\alpha_0, ..., \alpha_{n+1})$ and $\underline{a} := (a_0, ..., a_{n-1})$. Under these assumptions, we define $\mathbb{X}^2(G, \underline{\alpha}, \underline{a})$ to be the closure of the G-orbit of a sum of highest weights vectors in

$$\mathbb{P}\left(\bigoplus_{i=0}^{n-1} \bigoplus_{b=0}^{1+a_i} V(\omega_{\alpha_i} + b\omega_{\alpha_n} + (1 + a_i - b)\omega_{\alpha_{n+1}})\right).$$

Definition 5.29. *[44, Definition 4.1] Let K be a simple algebraic group over $\mathbb{C}$ and let γ, δ be two simple roots of K. The triple (K, γ, δ) is said to be smooth if it is one of the following 8 cases, up to exchanging γ and δ (with the notation of Bourbaki [5]).*

(1). $(A_m, \alpha_1, \alpha_m)$ with $m \geq 2$

(2). $(A_m, \alpha_i, \alpha_{i+1})$ with $m \geq 3$ and $i \in \{1, ..., m-1\}$

(3). $(B_m, \alpha_{m-1}, \alpha_m)$ with $m \geq 3$

(4). $(B_3, \alpha_1, \alpha_3)$

(5). $(C_m, \alpha_i, \alpha_{i+1})$ with $m \geq 2$ and $i \in \{1, ..., m-1\}$

(6). $(D_m, \alpha_{m-1}, \alpha_m)$ with $m \geq 4$

(7). $(F_4, \alpha_2, \alpha_3)$

(8). $(G_2, \alpha_1, \alpha_2)$

We say that the triple (type of K, γ, δ) is smooth of two-orbit type if it is one of the cases 3, 4, 5, 7 or 8 above.

Definition 5.30. *[44, Definition 4.3] Let K be a simple algebraic group over $\mathbb{C}$ and let β be a simple root of K and let R be a subset of simple roots of K, all distinct from β. Let n be a non-negative integer. Denote by L the Levi subgroup of the maximal parabolic* *subgroup* $P(\beta)$ *of* K*, then the semisimple part of* L *is a quotient by a finite central group of a product of simple groups* $L^1, ..., L^q$ *(with* $q \geq 0$*). The quadruple* (K, β, R, n) *is said to be smooth if*

(1) $n = 1, R = \{\gamma, \delta\}$ *such that* γ *and* δ *are simple roots of the same* L^k *such that the triple* (L^k, γ, δ) *is smooth;*

(2) or for any $k \in \{1, ..., q\}$*, at most one simple root of* L^k *is in* R*, and if* $\gamma \in R$ *is a simple root of* L^k*, then* L^k *is of type A or C and* γ *is a short extremal simple root of* L^k*.*

We can now define the restricted conditions that allow us to state the main theorem of [44].

Definition 5.31. *[44, Definition 4.4] Let* $X = \mathbb{X}^1(G, \beta, \underline{\alpha}, \underline{a})$ *as in Definition 5.28. Let* R_0 *be the maximal subset of* $\{\alpha_0, ..., \alpha_n\}$ *consisting of simple roots of* G_0*. We say that* X *satisfies the restricted condition* $(a), (b)$ *or* (c)*, respectively, if it satisfies all the following properties including* $(a), (b)$ *or* (c)*, respectively.*

(1) The quadruple (G_0, β, R_0, n) *is smooth.*

(2) If R_0 *is empty, then* G_0 *is the universal cover of the automorphism group of* $G/P(\omega_\beta)$*.*

(3) If $i < j$ *and* $a_i = a_j$*, then* $\alpha_j \in R_0$*. Moreover, if* α_i *and* α_j *are in* R_0*, we suppose them to be ordered with Bourbaki's notation as simple roots of* G_0*.*

(4) One of the three following cases occurs.

- *(a) We have* $n = t = 1$*,* α_0 *and* α_1 *are both simple roots of* G_1 *such that the triple* $(G_1, \alpha_0, \alpha_1)$ *is smooth; in particular,* $R_0 = \emptyset$ *and* $a_0 < a_1$*.*
  In the next two cases, the map $\{\alpha_0, ..., \alpha_n\} \setminus R_0 \to \{1, ..., t\}$ *is surjective and strictly increasing, and for any* $k \in \{1, ..., t\}$*, either* G_k *is isomorphic to some* SL_{d_k} *and* α_{i_k} *is the first simple root of* G_k*, or* G_k *is isomorphic to* $\mathbb{C}^*$ *or* $\{1\}$ *and* α_{i_k} *is the trivial simple root of* G_k*.*
- *(b) The simple root* α_n *is not trivial (in particular if* $a_{n-1} = a_n$*).*
- *(c) The simple root* α_n *is trivial (and then* $a_{n-1} < a_n$*).*

Definition 5.32. *[44, Definition 4.5] Let* $X = \mathbb{X}^2(G, \underline{\alpha}, \underline{a})$ *as in Definition 5.28. We say that* X *satisfies the restricted condition* $(a), (b)$ *or* (c)*, respectively, if it satisfies all the following properties including* $(a), (b)$ *or* (c)*, respectively.*

(1) We have $0 = a_0 < a_1 < \cdots < a_n$*.*

(2) The triple $(G_t, \alpha_n, \alpha_{n+1})$ *is smooth of two-orbit type; in particular,* α_n *and* α_{n+1} *are both simple roots of* G_t *and* $\alpha_0, ..., \alpha_{n-1}$ *are simple roots of* $G_0 \times G_1 \times \cdots \times G_{t-1}$*.*

(3) One of the three following cases occurs.

- *(a) We have* $n = 2$*,* $t = 1$ *and the triple* $(G_0, \alpha_0, \alpha_1)$ *is smooth.*
  In the two next cases: $t = n$*, the map* $\{\alpha_0, ..., \alpha_{n-1}\} \to \{0, ..., t-1\}$ *is surjective and strictly increasing; and for any* $i \in \{1, ..., t\}$*, either* G_i *is isomorphic to some* SL_{d_i} *and* α_i *is the first simple root of* G_i*, or* G_i *is isomorphic to* $\mathbb{C}^*$ *or* $\{1\}$ *and* α_i *is the trivial simple root of* G_i*.*

(b) The simple root α_{n-1} is not trivial.

(c) The simple root α_{n-1} is trivial.

Now, we can finally state the classification of smooth projective horospherical varieties of Picard rank two.

Theorem 5.33. *[44, Theorem 1.1] Let X be a smooth projective horospherical variety with Picard group $\mathbb{Z}^2$. Suppose that X is not the product of two varieties. Then X is isomorphic to one of the following horospherical varieties. In all cases, G is a product of simply connected simple groups, $\mathbb{C}^*$ and $\{1\}$.*

(0) G is simple and X is a homogeneous variety G/P where P is the intersection of two maximal (proper) parabolic subgroups of G containing the same Borel subgroup.

(1) X is one of the varieties $\mathbb{X}^1(G, \beta, \underline{\alpha}, \underline{a})$ as in Definition 5.28 with one of the restricted conditions $(a), (b)$ or (c).

(2) X is a variety $\mathbb{X}^2(G, \underline{\alpha}, \underline{a})$ as in Definition 5.28 with one of the restricted conditions $(a), (b)$ or (c).

Remark 5.34. [44, Remark 4.6] In Theorem 5.33, the decomposable projective bundles over projective spaces are the horospherical varieties X in Case (1) with restricted condition (b) or (c), and such that $R_0 = \emptyset$ and ω_β is the first simple root of $G_0 = \mathrm{SL}_{d_0}$ for some $d_0 \geq 2$ (and $0 < a_1 < \cdots < a_n$).

Remark 5.35. In Case (1a) one would have the product of two varieties if one allowed $a_1 = 0$. Here, one can compute the T-equivariant cobordism of the product using the Künneth formula 3.63.

Recall [44, Section 4.1] that the colored fans $\mathbb{F}^1$ and $\mathbb{F}^2$ of the horospherical varieties in Cases (1) and (2) of Theorem 5.33 are given as follows.

The colored fan $\mathbb{F}^1$ is the complete colored fan whose maximal colored cones are generated by all $u_0, ..., u_n$ except one and with all possible colors except β, where $(u_1, ..., u_n)$ is a basis of N and $u_0 = -u_1 - ... - u_n$. Recall also that the lattice N is of rank n which is the rank of the horospherical variety $\mathbb{X}^1(G, \beta, \underline{\alpha}, \underline{a})$.

The colored fan $\mathbb{F}^2$ is the complete colored fan whose maximal colored cones are generated by all $u_0, ..., u_r, v_1, ..., v_{s+1}$ except one u_i and one v_j where $(u_1, ..., u_r, v_1, ..., v_s)$ is a basis of N, $u_0 = -u_1 - ... - u_r$ and $v_{s+1} = a_1u_1 + ... + a_ru_r - v_1 - ... - v_s$. Furthermore, the maximal colored cones of the colored fan $\mathbb{F}^2$ contain all possible colors.

Proposition 5.36. *Let X be a smooth projective horospherical G-variety with Picard group $\mathbb{Z}^2$ and suppose that X is not the product of two varieties. Then the following statements are true.*

(i) All varieties X of rank one are coming from Cases (1a) and (1b).

(ii) Varieties of rank one coming from Case (1a) have finitely many T-stable curves.

(iii) Varieties of rank one coming from Case (1b) can potentially have infinitely many T-stable curves.

Proof. (i) There is a well known correspondence between closed G-orbits and maximal colored cones (cf. [29]). Using the previous description of the colored fans, one may observe that smooth projective horospherical varieties of Picard group $\mathbb{Z}^2$ which are of rank one can come from Case (1a) and (1b) and have two closed G-orbits. In Case (1c) one cannot construct an example with $n = 1$ and $t = 0$ because G_0 is by assumption neither $\mathbb{C}^*$ nor $\{1\}$. Furthermore, in Case (2) we require $n \geq 2$ by Definition 5.28. Thus, the only smooth projective horospherical varieties of Picard group $\mathbb{Z}^2$ of rank one come from Cases (1a) and (1b).

(ii) Let $X = \mathbb{X}^1(G, \beta, \underline{\alpha}, \underline{a})$ be a smooth projective horospherical variety with Picard group $\mathbb{Z}^2$ of Case (1a). In this case, the dense orbit is given by

$$G \cdot [v_{\omega_{\alpha_0}+\omega_\beta} + v_{\omega_{\alpha_1}+(1+a_1)\omega_\beta}] \subseteq \mathbb{P}(V(\omega_{\alpha_0} + \omega_\beta) \oplus V(\omega_{\alpha_1} + (1 + a_1)\omega_\beta))$$

where $v_{\omega_{\alpha_0}+\omega_\beta}$ and $v_{\omega_{\alpha_1}+(1+a_1)\omega_\beta}$ are the highest weight vectors of the correspoding representations associated to the highest weights. Similarly to the Picard number one case (cf. Section 5.2), we obtain the two closed G-orbits $Y \cong G/P_Y$ and $Z \cong G/P_Z$ where P_Y and P_Z are the stabilisers of $[\omega_{\alpha_0} + \omega_\beta]$ and $[v_{\omega_{\alpha_1}+(1+a_1)\omega_\beta}]$ in $\mathbb{P}(V(\omega_{\alpha_0} + \omega_\beta))$ and $\mathbb{P}(V(\omega_{\alpha_1} + (1 + a_1)\omega_\beta))$, respectively.

In order to determine the T-stable curves and the fixed point subschemes $X^{T'}$, we can again use diagram (5.1) where T' is a subtorus of codimension one in T. Furthermore, we can run the same strategy and proofs as in the case of Picard number one (cf. Section 5.2). In this case the difference of the two fundamental weights is given by

$$\chi := \omega_{\alpha_0} + \omega_\beta - (\omega_{\alpha_1} + (1 + a_1)\omega_\beta) = \omega_{\alpha_0} - \omega_{\alpha_1} - a_1\omega_\beta.$$

As we have seen in Example 5.24, we need to find a root which is a non-zero multiple of χ in order to find surfaces in $X^{T'}$ for some singular codimension one subtorus $T' \subseteq T$. We have $a_1 \neq 0$ in Case (1a) and therefore, we cannot find such a root because χ consists of a part coming from G_0 and another one coming from G_1. Thus, we conclude that there are only finitely many T-stable curves in Case (1a) of Theorem 5.33.

(iii) Now, we consider the smooth projective horospherical varieties of Picard group $\mathbb{Z}^2$ and of rank one coming from Case (1b). In that case we have $n = 1$ and $t = 0$. After having discussed Case (1a) we can make similar observations concerning the geometry for the examples coming from Case (1b). Consequently, we obtain $\chi = \omega_{\alpha_0} - \omega_{\alpha_1} - a_1\omega_\beta$, but in this case all the roots are roots of the same group G_0. Thus, one may be able to find a positive root which is a non-zero multiple of χ. This concludes the proof. □

Remark 5.37. For smooth projective horospherical varieties X of rank one with Picard group $\mathbb{Z}^2$ coming from Case (1a), the T-equivariant cobordism $\Omega^*_T(X)_{\mathbb{Z}[S_X^{-1}]}$ can be computed using Theorem 3.60 because there are only finitely many T-stable curves. As opposed to Case (1a), one needs to use Theorem 4.13 in order to compute the T-equivariant cobordism for smooth projective horospherical varieties of rank one with Picard group $\mathbb{Z}^2$ having infinitely many T-stable curves coming from Case (1b).

Using the algorithm from Section 5.2 which also works for smooth projective horospherical varieties X of rank one with Picard group $\mathbb{Z}^2$, we can check whether there occur some projective planes or Hirzebruch surfaces in the connected components of $X^{T'}$ for codimension one subtori $T' \subseteq T$. We will see in the upcoming examples that the degrees of the curves in the closed orbits can be way different from the ones occurring in the smooth projective horospherical varieties of Picard group $\mathbb{Z}$.

Now, we present some examples similar to the computations in Example 5.24 in Section 5.2.

Example 5.38. Using the notation of Bourbaki [5] concerning the roots and the fundamental weights, we consider certain examples.

(i) First, let $X = \mathbb{X}^1(\mathrm{Sp}_{2n}, \alpha_n, (\alpha_1, \alpha_{n-1}), (0,0))$ be the given smooth projective horospherical variety of Picard rank two from Case (1b). Here, we have

$$\begin{aligned}\chi &= \omega_1 - \omega_{n-1} = \varepsilon_1 - (\varepsilon_1 + ... + \varepsilon_{n-1}) \\ &= -\varepsilon_2 - ... - \varepsilon_{n-1}.\end{aligned}$$

For $3 \leq n \leq 4$ we can find a positive root which is a non-zero multiple of χ. We choose $n = 3$. Then we have $\alpha := -2\chi = 2\varepsilon_2$. We need to compute

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)} = \frac{2 \cdot 2\varepsilon_2}{(2\varepsilon_2, 2\varepsilon_2)} = \varepsilon_2$$

and thus, we obtain

$$(\alpha^\vee, \omega_1 + \omega_3) = (\varepsilon_2, \varepsilon_1 + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)) = 1$$

and

$$(\alpha^\vee, \omega_2 + \omega_3) = (\varepsilon_2, (\varepsilon_1 + \varepsilon_2) + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)) = 2.$$

This implies that we have 4 T-fixed points and that we obtain a Hirzebruch surface $\mathbb{F}_1$ in $X^{T'}$ for $T' = \mathrm{Ker}(\alpha)^0$. For $n = 4$ we have

$$\begin{aligned}\chi &= \omega_1 - \omega_3 = \varepsilon_1 - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \\ &= -\varepsilon_2 - \varepsilon_3\end{aligned}$$

and in this case, we choose $\alpha := -\chi$. We compute

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)} = \frac{2(\varepsilon_2 + \varepsilon_3)}{(\varepsilon_2 + \varepsilon_3, \varepsilon_2 + \varepsilon_3)} = \varepsilon_2 + \varepsilon_3$$

and hence, we have

$$(\alpha^\vee, \omega_1 + \omega_4) = (\varepsilon_2 + \varepsilon_3, \varepsilon_1 + (\varepsilon_1 + ... + \varepsilon_4)) = 2$$

and

$$(\alpha^\vee, \omega_3 + \omega_4) = (\varepsilon_2 + \varepsilon_3, 2\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3 + \varepsilon_4) = 4.$$

In this case, using the algorithm from Section 5.2, we obtain a Hirzebruch surface $\mathbb{F}_2$ in $X^{T'}$ for $T' = \mathrm{Ker}(\alpha)^0$.

(ii) Let $X = \mathbb{X}^1(E_8, \alpha_1, (\alpha_3, \alpha_2), (0,1))$ be the given smooth projective horospherical variety of Picard rank two from Case (1b). This leads to

$$\chi = (\omega_3 + \omega_1) - (\omega_2 + 2\omega_1) = \omega_3 - \omega_2 - \omega_1$$

$$= \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 + ... + \varepsilon_7 + 7\varepsilon_8) - \frac{1}{2}(\varepsilon_1 + ... + \varepsilon_7 + 5\varepsilon_8) - 2\varepsilon_8$$
$$= -\varepsilon_1 - \varepsilon_8.$$

Thus, we can choose the positive root $\alpha := -\chi = \varepsilon_1 + \varepsilon_8$. As above, we compute

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)} = \frac{2(\varepsilon_1 + \varepsilon_8)}{(\varepsilon_1 + \varepsilon_8, \varepsilon_1 + \varepsilon_8)} = \varepsilon_1 + \varepsilon_8$$

and hence, we obtain

$$(\alpha^\vee, \omega_3 + \omega_1) = (\varepsilon_1 + \varepsilon_8, \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 + ... + \varepsilon_7 + 11\varepsilon_8)) = 5$$

and

$$(\alpha^\vee, \omega_2 + 2\omega_1) = (\varepsilon_1 + \varepsilon_8, \frac{1}{2}(\varepsilon_1 + ... + \varepsilon_7 + 13\varepsilon_8)) = 7.$$

Similar to the previous case, this leads to a Hirzebruch surface $\mathbb{F}_2$ with a different embedding.

Remark 5.39. One can compute a plethora of examples for Case (1b) using the classification of smooth quadruples in [44, Appendix 6]. Even for $a_1 \neq 0$ one can see from Example 5.38 (ii) that surfaces occur in the fixed point subschemes $X^{T'}$ for codimension one subtori $T' \subseteq T$. The T-equivariant cobordism rings $\Omega^*_T(X)_\mathbb{Q}$ for all of these varieties can be described by Theorem 4.13.

Remark 5.40. A natural question to ask would be the generalisation of these computations to varieties of rank ≥ 2. Therefore, one could consider a smooth projective horospherical variety of rank two with Picard group $\mathbb{Z}^2$. In this case, there are three closed G-orbits, one dense G-orbit and three other G-orbits. One might be able to describe the T-geometry in the dense G-orbit using similar strategies as for the rank one case except that one would need to consider a fibration with fibres $(\mathbb{C}^*)^2$. On the other hand, one also has to describe the T-geometry in the three G-orbits which are neither open nor closed. These orbits are similarly given by G-orbits of a sum of weight vectors, but in general their T-geometry is very difficult to describe. Thus, the above computations cannot be easily extended to any smooth projective horospherical variety of Picard number two.

6 Equivariant multiplicities at isolated fixed points

In this section, we want to generalise some results for equivariant Chow groups from [7, Section 4] to equivariant algebraic cobordism. Furthermore, in this section we will work with the definition of T-filtrable schemes given in Section 3.1.

6.1 Equivariant multiplicities

Definition 6.1. *Let X be a scheme with a T-action. We call a T-fixed point $x \in X$ **nondegenerate** if the tangent space T_xX contains no nonzero fixed point. Equivalently, 0 is not a weight for the T-module T_xX. The weights of this module counted with their equivariant multiplicities will be called the **weights of x in X**.*

Remark 6.2. [7, Section 4.1] We have $T_x(X^T) = (T_xX)_0$ where $(T_xX)_0$ denotes the zero weight space. Therefore, any T-fixed point in a nonsingular T-variety is nondegenerate if and only if it is isolated. Thus, for the class of smooth projective and spherical varieties all T-fixed points are nondegenerate.

Before we start to prove the main analogues of [7, Section 4] we state two important statements which were proved by Krishna [31]. Let M be the character group of T. Recall that $S(T)[M^{-1}]$ is the graded ring obtained by inverting all non-zero linear forms $\sum_{j=1}^{n} m_j t_j$ which was described in more detail in Construction 3.43. For a smooth k-scheme X with a torus action, we denote $\Omega_T^*(X) \otimes_{S(T)} S(T)[M^{-1}]$ by $\Omega_T^*(X)[M^{-1}]$.

Proposition 6.3. *[31, Proposition 3.1] Let G be a linear algebraic group and $f : Y \to X$ be a regular G-equivariant embedding in $G - \mathbf{Sch}_k$ of pure codimension d and let $N_{Y/X}$ denote the equivariant normal bundle of Y inside X. Then one has*

$$f^* \circ f_*(\eta) = c_d^G(N_{Y/X})(\eta)$$

for every $\eta \in \Omega_*^G(Y)$.

Corollary 6.4. *[31, Corollary 7.3] Let X be a smooth T-filtrable variety with an action of a torus T and $i : X^T \to X$ be the inclusion of the fixed point subscheme. Then the pushforward map $i_* : \Omega_T^*(X^T)_{\mathbb{Q}} \to \Omega_T^*(X)_{\mathbb{Q}}$ becomes an isomorphism after base change to $S(T)_{\mathbb{Q}}[M^{-1}]$.*

We recall that the equivariant cobordism module of disconnected varieties is the direct sum of the equivariant cobordism modules of the connected components.

Definition 6.5. *Let X be a smooth T-filtrable variety with an action of a torus T. Further, let $[Y \to X] \in \Omega_T^*(X)_{\mathbb{Q}}$ and $x \in X$ be an isolated T-fixed point. We distinguish between isolated fixed points and connected components $F \subseteq X^T$ which are not an isolated point. For any isolated fixed point, we define the **equivariant multiplicity***

$$e_{x,X}[Y \to X] \in S(T)_{\mathbb{Q}}[M^{-1}]$$

of X at x to be given by the equality

$$[Y \to X] = i_* \left(\sum_{\substack{x \in X^T \\ \text{isolated}}} e_{x,X}[Y \to X][x \to x] + \sum_{F \subseteq X^T} e_F[F' \to F] \right)$$

which holds in $\Omega_T^*(X)_{\mathbb{Q}}[M^{-1}]$ for some $e_F \in S(T)_{\mathbb{Q}}[M^{-1}]$ and $[F' \to F] \in \Omega_T^*(F)_{\mathbb{Q}}$.

Lemma 6.6. *Let X be a smooth T-filtrable variety with a T-action. Furthermore, let $Y \subseteq X$ be a closed smooth T-filtrable subvariety. For the class $[f : Y \to X]$ in the $S(T)_{\mathbb{Q}}$-algebra $\Omega_T^*(X)_{\mathbb{Q}}$ and any T-fixed point $y \in Y$ which is nondegenerate in X we have*

$$e_{y,X}[Y \to X] = \frac{1}{c_1^T(L_{-\chi_1}) \cdots c_1^T(L_{-\chi_m})}$$

in $\Omega_T^*(X)_{\mathbb{Q}}[M^{-1}]$ where $\chi_1, ..., \chi_m$ are the weights of y in Y.

Proof. First, we remark that the assumption of $y \in Y^T$ being nondegenerate in X implies that y lies in X^T and is nondegenerate in Y. Next, we consider the equality

$$[Y \to Y] = \sum_{\substack{y' \in Y^T \\ \text{isolated}}} e_{y',Y}[Y \to Y][y' \to Y] + \sum_{F \subseteq Y^T} e_F[F' \to Y] \tag{6.1}$$

coming from Definition 6.5. For $j : Y^T \to Y$, we apply j^* on both sides. Using Proposition 6.3 and the Whitney sum formula, we obtain

$$[Y^T \to Y^T] = \sum_{\substack{y' \in Y^T \\ \text{isolated}}} e_{y',Y}[Y \to Y] \left(\prod_{\substack{\chi \text{ weights of} \\ y' \text{ in } Y}} c_1^T(L_{-\chi}) \right) [y' \to y'] + \sum_{F \subseteq Y^T} e_F[j^* F' \to F]$$

which leads to

$$e_{y',Y}[Y \to Y] = \left(\prod_{\substack{\chi \text{ weights of} \\ y' \text{ in } Y}} c_1^T(L_{-\chi}) \right)^{-1}$$

for all isolated $y' \in Y^T$. Now, we apply f_* to (6.1) and thus, we have

$$[Y \to X] = \sum_{\substack{y' \in Y^T \\ \text{isolated}}} e_{y',Y}[Y \to Y][y' \to X] + \sum_{F \subseteq Y^T} e_F[F' \to X]. \tag{6.2}$$

On the other hand, by Definition 6.5, we have the equality

$$[Y \to X] = \sum_{\substack{x \in X^T \\ \text{isolated}}} e_{x,X}[Y \to X][x \to X] + \sum_{\widetilde{F} \subseteq X^T} e_{\widetilde{F}}[\widetilde{F}' \to X].$$

Let $i : y \to X$ be the inclusion of the isolated fixed point y in X. Applying i^* to the two equations above implies

$$e_{y,Y}[Y \to Y] i^*[y \to X] = e_{y,X}[Y \to X] i^*[y \to X]. \tag{6.3}$$

Hence, comparing the coefficients leads to

$$e_{y,Y}[Y \to Y] = e_{y,X}[Y \to X]$$

which implies the claim. □

Next, we consider classes $[Y \to X]$ of the $S(T)_{\mathbb{Q}}$-algebra $\Omega_T^*(X)_{\mathbb{Q}}$ for which Y is not necessarily a closed smooth T-filtrable subscheme of X. This generalises [7, Proposition 4.3] in the setting of smooth T-filtrable varieties with a T-action.

Proposition 6.7. *Let X, Y be smooth T-filtrable varieties with a T-action such that $[f : Y \to X]$ is a class in the $S(T)_{\mathbb{Q}}$-algebra $\Omega_T^*(X)_{\mathbb{Q}}$. Let $x \in X$ be a nondegenerate fixed point. Assume further that all fixed points in the fibre $f^{-1}(x)$ are nondegenerate. Then we have*

$$e_{x,X}[Y \to X] = \sum_{\substack{y \in Y^T \\ f(y)=x}} e_{y,Y}[Y \to Y].$$

Proof. Let $j : U \to X$ be the inclusion of some open T-stable neighbourhood of x. By potential shrinking we may assume that x is the unique T-fixed point in X. Using Definition 6.5 in $\Omega_T^*(X)_{\mathbb{Q}}$, we obtain

$$[Y \to X] = \sum_{\substack{x \in X^T \\ \text{isolated}}} e_{x,X}[Y \to X][x \to X] + \sum_{\widetilde{F} \subseteq X^T} e_{\widetilde{F}}[\widetilde{F}' \to X].$$

We have $j^*[\widetilde{F}' \to X] = 0$ if $\mathrm{Im}(\widetilde{F}') \subseteq X$ does not contain x. Therefore, pulling back along j yields

$$[f^{-1}(U) \to U] = \sum_{x \in U^T} e_{x,X}[Y \to X][x \to U] = e_{x,X}[Y \to X][x \to U].$$

On the other hand, we have

$$[Y \to Y] = i_* \left(\sum_{\substack{y \in Y^T \\ \text{isolated}}} e_{y,Y}[Y \to Y][y \to y] + \sum_{F \subseteq Y^T} e_F[F' \to F] \right).$$

Applying the pushforward f_* to the equation results in

$$[Y \to X] = \left(\sum_{\substack{y \in Y^T \\ \text{isolated}}} e_{y,Y}[Y \to Y][y \to X] + \sum_{F \subseteq Y^T} e_F[F' \to X] \right).$$

Again, $j^*[F' \to X] = 0$ and $j^*[y' \to X] = 0$ for any $y' \in Y^T$ if $f(y') \neq x$. Thus, applying the pullback j^* yields

$$[f^{-1}(U) \to U] = \sum_{\substack{y \in Y^T \\ f(y)=x}} e_{y,Y}[Y \to Y][y \to U].$$

Due to the fact that $[x \to U] = [y \to U]$ holds in $\Omega_T^*(U)_{\mathbb{Q}}[M^{-1}]$ for any $y \in Y^T$ with $f(y) = x$, we obtain

$$\begin{aligned} e_{x,X}[Y \to X][x \to U] &= \sum_{\substack{y \in Y^T \\ f(y)=x}} e_{y,Y}[Y \to Y][y \to U] \\ &= \left(\sum_{\substack{y \in Y^T \\ f(y)=x}} e_{y,Y}[Y \to Y] \right) [y \to U]. \end{aligned}$$

Thus, the corresponding coefficients in $S(T)_{\mathbb{Q}}[M^{-1}]$ must coincide which implies the claim. □

Remark 6.8. The results in this section can be certainly proved for the coefficient ring $\mathbb{Z}[S_X^{-1}]$ (cf. Definition 3.58), but in fact one only needs condition (i) of Definition 3.56 in order to prove Corollary 6.4. Therefore, it would be enough to consider only condition (i) of Definition 3.56 for the subsequent Definition 3.58 which could then be used as the modified coefficient ring in this section.

6.2 Classes in IG(2, 5)

Example 6.9. We want to compute the pullback of the classes of IG(2, 5) in $S(T)_{\mathbb{Q}}^8$. Therefore, we first consider the Bialynicki-Birula decomposition coming from Brion's definition of T-filtrable varieties in [7, Section 3]. As a generic one-parameter subgroup we choose $\lambda : \mathbb{G}_m \to T, t \mapsto \mathrm{diag}(t^2, t, t^{-1}, t^{-2})$. Using the notation from Section 3.2, we obtain the cells

$$X_+(x_{45}, \lambda) = \left\{ \begin{pmatrix} a_1/a_4 & b_1/b_5 \\ a_2/a_4 & b_2/b_5 \\ a_3/a_4 & b_3/b_5 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \middle| -\frac{a_1}{a_4} + \frac{b_2}{b_5} = 0 \right\} = \mathbb{A}^5$$

$$X_+(x_{35}, \lambda) = \left\{ \begin{pmatrix} a_1/a_3 & b_1/b_5 \\ a_2/a_3 & b_2/b_5 \\ 1 & 0 \\ 0 & b_4/b_5 \\ 0 & 1 \end{pmatrix} \middle| -\frac{a_1}{a_3} - \frac{a_2 b_4}{a_3 b_5} = 0 \right\} = \mathbb{A}^4$$

$$X_+(x_{34}, \lambda) = \left\{ \begin{pmatrix} a_1/a_3 & b_1/b_4 \\ a_2/a_3 & b_2/b_4 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \middle| -\frac{a_2}{a_3} = 0 \right\} = \mathbb{A}_1^3$$

$$X_+(x_{25}, \lambda) = \left\{ \begin{pmatrix} a_1/a_2 & b_1/b_5 \\ 1 & 0 \\ 0 & b_3/b_5 \\ 0 & b_4/b_5 \\ 0 & 1 \end{pmatrix} \middle| -\frac{a_1}{a_2} - \frac{b_4}{b_5} = 0 \right\} = \mathbb{A}_2^3$$

$$X_+(x_{23}, \lambda) = \left\{ \begin{pmatrix} a_1/a_2 & b_1/b_3 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} = \mathbb{A}^2_1$$

$$X_+(x_{14}, \lambda) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & b_2/b_4 \\ 0 & b_3/b_4 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} = \mathbb{A}^2_2$$

$$X_+(x_{13}, \lambda) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & b_2/b_3 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} = \mathbb{A}^1$$

$$X_+(x_{12}, \lambda) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} = \mathbb{A}^0$$

which lead to the filtration $\mathbb{A}^0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_7 = \mathrm{IG}(2,5)$ where each X_i is given by attaching one more cell in reciprocal order. We recall the filtration (3.1), i.e. the X_i are T-stable closed subschemes of $\mathrm{IG}(2,5)$ for all $0 \leq i \leq 7$. Further, we remark that the $W_m = X_m \setminus X_{m-1}$, $0 \leq m \leq 7$, are precisely the Bialynicki-Birula cells which are the given affine spaces in our situation. The generators of $\Omega^T_*(\mathrm{IG}(2,5))_{\mathbb{Q}}$ are given by $[\widetilde{X}_i \to \mathrm{IG}(2,5)]$ (cf. Corollary 3.46) and we need to compute the weights acting on the tangent space of the fixed points contained in X_i which is done by finding affine stable neighbourhoods on which we can see the weights acting. Next, we give a list of the weights occurring for the different fixed points in their corresponding affine stable neighbourhoods where $\varepsilon_1, \varepsilon_2$ are given as in Example 4.29.

$$x_{45} \text{ in } X_7 \implies \text{weights: } \varepsilon_1 + \varepsilon_2, 2\varepsilon_2, \varepsilon_2, 2\varepsilon_1, \varepsilon_1$$

$$\begin{aligned} x_{35} \text{ in } X_7 &\implies \text{weights: } \varepsilon_2, -\varepsilon_2, 2\varepsilon_1, \varepsilon_1 + \varepsilon_2, \varepsilon_1 - \varepsilon_2 \\ x_{35} \text{ in } X_6 &\implies \text{weights: } \varepsilon_2, 2\varepsilon_1, \varepsilon_1 + \varepsilon_2, \varepsilon_1 - \varepsilon_2 \end{aligned}$$

$$\begin{aligned} x_{34} \text{ in } X_7 &\implies \text{weights: } \varepsilon_1, -\varepsilon_1, \varepsilon_1 + \varepsilon_2, 2\varepsilon_2, \varepsilon_2 - \varepsilon_1 \\ x_{34} \text{ in } X_6 &\implies \text{weights: } \varepsilon_1, \varepsilon_1 + \varepsilon_2, 2\varepsilon_2, \varepsilon_2 - \varepsilon_1 \\ x_{34} \text{ in } X_5 &\implies \text{weights: } \varepsilon_1, \varepsilon_1 + \varepsilon_2, 2\varepsilon_2 \end{aligned}$$

$$\begin{aligned} x_{25} \text{ in } X_7 &\implies \text{weights: } -\varepsilon_2, -2\varepsilon_2, 2\varepsilon_1, \varepsilon_1, \varepsilon_1 - \varepsilon_2 \\ x_{25} \text{ in } X_6 &\implies \text{weights: } -\varepsilon_2, 2\varepsilon_1, \varepsilon_1, \varepsilon_1 - \varepsilon_2 \\ x_{25} \text{ in } X_5 &\implies \text{weights: } 2\varepsilon_1, \varepsilon_1, \varepsilon_1 - \varepsilon_2 \\ x_{25} \text{ in } X_4 &\implies \text{weights: } 2\varepsilon_1, \varepsilon_1, \varepsilon_1 - \varepsilon_2 \end{aligned}$$

$$
\begin{aligned}
x_{23} \text{ in } X_7 &\implies \text{weights: } \varepsilon_1 - \varepsilon_2, -2\varepsilon_2, -\varepsilon_1 - \varepsilon_2, \varepsilon_1, -\varepsilon_1 \\
x_{23} \text{ in } X_6 &\implies \text{singular point} \\
x_{23} \text{ in } X_5 &\implies \text{singular point} \\
x_{23} \text{ in } X_4 &\implies \text{weights: } \varepsilon_1 - \varepsilon_2, \varepsilon_1, -\varepsilon_1 \\
x_{23} \text{ in } X_3 &\implies \text{weights: } \varepsilon_1 - \varepsilon_2, \varepsilon_1
\end{aligned}
$$

$$
\begin{aligned}
x_{14} \text{ in } X_7 &\implies \text{weights: } \varepsilon_2 - \varepsilon_1, -\varepsilon_1, -2\varepsilon_1, 2\varepsilon_2, \varepsilon_2 \\
x_{14} \text{ in } X_6 &\implies \text{weights: } \varepsilon_2 - \varepsilon_1, -\varepsilon_1, 2\varepsilon_2, \varepsilon_2 \\
x_{14} \text{ in } X_5 &\implies \text{singular point} \\
x_{14} \text{ in } X_4 &\implies \text{weights: } \varepsilon_2 - \varepsilon_1, 2\varepsilon_2, \varepsilon_2 \\
x_{14} \text{ in } X_3 &\implies \text{weights: } 2\varepsilon_2, \varepsilon_2 \\
x_{14} \text{ in } X_2 &\implies \text{weights: } 2\varepsilon_2, \varepsilon_2
\end{aligned}
$$

$$
\begin{aligned}
x_{13} \text{ in } X_7 &\implies \text{weights: } \varepsilon_2 - \varepsilon_1, -\varepsilon_1 - \varepsilon_2, -2\varepsilon_1, \varepsilon_2, -\varepsilon_2 \\
x_{13} \text{ in } X_6 &\implies \text{singular point} \\
x_{13} \text{ in } X_5 &\implies \text{singular point} \\
x_{13} \text{ in } X_4 &\implies \text{weights: } \varepsilon_2 - \varepsilon_1, \varepsilon_2, -\varepsilon_2 \\
x_{13} \text{ in } X_3 &\implies \text{singular point} \\
x_{13} \text{ in } X_2 &\implies \text{weights: } \varepsilon_2, -\varepsilon_2 \\
x_{13} \text{ in } X_1 &\implies \text{weights: } \varepsilon_2
\end{aligned}
$$

$$
\begin{aligned}
x_{12} \text{ in } X_7 &\implies \text{weights: } -\varepsilon_1, -\varepsilon_1 - \varepsilon_2, -2\varepsilon_1, -\varepsilon_2, -2\varepsilon_2 \\
x_{12} \text{ in } X_6 &\implies \text{singular point} \\
x_{12} \text{ in } X_5 &\implies \text{singular point} \\
x_{12} \text{ in } X_4 &\implies \text{singular point} \\
x_{12} \text{ in } X_3 &\implies \text{singular point} \\
x_{12} \text{ in } X_2 &\implies \text{weights: } -\varepsilon_2, -2\varepsilon_2 \\
x_{12} \text{ in } X_1 &\implies \text{weights: } -\varepsilon_2 \\
x_{12} \text{ in } X_0 &\implies \text{no weights}
\end{aligned}
$$

Besides that, we have the pullback $i^*[X_7 \to \mathrm{IG}(2,5)] = (1,1,1,1,1,1,1,1)$ for the inclusion $i : \mathrm{IG}(2,5)^T \to \mathrm{IG}(2,5)$. Furthermore, using Definition 6.5 and Lemma 6.6, one can compute the pullbacks of the fixed points which are given by

$$
\begin{aligned}
i^*_{x_{45}}[x_{45} \to \mathrm{IG}(2,5)] &= c_1^T(L_{-\varepsilon_1-\varepsilon_2})c_1^T(L_{-2\varepsilon_2})c_1^T(L_{-\varepsilon_2})c_1^T(L_{-2\varepsilon_1})c_1^T(L_{-\varepsilon_1}) \\
i^*_{x_{35}}[x_{35} \to \mathrm{IG}(2,5)] &= c_1^T(L_{-\varepsilon_2})c_1^T(L_{\varepsilon_2})c_1^T(L_{-2\varepsilon_1})c_1^T(L_{-\varepsilon_1-\varepsilon_2})c_1^T(L_{\varepsilon_2-\varepsilon_1}) \\
i^*_{x_{34}}[x_{34} \to \mathrm{IG}(2,5)] &= c_1^T(L_{-\varepsilon_1})c_1^T(L_{\varepsilon_1})c_1^T(L_{-\varepsilon_1-\varepsilon_2})c_1^T(L_{-2\varepsilon_2})c_1^T(L_{\varepsilon_1-\varepsilon_2}) \\
i^*_{x_{25}}[x_{25} \to \mathrm{IG}(2,5)] &= c_1^T(L_{\varepsilon_2})c_1^T(L_{2\varepsilon_2})c_1^T(L_{-2\varepsilon_1})c_1^T(L_{-\varepsilon_1})c_1^T(L_{\varepsilon_2-\varepsilon_1})
\end{aligned}
$$

$$
\begin{aligned}
i_{x_{23}}^*[x_{23} \to \mathrm{IG}(2,5)] &= c_1^T(L_{\varepsilon_2-\varepsilon_1})c_1^T(L_{2\varepsilon_2})c_1^T(L_{\varepsilon_1+\varepsilon_2})c_1^T(L_{-\varepsilon_1})c_1^T(L_{\varepsilon_1})\\
i_{x_{14}}^*[x_{14} \to \mathrm{IG}(2,5)] &= c_1^T(L_{\varepsilon_1-\varepsilon_2})c_1^T(L_{\varepsilon_1})c_1^T(L_{2\varepsilon_1})c_1^T(L_{-2\varepsilon_2})c_1^T(L_{-\varepsilon_2})\\
i_{x_{13}}^*[x_{13} \to \mathrm{IG}(2,5)] &= c_1^T(L_{\varepsilon_1-\varepsilon_2})c_1^T(L_{\varepsilon_1+\varepsilon_2})c_1^T(L_{2\varepsilon_1})c_1^T(L_{-\varepsilon_2})c_1^T(L_{\varepsilon_2})\\
i_{x_{12}}^*[x_{12} \to \mathrm{IG}(2,5)] &= c_1^T(L_{\varepsilon_1})c_1^T(L_{\varepsilon_1+\varepsilon_2})c_1^T(L_{2\varepsilon_1})c_1^T(L_{\varepsilon_2})c_1^T(L_{2\varepsilon_2})
\end{aligned}
$$

where x_{45} is the most attractive fixed point, i.e. the fixed point whose Bialynicki-Birula cell is open. Lastly, using Lemma 6.6 and by computing the weights on affine stable neighbourhoods of the fixed points, we deduce

$$
\begin{aligned}
i_{x_{12}}^*[X_0 \to \mathrm{IG}(2,5)] &= c_1^T(L_{\varepsilon_1})c_1^T(L_{\varepsilon_1+\varepsilon_2})c_1^T(L_{2\varepsilon_1})c_1^T(L_{\varepsilon_2})c_1^T(L_{2\varepsilon_2})\\
i_{x_{12}}^*[X_1 \to \mathrm{IG}(2,5)] &= c_1^T(L_{\varepsilon_1})c_1^T(L_{\varepsilon_1+\varepsilon_2})c_1^T(L_{2\varepsilon_1})c_1^T(L_{2\varepsilon_2})\\
i_{x_{13}}^*[X_1 \to \mathrm{IG}(2,5)] &= c_1^T(L_{\varepsilon_1-\varepsilon_2})c_1^T(L_{\varepsilon_1+\varepsilon_2})c_1^T(L_{2\varepsilon_1})c_1^T(L_{\varepsilon_2})\\
i_{x_{12}}^*[X_2 \to \mathrm{IG}(2,5)] &= c_1^T(L_{\varepsilon_1})c_1^T(L_{\varepsilon_1+\varepsilon_2})c_1^T(L_{2\varepsilon_1})\\
i_{x_{13}}^*[X_2 \to \mathrm{IG}(2,5)] &= c_1^T(L_{\varepsilon_1-\varepsilon_2})c_1^T(L_{\varepsilon_1+\varepsilon_2})c_1^T(L_{2\varepsilon_1})\\
i_{x_{14}}^*[X_2 \to \mathrm{IG}(2,5)] &= c_1^T(L_{\varepsilon_1-\varepsilon_2})c_1^T(L_{\varepsilon_1})c_1^T(L_{2\varepsilon_1})\\
i_{x_{12}}^*[X_2' \to \mathrm{IG}(2,5)] &= c_1^T(L_{\varepsilon_1+\varepsilon_2})c_1^T(L_{2\varepsilon_1})c_1^T(L_{2\varepsilon_2})\\
i_{x_{13}}^*[X_2' \to \mathrm{IG}(2,5)] &= c_1^T(L_{\varepsilon_1+\varepsilon_2})c_1^T(L_{2\varepsilon_1})c_1^T(L_{\varepsilon_2})\\
i_{x_{23}}^*[X_2' \to \mathrm{IG}(2,5)] &= c_1^T(L_{\varepsilon_1+\varepsilon_2})c_1^T(L_{\varepsilon_1})c_1^T(L_{2\varepsilon_2})
\end{aligned}
$$

where X_2 and X_2' are the two projective planes obtained by attaching the affine planes $X_2 \setminus X_1$ and $X_3 \setminus X_2$ to the projective line X_1, respectively. Therefore, X_3 is the union of the projective spaces X_2 and X_2' meeting in the projective line X_1. Let $\widetilde{X}_3$ be the normalisation of X_3 which is smooth. Then the pullback $i^*[\widetilde{X}_3 \to \mathrm{IG}(2,5)]$ is given by the sum of $i^*[X_2 \to \mathrm{IG}(2,5)]$ and $i^*[X_2' \to \mathrm{IG}(2,5)]$.

In the sequel, we set E_i to be the vector space generated by the first i basis vectors of $\mathbb{C}^5$. For the sake of completeness, we remark that X_0, X_1, X_2 and X_2' are given by

$$
\begin{aligned}
X_0 &= \{x_{12}\},\\
X_1 &= \{V_2 \in \mathrm{IG}(2,5) \mid E_1 \subseteq V_2\},\\
X_2 &= \{V_2 \in \mathrm{IG}(2,5) \mid E_1 \subseteq V_2 \subseteq E_4\},\\
X_2' &= \{V_2 \in \mathrm{IG}(2,5) \mid V_2 \subseteq E_3\}.
\end{aligned}
$$

Now, we will consider the singular subscheme $X_4 \subseteq \mathrm{IG}(2,5)$ which is obtained by attaching the $\mathbb{A}^3 = X_4 \setminus X_3$ containing the fixed point x_{25} to X_3. Geometrically, X_4 can be identified with a cone over a surface with only one singular point x_{12}. The pullback to smooth T-fixed points in X_4 works similar as in the previous cases. Therefore, we only consider the pullback to the singular fixed point x_{12}. One can compute the blow up of the point x_{12} in X_4 explicitly (cf. Section A.2) and check that there are four T-fixed points in the exceptional divisor E. Using Proposition 6.7, we need to compute the weights of the four T-fixed points in $E \subseteq \widetilde{X}_4$. These weights can be seen from the computation directly.

Using Proposition 6.7 and Definition 6.5 leads to

$$\begin{aligned}
i^*_{x_{12}}[\widetilde{X}_4 \to \mathrm{IG}(2,5)] &= e_{x_{12},\mathrm{IG}(2,5)}[\widetilde{X}_4 \to \mathrm{IG}(2,5)] i^*_{x_{12}}[x_{12} \to \mathrm{IG}(2,5)] \\
&= \left(\sum_{\substack{\tilde{x} \in \widetilde{X}_4^T \\ f(\tilde{x}) = x_{12}}} e_{\tilde{x},\widetilde{X}_4}[\widetilde{X}_4 \to \widetilde{X}_4] \right) i^*_{x_{12}}[x_{12} \to \mathrm{IG}(2,5)] \\
&= \frac{c_1^T(L_{\varepsilon_1}) c_1^T(L_{\varepsilon_1+\varepsilon_2}) c_1^T(L_{\varepsilon_2}) c_1^T(L_{2\varepsilon_2})}{c_1^T(L_{-\varepsilon_1}) c_1^T(L_{\varepsilon_2-\varepsilon_1})} \\
&+ \frac{c_1^T(L_{\varepsilon_1+\varepsilon_2}) c_1^T(L_{2\varepsilon_1}) c_1^T(L_{\varepsilon_2}) c_1^T(L_{2\varepsilon_2})}{c_1^T(L_{\varepsilon_1}) c_1^T(L_{\varepsilon_2-\varepsilon_1})} \\
&+ \frac{c_1^T(L_{\varepsilon_1}) c_1^T(L_{\varepsilon_1+\varepsilon_2}) c_1^T(L_{2\varepsilon_1}) c_1^T(L_{\varepsilon_2})}{c_1^T(L_{-\varepsilon_2}) c_1^T(L_{\varepsilon_1-\varepsilon_2})} \\
&+ \frac{c_1^T(L_{\varepsilon_1}) c_1^T(L_{\varepsilon_1+\varepsilon_2}) c_1^T(L_{2\varepsilon_1}) c_1^T(L_{2\varepsilon_2})}{c_1^T(L_{\varepsilon_2}) c_1^T(L_{\varepsilon_1-\varepsilon_2})}.
\end{aligned}$$

We remark that this element reduces to the correct one in Chow rings and that the pullback $i^*_{x_{12}}[\widetilde{X}_4 \to \mathrm{IG}(2,5)]$ is an element in $S(T)_\mathbb{Q}$. Alternatively, one could check that the geometric descriptions of X_4 and $\widetilde{X}_4$ are given by

$$\begin{aligned}
&X_4 = \{V_2 \in \mathrm{IG}(2,5) \mid E_2 \cap V_2 \neq 0\} \text{ and} \\
&\widetilde{X}_4 = \{(V_1, V_2, V_3) \in \mathbb{P}(\mathbb{C}^5) \times \mathrm{IG}(2,5) \times \mathrm{Gr}(3,5) \mid V_1 \subseteq E_2 \subseteq V_3 \subseteq V_1^\perp, V_1 \subseteq V_2 \subseteq V_3\}.
\end{aligned}$$

Now, we consider the closed subscheme $X_5 \subseteq \mathrm{IG}(2,5)$ which is obtained by attaching the cell $\mathbb{A}^3 = X_5 \setminus X_4$ containing the fixed point x_{34} to X_4. A short computation shows that the planes containing x_{12}, x_{13}, x_{14} and x_{12}, x_{13}, x_{23} are singular in X_5. Normalising yields X_4 and $X_4' := X_3 \cup (X_5 \setminus X_4)$. We remark that X_4' is given by the equations $e_4 \wedge e_5 = e_3 \wedge e_5 = e_2 \wedge e_5 = 0$ which implies

$$X_4' = \{V_2 \subseteq \mathbb{C}^5 \text{ isotropic} \mid V_2 \subseteq E_4\}.$$

One may observe that any isotropic subspace V_2 in E_4 has to remain isotropic when considering $\overline{V_2} := (V_2 + E_4^\perp)/E_4^\perp \subseteq E_4/E_4^\perp$, but since $E_4/E_4^\perp = \langle e_2, e_4 \rangle$ holds, we obtain

$$X_4' = \{V_2 \subseteq E_4 \mid V_2 \cap \langle e_1, e_3 \rangle \neq 0\}.$$

We claim that a resolution $\widetilde{X}_4'$ of X_4' is given by

$$\widetilde{X}_4' = \{(V_1, V_2, V_3) \in \mathbb{P}(E_4) \times X_4' \times \mathrm{Gr}(3, E_4) \mid V_1 \subseteq V_2 \cap \langle e_1, e_3 \rangle, V_3 \supseteq V_2 + \langle e_1, e_3 \rangle\}.$$

This is birational to X_4' via the second projection. Now, we consider the map

$$h : \widetilde{X}_4' \to \{(V_1, V_3) \mid V_1 \subseteq \langle e_1, e_3 \rangle, V_3 \supseteq \langle e_1, e_3 \rangle\} = \mathbb{P}^1 \times \mathbb{P}^1$$

which is a $\mathbb{P}^1$-fibration over $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore, $\widetilde{X}_4'$ is smooth and projective. The only singular point in X_4' is x_{13} and thus, we want to compute $i^*_{x_{13}}[\widetilde{X}_4' \to \mathrm{IG}(2,5)]$ using Proposition 6.7.

The T-fixed points in the exceptional divisor are given by

$$(E_1, \langle e_1, e_3\rangle, E_3), (E_1, \langle e_1, e_3\rangle, \langle e_1, e_3, e_4\rangle), (e_3, \langle e_1, e_3\rangle, E_3), (e_3, \langle e_1, e_3\rangle, \langle e_1, e_3, e_4\rangle).$$

Exemplarily, we compute the weights for the first T-fixed point in the exceptional divisor, i.e. for $\widetilde{x_1} := (E_1, \langle e_1, e_3\rangle, E_3)$. Therefore, we consider the morphism h and the tangent space $T_{h(\widetilde{x_1})}\mathbb{P}(e_1, e_3) \times \mathbb{P}(e_2, e_4) = T_{[1:0],[1:0]}\mathbb{P}^1 \times \mathbb{P}^1$ which leads to the weights $-\varepsilon_1$ and $-2\varepsilon_2$. The last weight can be seen in the tangent space $T_{\widetilde{x_1}}(h^{-1}(E_1, E_3)) = T_{[0:1]}\mathbb{P}(e_2, e_3)$. This leads to the weight ε_2. We summarise that the weights of $\widetilde{x_1}$ in $\widetilde{X}_4'$ are given by $-\varepsilon_1, \varepsilon_1$ and ε_2. The weights of the other T-fixed points in the exceptional divisor can be computed similarly. Therefore, for any T-fixed point $x \in X_5$ one can compute

$$i_x^*[\widetilde{X}_5 \to \mathrm{IG}(2,5)] = i_x^*[\widetilde{X}_4 \to \mathrm{IG}(2,5)] + i_x^*[\widetilde{X}_4' \to \mathrm{IG}(2,5)].$$

Lastly, we consider the singular subscheme $X_6 \subseteq X$ which is given by

$$X_6 = \{V_2 \subseteq \mathbb{C}^5 \text{ isotropic} \mid V_2 \cap \langle e_1, e_2, e_3\rangle \neq 0\}.$$

We claim that a resolution $\widetilde{X}_6$ of X_6 is given by

$$\widetilde{X}_6 = \{(V_1, V_2, V_4) \in \mathbb{P}(\mathbb{C}^5) \times X_6 \times \mathrm{Gr}(4,5) \mid V_1 \subseteq V_2 \cap E_3, V_4 \supseteq V_2 + E_3, V_4 \subseteq V_1^{\perp}\}.$$

Again, this is birational to X_6. Now, we want to show smoothness of $\widetilde{X}_6$. We consider the map

$$f : \widetilde{X}_6 \to \{V_4 \supseteq E_3\} = \mathbb{P}^1, (V_1, V_2, V_4) \mapsto V_4$$

whose fibres are

$$f^{-1}(V_4) = \{(V_1, V_2, V_4) \mid V_1 \subseteq E_3, V_1 \subseteq V_2 \subseteq V_4, V_4 \subseteq V_1^{\perp}\}$$

where $V_4 \subseteq V_1^{\perp} \Leftrightarrow V_1 \subseteq V_4^{\perp}$ holds. Consider now the projection

$$g : f^{-1}(V_4) \to \{V_1 \subseteq V_4^{\perp}\} \cong \mathbb{P}^1, (V_1, V_2, V_4) \mapsto V_1$$

which is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$ because $V_4^{\perp}$ is two-dimensional. Thus, $f^{-1}(V_4)$ is smooth and therefore, $\widetilde{X}_6$ is smooth and projective.

Now, we want to apply Proposition 6.7 to obtain the pullback $i_x^*[\widetilde{X}_6 \to \mathrm{IG}(2,5)]$ for the singular T-fixed points $x \in X_6$. The singular T-fixed points in X_6 are x_{12}, x_{13} and x_{23}. The T-fixed points in the exceptional divisor which map to x_{12} are given by (E_1, E_2, E_4) and $(e_2, E_2, \langle E_3, e_5\rangle)$. For the other two singular T-fixed points we obtain three T-fixed points in the exceptional divisor, e.g.

$$(e_1, \langle e_1, e_3\rangle, E_4), (e_3, \langle e_1, e_3\rangle, E_4) \text{ and } (e_3, \langle e_1, e_3\rangle, \langle E_3, e_5\rangle)$$

are the T-fixed points in the fibre of x_{13}. Exemplarily, we compute the weights for one of the T-fixed points in the fibre of x_{12}, i.e. $\tilde{x} := (E_1, E_2, E_4)$. Therefore, we consider the morphism f and the tangent space $T_{f(\tilde{x})}\mathbb{P}^1 = T_{[1:0]}\mathbb{P}(e_4, e_5)$. Thus, we obtain the weight $-\varepsilon_1 + \varepsilon_2$. Next, we need to compute the weights in $T_{\tilde{x}}(f^{-1}(E_4))$. Therefore, we consider the morphism g and the tangent space $T_{g(\tilde{x})}\mathbb{P}^1 = T_{[1:0]}\mathbb{P}(e_1, e_3)$ which leads to the weight $-\varepsilon_1$. Lastly, we consider the tangent space $T_{\tilde{x}}(g^{-1}(E_1))$ where $g^{-1}(E_1)$ is the set of two-dimensional spaces containing e_1 and contained in E_4. Thus, we obtain the last weights from $T_{\tilde{x}}(g^{-1}(E_1)) = T_{[1:0:0]}\mathbb{P}(e_2, e_3, e_4)$. This leads to the weights $-\varepsilon_2$ and $-2\varepsilon_2$. We summarise that the weights of $\tilde{x}$ in $\widetilde{X}_6$ are given by $-\varepsilon_1 + \varepsilon_2, -\varepsilon_1, -\varepsilon_2$ and $-2\varepsilon_2$. Similarly, one can compute all the other weights and apply Proposition 6.7 to finish the computation. This then determines the whole ring structure of $\Omega^*_T(\mathrm{IG}(2,5))_{\mathbb{Q}}$ and allows us to multiply classes.

Remark 6.10. Assuming we could determine the pullback at singular points using the equations given in Example 4.29 and the weights of the tangent spaces at smooth points as in T-equivariant Chow rings (cf. [7, Section 4]), we would be able to determine the class $[\widetilde{X}_4 \to \mathrm{IG}(2,5)]$ uniquely for an arbitrary T-equivariant resolution of singularities $\widetilde{X}_4$ of X_4. A long computation shows that one cannot even determine a unique class in T-equivariant K-theory only using the weights of the tangent spaces at smooth fixed points because of the multiplicative formal group law. In fact it is not even known whether these (not uniquely determined) classes correspond to the resolutions of singularities of X_4. The fact that one cannot determine the class $[\widetilde{X}_4 \to \mathrm{IG}(2,5)]$ uniquely is natural because two different resolutions of singularities determine two different classes in cobordism. For example, one could also consider another resolution of singularities of X_4 given by

$$\widetilde{X}_4^* = \{(V_1, V_2) \in \mathbb{P}(\mathbb{C}^5) \times \mathrm{IG}(2,5) \mid V_1 \subseteq \langle e_1, e_2 \rangle, V_2 \supseteq V_1 \text{ isotropic}\}$$

which is a $\mathbb{P}^2$-fibration over $\mathbb{P}^1$. The exceptional locus of $\widetilde{X}_4^*$ over X_4 is a $\mathbb{P}^1$ over the singular point x_{12}. A computation (cf. Section A.4) shows that the given pullbacks $i^*_{x_{12}}[\widetilde{X}_4 \to \mathrm{IG}(2,5)]$ and $i^*_{x_{12}}[\widetilde{X}_4^* \to \mathrm{IG}(2,5)]$ do not coincide, although they both reduce to the same element in T-equivariant Chow rings and in T-equivariant K-theory (cf. Section A.4).

To finish this section, we finally want to consider some examples in which we can use the refined coefficient ring (cf. Definition 3.58) which was already mentioned several times throughout the thesis. Using our technique, the upcoming examples which are not flag varieties are also new for Chow rings with the refined coefficient ring. Before analysing all the examples in detail, we will describe an algorithm with which one can determine all the occurring weights for the class of smooth projective horospherical varieties of Picard rank one which are not flag varieties.

Algorithm: By [43, Section 1.2-1.4] we know that if a smooth projective horospherical variety X of Picard rank one is not horospherical of rank 1, then X is homogeneous and more precisely, it is a flag variety. First, we recall the notation from Section 5.1. We denote by R^+ and R^- the positive and negative roots, respectively. For a subset $I \subseteq S$ of simple roots, let W_I be the group which is generated by reflections s_α for α in I. In addition, let $P_I = \coprod_{w \in W_I} BwB$ and $R^+_{P_I}$ $(R^-_{P_I})$ be the set of positive (negative) roots that can be written as sums of roots in I. Using [23, Section 9.1] in the case where X is a flag variety, we know that the weights of the B-fixed point in G/P_I are given by $R^- \setminus R^-_{P_I}$. If X is not homogeneous, then X is given by some triple (G, P_Y, P_Z) from Proposition 5.15. Exemplarily, we describe how to determine the weights of the B-fixed point in G/P_Y. The Weyl group action leads to the weights for the other T-fixed points in G/P_Y and a similar computation for G/P_Z concludes the determination of the weights of all T-fixed points in X. Now, we come to the computation of the weights of the B-fixed point in G/P_Y. First, we determine the weights coming from the flag variety G/P_Y using the previously described method. Then we need to determine the weights coming from curves connecting the B-fixed point in G/P_Y with T-fixed points in the other closed orbit G/P_Z. By construction (cf. Section 5.2), there is a T-stable curve connecting the B-fixed points of G/P_Y and G/P_Z. This implies that one weight of the B-fixed point in G/P_Y is given by $\omega_Z - \omega_Y$. Let I and J be the subsets of S associated to the parabolic subgroups P_Y and P_Z, respectively. Then the remaining weights are given by $s_\alpha \cdot (\omega_Z - \omega_Y)$ whenever $s_\alpha \in W_I$ and $s_\alpha \notin W_J$. This is the case because exactly for those s_α, we have $s_\alpha \cdot \omega_Y = \omega_Y$ and $s_\alpha \cdot \omega_Z \neq \omega_Z$. We remark that $s_\beta \cdot \omega_Z = \omega_Z$ and $s_\beta \cdot \omega_Y = \omega_Y$ hold if and only if $s_\beta \in W_I$ and $s_\beta \in W_J$.

Example 6.11. In this example, we investigate all the examples occurring in the classification of Pasquier (cf. Proposition 5.15) including the homogeneous cases (cf. [43, Section 1.3]) except the case of the projective space for which equivariant cobordism was already computed integrally (cf. [34]).

(i) First, we consider $X = \mathrm{IG}(2,5)$ of type (3) and recall the weights of the T-fixed points in X which are given by

$$\begin{aligned}
x_{45} \text{ in } X_7 &\implies \text{weights: } \varepsilon_1+\varepsilon_2, 2\varepsilon_2, \varepsilon_2, 2\varepsilon_1, \varepsilon_1\\
x_{35} \text{ in } X_7 &\implies \text{weights: } \varepsilon_2, -\varepsilon_2, 2\varepsilon_1, \varepsilon_1+\varepsilon_2, \varepsilon_1-\varepsilon_2\\
x_{34} \text{ in } X_7 &\implies \text{weights: } \varepsilon_1, -\varepsilon_1, \varepsilon_1+\varepsilon_2, 2\varepsilon_2, \varepsilon_2-\varepsilon_1\\
x_{25} \text{ in } X_7 &\implies \text{weights: } -\varepsilon_2, -2\varepsilon_2, 2\varepsilon_1, \varepsilon_1, \varepsilon_1-\varepsilon_2\\
x_{23} \text{ in } X_7 &\implies \text{weights: } \varepsilon_1-\varepsilon_2, -2\varepsilon_2, -\varepsilon_1-\varepsilon_2, \varepsilon_1, -\varepsilon_1\\
x_{14} \text{ in } X_7 &\implies \text{weights: } \varepsilon_2-\varepsilon_1, -\varepsilon_1, -2\varepsilon_1, 2\varepsilon_2, \varepsilon_2\\
x_{13} \text{ in } X_7 &\implies \text{weights: } \varepsilon_2-\varepsilon_1, -\varepsilon_1-\varepsilon_2, -2\varepsilon_1, \varepsilon_2, -\varepsilon_2\\
x_{12} \text{ in } X_7 &\implies \text{weights: } -\varepsilon_1, -\varepsilon_1-\varepsilon_2, -2\varepsilon_1, -\varepsilon_2, -2\varepsilon_2.
\end{aligned}$$

Next, we check conditions (i) and (ii) of Definition 3.56 for each of the occuring normal bundles $N_{x_i/X}$. We remark that our convention of taking a sign into the representation L_χ for some weight χ does not change the computations. First, we consider the normal bundle

$$\begin{aligned}
N_{x_{45}/X} &= L_{-\varepsilon_1-\varepsilon_2} \oplus L_{-2\varepsilon_2} \oplus L_{-\varepsilon_2} \oplus L_{-2\varepsilon_1} \oplus L_{-\varepsilon_1}\\
&= (L_{-\varepsilon_1} \oplus L_{-2\varepsilon_1}) \oplus (L_{-\varepsilon_2} \oplus L_{-2\varepsilon_2}) \oplus L_{-\varepsilon_1-\varepsilon_2}.
\end{aligned}$$

Using the notation from Definition 3.56, we see that the χ_i, $1 \leq i \leq 5$ are given by

$$-\varepsilon_1-\varepsilon_2, -2\varepsilon_2, -\varepsilon_2, -2\varepsilon_1, -\varepsilon_1$$

and the ψ_j, $1 \leq j \leq 3$, are given by $-\varepsilon_1-\varepsilon_2, -\varepsilon_1$ and $-\varepsilon_2$. We see that condition (i) is only fulfilled for all χ_i, $1 \leq i \leq 5$, if we invert $p = 2$. Furthermore, for condition (ii) we do not need to invert anything in this case. Similar computations for the other normal bundles show that we do not need to invert more than the prime $p = 2$. Indeed, as mentioned in Remark 6.8, it would have been enough to consider condition (i) in order to determine the classes which were discussed in Section 6, but in this case condition (ii) does not restrict the coefficient ring any further. To summarise, the multiplicative set S_X from Definition 3.58 is given by $S_X = 2\mathbb{Z}$ for the odd symplectic Grassmannian $X = \mathrm{IG}(2,5)$ which implies that we can obtain all the results describing the rational T-equivariant cobordism ring also for the T-equivariant cobordism ring $\Omega_T^*(\mathrm{IG}(2,5))_{\mathbb{Z}[\frac{1}{2}]}$.

(ii) Next, we compute the refined coefficient ring for the smooth projective horospherical G_2-variety X of type (5) from Proposition 5.15. We again need the weights of the T-fixed points in X. It suffices to compute the weights for the B-fixed points in the closed orbits because the weights of the other T-fixed points can be determined using the Weyl group action. First, we consider the first closed orbit

$$G_2/P(\omega_1) = G_2/P_{S\setminus\alpha_1} = G_2/P_{\alpha_2} \cong Y$$

where α_2 denotes the long root. We want to compute the weights of the T-fixed point $P_{\alpha_2}/P_{\alpha_2} \in G_2/P_{\alpha_2}$ in X by determining the weights in the closed orbit Y and the remaining ones using the diagram (5.1). The tangent space $T_{P_{\alpha_2}/P_{\alpha_2}}G_2/P_{\alpha_2}$ is given by $\mathfrak{g}_2/\mathfrak{p}_{\alpha_2}$ where $\mathfrak{g}_2$ and $\mathfrak{p}_{\alpha_2}$ denote the corresponding Lie algebras, respectively. Recall from [23, Section 9.1] that $\mathfrak{u}_{\alpha_2}^- = \mathfrak{g}_2/\mathfrak{p}_{\alpha_2}$ holds where $\mathfrak{u}_{\alpha_2}^-$ is the Lie algebra of the unipotent radical of the opposite parabolic subgroup of P_{α_2}. Using [23, Section 9.1] again we know that the weights of $T_{P_{\alpha_2}/P_{\alpha_2}}G_2/P_{\alpha_2}$ are given by $\alpha \in R^- \setminus R^-_{P_{\alpha_2}}$ where R^- are the negative roots and $R^-_{P_{\alpha_2}}$ are the negative roots generated by α_2. Thus, the weights of $T_{P_{\alpha_2}/P_{\alpha_2}}G_2/P_{\alpha_2}$ are given by

$$\{-\alpha_1, -\alpha_1-\alpha_2, -2\alpha_1-\alpha_2, -3\alpha_1-\alpha_2, -3\alpha_1-2\alpha_2\}.$$

We know that X is of dimension 7 by [16, Section 1.5]. Therefore, we need two more weights coming from T-stable curves connecting the T-fixed point $P_{\alpha_2}/P_{\alpha_2}$ with another T-fixed point in the other closed orbit Z in X. We can determine those using diagram (5.1). By construction there is a T-stable curve connecting the T-fixed points associated to the fundamental weights and hence, one weight of $P_{\alpha_2}/P_{\alpha_2}$ not coming from the closed orbit Y is given by

$$3\alpha_1 + 2\alpha_2 - (2\alpha_1 + \alpha_2) = \alpha_1 + \alpha_2.$$

Lastly, we need to find a reflection s_α, $\alpha \in R$, which leaves the T-fixed point $P_{\alpha_2}/P_{\alpha_2}$ fixed and moves the T-fixed point in the second closed orbit Z. There is only one reflection s_{α_2} which fixes the T-fixed point $P_{\alpha_2}/P_{\alpha_2}$. Thus, the last weight of $P_{\alpha_2}/P_{\alpha_2}$ is given by

$$\begin{aligned} s_{\alpha_2}\cdot(3\alpha_1+2\alpha_2-(2\alpha_1+\alpha_2)) &= s_{\alpha_2}\cdot(3\alpha_1+2\alpha_2) - s_{\alpha_2}\cdot(2\alpha_1+\alpha_2) \\ &= 3\alpha_1+\alpha_2-(2\alpha_1+\alpha_2) \\ &= \alpha_1. \end{aligned}$$

Using the Weyl group action we obtain roots as weights for the other T-fixed points in G_2/P_{α_2}.

The computations for the second closed orbit $G_2/P_{\alpha_1} \cong Z$ can be conducted similarly. The weights of the B-fixed point $P_{\alpha_1}/P_{\alpha_1} \in G_2/P_{\alpha_1}$ coming from the closed orbit Z are given by

$$\{-\alpha_2, -\alpha_1 - \alpha_2, -2\alpha_1 - \alpha_2, -3\alpha_1 - \alpha_2, -3\alpha_1 - 2\alpha_2\}.$$

Additionally, the two remaining weights coming from the T-stable curves connecting $P_{\alpha_1}/P_{\alpha_1}$ with another T-fixed point in Y are $-\alpha_1 - \alpha_2$ and $-2\alpha_1 - \alpha_2$.

Furthermore, we know from [5, Plate IX (VIII)] that the root lattice and the character lattice coincide for G_2 and therefore, condition (i) of Definition 3.56 is fulfilled for all occurring normal bundles without inverting any prime because all the roots are primitive. As opposed to the previous example concerning type (3), we need to invert $p = 2, 3$ in order to fulfill condition (ii) of Definition 3.56. Thus, the multiplicative set from Definition 3.58 is given by $S_X = (2, 3)\mathbb{Z}$. Using Remark 6.8 implies that we can obtain all the results from Section 6 also for the integral T-equivariant cobordism ring $\Omega^*_T(X)_{\mathbb{Z}}$. Contrary to the previous example of type (3), condition (ii) indeed restricts S_X even further. Hence, our main result, Theorem 4.13, is also valid for the T-equivariant cobordism ring $\Omega^*_T(X)_{\mathbb{Z}[\frac{1}{2},\frac{1}{3}]}$.

(iii) Let X be the smooth projective horospherical F_4-variety of type (4) from Proposition 5.15. A computation in Appendix A.5 shows that using the same arguments as in Example 6.11 (ii), one can conclude that condition (i) of Definition 3.56 is fulfilled for any occurring normal bundle without inverting any prime. In order to fulfill condition (ii) of Definition 3.56 for all occurring normal bundles, one has to invert $p = 2$ and thus, $S_X = 2\mathbb{Z}$. Therefore, making use of Remark 6.8, one can again deduce all the results from Section 6 also for the integral T-equivariant cobordism ring $\Omega^*_T(X)_{\mathbb{Z}}$. Furthermore, the main result (cf. Theorem 4.13) holds in particular for the T-equivariant cobordism ring $\Omega^*_T(X)_{\mathbb{Z}[\frac{1}{2}]}$.

(iv) Let X be the smooth projective horospherical variety of type $(B_3, P(\omega_1), P(\omega_3))$ from Proposition 5.15. From [16, Table 2] we know that

$$\dim X = 9,\ \dim G/P(\omega_1) = 5 \ \text{ and } \ \dim G/P(\omega_3) = 6$$

hold. As in Example 6.11 (ii) we know that the weights of the B-fixed point $P_{\{\alpha_2,\alpha_3\}}/P_{\{\alpha_2,\alpha_3\}}$ in $G/P_{\{\alpha_2,\alpha_3\}}$ are given by $\alpha \in R^- \setminus R^-_{P_{\{\alpha_2,\alpha_3\}}}$. Thus, the weights coming from the closed orbit Y are

$$\{-\alpha_1 - \alpha_2 - \alpha_3, -\alpha_1, -\alpha_1 - \alpha_2, -\alpha_1 - 2\alpha_2 - 2\alpha_3, -\alpha_1 - \alpha_2 - 2\alpha_3\}.$$

Doing similar computations to those in Appendix A.5 we obtain the remaining four weights in X which are given by

$$\begin{aligned} \omega_3 - \omega_1 &= \frac{1}{2}(\alpha_1 + 2\alpha_2 + 3\alpha_3) - (\alpha_1 + \alpha_2 + \alpha_3) = \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \\ s_{\alpha_2+\alpha_3}(\omega_3 - \omega_1) &= \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 + \varepsilon_3) \\ s_{\alpha_3}(\omega_3 - \omega_1) &= \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3) \\ s_{\alpha_2+2\alpha_3}(\omega_3 - \omega_1) &= \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3). \end{aligned}$$

Using the Weyl group action, we obtain the weights for the other T-fixed points in $G/P_{\{\alpha_2,\alpha_3\}}$.

Similar computations for the closed orbit $G/P(\omega_3) \cong Z$ lead to the weights

$$-\alpha_1 - \alpha_2 - \alpha_3, -\alpha_2 - \alpha_3, -\alpha_3, -\alpha_1 - 2\alpha_2 - 2\alpha_3, -\alpha_1 - \alpha_2 - 2\alpha_3, -\alpha_2 - 2\alpha_3$$

of the tangent space of $P_{\{\alpha_1,\alpha_2\}}/P_{\{\alpha_1,\alpha_2\}}$ in $G/P_{\{\alpha_1,\alpha_2\}}$. The remaining weights in X are

$$\begin{aligned} \omega_1 - \omega_3 &= \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3) \\ s_{\alpha_1+\alpha_2}(\omega_1 - \omega_3) &= \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 + \varepsilon_3) \\ s_{\alpha_1}(\omega_1 - \omega_3) &= \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3). \end{aligned}$$

Again, using the Weyl group action, we obtain the weights for the other T-fixed points in $G/P_{\{\alpha_1,\alpha_2\}}$.

These computations imply that condition (i) of Definition 3.56 is only fulfilled if we invert $p = 2$. Further, we do not need to invert anything additionally in order to fulfill condition (ii) of Definition 3.56. Similar computations for the other normal bundles imply that it suffices to invert $p = 2$ in order to obtain all our results for $\Omega_T^*(X)_{\mathbb{Z}[\frac{1}{2}]}$.

(v) The last case which is not homogeneous is given by smooth projective horospherical varieties $X(n)$ of type $(B_n, P(\omega_{n-1}), P(\omega_n))$, $n \geq 3$, from Proposition 5.15. Using again [16, Table 2] we know that the relevant dimensions are given by

$$\begin{aligned} \dim X(n) &= \frac{n(n+3)}{2} \\ \dim G/P(\omega_{n-1}) &= \frac{n(n+3)}{2} - 2 \\ \dim G/P(\omega_n) &= \frac{n(n+3)}{2} - n. \end{aligned}$$

As in the previous cases, the weights of the B-fixed point in Y are given by the roots $R^- \setminus R^-_{P_{\{\alpha_1,\dots,\alpha_{n-2},\alpha_n\}}}$. The negative roots R^- are

$$\begin{aligned} -\varepsilon_i &= \sum_{i\leq k\leq n} -\alpha_k \text{ for } 1 \leq i \leq n, \\ -\varepsilon_i + \varepsilon_j &= \sum_{i\leq k<j} -\alpha_k \text{ for } 1 \leq i < j \leq n, \\ -\varepsilon_i - \varepsilon_j &= \sum_{i\leq k<j} -\alpha_k + 2\sum_{j\leq k\leq n} -\alpha_k \text{ for } 1 \leq i < j \leq n. \end{aligned}$$

The cardinality of R^- is n^2 and furthermore, we will check that the cardinality of $R^-_{P_{\{\alpha_1,\dots,\alpha_{n-2},\alpha_n\}}}$ is $\frac{n(n-3)}{2} + 2$. It is obvious that there is only one root of the first type occurring in $R^-_{P_{\{\alpha_1,\dots,\alpha_{n-2},\alpha_n\}}}$ which is $-\alpha_n$. In addition, the only roots from the second type which do not lie in $R^-_{P_{\{\alpha_1,\dots,\alpha_{n-2},\alpha_n\}}}$ are given by $-\varepsilon_i + \varepsilon_n$

for $1 \leq i < n$, i.e. there are $\binom{n}{2} - (n-1)$ negative roots of the second type in $R^-_{P_{\{\alpha_1,\ldots,\alpha_{n-2},\alpha_n\}}}$. Lastly, there is no root from the third type in $R^-_{P_{\{\alpha_1,\ldots,\alpha_{n-2},\alpha_n\}}}$. Hence, we have

$$\begin{aligned} |R^-_{P_{\{\alpha_1,\ldots,\alpha_{n-2},\alpha_n\}}}| &= 1 + \left(\binom{n}{2} - (n-1)\right) \\ &= \frac{n(n-1)}{2} - n + 2 \end{aligned}$$

and thus, we conclude

$$\begin{aligned} |R^- \setminus R^-_{P_{\{\alpha_1,\ldots,\alpha_{n-2},\alpha_n\}}}| &= n^2 - \left(\frac{n(n-1)}{2} - n + 2\right) \\ &= \frac{n(n+3)}{2} - 2. \end{aligned}$$

Of course, this must coincide with the dimension of $Y \cong G/P(\omega_{n-1})$. Using the same methods as in previous examples, the remaining two weights of the B-fixed point of $G/P(\omega_{n-1})$ in $X(n)$ are given by

$$\begin{aligned} \omega_n - \omega_{n-1} &= \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n) - (\varepsilon_1 + \cdots + \varepsilon_{n-1}) \\ &= \frac{1}{2}(-\varepsilon_1 - \cdots - \varepsilon_{n-1} + \varepsilon_n) \\ s_{\alpha_n}(\omega_n - \omega_{n-1}) &= \frac{1}{2}(-\varepsilon_1 - \cdots - \varepsilon_n). \end{aligned}$$

Next, we consider the closed orbit $Z \cong G/P(\omega_n)$. Again, the weights of the B-fixed point in Z are given by $R^- \setminus R^-_{P_{\{\alpha_1,\ldots,\alpha_{n-1}\}}}$. The weights in $R^-_{P_{\{\alpha_1,\ldots,\alpha_{n-1}\}}}$ are of the form $-\varepsilon_i + \varepsilon_j$ for $1 \leq i < j \leq n$. Thus, we verify

$$|R^- \setminus R^-_{P_{\{\alpha_1,\ldots,\alpha_{n-1}\}}}| = n^2 - \binom{n}{2} = \frac{n(n+3)}{2} - n.$$

As in the previous case, we have one more weight which is given by

$$\omega_{n-1} - \omega_n = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n).$$

One can check that the remaining $n-1$ weights are exactly obtained by acting with the reflections $s_{\varepsilon_i - \varepsilon_n}$ for $1 \leq i \leq n-1$ because these reflections fix the fundamental weight ω_n associated to the closed orbit Z and move the fundamental weight ω_{n-1} associated to the closed orbit Y. Hence, the remaining weights are

$$s_{\varepsilon_i - \varepsilon_n}(\omega_{n-1} - \omega_n) = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{i-1} - \varepsilon_i + \varepsilon_{i+1} + \cdots + \varepsilon_n)$$

for all $1 \leq i \leq n-1$.

In both closed orbits, the Weyl group action leads to the weights of the other T-fixed points.

These computations imply again that condition (i) of Definition 3.56 is only fulfilled if we invert $p = 2$. Further, we do not need to invert anything additionally in order to fulfill condition (ii) of Definition 3.56. Similar computations for the other normal bundles imply that it suffices to invert $p = 2$ in order to obtain all our results for $\Omega_T^*(X(n))_{\mathbb{Z}[\frac{1}{2}]}$.

(vi) We are left with the homogeneous varieties in the classification of Pasquier [43, Theorem 1.7]. Case 1 of this theorem is of type $(A_m, \alpha_1, \alpha_m)$ for $m \geq 2$. In this case X is isomorphic to $\mathrm{SO}_{2m+2}/P(\omega_1)$ (cf. [43, Proposition 1.8]) which is again isomorphic to the orthogonal Grassmannian $\mathrm{OG}(1, 2m+2)$. This flag variety is of type D_{m+1} and thus, we will use this root system (cf. [5, Plate IV]). This Grassmannian is of dimension $2m$ and the weights of the B-fixed point are again given by $R^- \setminus R^-_{P_{\{\alpha_2,\dots,\alpha_{m+1}\}}}$. This complement is given by the negative roots $\{-\varepsilon_1 \pm \varepsilon_j\}$ for $2 \leq j \leq m+1$. The Weyl group action leads to the weights of the other T-fixed points. The connection index (cf. [5, Plate IV (VIII)]) is 4 and thus, we need to invert $p = 2$ in order to fulfill condition (i) of Definition 3.56 for all occurring normal bundles. Further, condition (ii) will be fulfilled for all normal bundles of Definition 3.56 after having inverted $p = 2$ and hence, we obtain all our results also for $\Omega_T^*(\mathrm{SO}_{2m+2}/P(\omega_1))_{\mathbb{Z}[\frac{1}{2}]}$ with $m \geq 2$.

(vii) Case 2 of Pasquier's [43, Theorem 1.7] is of type $(A_m, \alpha_i, \alpha_{i+1})$ for $m \geq 3$ and $1 \leq i \leq m-1$. By [43, Proposition 1.9], the resulting variety X is isomorphic to $\mathrm{Gr}(i+1, m+2) \cong \mathrm{SL}_{m+2}/P(\omega_{i+1})$. This flag variety is of type A_{m+1} and thus, we use [5, Plate I] in order to determine the weights of the B-fixed point. One can check that there are $m + 1 - i^2 + im$ weights which are again given by $R^- \setminus R^-_{P_{\{\alpha_1,\dots,\alpha_i,\alpha_{i+2},\dots,\alpha_{m+1}\}}}$. The Weyl group action leads to the weights of the other T-fixed points. Further, the connection index is $m+2$ (cf. [5, Plate I (VIII)]) and thus, we need to invert all primes occurring in the prime decomposition of $m + 2 = p_1^{\ell_1} \cdots p_k^{\ell_k}$ in order to fulfill condition (i) of Definition 3.56 for all normal bundles of the T-fixed points in $\mathrm{Gr}(i+1, m+2)$. In addition, condition (ii) will be fulfilled for all normal bundles of Definition 3.56 after having inverted the primes $p_1, \dots, p_k$ and hence, we obtain our results also for $\Omega_T^*(\mathrm{Gr}(i+1, m+2))_{\mathbb{Z}[\frac{1}{p_1},\dots,\frac{1}{p_k}]}$ with $m \geq 3$ and $1 \leq i \leq m-1$.

(viii) The last remaining case of [43, Theorem 1.7] is of type $(D_m, \alpha_{m-1}, \alpha_m)$ for some $m \geq 4$. By [43, Proposition 1.10], the variety X is isomorphic to the spinor variety $\mathrm{Spin}(2m+1)/P(\omega_m)$. This flag variety is of type B_m and the weights were already computed in Example 6.11 (v) because the orbit Z of the smooth projective horospherical variety of Picard number one of type $(B_m, P(\omega_{m-1}), P(\omega_m))$ coincides with the spinor variety $\mathrm{Spin}(2m+1)/P(\omega_m)$ for $m \geq 4$.

Remark 6.12. In the previous example, it was shown that one can refine the coefficient ring in the main results of this thesis for the odd symplectic Grassmannian $\mathrm{IG}(2, 5)$. Indeed, the computations in Example 6.11 (i) are similar for any odd symplectic Grassmannian $\mathrm{IG}(m, 2n+1)$ with integers $2 \leq m \leq n$ such that we can also prove our main results, Theorem 4.13 and Proposition 6.7, for the T-equivariant cobordism ring $\Omega_T^*(\mathrm{IG}(m, 2n+1))_{\mathbb{Z}[\frac{1}{2}]}$. The author did not check the previously mentioned results for $\mathrm{IG}(m, 2n+1)$ integrally which are non-trivial.

Remark 6.13. For the computations in Example 6.11 one sometimes has to take into account that the group Spin(n) is a two-fold covering group of SO(n) (cf. [40, Section 24 i.]) which implies that some characters in a maximal torus of Spin(n) might not be characters in a maximal torus of SO(n). Thus, we need to invert $p = 2$ in all those cases, but the computations show that this is already necessary in all the relevant cases. Therefore, all the previous computations are correct as they are stated.

A Appendix

In this chapter, we outsource several long computations which are not included in the main text.

A.1 T-stable curves in the flag variety G_2/P_α

Using the notation as in Example 5.8, we want to compute the degrees of the T-stable curves in G_2/P_α where α denotes the short simple root. We recall that the two generators s_α and s_β of the Weyl group of G_2 correspond to s and $sr = r^5 s$ in the dihedral group D_6, respectively, where s is the reflection at the y-axis and r the counterclockwise rotation by an angle of $\pi/3$. Further, one can verify that

$$\begin{aligned} s_{3\alpha+\beta} &= s_\alpha s_\beta s_\alpha \\ s_{\alpha+\beta} &= s_\beta s_\alpha s_\beta \\ s_{2\alpha+\beta} &= s_\alpha s_\beta s_\alpha s_\beta s_\alpha \\ s_{3\alpha+2\beta} &= s_\beta s_\alpha s_\beta s_\alpha s_\beta \end{aligned}$$

hold in the $W(G_2)$. In this case we have $R^+_{P_\alpha} = \{\alpha\}$ and thus, every class in W/W_α corresponds to one reflection indexed by a positive root where the trivial class in W/W_α corresponds to the root α. From Example 5.8 we know that there is a T-stable curve between each pair of the T-fixed points in G_2/P_α. Therefore, we have 15 T-stable curves. We start with the ones connecting $x(1)$ and $x(s_\gamma)$ for $\gamma \in R^+ \backslash \{\alpha\}$ which have degree $d(\gamma)$ by Lemma 5.3. We compute the degree of these curves using the definition in Section 5.1. We recall that the two simple roots are given by $\alpha = \varepsilon_1 - \varepsilon_2$ and $\beta = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$. This leads to the following degree

$$\begin{aligned} d(\beta) &= n_{\beta\beta} \frac{(\beta,\beta)}{(\beta,\beta)} \sigma(s_\beta) = 1 \cdot \sigma(s_\beta) \\ d(\alpha+\beta) &= n_{(\alpha+\beta)\beta} \frac{(\beta,\beta)}{(\alpha+\beta,\alpha+\beta)} \sigma(s_\beta) = 3 \cdot \sigma(s_\beta) \\ d(2\alpha+\beta) &= n_{(2\alpha+\beta)\beta} \frac{(\beta,\beta)}{(2\alpha+\beta,2\alpha+\beta)} \sigma(s_\beta) = 3 \cdot \sigma(s_\beta) \\ d(3\alpha+\beta) &= n_{(3\alpha+\beta)\beta} \frac{(\beta,\beta)}{(3\alpha+\beta,3\alpha+\beta)} \sigma(s_\beta) = 1 \cdot \sigma(s_\beta) \\ d(3\alpha+2\beta) &= n_{(3\alpha+2\beta)\beta} \frac{(\beta,\beta)}{(3\alpha+2\beta;3\alpha+2\beta)} \sigma(s_\beta) = 2 \cdot \sigma(s_\beta). \end{aligned}$$

We remark that $x(u) = x(u')$ if $u = u' \in W/W_\alpha$. Thus, acting on some T-fixed point $x(u), u \in W/W_\alpha$, by the reflection s_α will not change the T-fixed point. Now, we act with s_β on the curves and obtain 4 new T-stable curves, namely the curves connecting $x(s_\beta)$ and one of the T-fixed points

$$\begin{aligned} x(s_\beta s_{\alpha+\beta}) &= x(s_\alpha s_\beta) = x(s_{3\alpha+\beta}) \\ x(s_\beta s_{2\alpha+\beta}) &= x(s_\beta s_\alpha s_\beta s_\alpha s_\beta s_\alpha) = x(s_{3\alpha+2\beta}) \\ x(s_\beta s_{3\alpha+\beta}) &= x(s_\beta s_\alpha s_\beta s_\alpha) = x(s_{\alpha+\beta}) \\ x(s_\beta s_{3\alpha+2\beta}) &= x(s_\beta s_\beta s_\alpha s_\beta s_\alpha s_\beta) = x(s_{2\alpha+\beta}) \end{aligned}$$

which have degrees $3, 3, 1$ and 2, respectively, by Remark 5.5. Next, we act on the original 5 curves with $s_{\alpha+\beta}$ which leads to 3 new T-stable curves connecting $x(s_{\alpha+\beta})$ and one of the T-fixed points

$$\begin{aligned} x(s_{\alpha+\beta}s_{2\alpha+\beta}) &= x((s_\beta s_\alpha)^4) = x((s_\alpha s_\beta)^2) = x(s_{2\alpha+\beta}) \\ x(s_{\alpha+\beta}s_{3\alpha+\beta}) &= x((s_\beta s_\alpha)^3) = x(s_{3\alpha+2\beta}) \\ x(s_{\alpha+\beta}s_{3\alpha+2\beta}) &= x(s_\alpha s_\beta) = x(s_{3\alpha+\beta}) \end{aligned}$$

which have degrees $3, 1$ and 2, respectively. Similarly, we obtain the two new T-stable curves connecting $x(s_{2\alpha+\beta})$ and one of the T-fixed points

$$\begin{aligned} x(s_{2\alpha+\beta}s_\beta) &= x((s_\alpha s_\beta)^3) = x(s_{3\alpha+2\beta}) \\ x(s_{2\alpha+\beta}s_{3\alpha+\beta}) &= x(s_\alpha s_\beta) = x(s_{3\alpha+\beta}) \end{aligned}$$

and one last T-stable curve connecting $x(s_{3\alpha+\beta})$ and

$$x(s_{3\alpha+\beta}s_{\alpha+\beta}) = x((s_\alpha s_\beta)^3) = x(s_{3\alpha+2\beta}).$$

These three curves have degrees $1, 1$ and 3, respectively. We summarise that there are 6 T-stable curves of degree 1, 6 T-stable curves of degree 3 and 3 T-stable curves of degree 2 in G_2/P_α.

A.2 Blow up of X_4

In this section, we compute the mentioned blow up of the closed subscheme X_4 in the T-filtration of the odd symplectic Grassmannian $\mathrm{IG}(2, 5)$. One can see from Example 6.9 that x_{12} is the only singular T-fixed point and even the only singular point in the closed subscheme X_4. An intuitive way of thinking about the closed subscheme X_4 is to identify it with a cone over a surface where the latter contains four T-fixed points. Therefore, the exceptional divisor should also contain four T-fixed points. Now, we would like to determine the weights of the T-fixed points in the exceptional divisor of the blow up in order to use Proposition 6.7. Intuitively, one can simply take the weights of e.g. the smooth T-fixed point x_{13} in X_4 and determine the weight which comes from the line connecting it with the singular T-fixed point x_{12} in X_4. In this case, the weights are given by $\varepsilon_2 - \varepsilon_1, \varepsilon_2$ and $-\varepsilon_2$ where the one coming from the line connecting x_{13} and x_{12} is ε_2. This weight should have the opposite sign for the corresponding T-fixed point in the exceptional divisor and therefore, the educated guess would now be that there is one T-fixed point in the exceptional divisor with weights $\varepsilon_2 - \varepsilon_1, -\varepsilon_2$ and $-\varepsilon_2$. After we will have done the full computation concerning the blow up, we will see that this guess will be confirmed. This procedure can be done for all four smooth T-fixed points in X_4.

We consider an affine neighbourhood of the point x_{12} in $\mathrm{IG}(2, 5)$ which is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a_3/a_1 & b_3/b_2 \\ a_4/a_1 & b_4/b_2 \\ a_5/a_1 & b_5/b_2 \end{pmatrix}$$

where $a_4/a_1 = b_5/b_2$ holds because of the antisymmetric form.

We introduce new coordinates

$$\begin{aligned} x_1 &:= a_3/a_1 \\ x_2 &:= a_5/a_1 \\ x_3 &:= b_3/b_2 \\ x_4 &:= b_4/b_2 \\ x_5 &:= b_5/b_2 \end{aligned}$$

whose weights are given by $-\varepsilon_1, -2\varepsilon_1, -\varepsilon_2, -2\varepsilon_2$ and $-\varepsilon_1 - \varepsilon_2$, respectively. Restricting this affine neighbourhood to X_4 we obtain

$$Y = \operatorname{Spec} \frac{k[x_1, x_2, x_3, x_4, x_5]}{(x_1x_4 - x_3x_5, x_1x_5 - x_3x_2, x_5^2 - x_4x_2)}$$

whose Jacobian shows that $(0,0,0,0,0)$ is the only singular point in Y. The blow up is given by

$$\widetilde{Y} = \left\{ (x_1, x_2, x_3, x_4, x_5), [y_1 : y_2 : y_3 : y_4 : y_5] \in \mathbb{A}^5 \times \mathbb{P}^4 \;\middle|\; \begin{array}{l} x_iy_j = x_jy_i \ \forall i,j \\ x_5^2 - x_4x_2 = 0 \\ x_1x_5 - x_3x_2 = 0 \\ x_1x_4 - x_3x_5 = 0 \end{array} \right\}.$$

First, we work on the affine open $y_1 = 1$ and away from the exceptional set, such that $x_2 = x_1y_2, x_3 = x_1y_3, x_4 = x_1y_4$ and $x_5 = x_1y_5$ hold. For $x_1 \neq 0$ this leads to $y_4 = y_3^2y_2$ and $y_5 = y_3y_2$ which implies that there is only one T-fixed point in this chart, namely $(0,0,0,0,0), [1:0:0:0:0] =: \widetilde{x_{23}}$ which is in the exceptional divisor. The affine T-stable neighbourhood of $\widetilde{x_{23}}$ is given by

$$\mathbb{A}^3_{\widetilde{x_{23}}} = \left\{ (x_1, 0,0,0,0); \left[1 : \frac{y_2}{y_1} : \frac{y_3}{y_1} : \frac{y_3^2y_2}{y_1^3} : \frac{y_3y_2}{y_1^2} \right] \in \mathbb{A}^5 \times \mathbb{P}^4 \right\}$$

which leads to the weights $-\varepsilon_1, -\varepsilon_1$ and $-\varepsilon_2 + \varepsilon_1$.

Similarly, there is only one T-fixed point $(0,0,0,0,0), [0:1:0:0:0] =: \widetilde{x_{25}}$ for the affine chart $y_2 = 1$ with affine T-stable neighbourhood

$$\mathbb{A}^3_{\widetilde{x_{25}}} = \left\{ (0, x_2, 0,0,0); \left[\frac{y_1}{y_2} : 1 : \frac{y_1y_5}{y_2^2} : \frac{y_5^2}{y_2^2} : \frac{y_5}{y_1} \right] \in \mathbb{A}^5 \times \mathbb{P}^4 \right\}$$

which leads to the weights $-2\varepsilon_1, \varepsilon_1$ and $\varepsilon_1 - \varepsilon_2$.

Similar computations lead to the affine T-stable neighbourhood

$$\mathbb{A}^3_{\widetilde{x_{13}}} = \left\{ (0,0,x_3,0,0); \left[\frac{y_1}{y_3} : \frac{y_1^2y_4}{y_3^3} : 1 : \frac{y_4}{y_3} : \frac{y_1y_4}{y_3^2} \right] \in \mathbb{A}^5 \times \mathbb{P}^4 \right\}$$

of the T-fixed point $\widetilde{x_{13}} := (0,0,0,0,0), [0:0:1:0:0]$ in the chart $y_3 = 1$ and for which the weights are given by $-\varepsilon_2, -\varepsilon_1 + \varepsilon_2$ and $-\varepsilon_2$.

The last T-fixed point $\widetilde{x_{14}} := (0,0,0,0,0), [0:0:0:1:0]$ is in the chart given by

$y_4 = 1$. An affine T-stable neighbourhood is

$$\mathbb{A}^3_{\widetilde{x_{14}}} = \left\{(0,0,0,x_4,0); \left[\frac{y_3 y_5}{y_4^2} : \frac{y_5^2}{y_4^2} : \frac{y_3}{y_4} : 1 : \frac{y_5}{y_3}\right] \in \mathbb{A}^5 \times \mathbb{P}^4\right\}$$

which leads to the weights $-2\varepsilon_2, \varepsilon_2$ and $-\varepsilon_1 + \varepsilon_2$.

Lastly, there is no T-fixed point in the chart given by $y_5 = 1$. This concludes the computations because it is enough to consider the blow up of an open affine neighbourhood of the only singular point x_{12} in X_4. We also verified that our intuitive approach at the beginning of this section is confirmed by verifying the weights of $\widetilde{x_{13}}$.

A.3 Class of $\widetilde{X}_4$ in different equivariant cohomology theories

In this section, we will compare the fundamental class of $[X_4 \to \mathrm{IG}(2,5)]$ in different equivariant cohomology theories. We will start by analysing the class in equivariant Chow rings using Proposition 4.10 and assuming that we could only compute the equivariant multiplicities at smooth T-fixed points. For the given pullback map $i^* : \mathrm{CH}(\mathrm{IG}(2,5))_{\mathbb{Q}} \to \mathrm{CH}(\mathrm{IG}(2,5)^T)_{\mathbb{Q}}$, this would lead to

$$i^*[\widetilde{X}_4] = (f_{12}, 2(\varepsilon_1+\varepsilon_2)\varepsilon_1, 2\varepsilon_1\varepsilon_1, 2\varepsilon_2(\varepsilon_1+\varepsilon_2), 2\varepsilon_2\varepsilon_2, 0, 0, 0).$$

The equations coming from Proposition 4.10 containing f_{12} are given by (cf. Example 4.29)

$$\begin{aligned}
f_{12} &\equiv 0 \mod \varepsilon_1+\varepsilon_2,\\
f_{12} &\equiv f_{23} \equiv f_{25} \mod 2\varepsilon_1,\\
f_{12} &\equiv f_{13} \equiv f_{14} \mod 2\varepsilon_2,\\
f_{12} - 2f_{23} + f_{25} &\equiv 0 \mod 2\varepsilon_1^2,\\
f_{12} - 2f_{13} + f_{14} &\equiv 0 \mod 2\varepsilon_2^2.
\end{aligned}$$

These equations uniquely determine $f_{12} = (\varepsilon_1+\varepsilon_2)(2\varepsilon_1+2\varepsilon_2)$. It can be easily seen that all equations are satisfied. Lastly, we consider the equation for $i^*_{x_{12}}[\widetilde{X}_4]$ from Example 6.9 being reduced to Chow rings which is given by

$$\begin{aligned}
i^*_{x_{12}}[\widetilde{X}_4] =& \frac{\varepsilon_1(\varepsilon_1+\varepsilon_2)\varepsilon_2 2\varepsilon_2}{(-\varepsilon_1)(\varepsilon_2-\varepsilon_1)} + \frac{(\varepsilon_1+\varepsilon_2)2\varepsilon_1\varepsilon_2 2\varepsilon_2}{\varepsilon_1(\varepsilon_2-\varepsilon_1)}\\
&+ \frac{\varepsilon_1(\varepsilon_1+\varepsilon_2)2\varepsilon_1\varepsilon_2}{(-\varepsilon_2)(\varepsilon_1-\varepsilon_2)} + \frac{\varepsilon_1(\varepsilon_1+\varepsilon_2)2\varepsilon_1 2\varepsilon_2}{\varepsilon_2(\varepsilon_1-\varepsilon_2)}\\
=& \frac{(\varepsilon_1+\varepsilon_2)\varepsilon_2 2\varepsilon_2}{\varepsilon_1-\varepsilon_2} - \frac{(\varepsilon_1+\varepsilon_2)2\varepsilon_2 2\varepsilon_2}{\varepsilon_1-\varepsilon_2} - \frac{\varepsilon_1(\varepsilon_1+\varepsilon_2)2\varepsilon_1}{\varepsilon_1-\varepsilon_2} + \frac{2\varepsilon_1(\varepsilon_1+\varepsilon_2)2\varepsilon_1}{\varepsilon_1-\varepsilon_2}\\
=& \frac{(\varepsilon_1+\varepsilon_2)(2\varepsilon_2^2-2\varepsilon_1^2)}{\varepsilon_1-\varepsilon_2} + \frac{(\varepsilon_1+\varepsilon_2)(4\varepsilon_1^2-4\varepsilon_2^2)}{\varepsilon_1-\varepsilon_2}\\
=& \frac{2(\varepsilon_1+\varepsilon_2)(\varepsilon_2-\varepsilon_1)(\varepsilon_2+\varepsilon_1)}{\varepsilon_1-\varepsilon_2} + \frac{4(\varepsilon_1+\varepsilon_2)(\varepsilon_1-\varepsilon_2)(\varepsilon_1+\varepsilon_2)}{\varepsilon_1-\varepsilon_2}\\
=& -2(\varepsilon_1+\varepsilon_2)^2 + 4(\varepsilon_1+\varepsilon_2)^2\\
=& 2(\varepsilon_1+\varepsilon_2)^2
\end{aligned}$$

coinciding with the expression for f_{12} above.

Therefore, we see that we sometimes do not need the equivariant multiplicities at singular fixed points in order to compute classes in Chow rings. This situation will be completely different as soon as we compute in equivariant cohomology theories with more complicated formal group laws.

Next, we will have a look at the same situation in K-theory. Similarly, we would have

$$\begin{aligned}
&i^*_{x_{12}}[\widetilde{X}_4] = f_{12}\\
&i^*_{x_{13}}[\widetilde{X}_4] = c_1^T(L_{\varepsilon_1+\varepsilon_2})c_1^T(L_{2\varepsilon_1})\\
&i^*_{x_{14}}[\widetilde{X}_4] = c_1^T(L_{\varepsilon_1})c_1^T(L_{2\varepsilon_1})\\
&i^*_{x_{23}}[\widetilde{X}_4] = c_1^T(L_{2\varepsilon_2})c_1^T(L_{\varepsilon_1+\varepsilon_2})\\
&i^*_{x_{25}}[\widetilde{X}_4] = c_1^T(L_{\varepsilon_2})c_1^T(L_{2\varepsilon_2})\\
&i^*_{x_{34}}[\widetilde{X}_4] = i^*_{x_{35}}[\widetilde{X}_4] = i^*_{x_{45}}[\widetilde{X}_4] = 0
\end{aligned}$$

for the pullback $i^* : K_T^*(\mathrm{IG}(2,5))_{\mathbb{Q}} \to K_T^*(\mathrm{IG}(2,5)^T)_{\mathbb{Q}}$. In this case, we remark that $c_1^T(L_{\varepsilon_1+\varepsilon_2}) = c_1^T(L_{\varepsilon_1}) + c_1^T(L_{\varepsilon_2}) - \beta c_1^T(L_{\varepsilon_1})c_1^T(L_{\varepsilon_2})$ holds by the multiplicative formal group law. The equations are again given by

$$\begin{aligned}
&f_{12} \equiv f_{45} \mod c_1^T(L_{\varepsilon_1+\varepsilon_2}),\\
&f_{12} \equiv f_{23} \equiv f_{25} \mod c_1^T(L_{2\varepsilon_1}),\\
&f_{12} \equiv f_{13} \equiv f_{14} \mod c_1^T(L_{2\varepsilon_2}),\\
&(f_{12} - f_{23}) + \rho_{1/2}c_1^T(L_{2\varepsilon_1})(f_{25} - f_{12}) \equiv 0 \mod c_1^T(L_{2\varepsilon_1})^2,\\
&(f_{12} - f_{13}) + \rho_{1/2}c_1^T(L_{2\varepsilon_2})(f_{14} - f_{12}) \equiv 0 \mod c_1^T(L_{2\varepsilon_2})^2.
\end{aligned}$$

First, we need to determine $\rho_{1/2}c_1^T(L_{2\varepsilon_1}) \mod c_1^T(L_{2\varepsilon_1})^2$. In order to simplify the notation, we set $x := c_1^T(L_{\varepsilon_1}), y := c_1^T(L_{\varepsilon_2}), z := c_1^T(L_{2\varepsilon_1})$ and $z' := c_1^T(L_{2\varepsilon_2})$. By Remark 3.35 we have $\rho_{1/2}z = \frac{x}{z}$ and therefore, we need to express x in terms of z. By the multiplicative formal group law we have $x(2-\beta x) = z$ which leads to

$$x = \frac{1}{2}z + \frac{\beta}{8}z^2 + ...$$

which implies $\rho_{1/2}z \equiv \frac{1}{2} + \frac{\beta}{8}z \mod z^2$. Replacing $z = x(2-\beta x)$ leads to the epxression $\rho_{1/2}z \equiv \frac{1}{2} + \frac{\beta}{4}x \mod z^2$ because $x^2 \equiv 0 \mod z^2$. Now, we can start with the computation by considering the first equation $f_{12} \equiv 0 \mod c_1^T(L_{\varepsilon_1+\varepsilon_2})$ from which it follows that $f_{12} = c_1^T(L_{\varepsilon_1+\varepsilon_2}) \cdot a$ where $\deg a = 1$. Next, we have

$$f_{23} = (2y - \beta y^2)(x + y - \beta xy) \equiv (2y - \beta y^2)y \mod z$$

since $x \equiv 0 \mod z$ due to the fact that we consider rational coefficients. We conclude that

$$f_{12} \equiv (x + y - \beta xy)a \equiv ya \mod z$$

which implies that $a \equiv (2y - \beta y^2) \mod z$. This leads to $a = z' + z \cdot b$ with $\deg b = 0$.

The equation $f_{12} \equiv f_{25} \mod z$ will be fulfilled for any choice of b and hence, we do not get further restrictions on f_{12}. Further, we have

$$f_{13} = (x + y - \beta xy)(2x - \beta x^2) \equiv xz \mod z'$$

and

$$f_{12} = (x + y - \beta xy)(z' + z \cdot b) \equiv xz \cdot b \mod z'$$

which implies that $b = 1 + z' \cdot c$ with $\deg c = -1$. By now, we know that f_{12} is of the form

$$f_{12} = (x + y - \beta xy))a = (x + y - \beta xy)(z' + zb) = (x + y - \beta xy)(z' + z + zz'c)$$

with $\deg c = -1$. It remains to check the two surface equations. We have

$$\begin{aligned} f_{12} &= (x + y - \beta xy)(2y - \beta y^2 + 2x - \beta x^2 + (2y - \beta y^2)(2x - \beta x^2)c) \\ &\equiv (x + y - \beta xy)(2y - \beta y^2 + 2x + (2y - \beta y^2)2xc) && \mod z^2 \\ &\equiv 2y^2 - \beta y^3 + 2xy + 4xy^2 c - 2\beta xy^3 c + 2xy - \beta xy^2 - 2\beta xy^2 + \beta^2 xy^3 && \mod z^2 \end{aligned}$$

which leads to the equation

$$\begin{aligned} &(f_{12} - f_{23}) + \rho_{1/2} z(f_{25} - f_{12}) \\ \equiv&(2y^2 - \beta y^3 + 2xy + 4xy^2 c - 2\beta xy^3 c + 2xy - \beta xy^2 - 2\beta xy^2 + \beta^2 xy^3) \\ &- (2y - \beta y^2)(x + y - \beta xy) + \left(\frac{1}{2} + \frac{\beta}{4}x\right)\Big(y(2y - \beta y^2) \\ &-(2y^2 - \beta y^3 + 2xy + 4xy^2 c - 2\beta xy^3 c + 2xy - \beta xy^2 - 2\beta xy^2 + \beta^2 xy^3)\Big) \mod z^2 \\ \equiv&\frac{1}{2}\Big((2y^2 - \beta y^3 + 2xy + 4xy^2 c - 2\beta xy^3 c) + (2xy - \beta xy^2) + (-2\beta xy^2 + \beta^2 xy^3)\Big) \\ &- \frac{\beta}{2}xy^2 + \frac{\beta^2}{4}xy^3 - 2xy + \beta xy^2 - 2y^2 + \beta y^3 + 2\beta xy^2 - \beta^2 xy^3 \\ &+ y^2 - \frac{\beta}{2}y^3 + \frac{\beta}{2}xy^2 - \frac{\beta^2}{4}xy^3 \mod z^2 \\ \equiv&(1 - 2 + 1)y^2 + \left(-\frac{\beta}{2} + \beta - \frac{\beta}{2}\right)y^3 + (1 + 1 - 2)xy \\ &+ \left(2c - \frac{\beta}{2} - \beta - \frac{\beta}{2} + \beta + 2\beta + \frac{\beta}{2}\right)xy^2 + \left(-\beta c + \frac{\beta^2}{2} + \frac{\beta^2}{4} - \beta^2 - \frac{\beta^2}{4}\right)xy^3 \mod z^2 \\ \equiv&\left(2c + \frac{3\beta}{2}\right)xy^2 + \left(-\frac{\beta^2}{2} - \beta c\right)xy^3 \mod z^2 \end{aligned}$$

and lastly, we obtain

$$\begin{aligned} f_{12} &= (x + y - \beta xy)\Big(2y - \beta y^2 + 2x - \beta x^2 + (2y - \beta y^2)(2x - \beta x^2)c\Big) \\ &\equiv 2xy - \beta x^2 y - 2\beta x^2 y + \beta^2 x^3 y + 2x^2 - \beta x^3 + 2xy + 4x^2 yc - 2\beta x^3 yc \mod (z')^2 \end{aligned}$$

and hence, using the identity $\rho_{1/2}z' = \left(\frac{1}{2} + \frac{\beta}{4}y\right)$, we have

$$
\begin{aligned}
&(f_{12} - f_{13}) + \rho_{1/2}z'(f_{14} - f_{12}) \\
\equiv&\frac{1}{2}\left(2xy - \beta x^2y - 2\beta x^2 y + \beta^2 x^3 y + 2x^2 - \beta x^3 + 2xy + 4x^2yc - 2\beta x^3 yc\right) \\
&- (x + y - \beta xy)(2x - \beta x^2) + \left(\frac{1}{2} + \frac{\beta}{4}y\right)\left(x(2x - \beta x^2)\right) \\
&- \frac{\beta}{4}y\left(2x^2 - \beta x^3\right) \mod (z')^2 \\
\equiv&\frac{1}{2}\left(2xy - \beta x^2y - 2\beta x^2 y + \beta^2 x^3 y + 2x^2 - \beta x^3 + 2xy + 4x^2yc - 2\beta x^3 yc\right) \\
&- 2x^2 - 2xy + 2\beta x^2 y + \beta x^3 + \beta x^2 y - \beta^2 x^3 y + x^2 - \frac{\beta}{2}x^3 + \frac{\beta}{2}x^2 y - \frac{\beta^2}{4}x^3 y \\
&- \frac{\beta}{2}x^2 y + \frac{\beta^2}{4}x^3 y \mod (z')^2 \\
\equiv&(1 - 2 + 1)x^2 + \left(-\frac{\beta}{2} + \beta - \frac{\beta}{2}\right)x^3 + (1 + 1 - 2)xy \\
&+ \left(-\frac{3\beta}{2} + 2c + 2\beta + \beta + \frac{\beta}{2} - \frac{\beta}{2}\right)x^2 y + \left(\frac{\beta^2}{2} - \beta c - \beta^2 - \frac{\beta^2}{4} + \frac{\beta^2}{4}\right)x^3 y \mod (z')^2 \\
\equiv&\left(\frac{3\beta}{2} + 2c\right)x^2 y + \left(-\frac{\beta^2}{2} - \beta c\right)x^3 y \mod (z')^2.
\end{aligned}
$$

The previous two equations give more constraints on c, but they are far away from being uniquely determined. Therefore, we cannot only use the equations and the equivariant multiplicities at smooth fixed points in order to determine the class of some resolution of singularities of X_4 uniquely in rational T-equivariant K-theory.

A.4 Comparison of $[\widetilde{X}_4^*]$ and $[\widetilde{X}_4]$ in different cohomology theories

Next, we compare the class of $\widetilde{X}_4^*$ from Remark 6.10 with the previously described class of $\widetilde{X}_4$. Recall that $\widetilde{X}_4^*$ is given by

$$\widetilde{X}_4^* = \{(V_1, V_2) \in \mathbb{P}(\mathbb{C}^5) \times \mathrm{IG}(2,5) \mid V_1 \subseteq \langle e_1, e_2\rangle, V_2 \supseteq V_1 \text{ isotropic}\}$$

which is a $\mathbb{P}^2$-fibration over $\mathbb{P}^1$. The exceptional locus of $\widetilde{X}_4^*$ over X_4 is a $\mathbb{P}^1$ over the singular point x_{12} and the two T-fixed points in the exceptional divisor are $x_1 := (e_1, E_2)$ and $x_2 := (e_2, E_2)$. We consider the $\mathbb{P}^2$-fibration

$$g : \widetilde{X}_4^* \to \{(V_1) \subseteq \langle e_1, e_2\rangle\} = \mathbb{P}^1$$

and the tangent space $T_{g(x_1)}\mathbb{P}(e_1, e_2) = T_{[1:0]}\mathbb{P}^1$ which leads to the first weight $\varepsilon_2 - \varepsilon_1$. The remaining two weights can be seen in the tangent space $T_{x_1}(g^{-1}(e_1)) = T_{[1:0:0]}\mathbb{P}(e_2, e_3, e_4)$. This leads to the weights $-\varepsilon_2$ and $-2\varepsilon_2$. Similarly, we obtain the weights $\varepsilon_1 - \varepsilon_2, -\varepsilon_1$ and $-2\varepsilon_1$ for the T-fixed point x_2.

Let $i_{x_{12}} : x_{12} \to \mathrm{IG}(2,5)$ be the inclusion of the T-fixed point x_{12}. First, we determine the class $[\widetilde{X}_4^*]$ in rational T-equivariant Chow rings. We only need to compute

$$\begin{aligned} i_{x_{12}}^*[\widetilde{X}_4^*] &= \frac{\varepsilon_1(\varepsilon_1+\varepsilon_2)2\varepsilon_1\varepsilon_2 2\varepsilon_2}{(\varepsilon_1-\varepsilon_2)\varepsilon_2 2\varepsilon_2} + \frac{\varepsilon_1(\varepsilon_1+\varepsilon_2)2\varepsilon_1\varepsilon_2 2\varepsilon_2}{(\varepsilon_2-\varepsilon_1)\varepsilon_1 2\varepsilon_1} \\ &= (\varepsilon_1+\varepsilon_2)\frac{2\varepsilon_1^2-2\varepsilon_2^2}{\varepsilon_1-\varepsilon_2} \\ &= 2(\varepsilon_1+\varepsilon_2)^2 \end{aligned}$$

which is the same as $i_{x_{12}}^*[\widetilde{X}_4]$ in rational T-equivariant Chow rings.

Similarly, we want to compare the two classes $[\widetilde{X}_4]$ and $[\widetilde{X}_4^*]$ in rational T-equivariant K-theory. We set $x := c_1^T(L_{\varepsilon_1})$ and $y := c_1^T(L_{\varepsilon_2})$ as in previous computations. We remark that $c_1^T(L_{\varepsilon_2-\varepsilon_1}) = a \cdot c_1^T(L_{\varepsilon_1-\varepsilon_2})$ for some infinite power series $a \in K_T^*(\mathrm{IG}(2,5))_{\mathbb{Q}}$ which is invertible because the constant term is -1. This leads to the already simplified expression

$$\begin{aligned} i_{x_{12}}^*[\widetilde{X}_4^*] &= \frac{x(2x-\beta x^2)(x+y-\beta xy)}{c_1^T(L_{\varepsilon_1-\varepsilon_2})} + \frac{y(2y-\beta y^2)(x+y-\beta xy)}{a\cdot c_1^T(L_{\varepsilon_1-\varepsilon_2})} \\ &= (x+y-\beta xy)\frac{x(2x-\beta x^2)+a^{-1}y(2y-\beta y^2)}{c_1^T(L_{\varepsilon_1-\varepsilon_2})}. \end{aligned}$$

We remark that we obtain $c_1^T(L_{-\varepsilon_1}) = b\cdot c_1^T(L_{\varepsilon_1})$ and $c_1^T(L_{-\varepsilon_2}) = c\cdot c_1^T(L_{\varepsilon_2})$ for infinite power series $b, c \in K_T^*(\mathrm{IG}(2,5))_{\mathbb{Q}}$ which are again invertible due to their constant terms -1. Following [37, Section 4.2.1] we have

$$[-1]_m(u) = \frac{-u}{1-\beta u}$$

where $[-1]_m$ denotes the inverse in the multiplicative formal group law. Thus, we obtain $b^{-1} = -1+\beta x$ and $c^{-1} = -1+\beta y$.

Next, we consider the equation for $i_{x_{12}}^*[\widetilde{X}_4]$ from Example 6.9 being reduced to K-theory which is then given by

$$\begin{aligned} i_{x_{12}}^*[\widetilde{X}_4] =& \frac{x(x+y-\beta xy)y(2y-\beta y^2)}{xb(c_1^T(L_{\varepsilon_1-\varepsilon_2}))a} + \frac{(x+y-\beta xy)(2x-\beta x^2)y(2y-\beta y^2)}{xc_1^T(L_{\varepsilon_1-\varepsilon_2})a} \\ &+ \frac{x(x+y-\beta xy)(2x-\beta x^2)y}{ycc_1^T(L_{\varepsilon_1-\varepsilon_2})} + \frac{x(x+y-\beta xy)(2x-\beta x^2)(2y-\beta y^2)}{yc_1^T(L_{\varepsilon_1-\varepsilon_2})} \\ =&(x+y-\beta xy)\left(\frac{a^{-1}b^{-1}y(2y-\beta y^2)+(2-\beta x)y(2y-\beta y^2)a^{-1}}{c_1^T(L_{\varepsilon_1-\varepsilon_2})}\right. \\ &\left.+\frac{c^{-1}x(2x-\beta x^2)+x(2x-\beta x^2)(2-\beta y)}{c_1^T(L_{\varepsilon_1-\varepsilon_2})}\right) \\ =&\frac{(x+y-\beta xy)}{c_1^T(L_{\varepsilon_1-\varepsilon_2})}\Big(x(2x-\beta x^2)(-1+\beta y+2-\beta y) \\ &+a^{-1}y(2y-\beta y^2)(-1+\beta x+2-\beta x)\Big) \\ =&\frac{(x+y-\beta xy)}{c_1^T(L_{\varepsilon_1-\varepsilon_2})}\Big(x(2x-\beta x^2)+a^{-1}y(2y-\beta y^2)\Big). \end{aligned}$$

This implies $i^*_{x_{12}}[\widetilde{X}_4] = i^*_{x_{12}}[\widetilde{X}^*_4]$ and therefore, the classes $i^*[\widetilde{X}_4]$ and $i^*[\widetilde{X}^*_4]$ coincide in $K^*_T(\mathrm{IG}(2,5)^T)_{\mathbb{Q}}$.

Lastly, we consider those two classes in cobordism. We set again $x := c_1^T(L_{\varepsilon_1})$ and $y := c_1^T(L_{\varepsilon_2})$, but this time these are Chern classes in equivariant cobordism. Similarly, we have $c_1^T(L_{\varepsilon_2-\varepsilon_1}) = a \cdot c_1^T(L_{\varepsilon_1-\varepsilon_2})$ for some invertible infinite power series $a \in \Omega^*_T(\mathrm{IG}(2,5))_{\mathbb{Q}}$. This leads to

$$\begin{aligned} i^*_{x_{12}}[\widetilde{X}^*_4] &= \frac{xF_\Omega(x,x)F_\Omega(x,y)}{c_1^T(L_{\varepsilon_1-\varepsilon_2})} + \frac{yF_\Omega(y,y)F_\Omega(x,y)}{a\cdot c_1^T(L_{\varepsilon_1-\varepsilon_2})} \\ &= F_\Omega(x,y)\frac{xF_\Omega(x,x)+a^{-1}yF_\Omega(y,y)}{F_\Omega(x,[-1]_\Omega y)}. \end{aligned}$$

Again, we have $c_1^T(L_{-\varepsilon_1}) = b\cdot c_1^T(L_{\varepsilon_1})$ and $c_1^T(L_{-\varepsilon_2}) = c\cdot c_1^T(L_{\varepsilon_2})$ for invertible infinite power series $b, c \in \Omega^*_T(\mathrm{IG}(2,5))_{\mathbb{Q}}$. Following [37, Section 2.5.1] we have

$$[-1]_\Omega x = -x + a_{11}x^2 - a_{11}^2x^3 + (a_{11}^3 + a_{11}a_{21} + 2a_{31} - a_{22})x^4 + \text{higher order terms}$$

where $[-1]_\Omega$ denotes the inverse in the universal formal group law. Thus, one can check that we obtain

$$\begin{aligned} b^{-1} &= -1 - a_{11}x + (a_{22} - a_{11}a_{21} - 2a_{31})x^3 + \text{higher order terms} \\ c^{-1} &= -1 - a_{11}y + (a_{22} - a_{11}a_{21} - 2a_{31})y^3 + \text{higher order terms.} \end{aligned}$$

Further, we have

$$\begin{aligned} a^{-1} &= -1 - a_{11}c_1^T(L_{\varepsilon_1-\varepsilon_2}) + (a_{22} - a_{11}a_{21} - 2a_{31})c_1^T(L_{\varepsilon_1-\varepsilon_2})^3 + \text{higher order terms} \\ &= -1 - a_{11}F(x,[-1]_\Omega y) + (a_{22} - a_{11}a_{21} - 2a_{31})F(x,[-1]_\Omega y)^3 + \text{higher order terms} \\ &= -1 - a_{11}(x + cy + a_{11}xcy + ...) + (a_{22} - a_{11}a_{21} - 2a_{31})(x + cy + a_{11}xcy + ...)^3 \\ &\quad + \text{higher order terms} \end{aligned}$$

Now, we consider the equation for $i^*_{x_{12}}[\widetilde{X}_4]$ from Example 6.9 which is given by

$$\begin{aligned} i^*_{x_{12}}[\widetilde{X}_4] =& \frac{xF_\Omega(x,y)yF_\Omega(y,y)}{xbF_\Omega(x,[-1]_\Omega y)a} + \frac{F_\Omega(x,y)F_\Omega(x,x)yF_\Omega(y,y)}{xF_\Omega(x,[-1]_\Omega y)a} \\ &+ \frac{xF_\Omega(x,y)F_\Omega(x,x)y}{ycF_\Omega(x,[-1]_\Omega y)} + \frac{xF_\Omega(x,y)F_\Omega(x,x)F_\Omega(y,y)}{yF_\Omega(x,[-1]_\Omega y)} \\ =& \frac{F_\Omega(x,y)}{F_\Omega(x,[-1]_\Omega y)}\Big(xF_\Omega(x,x)(c^{-1} + (2 + a_{11}y + 2a_{21}y^2 + (2a_{31} + a_{22})y^3 + ...)) \\ &+ yF_\Omega(y,y)a^{-1}(b^{-1} + (2 + a_{11}x + 2a_{21}x^2 + (2a_{31} + a_{22})x^3 + ...))\Big) \\ =& \frac{F_\Omega(x,y)}{F_\Omega(x,[-1]_\Omega y)}\Big(xF_\Omega(x,x)(1 + 2a_{21}y^2 + (2a_{22} - a_{11}a_{21})y^3 + ...) \\ &+ yF_\Omega(y,y)a^{-1}(1 + 2a_{21}x^2 + (2a_{22} - a_{11}a_{21})x^3 + ...)\Big). \end{aligned}$$

In order to have the equality $i^*_{x_{12}}[\widetilde{X}_4] = i^*_{x_{12}}[\widetilde{X}^*_4]$, the expression

$$\begin{aligned}
&xF_\Omega(x,x)(2a_{21}y^2+(2a_{22}-a_{11}a_{21})y^3+...)\\
&\ +yF_\Omega(y,y)a^{-1}(2a_{21}x^2+(2a_{22}-a_{11}a_{21})x^3+...)\\
=&xF_\Omega(x,x)(2a_{21}y^2+(2a_{22}-a_{11}a_{21})y^3+...)\\
&\ +yF_\Omega(y,y)(-1-a_{11}(x+cy+a_{11}xcy+...)\\
&\ +(a_{22}-a_{11}a_{21}-2a_{31})(x+cy+a_{11}xcy+...)^3+...)(2a_{21}x^2+(2a_{22}-a_{11}a_{21})x^3+...)
\end{aligned}$$

would have to vanish. We now compare the coefficients of the monomial x^2y^3 and keep in mind that

$$c=-1+a_{11}y-a_{11}^2y^2+\text{higher order terms}$$

holds. This leads to

$$x(2x)\left(2a_{22}-a_{11}a_{21}y^3\right)+2a_{21}x^2\left(y(2y)(-a_{11}(-1)y)+y(a_{11}y^2)(-1)\right)=4a_{22}x^2$$

which implies that $i^*_{x_{12}}[\widetilde{X}_4]\neq i^*_{x_{12}}[\widetilde{X}_4^*]$ holds in cobordism and therefore, the classes $i^*[\widetilde{X}_4]$ and $i^*[\widetilde{X}_4^*]$ do not coincide in cobordism.

A.5 Refined coefficient ring for the horospherical F_4-variety

In this section, we want to compute the multiplicative set S_X from Definition 3.58 for the smooth projective horospherical F_4-variety X of type (4) from Proposition 5.15. A similar computation has been done in Example 6.11 (ii) for the smooth projective horospherical G_2-variety of type (5). As in the just mentioned example, we want to compute the weights of the T-fixed points in X. Let R be the root system of F_4, S the simple roots and s_α the corresponding reflection in the Weyl group for some root $\alpha\in R$. The simple roots of F_4 are given by

$$\alpha_1=\varepsilon_2-\varepsilon_3,\ \alpha_2=\varepsilon_3-\varepsilon_4,\ \alpha_3=\varepsilon_4,\ \alpha_4=\frac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)$$

by [5, Plate VIII (II)]. Furthermore, recall [5, Plate VIII (II)] that the positive roots are

$$\varepsilon_i\ (1\leq i\leq 4),\ \varepsilon_i\pm\varepsilon_j\ (1\leq i<j\leq 4),\ \frac{1}{2}(\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4).$$

For later use, we need to express all the 24 positive roots in terms of the simple roots:

$$\begin{aligned}
\varepsilon_1&=\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4\\
\varepsilon_2&=\alpha_1+\alpha_2+\alpha_3\\
\varepsilon_3&=\alpha_2+\alpha_3\\
\varepsilon_4&=\alpha_3\\
\varepsilon_1+\varepsilon_2&=2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4\\
\varepsilon_1-\varepsilon_2&=\alpha_2+2\alpha_3+2\alpha_4\\
\varepsilon_1+\varepsilon_3&=\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4\\
\varepsilon_1-\varepsilon_3&=\alpha_1+\alpha_2+2\alpha_3+2\alpha_4\\
\varepsilon_1+\varepsilon_4&=\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4\\
\varepsilon_1-\varepsilon_4&=\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4
\end{aligned}$$

$$\begin{aligned}
\varepsilon_2 + \varepsilon_3 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 \\
\varepsilon_2 - \varepsilon_3 &= \alpha_1 \\
\varepsilon_2 + \varepsilon_4 &= \alpha_1 + \alpha_2 + 2\alpha_3 \\
\varepsilon_2 - \varepsilon_4 &= \alpha_1 + \alpha_2 \\
\varepsilon_3 + \varepsilon_4 &= \alpha_2 + 2\alpha_3 \\
\varepsilon_3 - \varepsilon_4 &= \alpha_2 \\
\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4 \\
\frac{1}{2}(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4) &= \alpha_2 + 2\alpha_3 + \alpha_4 \\
\frac{1}{2}(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4) &= \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 \\
\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4) &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 \\
\frac{1}{2}(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4) &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\
\frac{1}{2}(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4) &= \alpha_2 + \alpha_3 + \alpha_4 \\
\frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4) &= \alpha_3 + \alpha_4 \\
\frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4) &= \alpha_4.
\end{aligned}$$

We first consider the closed orbit

$$F_4/P(\omega_2) = F_4/P_{\{S\setminus\alpha_2\}} = F_4/P_{\{\alpha_1,\alpha_3,\alpha_4\}} \cong Y.$$

Using [23, Section 9.1] the weights of the B-fixed point in $F_4/P_{\{\alpha_1,\alpha_3,\alpha_4\}}$ are given by $R^- \setminus R^-_{P_{\{\alpha_1,\alpha_3,\alpha_4\}}}$ where R^- and $R^-_{P_{\{\alpha_1,\alpha_3,\alpha_4\}}}$ are the negative roots and the negative roots generated by $\{\alpha_1, \alpha_3, \alpha_4\}$, respectively. Thus, the weights are given by all the negative roots except $\{-\alpha_1, -\alpha_3, -\alpha_4, -\alpha_3 - \alpha_4\}$. This also implies that Y has dimension 20. As in Example 6.11 (ii) we recall (cf. [16, Section 1.5]) that X has dimension 23. Thus, we need 3 more weights not coming from the closed orbit Y. Using diagram (5.1) we know that there is a T-stable curve connecting the T-fixed points $P_{\{\alpha_1,\alpha_3,\alpha_4\}}/P_{\{\alpha_1,\alpha_3,\alpha_4\}} \in Y$ and $P_{\{\alpha_1,\alpha_2,\alpha_4\}}/P_{\{\alpha_1,\alpha_2,\alpha_4\}} \in Z$. The corresponding weight for the B-fixed point in Y is given by

$$\begin{aligned}
\omega_3 - \omega_2 &= (2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4) - (3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 4\alpha_4) \\
&= -\alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4 \\
&= \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4)
\end{aligned}$$

where ω_3 and ω_2 are the corresponding fundamental weights. Again, we need to find reflections s_α such that the B-fixed point in Y stays fixed. The only reflections with this property are $s_{\alpha_1}, s_{\alpha_3}, s_{\alpha_4}$ and $s_{\alpha_3+\alpha_4}$. Further, the reflections s_{α_1} and s_{α_4} also leave the B-fixed point in Z fixed. Thus, the only remaining weights of the B-fixed point in Y are given by

$$\begin{aligned}
s_{\alpha_3} \cdot (\omega_3 - \omega_2) &= s_{\alpha_3} \cdot (2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4) - (3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 4\alpha_4) \\
&= s_{\alpha_3} \cdot \left(\frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4)\right) \\
&= \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \\
&= -\alpha_1 - 2\alpha_2 - 3\alpha_3 - \alpha_4
\end{aligned}$$

and

$$\begin{aligned}
s_{\alpha_3+\alpha_4} \cdot (\omega_3 - \omega_2) &= s_{\alpha_3+\alpha_4} \cdot (2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4) - (3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 4\alpha_4) \\
&= s_{\alpha_3+\alpha_4} \cdot \left(\frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4)\right) \\
&= -\varepsilon_1 \\
&= -\alpha_1 - 2\alpha_2 - 3\alpha_3 - 2\alpha_4.
\end{aligned}$$

The weights for the other T-fixed points can be obtained by the Weyl group action.

Similarly, the weights for the B-fixed point in the closed orbit Z are given by all the negative roots except $\{-\alpha_1, -\alpha_2, -\alpha_4, -\alpha_1 - \alpha_2\}$ and the weights

$$\begin{aligned}
\omega_2 - \omega_3 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 \\
s_{\alpha_2}(\omega_2 - \omega_3) &= \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 \\
s_{\alpha_1+\alpha_2}(\omega_2 - \omega_3) &= \alpha_2 + 2\alpha_3 + \alpha_4
\end{aligned}$$

coming from the T-stable curves connecting the two closed orbits.

All the weights occurring are indeed roots and since the root lattice and the character lattice coincide for F_4 (cf. [5, Plate VIII (VIII)]), we conclude that condition (i) of Definition 3.56 is fulfilled for all occurring normal bundles without inverting any prime. Condition (ii) of Definition 3.56 is only fulfilled after inverting $p = 2$. We do not need to invert $p = 3$, even though this coefficient appears in some of the roots, because for any pair of primitive characters, one can choose the remaining two basis elements such that one can avoid having to invert $p = 3$. Hence, the multiplicative set from Definition 3.58 is given by $S_X = 2\mathbb{Z}$.

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