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Chow–Witt Groups, Tensor Triangular Geometry, and Milnor–Witt Motives

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Can't is a word none should speak without blushing.

Edgar Albert Guest

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Chapter 1

Introduction

1.1 Motivation

Chow–Witt groups were introduced by Barge and Morel in [BM00]. The motivating question when a projective A -module is isomorphic to the direct sum $Q \oplus A$ for some projective A -module Q led to the construction of the *Euler class* $\tilde{c}_n(E)$ of a vector bundle E . It is a refinement of the top Chern class $c_n(E)$ in the theory of Chow groups, see e.g. [Ful98]. To answer the above question, one first asks whether this Euler class is 0, see [BM00, Fas08].

Let X be a regular scheme. The refinement from Chow groups $\mathrm{CH}(X)$ to Chow–Witt groups $\widetilde{\mathrm{CH}}(X)$ includes replacing Milnor K -groups at certain points of the construction with Minor–Witt K -groups. In degree zero, this is equivalent to replacing K_0 -groups by Grothendieck–Witt groups GW .

We apply the idea of such refinements, i.e. of “decorating with tilde”, to the construction of tensor triangular Chow groups $\mathrm{CH}_\Delta(\mathcal{T})$ introduced in [Bal13] to define *tensor triangular Chow–Witt groups* $\widetilde{\mathrm{CH}}_\Delta(\mathcal{T})$, where \mathcal{T} is a sufficiently “nice” tensor triangulated category. This way, we obtain a generalization of tensor triangular Chow groups, but, on the other hand, these groups generalize the definition of classical Chow–Witt groups of a scheme as introduced in [BM00]. This refinement clears the way for new areas apart from algebraic geometry. Thus, it is only natural to ask what Chow–Witt groups of tensor triangulated categories such as the stable module category kG -stab might be for initial cases of k and G . In future work, a refinement of the Euler class would be an interesting direction. For this, the notion of a vector bundle would have to be generalized to the tensor triangular setting first.

In [BC+22], Bachmann, Calmès, Déglise, Fasel, and Østvær refine Voevodsky’s derived category of motives $\mathrm{DM}(k; R)$ (see [MVW06, Voe00]) and construct the *derived category of Milnor–Witt motives* $\widetilde{\mathrm{DM}}(k; R)$. In a first attempt to understand the structure of this category, restriction to Milnor–Witt motives of 0-dimensional schemes – the *Artin Milnor–Witt motives* – will already yield interesting results that are motivated by the non-oriented results from [BG23a]. These results then serve as a basis to compute first Balmer spectra of subcategories of

$\widetilde{\text{DM}}(k; R)$, in this case of the subcategory of Artin Milnor–Witt motives. Other subcategories are promising topics for future work. For instance, first steps in the direction of Tate Milnor–Witt motives are taken in [FY23].

1.2 Outline

We provide an outline of this work.

To begin with, **Chapter 2** introduces the reader to Balmer’s tensor triangular geometry ([Bal02, Bal05b, Bal07]). Basic concepts of rigid tensor triangulated categories, such as the Balmer spectrum and filtration by dimension of support, are recalled. In particular, we prove that certain constructions such as shifted dualities, restriction to filtration components, idempotent completion, and a decomposition statement behave well with duality (e.g. Lemma 2.3.5, Lemma 2.3.7). Moreover, we recall that a rigid tensor triangulated category can be turned into a triangulated category with duality (Example 2.2.6 (i)).

In **Chapter 3**, the theory of (Grothendieck–)Witt groups in the setting of triangulated categories with duality ([Bal00, Wal03, Sch17]) and of exact categories with duality (and weak equivalences, respectively) ([Sch10a, Sch10b, FS09]) is recalled. We focus on the various localization statements by Balmer and Walter, as well as by Schlichting, Fasel, and Srinivas, where the latter two proved these sequences of the different settings to be equivalent (Theorem 3.3.4).

Building the core of this work’s first part, **Chapter 4** gives the central construction of tensor triangular Chow–Witt groups after tensor triangular Chow groups ([Bal13, Kl16a]) and classical Chow–Witt groups ([BM00, Fas08]) are recalled. In Section 4.3, tensor triangular oriented cocycles, coboundaries, and finally Chow–Witt groups $\widehat{\text{CH}}_\Delta$ are defined (Definition 4.3.3, Definition 4.3.5) using the localization sequence of (Grothendieck–)Witt groups of triangulated categories with duality. The central theorem (Theorem 4.4.5) then follows mandatorily in Section 4.4: We show that for the rigid tensor triangulated category $\text{D}^{\text{perf}}(X)$, X being a regular scheme, its tensor triangular Chow–Witt group coincides with the classical Chow–Witt group of X introduced by Barge and Morel. The main tool here will be the comparison between the filtration by dimension of support in the tensor triangular setting and the filtration by codimension of homological support in the algebro-geometric setting. After presenting a computation for $X = \text{Spec}(k)$ by hand, we prove functorial properties of the given construction in Section 4.5. Dropping some assumptions on the tensor triangulated category, one can obtain a more general definition of tensor triangular Chow–Witt groups, but at the price of introducing models. We sketch this idea in Section 4.6.

Tensor triangular geometry grants, in particular, access to the field of representation theory. Hence, **Chapter 5** discusses the application of tensor triangular Chow–Witt groups to the stable module category $kG\text{-stab}$ for a field k dividing the order of G . At first, basic notions and results are recalled. In Section 5.2, the most prominent result is the computation of the Grothendieck Witt group of

kG -stab as $\mathrm{GW}(kG\text{-mod})$ divided by the subgroup generated by projective kG -modules (Theorem 5.2.3). Deriving a dévissage result from various results in the literature (Proposition 5.2.6), we will moreover be able to reduce the calculation to a quotient of the Grothendieck–Witt group of the base field. We prove the same results for Witt groups.

Let $p \neq 2$ and k of characteristic p . The first concrete example of $\widetilde{\mathrm{CH}}_{\Delta}(kG\text{-stab})$ treats the cyclic group $G = \mathbb{Z}/p^n\mathbb{Z}$. Using previous results, we show that the 0-th tensor triangular Chow–Witt group of kG -stab is $\mathbb{Z}/p^n\mathbb{Z}$ and vanishes in all other degrees if k is finite or algebraically closed (Example 5.3.3, Example 5.3.4). In particular, in these cases, Chow and Chow–Witt groups coincide. For $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, the 0-th Chow–Witt group is $\mathbb{Z}/p^2\mathbb{Z}$ and all degrees $\neq 1$ vanish if k is algebraically closed or finite (Example 5.5.2, Example 5.5.3), just as in the case of Chow groups. For degree 1, we give partial results.

From **Chapter 6** on, we dive into the world of motives. First, Section 6.1 reviews the construction of Voevodsky’s derived category of motives $\mathrm{DM}(k; R)$ following [MVW06, Voe00]. Then, we recall the main results and tools of [BG23a] in Section 6.2, which essentially breaks down to the equivalence of the derived categories of Artin motives $\mathrm{DAM}(k; R)$, of permutation modules $\mathrm{D}\mathrm{Perm}(\Gamma; R)$, and of cohomological Mackey functors $\mathrm{D}(\mathrm{Mack}_R^{\mathrm{coh}}(\Gamma))$ (Corollary 6.2.12).

In **Chapter 7**, we begin to set up the stage to refine the results from [BG23a] to the oriented setting. For this, Section 7.1 recalls the construction of the derived category of Milnor–Witt motives $\widetilde{\mathrm{DM}}(k; R)$ from [BC+22] before investigating the special case of finite Milnor–Witt correspondences generated by 0-dimensional schemes $\widetilde{\mathrm{Cor}}_k^0$ in Section 7.2. Then, Section 7.3 establishes a first refinement of [BG23a], namely the equivalence between the tensor triangulated categories $\widetilde{\mathrm{DAM}}^{\mathrm{gm}}(k; R)$ and $\mathrm{K}_b((\widetilde{\mathrm{Cor}}_k^0)^{\sharp})$ (Proposition 7.3.3). The general version for non-geometric motives is shown in the case that Conjecture 7.3.10 holds (Corollary 7.3.12).

The remaining equivalence fitting into the diagram

$$\begin{array}{ccc} \widetilde{\mathrm{DAM}}(k; R) & \xleftarrow{\cong} & \mathrm{D}(\mathrm{Sh}_{\mathrm{Nis}}(\widetilde{\mathrm{Cor}}_{k,R}^0)) \\ & & \downarrow \cong \\ & & \mathrm{D}(\widetilde{\mathrm{Mack}}_R^{\mathrm{coh}}(k)) \end{array}$$

is established in **Chapter 8**. First, Section 8.1 introduces the category $\widetilde{\Omega}_R(k)$, which is closely related to classical span categories (Construction 8.1.2). Using the construction of $\widetilde{\Omega}_R(k)$, the category of (cohomological) Milnor–Witt Mackey functors in the algebro-geometric setting $\widetilde{\mathrm{Mack}}_R^{\mathrm{coh}}(k)$ is defined (Definition 8.2.1) and the vertical equivalence (Remark 8.2.4) is shown in Section 8.2 under the assumption that a certain factorization is unique (Conjecture 8.2.2).

In **Chapter 9**, examples of the Balmer spectrum of $\widetilde{\text{DAM}}^{\text{gm}}(k; R)$ are computed. If k is an algebraically closed field, the spectrum is shown to coincide with the spectrum of $\text{DAM}^{\text{gm}}(k; R)$, which is merely a point (Corollary 9.1.2). In Section 9.2, we then present first steps in the computation of the spectrum of $\widetilde{\text{DAM}}^{\text{gm}}(k; K)$ for $k = \mathbb{R}$ and K a field of characteristic 2. In particular, we show that it contains the tensor prime ideal $\langle \widetilde{M}(\mathbb{C}) \rangle$, which denotes the thick tensor ideal generated by the complex given by $\text{Spec}(\mathbb{C})$ concentrated in degree 0 (Lemma 9.2.8).

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Chapter 2

Tensor Triangular Geometry and Dualities

In this chapter, we first review Balmer’s tensor triangular geometry before explaining how the concept of dualities can be included. The main references for most of the contents of this chapter are [Bal02, Bal05b, Bal07, Bal10b]. We refer to [GN03] for the concept of a dualizing pairing. For readers interested in the foundations of triangulated categories, see for example [Wei95], whereas readers who would like to read more about monoidal structures are referred to [MacL71, Chapter VII].

Convention 2.0.1. Categories we consider in this work will always assumed to be essentially small.

2.1 Foundations

First, we define the objects of study, namely the *tensor triangulated categories* and functors between them.

Definition 2.1.1. (i) A *tensor triangulated category* $\mathcal{T} = (\mathcal{T}, \otimes, \mathbb{1})$ is a triangulated category together with a symmetric monoidal structure $\otimes : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$ with unit object $\mathbb{1}$ such that \otimes is triangulated in both variables. We usually denote the translation functor of \mathcal{T} by T .

In particular, for every a, b in \mathcal{T} , we have natural isomorphisms $T(a) \otimes b \cong T(a \otimes b)$ and $a \otimes T(b) \cong T(a \otimes b)$ such that $T(a) \otimes T(b) \cong T(T(a) \otimes b) \cong T^2(a \otimes b)$ and $T(a) \otimes T(b) \cong T(a \otimes T(b)) \cong T^2(a \otimes b)$ only differ by sign.

(ii) A triangulated functor $F : \mathcal{T} \rightarrow \mathcal{S}$ between tensor triangulated categories \mathcal{T} and \mathcal{S} is called a *tensor triangulated functor* if it commutes with the tensor product up to isomorphism and fulfills $F(\mathbb{1}_{\mathcal{T}}) = \mathbb{1}_{\mathcal{S}}$.

Remark 2.1.2. In [May01, Chapter 4], May gives a different definition of a tensor triangulated category under the name *closed monoidal category with “compatible”*

triangulation; also, Hovey, Palmieri, and Strickland give a different definition in [HPS97, Definition A.2.1]. The equivalence of the latter to May's axioms (TC1) and (TC2) is shown in [May01, Remark 4.3], but he established additional axioms (T3)-(T5) to carry more information about the product. The axioms (TC1) and (TC2) also suffice for our purposes and are equivalent to Balmer's definition above, see [Bal10b, Remark 4].

We now explain how to endow triangulated categories with a duality structure. For this, we need the following type of functors.

Definition 2.1.3. Let $\delta = \pm 1$ and \mathcal{C}, \mathcal{D} be triangulated categories. A *contravariant δ -triangulated functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ is a contravariant additive functor satisfying $FT_{\mathcal{C}}^{-1} = T_{\mathcal{D}}F$ such that if

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T_{\mathcal{C}}(A)$$

is an exact triangle in \mathcal{C} , then

$$FC \xrightarrow{Fv} FB \xrightarrow{Fu} FA \xrightarrow{\delta \cdot T_{\mathcal{D}}(Fw)} T_{\mathcal{D}}(FC)$$

is an exact triangle in \mathcal{D} .

Now, we can define *triangulated categories with duality* and specify functors between such.

Definition 2.1.4. (i) Let $\delta = \pm 1$. A *triangulated category with δ -duality* $(\mathcal{T}, *, \delta, \eta)$ is a triangulated category \mathcal{T} together with a δ -triangulated functor $* : \mathcal{T}^{\text{op}} \rightarrow \mathcal{T}$ and an isomorphism of triangulated functors $\eta : \text{id}_{\mathcal{T}} \xrightarrow{\cong} * \circ *$ such that $\eta_A^* \circ \eta_{A^*} = \text{id}_{A^*}$ and $\eta_{T(A)} = T(\eta_A)$ for all objects A of \mathcal{T} . Here, $*$ is called the *duality functor* and η is called the *double dual identification*.

(ii) A *duality-preserving functor* between triangulated categories with dualities $(\mathcal{T}, *, \delta, \eta)$ and $(\mathcal{T}', *, \delta', \eta')$ is a pair (F, μ) with $F : \mathcal{T} \rightarrow \mathcal{T}'$ a triangulated functor and $\mu : F \circ * \rightarrow *' \circ F$ an isomorphism of triangulated functors such that the diagram

$$\begin{array}{ccc} FA & \xrightarrow{F\eta_A} & F(A^{**}) \\ \eta'_{FA} \downarrow & & \downarrow \mu_{A^*} \\ (FA)^{**'} & \xrightarrow{\mu_A^{*'}} & F(A^*)^{*'} \end{array}$$

commutes for all objects A in \mathcal{T} .

Notation 2.1.5. We will sometimes write $(\mathcal{T}, *, \eta)$ or even $(\mathcal{T}, *)$ instead of $(\mathcal{T}, *, \delta, \eta)$ when δ and η are clear from the context (or $\delta = 1$). Often we will say *triangulated category with duality* and omit δ from the name.

Remark 2.1.6. The notion of the unit element $\mathbf{1}$ is defined in detail in [MacL71] as an element together with natural isomorphisms α, β such that, for every object

a , we have $a \otimes \mathbb{1} \cong^{\alpha} a$ and $\mathbb{1} \otimes a \cong^{\beta} a$. Here, a unit element is unique up to (not necessarily unique) isomorphism: Suppose there is another unit $\mathbb{1}'$ together with isomorphisms α' and β' as above. Then, $\mathbb{1} \cong^{\alpha'} \mathbb{1} \otimes \mathbb{1}' \cong^{\beta} \mathbb{1}'$ and $\mathbb{1} \cong^{\beta'} \mathbb{1}' \otimes \mathbb{1} \cong^{\alpha} \mathbb{1}'$.

Remark 2.1.7. There are also related notions such as *abelian categories with duality* or *exact categories with duality* which we will encounter later. In general, the concept of duality includes a duality functor compatible in a certain sense with the given structure and a double dual identification, see [Sch10a, Definition 3.1] for the definition of a *category with duality*.

Example 2.1.8. (i) In many cases, the duality we consider on triangulated categories will be (derived from) $\mathrm{Hom}(-, \mathbb{1})$, such as in the derived category of finitely generated R -modules for $\mathbb{1} = R$, the derived category of perfect complexes with $\mathbb{1} = \mathcal{O}_X$, where X is a regular scheme (see [Sch10a, Examples 2.2 and 2.3], compare Remark 2.1.19), or the category of kG -modules with $\mathbb{1} = k$. More details on these particular triangulated categories with duality will follow in Section 2.2.

- (ii) Note that $\mathrm{Hom}_R(-, R)$ does, in general, *not* equip the category $R\text{-Mod}$ of R -modules with a duality since we do not have a double dual identification in general. For instance, if $R = k$ is a field, the cardinality of the dual (and therefore also the double dual) of an infinite-dimensional vector space V is higher than its own cardinality. Hence, $(V^*)^* \not\cong V$.

Neither does $\mathrm{Hom}_R(-, R)$ equip the category of finite length R -modules with a duality in general. Let $R = \mathbb{Z}$ and $n > 1$. Then, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$ contains only the zero morphism. To see this, take a \mathbb{Z} -module homomorphism $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}$. We have

$$0 = f(0) = f(n) = n \cdot f(1),$$

and, since \mathbb{Z} is an integral domain, $f(1) = 0$. As a consequence, $((\mathbb{Z}/n\mathbb{Z})^*)^* = (\{0\})^* = \{0\} \not\cong \mathbb{Z}/n\mathbb{Z}$.

- (iii) If A is a local 0-dimensional ring that is a finite dimensional k -algebra for some field k , $\mathrm{Hom}_A(\mathrm{Hom}_A(M, A), A) \not\cong M$ in general, which can be fixed by choosing the duality $\mathrm{Hom}_k(-, k)$; for more details see [Eis95, Chapter 21].

Remark 2.1.9. Note that [CH09] denotes a *strong duality* a duality such that η from Definition 2.1.4 is an isomorphism, but we assume all triangulated categories to be equipped with such a *strong duality*. To understand the difference between Definition 2.1.4 and [CH09, Definition 2.1.1] in detail, the reader is referred to [CH09, Remark 2.1.2].

The following lemma is straightforward.

Lemma 2.1.10. *Let $F : (\mathcal{T}, *) \rightarrow (\mathcal{L}, *')$ be a duality-preserving functor between triangulated categories with duality. Let \mathcal{J} be a triangulated subcategory of \mathcal{T} that*

we consider a triangulated category with duality when equipped with the restricted duality. Let $F(\mathcal{J}) \subset \mathcal{K}$ for some triangulated subcategory of \mathcal{L} , considered a triangulated category with restricted duality. Then,

- (i) The restriction $F|_{\mathcal{J}} : \mathcal{J} \rightarrow \mathcal{L}$ is duality-preserving.
- (ii) The induced duality turns the Verdier quotient \mathcal{T}/\mathcal{J} into a triangulated category with duality.
- (iii) The functor on the quotients $\mathcal{T}/\mathcal{J} \rightarrow \mathcal{L}/\mathcal{K}$ is duality-preserving.

Now, we want to introduce *tensor triangulated categories with duality*. For this, we use the following notion from [GN03].

Definition 2.1.11. A *product* between triangulated categories \mathcal{K} and \mathcal{L} with codomain \mathcal{M} is a bi-covariant functor $\otimes : \mathcal{K} \times \mathcal{L} \rightarrow \mathcal{M}$ such that

- (i) the functor \otimes is 1-triangulated in both variables and
- (ii) the following diagram is skew-commutative for A in \mathcal{K} and B in \mathcal{L} , i.e.

$$T_{\mathcal{M}}(\mathfrak{r}_{A,B}) \circ \mathfrak{l}_{A,T(B)} = -T_{\mathcal{M}}(\mathfrak{l}_{A,B}) \circ \mathfrak{r}_{T(A),B}$$

$$\begin{array}{ccc} T_{\mathcal{K}}(A) \otimes T_{\mathcal{L}}(B) & \xrightarrow{\mathfrak{l}_{A,T(B)}} & T_{\mathcal{M}}(A \otimes T_{\mathcal{L}}(B)) \\ \mathfrak{r}_{T(A),B} \downarrow & & \downarrow T_{\mathcal{M}}(\mathfrak{r}_{A,B}) \\ T_{\mathcal{M}}(T_{\mathcal{K}}(A) \otimes B) & \xrightarrow{T_{\mathcal{M}}(\mathfrak{l}_{A,B})} & T_{\mathcal{M}}^2(A \otimes B), \end{array}$$

where $\mathfrak{r}_{A,B} : A \otimes T_{\mathcal{L}}(B) \xrightarrow{\cong} T_{\mathcal{M}}(A \otimes B)$ and $\mathfrak{l}_{A,B} : T_{\mathcal{K}}(A) \otimes B \xrightarrow{\cong} T_{\mathcal{M}}(A \otimes B)$.

Definition 2.1.12. Let \mathcal{K}, \mathcal{L} , and \mathcal{M} be triangulated categories with dualities and $\otimes : \mathcal{K} \times \mathcal{L} \rightarrow \mathcal{M}$ a product between their underlying triangulated categories in the sense of Definition 2.1.11. It is called a *dualizing pairing* between \mathcal{K}, \mathcal{L} , and \mathcal{M} if for all objects A in \mathcal{K} and B in \mathcal{L} there are isomorphisms $\mu_{A,B} : A^{*\mathcal{K}} \otimes B^{*\mathcal{L}} \xrightarrow{\cong} (A \otimes B)^{*\mathcal{M}}$ functorial in A and B which make the following diagrams

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\eta_A^{\mathcal{K}} \otimes \eta_B^{\mathcal{L}}} & A^{*\mathcal{K}} \otimes B^{*\mathcal{L}} \\ \eta_{A \otimes B}^{\mathcal{M}} \downarrow & & \downarrow \mu_{A,B} \\ (A \otimes B)^{*\mathcal{M}} & \xrightarrow{(\mu_{A,B})^{*\mathcal{M}}} & (A^{*\mathcal{K}} \otimes B^{*\mathcal{L}})^{*\mathcal{M}} \end{array}$$

and

$$\begin{array}{ccccc} T_{\mathcal{M}}(T_{\mathcal{K}}(A)^{*\mathcal{K}} \otimes B^{*\mathcal{L}}) & \longleftarrow & A^{*\mathcal{K}} \otimes B^{*\mathcal{L}} & \longrightarrow & T_{\mathcal{M}}(A^{*\mathcal{K}} \otimes T_{\mathcal{L}}(B)^{*\mathcal{L}}) \\ T_{\mathcal{M}}(\mu_{T(A),B}) \downarrow & & \downarrow \mu_{A,B} & & \downarrow T(\mu_{A,T(B)}) \\ T_{\mathcal{M}}((T_{\mathcal{K}}(A) \otimes B)^{*\mathcal{M}}) & \longleftarrow & (A \otimes B)^{*\mathcal{M}} & \longrightarrow & T_{\mathcal{M}}((A \otimes T_{\mathcal{L}}(B))^{*\mathcal{M}}) \end{array}$$

commute, where η denotes the double dual identification.

Definition 2.1.13. Let $\delta = \pm 1$. The tuple $(\mathcal{T}, \otimes, \mathbb{1}, *, \eta)$ is called a *tensor triangulated category with δ -duality* if

- (i) $(\mathcal{T}, \otimes, \mathbb{1})$ is a tensor triangulated category as in Definition 2.1.1,
- (ii) $(\mathcal{T}, *, \eta, \delta)$ is a triangulated category with δ -duality as in Definition 2.1.4,
- (iii) $\otimes : (\mathcal{T}, *, \eta) \times (\mathcal{T}, *, \eta) \rightarrow (\mathcal{T}, *, \eta)$ is a dualizing pairing as in Definition 2.1.12.

Remark 2.1.14. This definition is more than we actually need later. In Subsection 4.3.2, we will merely need rigid tensor triangulated categories that can be considered triangulated categories with dualities without assuming a compatibility in the sense of a dualizing pairing.

Definition 2.1.15. A *duality-preserving functor* between tensor triangulated categories with dualities is a functor that is

- (i) a tensor functor as in Definition 2.1.1 (ii) when considered a functor between the underlying tensor categories, and
- (ii) a duality-preserving functor as in Definition 2.1.4 (ii) when considered a functor between the underlying triangulated categories with duality (thus, in particular, triangulated).

With the definition of tensor triangulated categories with duality at hand, we now turn to the main construction in the setting of tensor triangular geometry: the *Balmer spectrum* ([Bal05b, Definition 2.1]). The idea behind it is to generalize the notion of (prime) ideals of a ring to the categorical setting; this way, well-known methods from the algebro-geometric setting can be imitated.

Definition 2.1.16. Let $(\mathcal{T}, \otimes, \mathbb{1})$ be a tensor triangulated category. A thick triangulated subcategory $\mathcal{J} \subset \mathcal{T}$ is called a *tensor ideal* if $\mathcal{T} \otimes \mathcal{J} \subset \mathcal{J}$. It is called *prime* if $\mathcal{J} \neq \mathcal{T}$ and

$$A \otimes B \in \mathcal{J} \Rightarrow ((A \in \mathcal{J}) \text{ or } (B \in \mathcal{J})).$$

Definition-Lemma 2.1.17. For an essentially small tensor triangulated category $\mathcal{T} = (\mathcal{T}, \otimes, \mathbb{1})$ we define its (Balmer) spectrum as the set

$$\mathrm{Spc}(\mathcal{T}) := \{\mathcal{P} \subset \mathcal{T} \mid \mathcal{P} \text{ is a tensor prime ideal}\}.$$

It is a topological space by defining the basic closed subsets as the sets of the form

$$\mathrm{supp}(A) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \mid A \notin \mathcal{P}\} \subset \mathrm{Spc}(\mathcal{T}),$$

for every object A in \mathcal{T} .

We now introduce the category $\mathrm{D}^{\mathrm{perf}}(X)$ (see [TT90, Section 2]), which will serve as our running example and is of vital importance when comparing classical Chow–Witt groups to the new construction of tensor triangular Chow–Witt groups in Section 4.3.

Definition 2.1.18. ([SGA6, Définition 2.1]) Let X be a scheme. A *perfect complex* of \mathcal{O}_X -modules is a complex locally quasi-isomorphic to a bounded complex of algebraic vector bundles.

We denote by $D^{\text{perf}}(X)$ the derived category of perfect complexes equipped with the derived tensor product $\otimes_{\mathcal{O}_X}$, which is a tensor triangulated category by [Tho97, Theorem 3.15].

Remark 2.1.19. Recall that, if we have a regular noetherian scheme X , there is a categorical equivalence between $D^{\text{perf}}(X)$ and the bounded derived category $D^b(\text{Coh}(X)) =: D^b(X)$ of coherent sheaves on X .

Moreover, these categories are equivalent to $D^b(\text{Vect}(X))$ if X is regular, where $\text{Vect}(X)$ is the exact category of algebraic vector bundles, i.e., of locally free sheaves, on X .

The category $P(X)$ of coherent locally free sheaves on X is contained in $\text{Vect}(X)$. From [Bal99], we know that for regular X there is a categorical equivalence $D^b(\text{Coh}(X)) \simeq D^b(P(X))$. (Here, *regular* means that every coherent \mathcal{O}_X -module has a finite resolution by locally free coherent \mathcal{O}_X -modules, see [Bal99, Remark 2.8]). In conclusion, there are equivalences

$$D^{\text{perf}}(X) \simeq D^b(\text{Coh}(X)) \simeq D^b(\text{Vect}(X)) \simeq D^b(P(X))$$

when X is regular.

Example 2.1.20. ([Bal05b, Corollary 5.6]) Let X be a topologically noetherian scheme. There is a homeomorphism $X \cong \text{Spc}(D^{\text{perf}}(X))$ of the underlying topological space of X and the Balmer spectrum of $D^{\text{perf}}(X)$, given explicitly by

$$\begin{aligned} \sigma : X &\rightarrow \text{Spc}(D^{\text{perf}}(X)) \\ x &\mapsto \{A^\bullet \in D^{\text{perf}}(X) \mid A_x^\bullet \cong 0 \text{ in } D^{\text{perf}}(\mathcal{O}_{X,x})\} \subset D^{\text{perf}}(X). \end{aligned}$$

Under this homeomorphism, the support $\text{supp}(A^\bullet)$ corresponds to the *homological support* $\text{supph}(A^\bullet)$, see Definition 4.2.5.

Remark 2.1.21. Tensor triangular geometry is a relatively new topic. It is a difficult problem to compute Balmer spectra, which is why there is still a wide range of open questions.

Ahead of time, before the language of tensor triangular geometry was introduced, Hopkins and Smith already determined the spectrum of the topological stable homotopy category of finite spectra SH^{fin} in [HS98] (compare [Bal10a, Corollary 9.5]).

Many results in this direction are due to Paul Balmer. For example, the spectrum of the bounded derived category of kG -modules was shown to be equivalent to the homogeneous spectrum of the cohomology ring $H^\bullet(G, k)$ in [Bal05b, Corollary 5.10]. An overview of open problems is given in [Bal10b], where Balmer describes, for example, the computation of $\text{Spc}(\text{SH}_{\text{gm}}^{\text{A}^1}(S))$ as a “long-term challenge” ([Bal10b, p. 18]). A first step is given in [HO18, Theorem 1.1].

Recent progress in the field of motivic homotopy theory includes the computations of the spectra of the derived category of Artin-Tate motives over \mathbb{R} with integral coefficients in [BG22a] by Balmer and Gallauer or the triangulated category of Tate motives with integral coefficients over certain algebraically closed base fields by Gallauer in [Gal19, Theorem 8.6]. In [BG22b] and [BG23b], a description of the spectrum of the derived category of permutation modules over a finite group with coefficients in a field of positive characteristic is given. The extension to profinite groups has recently been published in the preprint [BG24]. In [Pe13], Peter proved that the spectrum of mixed Tate motives with rational coefficients $\mathrm{DTM}(k, \mathbb{Q})$ is a point.

Further results and progress in computing Balmer spectra in the field of algebraic geometry are, for example, [DS14, Theorem 4.7], [Ste14, Theorem 7.7], and [Hal16, Theorem 1.2]. In modular representation theory, examples are the spectrum of the stable module category in [BCR97]. For more examples, the reader is referred to [Bal19].

Having recalled the basic concepts of tensor triangular geometry, we can direct ourselves in more detail toward the matter of duality in the following subsection.

2.2 Rigid Tensor Triangulated Categories

In Definition 2.1.13, we introduced triangulated categories with duality. We mentioned that, in most cases, the duality we are interested in is inherited from some Hom-functor. This idea is captured by the notion of *rigid* tensor triangulated categories. These will be our main object of study and are introduced in this subsection.

Our overall goal of defining tensor triangular Chow–Witt groups is inspired by the construction of tensor triangular Chow groups by Balmer and Klein [Bal13], [Kl16a]. This idea is led by the concept of a filtration of the given tensor triangulated category by dimension of support (cf. Section 2.3)

$$\mathcal{T}_{(-\infty)} \subset \dots \subset \mathcal{T}_{(n-1)} \subset \mathcal{T}_{(n)} \subset \dots \subset \mathcal{T}_{(\infty)},$$

where we define $\mathcal{T}_{(n)} := \{a \in \mathcal{T} \mid \dim(\mathrm{supp}(a)) \leq n\}$ for a dimension function \dim in the sense of Definition 2.3.1 and $n \in \mathbb{Z}$.

We would like to use the same approach, but, in general, supports do not behave well with duality, i.e. the support of an object might differ from the support of its dual. Balmer shows that the assumption of the tensor triangulated category being *rigid* (Definition 2.2.1) fixes this problem. He proves that, in this case, the support of an object equals the support of its dual, which is one of the reasons why we introduce this notion (see [Bal07, Chapter 2]).

Definition 2.2.1. A tensor triangulated category $(\mathcal{T}, \otimes, \mathbf{1})$ is called *rigid* if there is a bi-triangulated functor $\mathrm{hom} : \mathcal{T}^{\mathrm{op}} \times \mathcal{T} \rightarrow \mathcal{T}$ such that

(i) there are natural isomorphisms

$$\mathrm{Hom}_{\mathcal{T}}(a \otimes b, c) \cong \mathrm{Hom}_{\mathcal{T}}(a, \mathrm{hom}(b, c))$$

for all objects a, b , and c in \mathcal{T} , and

(ii) every object a is *rigid*, that is, the natural morphism

$$\mathrm{hom}(a, \mathbb{1}) \otimes b \rightarrow \mathrm{hom}(a, b)$$

is an isomorphism for all objects b of \mathcal{T} .

The functor hom is often called the *internal Hom*. We denote the rigid tensor triangulated category by $(\mathcal{T}, \otimes, \mathbb{1}, \mathrm{hom})$.

Remark 2.2.2. In [PS14, Definition 2.1], another description of *rigid* (or *(strongly) dualizable*) objects in tensor triangulated (or *closed symmetric monoidal*) categories is given: An object a in $(\mathcal{T}, \otimes, \mathbb{1})$ is *rigid* if there exists a dual a^* together with evaluation and coevaluation maps

$$\eta : \mathbb{1} \rightarrow a \otimes a^* \text{ and } \varepsilon : a^* \otimes a \rightarrow \mathbb{1}$$

satisfying

$$(\mathrm{id}_a \otimes \varepsilon)(\eta \otimes \mathrm{id}_a) = \mathrm{id}_a \text{ and } (\varepsilon \otimes \mathrm{id}_{a^*})(\mathrm{id}_{a^*} \otimes \eta) = \mathrm{id}_{a^*}.$$

The definitions are equivalent (see [PS14, p.7]) in the following sense: An object a is rigid with dual $\mathrm{hom}(a, \mathbb{1})$ in the sense of [PS14, Definition 2.1] if and only if the map

$$\mathrm{hom}(a, \mathbb{1}) \otimes a \rightarrow \mathrm{hom}(a, a)$$

is an isomorphism.

Remark 2.2.3. Definition 2.2.1 (i) states that (\otimes, hom) is an adjoint couple, and we call Definition 2.2.1 (i) the *adjunction isomorphism*.

As an important example, we will now show how rigid tensor triangulated categories yield triangulated categories with duality. As a first step, we recall some notions on objects L that twist dualities.

Definition 2.2.4. An object L of a tensor triangulated category $(\mathcal{T}, \otimes, \mathbb{1})$ is called *\otimes -invertible* if there exists an object M of \mathcal{T} and an isomorphism $L \otimes M \xrightarrow{\cong} \mathbb{1}$.

These objects satisfy some nice technical properties.

Lemma 2.2.5. *Let $(\mathcal{T}, \otimes, \mathbb{1}, \mathrm{hom})$ be a rigid tensor triangulated category.*

(i) *If K and L are \otimes -invertible objects, then so is $K \otimes L$.*

(ii) *For a \otimes -invertible object L in a (not necessarily rigid) tensor triangulated category we have $\mathrm{supp}(L) = \mathrm{Spc}(\mathcal{T})$.*

Proof. Part (i) can be seen as follows: For \otimes -invertible objects K and L there exist objects M and M' such that $K \otimes M \cong \mathbb{1}$ resp. $L \otimes M' \cong \mathbb{1}$. It then follows by associativity of \otimes that $(K \otimes L) \otimes (M' \otimes M) \cong K \otimes (L \otimes M') \otimes M \cong K \otimes \mathbb{1} \otimes M \cong K \otimes M \cong \mathbb{1}$.

For part (ii), assume that $\text{supp}(L) \neq \text{Spc}(\mathcal{T})$, which is equivalent to $\{\mathcal{P} \in \text{Spc}(\mathcal{T}) \mid L \in \mathcal{P}\}$ being nonempty, i.e., there exists a tensor prime ideal $\mathcal{P} \in \text{Spc}(\mathcal{T})$ such that $L \in \mathcal{P}$. Since \mathcal{P} is a thick tensor ideal we have that, in particular, $L \otimes M \in \mathcal{P}$ for its \otimes -inverse M . But since $L \otimes M \cong \mathbb{1}$, \mathcal{P} is not proper and thus cannot be a tensor prime ideal. \square

Example 2.2.6. This example shows how to turn rigid tensor triangulated categories into triangulated categories with dualities.

- (i) Let $(\mathcal{T}, \otimes, \mathbb{1}, \text{hom})$ be a rigid tensor triangulated category. We can turn it into a triangulated category with 1-duality as follows: Set $*$: $\mathcal{T}^{\text{op}} \rightarrow \mathcal{T}$ as $*$:= $\text{hom}(-, \mathbb{1})$, which is 1-triangulated by Definition 2.2.1. The double dual identification $\eta : \text{id}_{\mathcal{T}} \xrightarrow{\cong} * \circ *$ is defined as the image of the evaluation map

$$\text{ev} : A \otimes \text{hom}(A, \mathbb{1}) \rightarrow \mathbb{1}$$

under the adjunction isomorphism Definition 2.2.1 (i). The natural transformation η is indeed an isomorphism, see [HPS97, Theorem A.2.5 (b)]. The fact that $(\mathcal{T}, \text{hom}(-, \mathbb{1}))$ is a triangulated category with duality follows from [CH09, Corollary 3.2.4].

- (ii) Let L be a fixed \otimes -invertible object. More generally, we can turn the rigid tensor triangulated category $(\mathcal{T}, \otimes, \mathbb{1}, \text{hom})$ into a triangulated category with duality by choosing the duality $D_L := \text{hom}(-, L)$, which amounts to tensoring $\text{hom}(-, \mathbb{1})$ with L in the setting above since $\text{hom}(-, \mathbb{1}) \otimes L \cong \text{hom}(-, L)$ by rigidity. The reader is referred to [CH09, Corollary 3.2.4] for more details.
- (iii) Let X be a quasi-compact, quasi-separated scheme (e.g. noetherian). The category $\text{D}^{\text{perf}}(X)$ is a rigid tensor triangulated category with the left derived tensor product $\otimes_{\mathcal{O}_X}^L$ and the right derived Hom-sheaf $R\text{Hom} := \text{hom}$ as internal Hom, see [Bal07, Proposition 4.1]. In particular, it can be turned into a triangulated category with duality as above. For a regular scheme X , $\text{D}^{\text{perf}}(X)$ is equivalent to $\text{D}^b(X)$ as stated in Remark 2.1.19.
- (iv) Let k be a field of positive characteristic and G a finite group whose order is divisible by $\text{char}(k)$. Then, the stable category $kG\text{-stab}$ is a rigid tensor triangulated category with tensor product \otimes_k and internal Hom $\text{hom} := \text{Hom}_k$, see [Bal07, Proposition 4.2]. A detailed construction of the category $kG\text{-stab}$ will follow in Section 5.1.
- (v) As stated in [Bal07, Remark 2.2], rigid tensor triangulated categories often arise as subcategories of compact objects of some bigger tensor triangulated

category. For example, if X is quasi-compact and quasi-separated, $D^{\text{perf}}(X)$ is the subcategory of compact objects in the derived category $D_{\text{Qcoh}}^+(X)$ of cohomologically bounded below complexes with quasi-coherent cohomology. The latter is an example of a tensor triangulated category that is *not* rigid.

Given a triangulated category with duality, we can construct *shifted dualities*. We first recall Balmer's definition of this notion.

Definition 2.2.7. Let $(\mathcal{T}, *, \eta, \delta)$ be a triangulated category with duality. We define the associated triangulated category with *shifted duality* as

$$T(\mathcal{T}, *, \eta, \delta) := (\mathcal{T}, T \circ *, -\delta \cdot \eta, -\delta),$$

where T is the shift functor of \mathcal{T} . Moreover, we set

$$T^{-1}(\mathcal{T}, *, \eta, \delta) := (\mathcal{T}, T^{-1} \circ *, \delta \cdot \eta, \delta).$$

Remark 2.2.8. For example, if $\delta = 1$, then

$$T^n(\mathcal{T}, *, \eta, \delta) := (\mathcal{T}, T^n \circ *, (-1)^{\frac{n(n+1)}{2}} \eta, (-1)^n \cdot \delta)$$

is a triangulated category with $(-1)^n \delta$ -duality for $n \in \mathbb{Z}$ by [Bal00, Remark and Definition 2.9].

So, for each $n \in \mathbb{Z}$, we obtain a triangulated category with duality associated to $(\mathcal{T}, *, \eta, \delta)$. In particular, if $(\mathcal{T}, \otimes, \text{hom})$ is a rigid tensor triangulated category, $T^n(\mathcal{T}, D_L, \eta_L)$ is a triangulated category with duality for a fixed \otimes -invertible object L in \mathcal{T} .

This subsection has introduced the notion of rigid tensor triangulated categories which embodies the idea of allowing mainly Hom-dualities when talking about triangulated categories with duality. We have seen examples and further notions as a preparation to talk about filtrations by dimension of support in the next section.

2.3 Dimension Functions and Filtrations

The reason why we introduced rigid tensor triangulated categories above was that we wanted the dualities to behave well with the support, in particular, with the filtration of our tensor triangulated category we are about to construct. In order to show this, we first introduce notions of *dimension* before constructing the filtration components.

Definition 2.3.1. A *dimension function* on a tensor triangulated category $\mathcal{T} = (\mathcal{T}, \otimes, \mathbb{1})$ is a map

$$\dim : \text{Spc}(\mathcal{T}) \rightarrow \mathbb{Z} \cup \{\pm\infty\}$$

satisfying the following properties:

- (i) For tensor prime ideals $\mathcal{Q} \subset \mathcal{P}$ of \mathcal{T} , we have $\dim(\mathcal{Q}) \leq \dim(\mathcal{P})$.

(ii) If $\mathcal{Q} \subset \mathcal{P}$ and $\dim(\mathcal{Q}) = \dim(\mathcal{P})$, it follows that $\mathcal{Q} = \mathcal{P}$.

Convention 2.3.2. All categories are essentially small as in Convention 2.0.1. Moreover, let the Balmer spectrum be noetherian for any tensor triangulated category. Tensor triangulated categories will be equipped with a dimension function \dim . Given a rigid tensor triangulated category $(\mathcal{T}, \otimes, \mathbb{1}, \text{hom})$, we fix a \otimes -invertible object L . When considered a triangulated category with duality, \mathcal{T} will be equipped with the duality $D_L := \text{hom}(-, L)$ as in Example 2.2.6 (ii).

Remark 2.3.3. We will see later that the most important examples for us will be, on the one hand, the opposite of the Krull-codimension $-\text{codim}_{\text{Krull}}$, equipping Example 2.2.6 (iii), i.e. $D^{\text{perf}}(X)$ for X quasi-compact and quasi-separated, with a dimension function, see [Bal07, Example 3.3]

On the other hand, the Krull dimension \dim_{Krull} will be our choice of a dimension function for kG -stab from Example 2.2.6 (iv), see Definition 5.1.24

For further details, the reader is referred to [Bal07, Examples 3.2 and 3.3].

Definition 2.3.4. For a tensor triangulated category $\mathcal{T} = (\mathcal{T}, \otimes, \mathbb{1})$ with a dimension function \dim , we define the *dimension* of a subset $V \subset \text{Spc}(\mathcal{T})$ as $\dim(V) := \sup\{\dim(\mathcal{P}) \mid \mathcal{P} \in V\}$. For any $n \in \mathbb{Z} \cup \{\pm\infty\}$, we set

$$\mathcal{T}_{(n)} := \{A \in \mathcal{T} \mid \dim(\text{supp}(A)) \leq n\}$$

yielding a thick tensor ideal in \mathcal{T} by the properties of support ([Bal07, Proposition 1.3]) and dimension function (Definition 2.3.1).

We often call $\mathcal{T}_{(n)}$ the *n-th filtration component* of the tensor triangulated category \mathcal{T} . Moreover, we define $\text{Spc}(\mathcal{T})_n$ as the set of points \mathcal{Q} in $\text{Spc}(\mathcal{T})$ with $\dim(\mathcal{Q}) = n$.

Lemma 2.3.5. *Let $(\mathcal{T}, \otimes, \mathbb{1}, \text{hom}, \dim)$ be a rigid tensor triangulated category with dimension function that we consider a triangulated category with the duality $D_L := \text{hom}(-, L)$ for a fixed \otimes -invertible object L in \mathcal{T} following Example 2.2.6 (i)/(ii). Let $n \in \mathbb{Z} \cup \{\pm\infty\}$.*

Then, the thick tensor ideal $\mathcal{T}_{(n)}$ is a triangulated category with duality when equipped with the duality induced by D_L .

Proof. Since $\mathcal{T}_{(n)}$ is a tensor prime ideal, it is in particular a full triangulated subcategory. It follows from [Bal07, Proposition 2.7] that for any object a in \mathcal{T} , we have $\text{supp}(a) \cong \text{supp}(\text{hom}(a, \mathbb{1}))$, and thus especially $\dim(\text{supp}(a)) = \dim(\text{supp}(\text{hom}(a, \mathbb{1})))$, which means that if a is in $\mathcal{T}_{(n)}$, then so is $\text{hom}(a, \mathbb{1})$. More generally, by [Bal07, Proposition 2.7] and Lemma 2.2.5, we have $\text{supp}(\text{hom}(a, L)) = \text{supp}(a) \cap \text{supp}(L)$ and $\text{supp}(L) = \text{Spc}(\mathcal{T})$ as subspaces of $\text{Spc}(\mathcal{T})$, yielding that for a in $\mathcal{T}_{(n)}$ and L \otimes -invertible in \mathcal{T} ,

$$\begin{aligned} \dim(\text{supp}(\text{hom}(a, L))) &= \dim(\text{supp}(a) \cap \text{supp}(L)) \\ &= \dim(\text{supp}(a) \cap \text{Spc}(\mathcal{T})) = \dim(\text{supp}(a)). \end{aligned}$$

That is, if a is in $\mathcal{T}_{(n)}$, so is $\text{hom}(a, L)$. The duality functor $\text{hom}(-, L)$ and the double dual identification from Example 2.2.6 therefore restrict to $\mathcal{T}_{(n)}$, equipping it with a duality structure. \square

Remark 2.3.6. The above lemma states that for a rigid tensor triangulated category $(\mathcal{T}, \otimes, \mathbf{1}, \text{hom})$, we obtain triangulated categories with duality $\mathcal{T}_{(n)}$ equipped with the restricted duality of $D_L := \text{hom}(-, L)$, allowing us to define Witt groups and Grothendieck–Witt groups of the filtration components $\mathcal{T}_{(n)}$ later.

Apart from the filtration by dimension of support, we will also need the technique of idempotent completion of a (tensor) triangulated category later since the quotients of the filtration components are not necessarily idempotent complete even if the category \mathcal{T} is in the first place, see [Bal07, Remark 1.14].

We know from [BS01] and [Bal07, Proposition 2.15] that applying the usual idempotent completion $(-)^{\natural}$ of an additive category to a (tensor) triangulated category naturally yields a (tensor) triangulated category again. In our setting, we need the idempotent completion to moreover behave well with dualities.

Lemma 2.3.7. *The idempotent completion \mathcal{T}^{\natural} of a triangulated category with duality $\mathcal{T} = (\mathcal{T}, \otimes, \mathbf{1}, *, \eta)$ is again a triangulated category with duality.*

Proof. Recall that $(\mathcal{T})^{\natural}$ is naturally already a triangulated category by [BS01]. It inherits a duality structure

$$\tilde{*} : ((\mathcal{T})^{\natural})^{\text{op}} \rightarrow (\mathcal{T})^{\natural}$$

given by

$$(A, a) \mapsto (A^*, a^*), f : [(A, a) \rightarrow (B, b)] \mapsto [f^{\tilde{*}} : (B^*, b^*) \rightarrow (A^*, a^*)],$$

where $f^{\tilde{*}} = f^*$. Note that the dual of an idempotent morphism is still idempotent since we have $a^* \circ a^* = (a \circ a)^* = a^*$. Also, $f^{\tilde{*}}$ actually is a morphism in the idempotent completion since we have $b \circ f = f = f \circ a$ by definition of the idempotence of f and thus obtain the desired idempotence of $f^{\tilde{*}}$ by applying the duality functor $*$.

Moreover, η induces a natural transformation

$$\tilde{\omega} : \mathbf{1}_{\mathcal{T}^{\natural}} \rightarrow \tilde{*} \circ \tilde{*}$$

defined elementwise as $\tilde{\omega}_{(A,a)} := \eta_A \times \eta_a$. Hence, \mathcal{T}^{\natural} is a triangulated category with duality. \square

Moreover, we need idempotent completion to behave well with the structure of a rigid tensor triangulated category as well as with Verdier localization.

Lemma 2.3.8. ([Bal07, Proposition 2.15])

- (i) *The idempotent completion of a rigid tensor triangulated category is again a rigid tensor triangulated category. The idempotent completion functor is a duality-preserving functor when regarding the rigid tensor triangulated category a triangulated category with duality (see Example 2.2.6 (i) and (ii)).*

(ii) *The Verdier localization of a rigid tensor triangulated category at a thick tensor ideal is again a rigid tensor triangulated category. The localization functor is a duality-preserving functor when regarding the rigid tensor triangulated category a triangulated category with duality (see Example 2.2.6 (i) and (ii)).*

Recall from Definition 2.3.4 that, for a rigid tensor triangulated category with a dimension function $(\mathcal{T}, \otimes, \mathbf{1}, \text{hom}, \text{dim})$, we defined filtration components

$$\mathcal{T}_{(n)} := \{A \in \mathcal{T} \mid \text{dim}(\text{supp}(A)) \leq n\}$$

for $n \in \mathbb{Z} \cup \{\pm\infty\}$, which yield a filtration by thick tensor ideals

$$\mathcal{T}_{(-\infty)} \subset \dots \subset \mathcal{T}_{(n-1)} \subset \mathcal{T}_{(n)} \subset \dots \subset \mathcal{T}_{(\infty)} = \mathcal{T}.$$

It lifts to a filtration of \mathcal{T} when considered a triangulated category with duality with each component being a triangulated category with duality again (cf. Lemma 2.3.5):

$$\mathcal{T}_{(-\infty)} \subset \dots \subset (\mathcal{T}_{(n-1)}, D_L|_{\mathcal{T}_{(n-1)}}, \eta) \subset (\mathcal{T}_{(n)}, D_L|_{\mathcal{T}_{(n)}}, \eta) \subset \dots \subset \mathcal{T}_{(\infty)}.$$

The inclusion functors are duality-preserving functors since all the dualities are induced by $\text{hom}(-, L)$.

For a tensor prime ideal $\mathcal{P} \in \text{Spc}(\mathcal{T})$, let $\mathcal{T}_{\mathcal{P}} := (\mathcal{T}/\mathcal{P})^{\natural}$ and let $\text{FL}(\mathcal{T}_{\mathcal{P}}) := \{a \in \mathcal{T}_{\mathcal{P}} \mid \text{supp}(a) \subset *\}$, where $*$ is the unique closed point of $\text{Spc}(\mathcal{T}_{\mathcal{P}})$. See [Bal07, Definition 3.7] for more details.

For future reference, we recall the following important decomposition statement.

Theorem 2.3.9. ([Bal07, Theorem 3.24]) *Let $(\mathcal{T}, \otimes, \mathbf{1}, \text{hom}, \text{dim})$ be an essentially small, rigid tensor triangulated category equipped with a dimension function such that $\text{Spc}(\mathcal{T})$ is noetherian. Then, for all $n \in \mathbb{Z}$, we have a triangulated equivalence*

$$(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})^{\natural} \rightarrow \bigoplus_{\substack{\mathcal{P} \in \text{Spc}(\mathcal{T}) \\ \text{dim}(\mathcal{P})=n}} \text{FL}(\mathcal{T}_{\mathcal{P}}).$$

Corollary 2.3.10. *The above equivalence passes to an equivalence of triangulated categories with duality when considering the rigid tensor triangulated category $(\mathcal{T}, \otimes, \mathbf{1}, \text{hom})$ a triangulated category with duality $(\mathcal{T}, D_L, \eta_L)$ as in Example 2.2.6 (i)/(ii) for L a \otimes -invertible object of \mathcal{T} .*

Proof. It follows from Example 2.2.6 (ii) that $(\mathcal{T}, D_L, \eta_L)$ is indeed a triangulated category with duality as well as $(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})^{\natural}$ by the above considerations of this section, see e.g. Lemma 2.3.8. The duality \overline{D}_L on the triangulated category on the right-hand side is induced by the localization functors $q_{\mathcal{P}} : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{P}$ for each

$\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$. Applying [Bal07, Proposition 2.15], we see that for all objects a we have

$$\overline{D}_L(a) = \mathrm{hom}(q_{\mathcal{P}}(a), q_{\mathcal{P}}(L)) \cong q_{\mathcal{P}}(\mathrm{hom}(a, L)) = q_{\mathcal{P}}(D_L(a)).$$

The triangulated equivalence from the statement is induced by the functor

$$\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)} \rightarrow \bigoplus_{\substack{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \\ \dim(\mathcal{P})=n}} \mathrm{Min}(\mathcal{T}/\mathcal{P}), \quad a \mapsto q_{\mathcal{P}}(a).$$

We can thus deduce from Lemma 2.3.8 that the equivalence is a duality-preserving functor. \square

In this subsection, the central idea of tensor triangular geometry to filtrate by dimension of support was introduced. We validated that this construction, as well as important tools such as idempotent completion or Verdier localization, behave well with dualities.

The whole section intended to introduce the reader to tensor triangular geometry, starting with the basic definition of its objects of study, the functors between them, and main concepts such as tensor prime ideals and the Balmer spectrum $\mathrm{Spc}(\mathcal{T})$ of a tensor triangulated category. After that, rigid tensor triangulated categories were introduced and portrayed as our central way of introducing a duality structure on tensor triangulated categories inherited by the internal Hom. Examples have been given and shifted dualities have been defined. Eventually, dimension functions embodied the last ingredient to filtrate a given tensor triangulated category by dimension of support. This construction and moreover idempotent completion and Verdier localization have been shown to be compatible with our concept of duality.

The following section will start from a different point of view and introduce Grothendieck–Witt and Witt groups as a generalization of classical algebraic K -theory, which will be a main tool to generalize the construction of tensor triangular Chow groups to tensor triangular Chow–Witt groups in Section 4.2.

Chapter 3

Grothendieck–Witt and Witt Groups

The generalization from Chow groups to Chow–Witt groups in the classical algebro-geometric case involves, in particular, the passage from Milnor K -theory to Milnor–Witt K -theory. In degree 0, this nails down to the passage from the Grothendieck group K_0 to the Grothendieck–Witt group GW .

We assume that the reader is familiar with the theory of Grothendieck–Witt and Witt groups of fields and rings. As an introduction, we recommend [Lam05, MH73, Scha85].

In this chapter, we introduce the notions of Grothendieck–Witt groups and Witt groups of triangulated categories with duality as well as of complicial exact categories with weak equivalences and duality. We introduce basic notions in these settings and compare the definitions. In the last section, we examine the different localization theorems that will later serve as a tool to show that the definition of Chow–Witt groups of tensor triangulated categories indeed generalizes the classical one of schemes.

In the triangulated case, we follow the references [Bal00, Wal03, Sch17], whereas for the exact case [Sch10a, Sch10b, FS09] are our main references.

Convention 3.0.1. As stated above, all categories will be considered essentially small if not stated otherwise. In this chapter, we assume that any triangulated category \mathcal{T} or exact category \mathcal{E} contains $\frac{1}{2}$, i.e., the abelian group $\mathrm{Hom}(A, B)$ is uniquely divisible by 2 for all objects A and B in \mathcal{T} (or \mathcal{E} , respectively). We also say that \mathcal{T} is $\mathbb{Z}[\frac{1}{2}]$ -linear. For the example of schemes we will consider later, all base fields k will have a characteristic $\neq 2$ (compare Remark 3.1.4).

3.1 (Grothendieck–)Witt Groups of Triangulated Categories

We start with the theory of (Grothendieck–)Witt groups of triangulated categories with duality. The constructions of this chapter do not rely on any tensor structure.

A rigid tensor triangulated category may be turned into a triangulated category with duality as in Example 2.2.6 (i)/(ii), to which one can apply GW or W. We will proceed like this in Section 4.2. Now, we begin with some conventions before introducing the notions necessary to define (Grothendieck–)Witt groups of triangulated categories with duality.

Witt groups and Grothendieck–Witt groups of fields and rings intend to capture the behavior of symmetric bilinear forms on vector spaces and modules, respectively, which can be described briefly as symmetric isomorphisms to their dual. To generalize this definition for triangulated categories, one needs to define *symmetric objects*.

Definition 3.1.1. For objects A, B of a triangulated category with duality $\mathcal{T} = (\mathcal{T}, *, \eta)$, the *transpose* of a morphism $u : A \rightarrow B^*$ is defined as $u^t := u^* \circ \eta_B$. A morphism $w : A \rightarrow A^*$ is *symmetric* if $w^t = w$. A *symmetric object* of \mathcal{T} is a pair (A, w) , where $w : A \xrightarrow{\cong} A^*$ is a symmetric isomorphism. Two symmetric objects (A, w) and (B, v) are *isometric* if there is an isomorphism $r : A \xrightarrow{\cong} B$ such that $w = r^*vr$.

We denote by $\text{SymOb}(\mathcal{T})$ the free abelian group of isometry classes of symmetric objects of \mathcal{T} with the orthogonal sum as group operation, where the orthogonal sum $(A, \alpha) \oplus (B, \beta)$ of symmetric spaces (A, α) and (B, β) is defined as

$$\alpha \oplus \beta := \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} : A \oplus B \rightarrow A^* \oplus B^*.$$

Remark 3.1.2. The symmetric objects together with their isomorphisms form a groupoid. Note that a duality-preserving functor $(F, \mu) : (\mathcal{A}, *) \rightarrow (\mathcal{B}, \tilde{*})$ between triangulated categories with duality induces a morphism of groupoids sending $w : A \xrightarrow{\cong} A^*$ to $\mu_A \circ Fw : FA \xrightarrow{\cong} F(A^*) \cong F(A)^{\tilde{*}}$.

Definition-Lemma 3.1.3. [[Bal00], Theorem 2.6] *Let $(\mathcal{T}, *, \eta)$ be a triangulated category with duality.*

- (i) *For any morphism $u : T^{-1}(A) \rightarrow A^*$ that is symmetric with respect to the (-1) -st shifted duality, there exists an exact triangle of the form*

$$T^{-1}(A) \xrightarrow{u} A^* \xrightarrow{\phi^{-1}v^*} B \xrightarrow{v} A.$$

Here, $\phi : B \xrightarrow{\cong} B^$ is an isomorphism with respect to the 0-th shifted duality and (B, ϕ) is uniquely determined by (A, u) up to isomorphism. We define $\text{cone}(A, u) := (B, \phi)$.*

- (ii) *We call a symmetric object metabolic if it is isomorphic to some $\text{cone}(A, u)$. In the above setting, A^* is called a Lagrangian of (B, ϕ) .*

- (iii) *A hyperbolic object is a symmetric object $\mathbb{H}(A)$ for some object A of the form*

$$\mathbb{H}(A) := (A \oplus A^*, \begin{bmatrix} 0 & 1_{A^*} \\ \eta_A & 0 \end{bmatrix}) = \text{cone}(A, 0).$$

Remark 3.1.4. Note that we usually assume that $\text{char}(k) \neq 2$ (or $\frac{1}{2}$, i.e. 2, being invertible). If 2 is invertible, *quadratic* and symmetric forms coincide as well as *split-metabolic* and hyperbolic objects (see [Bal05a, Remark 36]) for the category of finitely generated projective R -modules.

However, if 2 is not invertible, this is not always the case, see [Bal05a, Example 38] for an example of a split-metabolic object that is not hyperbolic.

There has been progress for when 2 is not invertible. For example, in [Sch10a] and [Sch10b], assumptions on the characteristic of the base field when considering (Grothendieck–)Witt groups of exact categories are dropped already.

Definition 3.1.5. Let $\mathcal{T} = (\mathcal{T}, *, \eta)$ be a triangulated category with duality.

- (i) The *Grothendieck–Witt group* $\text{GW}(\mathcal{T}) = \text{GW}(\mathcal{T}, *, \eta)$ is the quotient of $\text{SymOb}(\mathcal{T})$ divided by the following relations:
 - (a) $[(A, \alpha) \perp (B, \beta)] = [A, \alpha] + [B, \beta]$ and
 - (b) $[\text{cone}(Y, f)] = [\mathbb{H}(Y)]$.
- (ii) The *Witt group* $\text{W}(\mathcal{T}, *, \eta)$ is the quotient of $\text{GW}(\mathcal{T})$ by the subgroup \mathbb{H} generated by the hyperbolic classes, that is $\text{W}(\mathcal{T}) := \text{GW}(\mathcal{T})/\mathbb{H}$.
- (iii) For $n \in \mathbb{Z}$, the n -th *shifted Grothendieck–Witt group* $\text{GW}^{[n]}(\mathcal{T})$ (and *Witt group* $\text{W}^{[n]}(\mathcal{T})$) is the usual Grothendieck–Witt group (and Witt group, respectively) equipped with the n -th shifted duality $T^n \circ *$ from Definition 2.2.7.
- (iv) Let $*$ = D_L for a \otimes -invertible object in \mathcal{T} , for example in the case of Example 2.2.6 (ii). In this special case, we denote by $\text{GW}(\mathcal{T}, L) := \text{GW}(\mathcal{T}, D_L, \eta_L)$ and $\text{W}(\mathcal{T}, L) := \text{W}(\mathcal{T}, D_L, \eta_L)$ the Grothendieck–Witt group and Witt group *twisted by* L , respectively.

Remark 3.1.6. (i) Note that symmetric objects from [Wal03] coincide with *symmetric spaces* from [Bal00]. In [Wal03, p.6], an equivalent definition of $\text{W}(\mathcal{T})$ is given via relations that immediately shows the equivalence of Definition 3.1.5 below to Balmer’s definition of triangular Witt groups in [Bal00, Definition 2.13]. In total, (Grothendieck–)Witt groups of triangulated categories of Walter ([Wal03]), Schlichting ([Sch17]), and Balmer ([Bal00]) coincide by [Sch17, Remark 3.14].

- (ii) Readers familiar with the notion of *higher* Grothendieck–Witt groups, e.g. of exact categories as we will use later, must not confuse these with *shifted* Grothendieck–Witt groups, in this case of triangulated categories, for there are no higher Grothendieck–Witt groups of triangulated categories. More details on this comparison can be found in [Sch10a, Remark 4.14].

Lemma 3.1.7. *Given a duality-preserving functor between triangulated categories with duality $(F, \varphi) : (\mathcal{T}, *_{\mathcal{T}}) \rightarrow (\mathcal{S}, *_{\mathcal{S}})$ as in Definition 2.1.4, i.e. $\varphi : F \circ *_{\mathcal{T}} \simeq$*

$*_{\mathcal{S}} \circ F$ is a triangulated equivalence, we obtain induced group homomorphisms for $n \in \mathbb{Z}$

$$\overline{F} : \mathrm{GW}^{[n]}(\mathcal{T}, *_{\mathcal{T}}) \rightarrow \mathrm{GW}(\mathcal{S}, *_{\mathcal{S}})$$

and

$$\overline{F} : \mathrm{W}^{[n]}(\mathcal{T}, *_{\mathcal{T}}) \rightarrow \mathrm{W}(\mathcal{S}, *_{\mathcal{S}}).$$

In particular, if $*_{\mathcal{T}} = D_L$ and $*_{\mathcal{S}} = D_{L'}$ for fixed \otimes -invertible objects L and L' of \mathcal{T} and \mathcal{S} , respectively, we obtain group homomorphisms $\overline{F} : \mathrm{GW}^{[n]}(\mathcal{T}, L) \rightarrow \mathrm{GW}(\mathcal{S}, L')$ and $\overline{F} : \mathrm{W}^{[n]}(\mathcal{T}, L) \rightarrow \mathrm{W}(\mathcal{S}, L')$.

If F is moreover a triangulated equivalence, \overline{F} is an isomorphism.

Proof. For symmetric objects (A, a) of $\mathrm{GW}(\mathcal{T}, *_{\mathcal{T}})$, we define

$$\overline{F}(A, a) = (FA, \varphi_a \circ Fa).$$

The mapping preserves metabolic objects and Lagrangians. Hence, it induces a morphism on GW . In particular, it preserves hyperbolic classes (cf. [Sch10a, Section 3.1]) and, thus, induces morphisms on Witt groups. The “moreover” statement follows immediately from the concrete description of \overline{F} . The case of nonzero n follows analogously. \square

Having seen the definition of (Grothendieck–)Witt groups of triangulated categories with duality, we will introduce the same notion for certain types of exact categories in the next section.

3.2 (Grothendieck–)Witt Groups of Exact Categories

We are now familiar with the definition of (Grothendieck–)Witt groups of triangulated categories with duality. One can also define Grothendieck–Witt and Witt groups of exact categories with duality as in [Sch10a]. The two constructions agree in the sense of the forthcoming Proposition 3.2.14. Note that Schlichting uses the word *symmetric space* for what we call a symmetric object here.

In Section 4.2, we will need both notions to prove that our tensor triangular Chow–Witt groups indeed generalize the classical Chow–Witt groups.

For the convenience of the reader, we briefly recall the definition of exact categories with duality.

Definition 3.2.1. We call a triple $(\mathcal{E}, *, \eta)$ an *exact category with duality* if \mathcal{E} is an exact category in the sense of Quillen, $*$: $\mathcal{E} \rightarrow \mathcal{E}^{\mathrm{op}}$ is an exact functor, and $\eta_A : A \xrightarrow{\cong} A^{**}$ is a natural isomorphism fulfilling $\mathrm{id}_{A^*} = \eta_A^* \circ \eta_A$ for all objects A in \mathcal{E} , called the double dual identification.

As in the case of triangulated categories, we now want to introduce symmetric objects on exact categories with duality.

Definition 3.2.2. Let $(\mathcal{E}, *, \eta)$ be an exact category with duality. A *symmetric object* is a pair (M, ϕ) , where M is an object of \mathcal{E} and $\phi : M \xrightarrow{\cong} M^*$ a symmetric isomorphism, i.e. an isomorphism satisfying $\phi^* \eta = \phi$. An *isometry* between two symmetric objects (M, ϕ) and (N, ψ) is an isomorphism $f : M \xrightarrow{\cong} N$ such that $\phi = f^* \psi f$.

We denote the Grothendieck group of the abelian monoid of isometry classes of symmetric objects of \mathcal{E} with the orthogonal sum as group operation by $\text{SymOb}(\mathcal{E})$.

Remark 3.2.3. In Definition 3.1.1 and Definition 3.2.2, we defined symmetric objects as objects A equipped with a symmetric isomorphism to its dual $\varphi : A \xrightarrow{\cong} A^*$. In the example of vector spaces over a field K , there is moreover the notion of a *symmetric bilinear form*. By definition, this is a map

$$B : V \times V \rightarrow K$$

satisfying the following three conditions:

- (i) $B(x, y) = B(y, x)$,
- (ii) $B(x + y, z) = B(x, z) + B(y, z)$,
- (iii) $B(\lambda x, y) = \lambda B(x, y)$ for all $x, y, z \in V$ and $\lambda \in K$.

Given a symmetric object (V, φ) , we obtain a symmetric bilinear form by setting $B(x, y) := \varphi(x)(y)$. Conversely, a symmetric bilinear form B yields a symmetric object (V, φ) with $\varphi(x) := B(x, -)$. Hence, we can switch between symmetric isomorphisms to its dual and symmetric bilinear forms. For more details, see [Scha85, Chapter 1, §2].

Definition 3.2.4. Let $(\mathcal{E}, *, \eta)$ be an exact category with duality.

- (i) A *totally isotropic subobject* of a symmetric object (M, φ) is an admissible monomorphism $i : L \rightarrow M$ such that $0 = i^* \varphi i$, and such that $L \rightarrow \ker(i^* \varphi) \subset M$ is also an admissible monomorphism.
- (ii) A *Lagrangian* of a symmetric object (M, φ) is a totally isotropic subobject $L \xrightarrow{i} M$ such that

$$L \xrightarrow{i} M \xrightarrow{i^* \varphi} L^*$$

is an exact sequence in \mathcal{E} .

- (iii) A symmetric object (M, φ) is called *metabolic* if it has a Lagrangian.
- (iv) A *hyperbolic* object is a symmetric object $\mathbb{H}(A)$ for some object A of the form

$$\mathbb{H}(A) := (A \oplus A^*, \begin{bmatrix} 0 & 1_{A^*} \\ \eta_A & 0 \end{bmatrix}).$$

Definition 3.2.5. Let $(\mathcal{E}, *)$ be an exact category with duality. We define the *Grothendieck–Witt group* $\mathrm{GW}_0(\mathcal{E}, *) := \mathrm{GW}(\mathcal{E}, *)$ as the quotient of $\mathrm{SymOb}(\mathcal{E})$ divided by the relation $[(M, \varphi)] = [\mathbb{H}(L)]$ for metabolic objects $[(M, \varphi)]$ with Lagrangian L .

The *Witt group* $\mathrm{W}_0(\mathcal{E}, *) := \mathrm{W}(\mathcal{E}, *)$ is the quotient of $\mathrm{GW}_0(\mathcal{E}, *)$ modulo the relation $[\mathbb{H}(A)] = 0$ for any hyperbolic object $\mathbb{H}(A)$.

Example 3.2.6. Given a ring R with involution and a scheme X together with a line bundle L , the category of finitely generated projective right R -modules $\mathrm{P}(R)$ and the category of algebraic vector bundles $\mathrm{Vect}(X)$ are exact categories with duality via the dualities $\mathrm{Hom}_R(-, R)$ and $\mathrm{Hom}_{\mathcal{O}_X}(-, L)$, respectively, compare [Sch10a, Examples 2.2 and 2.3].

We now verify the compatibility of the definition of Grothendieck–Witt groups of exact categories with duality and triangulated categories with duality. The argument in particular implies that the relation $[(M, \varphi)] = [\mathbb{H}(L)]$ is always fulfilled in the split-exact category $\mathrm{P}(k)$ of finite-dimensional k -vector spaces as stated in [QSS79, Theorem 5.6].

Proposition 3.2.7. [[Wal03, Theorem 6.1], [Bal01, Theorem 4.3]] *Let k be a field of characteristic $\neq 2$. Then, we have a group isomorphism of Grothendieck–Witt groups*

$$\mathrm{GW}(k) := \mathrm{GW}(\mathrm{P}(k), \mathrm{Hom}_k(-, k)) \cong \mathrm{GW}(\mathrm{D}^{\mathrm{perf}}(\mathrm{Spec}(k)), \mathrm{hom}(-, \mathbb{1}))$$

as well as a group isomorphism

$$\mathrm{W}(k) := \mathrm{W}(\mathrm{P}(k), \mathrm{Hom}_k(-, k)) \cong \mathrm{W}(\mathrm{D}^{\mathrm{perf}}(\mathrm{Spec}(k)), \mathrm{hom}(-, \mathbb{1})).$$

Remark 3.2.8. In the above proposition, the isomorphisms generalizes to isomorphisms

$$\mathrm{GW}(\mathrm{P}(k), \mathrm{Hom}_k(-, L)) \cong \mathrm{GW}(\mathrm{D}^{\mathrm{perf}}(\mathrm{Spec}(k)), \mathrm{hom}(-, L))$$

(and for Witt groups respectively) for any line bundle L on $\mathrm{Spec}(k)$ corresponding to an invertible k -vector space L , i.e. a vector space of dimension 1 over k . Note that $\mathrm{GW}(k, L) \cong \mathrm{GW}(k)$ and $\mathrm{W}(k, L) \cong \mathrm{W}(k)$ for any invertible k -vector space L , but this isomorphism is non-canonical. In the theory of Chow–Witt groups, twists of this sort can be interpreted as local orientations, see [Deg23, Remark 2.1.15].

We will need additional structure in the form of *weak equivalences*, similar to the idea of Waldhausen categories in algebraic K -theory, in order to formulate localization sequences.

Definition 3.2.9. (i) An *exact category with weak equivalences* (\mathcal{E}, w) consists of an exact category \mathcal{E} in the sense of [Qui73] (compare [Kel96, Chapter 4]) and a set of morphisms w that contains all identity morphisms, is closed under isomorphisms, composition, retracts, pushouts along admissible monomorphisms, pullbacks along admissible epimorphisms, and satisfies the 2-out-of-3 property. We call morphisms in w *weak equivalences*.

- (ii) An *exact category with weak equivalences and duality* is a collection of data $(\mathcal{E}, w, *, \eta)$ such that (\mathcal{E}, w) is an exact category with weak equivalences and $(\mathcal{E}, *, \eta)$ is an exact category with (not necessarily strong) duality such that
- (i) $*$: $(\mathcal{E}^{\text{op}}, w) \rightarrow (\mathcal{E}, w)$ is exact, hence, in particular, $*(w) \subset w$, and
 - (ii) $\eta_X \in w$ for all objects X .

Remark 3.2.10. If $(\mathcal{E}, w, *, \eta)$ is an exact category with weak equivalences and duality, the subcategory $w\mathcal{E}$ of weak equivalences is an exact category with duality.

The example of chain complexes of coherent locally free sheaves on a scheme X will play an important role in showing the agreement of algebro-geometric Chow–Witt groups and tensor triangular Chow–Witt groups. It can be considered an exact category with weak equivalences and duality in the following way.

Example 3.2.11. Let k be a field, X be a k -scheme, and L a line bundle on X . By [Sch11, Example 3.2.4] the exact category with duality of coherent locally free sheaves on a scheme X

$$(P(X), \text{Hom}_{\mathcal{O}_X}(-, L))$$

yields an exact category with weak equivalences and duality (see [Sch10b, Section 6.1])

$$(\text{Ch}^b(P(X)), \text{quis}, \text{Hom}_{\mathcal{O}_X}(-, L))$$

by fixing the quasi-isomorphisms $w := \text{quis}$, which is moreover *complicial* in the sense of [Sch11, Definition 3.2.2]. We define for $n \in \mathbb{N}$ the subcategory of complexes with homology supported in codimension $\geq n$

$$\text{Ch}^b(P(X))^{(n)} := \{P_{\bullet} \in \text{Ch}^b(P(X)) \mid \text{codim}(\text{supph}(P_{\bullet})) \geq n\}$$

(see Definition 4.2.5 for the definition of the homological support). By $\text{quis}^{n'}$, we denote the morphisms in $\text{Ch}^b(P(X))^{(n)}$ with cone in $\text{Ch}^b(P(X))^{(n')}$ for $n' \geq n$. Then, for any $n' \geq n$,

$$(\text{Ch}^b(P(X))^{(n)}, \text{quis}^{n'}, \text{Hom}_{\mathcal{O}_X}(-, L))$$

is a $(\mathbb{Z}[\frac{1}{2}]$ -)complicial exact category with weak equivalences and duality, compare [FS09, Definition 6]. We denote the triangulated category where we invert the quasi-isomorphisms $\text{quis}^{n'}$ by $\text{Ch}^b(P(X))^{(n)}[(\text{quis}^{n'})^{-1}]$.

Remark 3.2.12. In a preprint version of [Sch17], Schlichting used the language of complicial exact categories with weak equivalences and duality as we do here. The published version is written in the setting of dg-categories, but Schlichting explains in the introduction of *loc. cit.* (p.1733) that, in practice, all complicial exact categories have a *dg-enhancement*. They might have different exact structures, but this does not affect the GW-groups. For this reason, our references to [Sch17] remain valid despite the fact that we work in the complicial setting. For further details, the reader is referred to [Sch10b, Sch17].

Analogously to the triangulated case, one can define for $n \in \mathbb{Z}$ the shifted Grothendieck–Witt groups $\mathrm{GW}_0^{[n]}(\mathcal{E}, w, *, \eta) := \mathrm{GW}^{[n]}(\mathcal{E}, w, *, \eta) =: \mathrm{GW}^{[n]}(\mathcal{E})$ and Witt groups $\mathrm{W}_0^{[n]}(\mathcal{E}, w, *, \eta) := \mathrm{W}^{[n]}(\mathcal{E}, w, *, \eta) =: \mathrm{W}^{[n]}(\mathcal{E})$ of a complicial exact category with weak equivalences and duality. We will not give the explicit definition but refer the reader to [Sch17].

Now, we discuss the compatibility between the concept of GW of exact categories with duality and exact categories with weak equivalences and duality, yielding an intuitive result well-known for more theories such as algebraic K -theory.

The result is stated in a more general way. Expanding the definitions we just gave, one can define *higher (shifted) Grothendieck–Witt groups* $\mathrm{GW}_m^{[n]}(\mathcal{E})$, $m \geq 1$, for complicial exact categories with duality (see [Sch10a, Section 4]) as well as for exact categories with weak equivalences and duality (see [Sch10b, Section 2.7]).

Proposition 3.2.13. ([Sch10b, Proposition 6]) *Let $(\mathcal{E}, *)$ be an exact category with duality, $m \in \mathbb{N}$. There are isomorphisms of (higher) Grothendieck–Witt groups induced by the inclusion as complexes concentrated in degree 0*

$$\mathrm{GW}_m(\mathcal{E}, *) \xrightarrow{\cong} \mathrm{GW}_m^{[0]}(\mathrm{Ch}^b(\mathcal{E}), \mathrm{quis}, *).$$

Moreover, we have the following agreement between Grothendieck–Witt groups of exact categories with weak equivalences and duality, and of triangulated categories.

Proposition 3.2.14. ([Sch17, Proposition 3.8]) *For an exact category with duality $(\mathcal{E}, *)$, $n, n' \in \mathbb{N}$, $n' \geq n$, $i \in \mathbb{Z}$, we have isomorphisms of (shifted) Grothendieck–Witt groups*

$$\mathrm{GW}_0^{[i]}(\mathrm{Ch}^b(\mathcal{E})^{(n)}, \mathrm{quis}^{n'}, *) \cong \mathrm{GW}_0^{[i]}(\mathrm{Ch}^b(\mathcal{E})^{(n)}[(\mathrm{quis}^{n'})^{-1}]).$$

Remark 3.2.15. Let X be a scheme and L a line bundle on X . We follow the idea of Fasel and Srinivas ([FS09]) in Theorem 3.3.4. For $\mathrm{Ch}^b(P(X))^{(n)}[(\mathrm{quis}^{n'})^{-1}]$, they use the notation $\mathrm{D}^b(P(X))^{n/n'}$. Moreover, they denote by $\mathrm{GW}_m(\mathrm{D}^b(P(X))^{n/n'})$ the m -th higher Grothendieck–Witt group of the complicial exact category with duality $(\mathrm{Ch}^b(P(X))^{(n)}, \mathrm{quis}^{n'}, \mathrm{Hom}_{\mathcal{O}_X}(-, L))$. In our case, we consider this notation misleading since it suggests the existence of higher Grothendieck–Witt groups of triangulated categories with duality. However, so far we do not have such a definition, hence the change of notation.

Note that, as in the case of rings, negative Grothendieck–Witt groups ([Sch17]) of complicial exact categories with weak equivalences and duality can be defined and recover Witt groups in the following sense.

Proposition 3.2.16. ([Sch17, Proposition 6.3]) *Let $\mathcal{A} = (\mathcal{A}, w, *, \eta)$ be a complicial exact category with weak equivalences and duality, $m < 0$, and $n \in \mathbb{Z}$. Then, there are isomorphisms*

$$\mathrm{GW}_m^{[n]}(\mathcal{A}, w) \cong \mathrm{GW}_{m-1}^{[n-1]}(\mathcal{A}, w)$$

and

$$\mathrm{W}^{[n-m]}(\mathcal{A}, w) \cong \mathrm{GW}_m^{[n]}(\mathcal{A}, w).$$

Together with [Bal01, Theorem 4.3] this yields the following comparison.

Corollary 3.2.17. *Let $(\mathcal{E}, *)$ be an exact category with duality, $n, n' \in \mathbb{N}$, $n' \geq n$, $m < 0$, and $i \in \mathbb{Z}$*

$$\begin{aligned} \mathrm{GW}_m^{[i]}(\mathrm{Ch}^b(\mathcal{E})^{(n)}, \mathrm{quis}^{n'}, *) &\cong \mathrm{W}^{[i-m]}(\mathrm{Ch}^b(\mathcal{E})^{(n)}, \mathrm{quis}^{n'}, *) \\ &\cong \mathrm{W}^{[i-m]}(\mathrm{Ch}^b(\mathcal{E})^{(n)}[(\mathrm{quis}^{n'})^{-1}]). \end{aligned}$$

In a nutshell, this section introduced the notion of (Grothendieck–)Witt groups of exact categories with duality as well as of complicial exact categories with weak equivalences and duality. For an exact category with duality $(\mathcal{E}, *)$, we have seen that its (Grothendieck–)Witt groups coincide with the ones for the category with weak equivalences and duality $(\mathrm{Ch}^b(\mathcal{E}), \mathrm{quis}, *, \eta)$ as well as with the (Grothendieck–)Witt groups of triangulated categories with duality after inverting the quasi-isomorphisms.

These concepts and their connection will be essential in the proof of the central agreement theorem of tensor triangular Chow–Witt groups and classical Chow–Witt groups in Section 4.2. The main ingredients will be the different forms of localization sequences arising from the different GW/W-theories we have seen so far in this chapter.

3.3 Localization

As already mentioned, one central idea of the agreement between triangular and algebro-geometric Chow–Witt groups lies in the agreement of the different localization sequences. To lay the foundations for this, we discuss localization sequences for GW and W in the different settings of triangulated categories and exact categories.

We begin with the triangulated setting in which a lot of work on Witt groups has been done by Balmer, e.g. in [Bal00, Bal01]. Although Walter did not publish the article [Wal03] in which he stated the localization theorem for Grothendieck–Witt and Witt groups of triangulated categories, we still rely on this source since Grothendieck–Witt groups play a more central role here.

Let us first recall the definition of *exact sequences* in the triangulated setting.

Definition 3.3.1. We say that a sequence $\mathcal{J} \rightarrow \mathcal{T} \rightarrow \mathcal{L}$ of triangulated categories with duality and duality-preserving functors is a *localization* or *exact sequence* if $\mathcal{L} = S^{-1}\mathcal{T}$ for some class S of morphisms and \mathcal{J} is the full subcategory of \mathcal{T} spanned by those objects that become isomorphic to 0 in \mathcal{L} .

Theorem 3.3.2. ([Bal00, Theorem 6.2 and Corollary/Definition 5.16]) *If we are given a localization of triangulated categories with duality $\mathcal{J} \xrightarrow{j} \mathcal{T} \xrightarrow{q} S^{-1}\mathcal{T}$ such that $A \oplus B \cong B$ implies $A \cong 0$ in $S^{-1}\mathcal{T}$ (i.e. $S^{-1}\mathcal{T}$ is weakly cancellative), $\frac{1}{2} \in \mathcal{T}$,*

and \mathcal{T} satisfies the axiom (TR4⁺) in [Bal00, Section 1], we obtain a long exact sequence for $n \in \mathbb{Z}$

$$\begin{aligned} \dots &\rightarrow \mathbb{W}^{[n-1]}(S^{-1}\mathcal{T}) \xrightarrow{d^{n-1}} \mathbb{W}^{[n]}(\mathcal{J}) \xrightarrow{\mathbb{W}^{n(j)}} \mathbb{W}^{[n]}(\mathcal{T}) \\ &\xrightarrow{\mathbb{W}^{[n](q)}} \mathbb{W}^{[n]}(S^{-1}\mathcal{T}) \xrightarrow{d^n} \mathbb{W}^{[n+1]}(\mathcal{J}) \rightarrow \dots, \end{aligned}$$

where d^n is the well-defined homomorphism

$$d^n : \mathbb{W}^{[n]}(S^{-1}(\mathcal{T})) \rightarrow \mathbb{W}^{[n+1]}(\mathcal{J}), x \mapsto [\text{cone}(A, s)]$$

with (A, s) being any S -space (see [Bal00], Definition 5.4) such that $x = [q(A), q(s)]$.

Walter then derives the following localization theorem involving Grothendieck–Witt groups.

Theorem 3.3.3. ([Wal03, Localization Theorem 2.4]) *For $n \in \mathbb{N}$, Balmer’s long exact sequence from above extends to an exact sequence*

$$\text{GW}^{[n]}(\mathcal{J}) \rightarrow \text{GW}^{[n]}(\mathcal{T}) \rightarrow \text{GW}^{[n]}(S^{-1}\mathcal{T}) \xrightarrow{d^n} \mathbb{W}^{[n+1]}(\mathcal{J}) \rightarrow \mathbb{W}^{[n+1]}(\mathcal{T}) \rightarrow \dots$$

Now, to establish a localization sequence in the setting of complicial exact categories with weak equivalences and duality, it is not necessary for our purposes to state it in the full generality of [Sch17, Theorem 6.6] but only for $(\text{Ch}^b(P(X))^{(n)}, \text{quis})$ and $(\text{Ch}^b(P(X))^{(n)}, \text{quis}^{n'})$ from Example 3.2.11.

Theorem 3.3.4. ([FS09, Theorem 8], [Sch17, Theorem 6.6]) *For a k -scheme X with $\text{char}(k) \neq 2$, a line bundle L , and $n, n' \in \mathbb{N}$ such that $n' \geq n$, consider the functors of complicial exact categories with weak equivalences and duality*

$$(\text{Ch}^b(P(X))^{(n')}, \text{quis}) \xrightarrow{I_{\text{Ch}}} (\text{Ch}^b(P(X))^{(n)}, \text{quis}) \xrightarrow{Q_{\text{Ch}}} (\text{Ch}^b(P(X))^{(n)}, \text{quis}^{n'}),$$

where all dualities are derived from $\text{Hom}_{\mathcal{O}_X}(-, L)$, see Example 3.2.11.

They yield a long exact sequence for all $i, m \in \mathbb{Z}$

$$\begin{aligned} \dots &\rightarrow \text{GW}_m^{[i]}(\text{Ch}^b(P(X))^{(n')}, \text{quis}) \xrightarrow{i_{\text{Ch}}^{i,m}} \text{GW}_m^{[i]}(\text{Ch}^b(P(X))^{(n)}, \text{quis}) \\ &\xrightarrow{q_{\text{Ch}}^{i,m}} \text{GW}_m^{[i]}(\text{Ch}^b(P(X))^{(n)}, \text{quis}^{n'}) \\ \dots &\xrightarrow{d_{\text{Ch}}^m} \text{GW}_{m-1}^{[i]}(\text{Ch}^b(P(X))^{(n')}, \text{quis}) \xrightarrow{i_{\text{Ch}}^{i,m-1}} \text{GW}_{m-1}^{[i]}(\text{Ch}^b(P(X))^{(n)}, \text{quis}) \rightarrow \dots \end{aligned}$$

The sequence starting at $\text{GW}_0^{[i]}(\text{Ch}^b(P(X))^{(n')})$ coincides with the sequence

$$\begin{aligned} &\text{GW}^{[i]}(\text{Ch}^b(P(X))^{(n')}[(\text{quis})^{-1}]) \xrightarrow{i^i} \text{GW}^{[i]}(\text{Ch}^b(P(X))^{(n)}[(\text{quis})^{-1}]) \\ &\xrightarrow{q^i} \text{GW}^{[i]}(\text{Ch}^b(P(X))^{(n)}[(\text{quis}^{n'})^{-1}]) \xrightarrow{d^i} \mathbb{W}^{[i+1]}(\text{Ch}^b(P(X))^{(n')}[(\text{quis})^{-1}]) \rightarrow \dots \end{aligned}$$

of triangulated categories with duality from Theorem 3.3.3 for the localization

$$\text{Ch}^b(P(X))^{(n')}[(\text{quis})^{-1}] \rightarrow \text{Ch}^b(P(X))^{(n)}[(\text{quis})^{-1}] \rightarrow \text{Ch}^b(P(X))^{(n)}[(\text{quis}^{n'})^{-1}].$$

In this chapter, we have seen localization sequences for Grothendieck–Witt and Witt groups in different settings. On the one hand, Balmer and Walter give a localization sequence for triangulated categories with duality. On the other hand, Fasel, Srinivas, and Schlichting show localization for complicit exact categories with weak equivalences and duality. These sequences coincide by construction, as Fasel and Srinivas state.

Having introduced (Grothendieck–)Witt groups and localization for triangulated and exact categories in this chapter and having shown their agreement, we now have enough foundation to generalize the definition of Chow–Witt groups of schemes to Chow–Witt groups of rigid tensor triangulated categories in the next chapter and prove the agreement of these two approaches.

Chapter 4

Tensor Triangular Chow–Witt Groups

In this work’s central chapter, we construct *tensor triangular Chow–Witt groups* and justify their name by showing the equivalence to Chow–Witt groups of a regular scheme X for the derived category of perfect complexes $D^{\text{perf}}(X)$.

In order to do so, we refer to two underlying generalization concepts. On the one hand, we let the construction be inspired by the passage from classical Chow groups to classical Chow–Witt groups. On the other hand, Balmer’s definition and Klein’s work on tensor triangular Chow groups ([Bal13, Kl16a, Kl16b]) will lead the way to lift the algebro-geometric definition to the tensor triangular setting.

Consequently, this chapter needs to begin with recalling these underlying ideas. In Section 4.1, Balmer’s idea of tensor triangular Chow groups and Klein’s work on the agreement with the algebro-geometric definition are reconstructed. After that, Section 4.2 will give an overview of Chow–Witt groups of a scheme, before we can sketch the idea and give the concrete definition of tensor triangular Chow–Witt groups in Section 4.3, but with the restriction of certain quotients being idempotent complete. Section 4.4 is then dedicated to the proof of the agreement theorem Theorem 4.4.5, making use of the foundations we presented in Chapter 2 and Chapter 3. Section 4.5 treats the functoriality of the construction before Section 4.6 focuses on the case when the quotients mentioned above are not idempotent complete, which will lead to the introduction of *models*.

4.1 Tensor Triangular Chow Groups

A standard reference for classical Chow groups of schemes is [Ful98]. Let us first recall the definition of tensor triangular Chow groups that was first presented by Paul Balmer in [Bal13] and elaborated by Sebastian Klein in [Kl16a] and [Kl16b]. We closely follow [Kl16a] here.

The rough idea is to use a definition of classical Chow groups based on a long exact sequence in K -theory alternative to the original one of cycles modulo rational equivalence (compare [Ful98, Section 1.3]). This sequence comes from

a filtration of the abelian category of coherent sheaves $\text{Coh}(X)$ on a separated, non-singular scheme X of finite type over a base field k by codimension of support. This filtration can be transferred to the tensor triangular setting as introduced in Section 2.3. This way, the definition of Chow groups as the cohomology of the *Rost–Schmid complex* at a certain point can be generalized.

Convention 4.1.1. Let \mathcal{T} be an essentially small rigid tensor triangulated category equipped with a dimension function \dim and such that $\text{Spc}(\mathcal{T})$ is noetherian. Moreover, let X be a regular, separated scheme of finite type over a field k .

Klein begins with considering a filtration of the abelian category of coherent sheaves $\text{Coh}(X)$

$$\dots \subset M^n \subset M^{n-1} \subset \dots \subset M^0 = \text{Coh}(X),$$

where M^n is the abelian subcategory of $\text{Coh}(X)$ with $\text{codim}(\text{supp}(A)) \geq n$ for A in $\text{Coh}(X)$. Since the components of this filtration are moreover *Serre subcategories* of the respective next component, one obtains quotient abelian categories M^n/M^{n+1} and, thus, short exact sequences

$$M^{n+1} \rightarrow M^n \rightarrow M^n/M^{n+1}.$$

These yield a long exact sequence in K -theory

$$\begin{aligned} \dots \rightarrow K_m(M^{n+1}) &\xrightarrow{i_m^n} K_m(M^n) \xrightarrow{q_m^n} K_m(M^n/M^{n+1}) \\ &\xrightarrow{b_m^n} K_{m-1}(M^{n+1}) \xrightarrow{i_{m-1}^n} K_{m-1}(M^n) \xrightarrow{q_{m-1}^n} K_{m-1}(M^n/M^{n+1}) \rightarrow \dots \end{aligned}$$

One considers the composition

$$d_1 : K_1(M^{n-1}/M^n) \xrightarrow{b_1^{n-1}} K_0(M^n) \xrightarrow{q_0^n} K_0(M^n/M^{n+1})$$

and uses Quillen’s theorem [Qui73, Chapter 7, Proposition 5.14] to prove that the image of d_1 equals the subgroup of codimension- n -cocycles rationally equivalent to 0. That means, one has

$$Z^n(X)/\text{Im}(d_1) \cong \text{CH}^n(X),$$

where $Z^n(X) = K_0(M^n/M^{n+1})$, compare [Kl16a, Remark 3.1.3]. Moreover, this proves that, by using the exactness of the above long exact sequence in K -theory of abelian categories, we have

$$\text{Im}(d_1) = \text{Im}(q_0^n \circ b_1^{n-1}) = q_0^n(\text{Im}(b_1^{n-1})) = q_0^n(\ker(i_0^{n-1})).$$

Thus, the codimension- n -cocycles rationally equivalent to 0 are, moreover, equivalent to $q_0^n(\ker(i_0^{n-1}))$, which gives us an inspiration for the tensor triangular definition.

The maps i_0^n and q_0^n are straightforward to translate into the tensor triangular setting. For a tensor triangulated category \mathcal{T} with dimension function, we obtain a filtration by dimension of support as mentioned in Section 2.3

$$\mathcal{T}_{(-\infty)} \subset \dots \subset \mathcal{T}_{(n-1)} \subset \mathcal{T}_{(n)} \subset \dots \subset \mathcal{T}_{(\infty)} = \mathcal{T}.$$

At first sight, this mimics the filtration of the category $\text{Coh}(X)$ of coherent sheaves, but keep in mind that $\text{Coh}(X)$ is *not* a tensor triangulated category since it is not even triangulated. Moreover, one has to be careful about the indices since the sign will change when passing to the tensor triangular setting later.

For the agreement with the algebro-geometric case, we will make use of the passage to its bounded derived category $D^b(\text{Coh}(X))$, but first, let us consider the following diagram of triangulated categories and functors coming from the above filtration

$$\begin{array}{ccc} \mathcal{T}_{(n)} & \xrightarrow{I_{\mathcal{T}}^{n+1}} & \mathcal{T}_{(n+1)} \\ \downarrow Q_{\mathcal{T}}^n & & \\ \mathcal{T}_{(n)}/\mathcal{T}_{(n-1)} & & \\ \downarrow J_{\mathcal{T}}^n & & \\ (\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})^{\natural} & & \end{array}$$

for each $n \in \mathbb{Z}$.

Here, $I_{\mathcal{T}}^{n+1}$ is the inclusion functor, $Q_{\mathcal{T}}^n$ the localization functor, and $J_{\mathcal{T}}^n$ the idempotent completion functor.

We will later explain why we need to include the idempotent completion $J_{\mathcal{T}}^n$ in this process, but first, recall that the idempotent completion of a triangulated category naturally inherits the structure of a triangulated category ([BS01]). Now, one can apply K_0 to this diagram to obtain

$$\begin{array}{ccc} K_0(\mathcal{T}_{(n)}) & \xrightarrow{i_{\mathcal{T}}^{n+1}} & K_0(\mathcal{T}_{(n+1)}) \\ \downarrow q_{\mathcal{T}}^n & & \\ K_0(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}) & & \\ \downarrow j_{\mathcal{T}}^n & & \\ K_0((\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})^{\natural}), & & \end{array} \quad (4.1)$$

which we call the *defining diagram* of tensor triangular Chow groups. Inspired by the alternative definition of cocycles rationally equivalent to 0 as $q_0^n(\ker(i_0^{n-1}))$, triangular algebraic cocycles and Chow groups are defined as follows.

Definition 4.1.2. For an essentially small rigid tensor triangulated category $(\mathcal{T}, \otimes, \mathbf{1})$ equipped with a dimension function \dim such that $\text{Spc}(\mathcal{T})$ is noetherian, its n -th tensor triangular Chow group, $n \in \mathbb{Z}$, is defined as

$$\text{CH}_{\Delta}^n(\mathcal{T}) := Z_{\Delta}^n(\mathcal{T}) / j_{\mathcal{T}}^n \circ q_{\mathcal{T}}^n(\ker(i_{\mathcal{T}}^{n+1})),$$

where the n -th tensor triangular cocycles are given as follows

$$Z_{\Delta}^n(\mathcal{T}) := K_0((\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})^{\natural})$$

and $j_{\mathcal{T}}^n, q_{\mathcal{T}}^n$ and $i_{\mathcal{T}}^{n+1}$ are the morphisms from Diagram 4.1.

To see the agreement of this definition to the original Chow Witt groups of a regular scheme X for the tensor triangulated category $D^{\text{perf}}(X)$, see Definition 2.1.18, there are a few things left to show.

First, Klein proves that tensor triangular cocycles and algebraic cocycles are equivalent for that special case. He makes use of theorems on the algebraic geometric side as well as on the tensor triangular side as follows.

Quillen’s dévissage theorem yields equivalences for all $n \geq 0$

$$K_0(M^n/M^{n+1}) \cong \bigoplus_{x \in X^{(n)}} K_0(k(x)) \cong \bigoplus_{x \in X^{(n)}} \mathbb{Z} \cong Z^n(X),$$

showing that the Grothendieck group of the quotient M^n/M^{n+1} obtained by the filtration of $\text{Coh}(X)$ by codimension of support equals the usual cocycle groups $Z^n(X)$ of a regular scheme X . This is an elegant way to overcome the concrete description of algebraic cocycles and instead use an alternative definition that we can translate more directly into the tensor triangular settings.

In the tensor triangular case, using Balmer’s Theorem 2.3.9, we obtain an equivalence after applying K_0

$$Z_{\Delta}^n(\mathcal{T}) = K_0((\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})^{\natural}) \cong \bigoplus_{\substack{\mathcal{P} \in \text{Spc}(\mathcal{T}) \\ \dim(\mathcal{P})=n}} K_0(\text{Min}(\mathcal{T}_{\mathcal{P}})).$$

The necessity of the idempotent completion for the equivalence in Theorem 2.3.9 now motivates us to include the idempotent completion functors $J_{\mathcal{T}}^n$ and the maps $j_{\mathcal{T}}^n$, which will become the identity in the algebro-geometric case for regular X (see below).

Eventually, Klein proves the following.

Lemma 4.1.3. ([Kl16a, Corollary 3.1.9]) *For a noetherian, regular scheme X and $\mathcal{T} := D^{\text{perf}}(X)$ equipped with the dimension function $-\text{codim}_{K_{\text{rull}}}$, there is an isomorphism for all $n \geq 0$*

$$Z_{\Delta}^{-n}(\mathcal{T}) \xrightarrow{\cong} Z^n(X).$$

After having generalized the definition of algebraic cocycles and seen their agreement, the second problem in the construction is to compare the defining diagram in the algebro-geometric setting to the tensor triangulated case. We at least have homomorphisms (the diagonal maps in Diagram 4.2) between the diagrams given by the formula

$$[C^{\bullet}] \mapsto \sum_i (-1)^i [H^i(C^{\bullet})],$$

yielding the following diagram

$$\begin{array}{ccccc}
 K_0(\mathcal{D}^b(X)_{(n)}) & \xrightarrow{i_{\mathcal{T}}^{n+1}} & K_0(\mathcal{D}^b(X)_{(n+1)}) & & (4.2) \\
 \downarrow q_{\mathcal{T}}^n & \searrow & \downarrow & \searrow & \\
 K_0(\mathcal{D}^b(X)_{(n)}/\mathcal{D}^b(X)_{(n-1)}) & & K_0(M^{-n}) & \xrightarrow{i_0^{-n-1}} & K_0(M^{-n-1}) \\
 \downarrow j_{\mathcal{T}}^n & \searrow & \downarrow q_0^{-n} & & \\
 K_0((\mathcal{D}^b(X)_{(n)}/\mathcal{D}^b(X)_{(n-1)})^\natural) & & K_0(M^{-n}/M^{-n+1}) & &
 \end{array}$$

For the justification of the construction of tensor triangular Chow groups, it is now left to show that the diagonal arrows are isomorphisms and the map $j_{\mathcal{T}}^n$ is the identity. For this, Klein applies the following theorem.

Theorem 4.1.4. ([Kel99, Section 1.15]) *Let \mathcal{A} be a Serre subcategory of an abelian category \mathcal{B} and let the following criterion be satisfied:*

If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence in \mathcal{B} with A in \mathcal{A} , then there is a commutative diagram with exact rows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\
 & & \text{id} \downarrow & & f \downarrow & & g \downarrow \\
 0 & \longrightarrow & A & \longrightarrow & A' & \longrightarrow & A'' \longrightarrow 0
 \end{array}$$

such that A', A'' are in \mathcal{A} .

Then, the inclusion induces a triangulated equivalence of triangulated categories $\mathcal{D}^b(\mathcal{A}) \xrightarrow{\simeq} \mathcal{D}_{\mathcal{A}}^b(\mathcal{B})$, where $\mathcal{D}_{\mathcal{A}}^b(\mathcal{B}) \subset \mathcal{D}^b(\mathcal{B})$ is the full subcategory of complexes with homology in \mathcal{A} .

Moreover, the sequence

$$\mathcal{D}^b(\mathcal{A}) \rightarrow \mathcal{D}^b(\mathcal{B}) \rightarrow \mathcal{D}^b(\mathcal{B}/\mathcal{A})$$

is exact, i.e. $\mathcal{D}^b(\mathcal{B}/\mathcal{A}) \simeq \mathcal{D}^b(\mathcal{B})/\mathcal{D}^b(\mathcal{A})$.

The conditions are satisfied by [Kl16a, Lemma 3.2.5] if X is regular, so, one obtains an equivalence

$$\mathcal{D}^b(M^{-n}) \simeq \mathcal{D}_{M^{-n}}^b(\text{Coh}(X)) = \mathcal{D}^b(X)_{(n)}$$

for all n , where $\mathcal{D}_{M^{-n}}^b(\text{Coh}(X))$ consists of the complexes in $\mathcal{D}^b(\text{Coh}(X))$ with homology in M^{-n} . Moreover, there is an equivalence of triangulated categories

$$\mathcal{D}^b(M^{-n}/M^{-n+1}) \simeq \mathcal{D}^b(M^{-n})/\mathcal{D}^b(M^{-n+1}).$$

This again establishes equivalences

$$\mathcal{D}^b(X)_{(n)}/\mathcal{D}^b(X)_{(n-1)} \simeq \mathcal{D}^b(M^{-n}/M^{-n+1}).$$

Moreover, we know that there are isomorphisms $K_0(\mathbf{D}^b(\mathcal{A})) \cong K_0(\mathcal{A})$ for abelian categories \mathcal{A} (see e.g. [Wei13, Chapter II, Theorem 9.2.2] which is essentially [SGA6, I.6.4] for Waldhausen categories or [Nee05, Theorem 3] for triangulated categories). In particular, this holds for all components of the filtration of $\mathrm{Coh}(X)$ and its quotients since Serre quotients of abelian categories are abelian. This proves that the diagonal maps in Diagram 4.2) are isomorphisms.

Moreover, the derived category of an abelian category is always idempotent complete, hence so is $\mathbf{D}^b(X)_{(n)}/\mathbf{D}^b(X)_{(n-1)}$ for all n by the above equivalences. As a consequence, $J_{\mathcal{T}}^n$ and thus also $j_{\mathcal{T}}^n$ are the identity.

In [Kl16a, Theorem 3.2.6], the results are summarized and the agreement of tensor triangular and classical Chow groups is shown for the category $\mathbf{D}^{\mathrm{perf}}(X)$ for a separated, regular scheme of finite type.

We have seen in this section the construction of tensor triangular Chow groups and the agreement with the classical definition for schemes when considering the tensor triangulated category $\mathbf{D}^{\mathrm{perf}}(X)$. The main idea was to use an alternative definition of Chow groups as the cohomology of a chain complex arising from the filtration of the category $\mathrm{Coh}(X)$ of coherent sheaves by codimension of support. This filtration can be translated to the tensor triangular setting, giving rise to the definition of tensor triangular cocycles and coboundaries, and, eventually, of tensor triangular Chow groups.

In the next section, basics on Chow–Witt groups of a scheme will be recalled serving as the second inspiration of how Chow–Witt groups of a tensor triangulated category could be constructed.

4.2 Chow–Witt Groups of a Scheme

Chow–Witt groups were originally constructed by Barge and Morel in [BM00] to answer the question of when a projective A -module is isomorphic to the direct sum $Q \oplus A$ for some projective A -module Q . To approach this problem, they constructed the *Euler class* $\tilde{c}_n(E)$ of a vector bundle E as a refinement of the top Chern class $c_n(E)$. One solution to the problem now breaks down to determining whether the Euler class (or Chern class) is zero, see [BM00] for more details. More information on Chow–Witt groups of schemes can be found in e.g. [Fas07, Fas08]. We will mainly follow the latter here.

Convention 4.2.1. Let X be a separated, regular scheme of finite type over a perfect field k of characteristic $\neq 2$.

The Chow–Witt groups of X can be defined as the cohomology of a complex obtained as the fiber product

$$\begin{array}{ccc} C^i(X, G^j) & \longrightarrow & C^i(X, I^j) \\ \downarrow & & \downarrow \\ C^i(X, K_j^{\mathbf{M}}) & \longrightarrow & C^i(X, I^j/I^{j+1}), \end{array}$$

whose components we will define below. If twists by a graded line bundle L are included, the square changes as follows

$$\begin{array}{ccc} C^i(X, G^j, L) & \longrightarrow & C^i(X, I^j, L) \\ \downarrow & & \downarrow \\ C^i(X, K_j^M) & \longrightarrow & C^i(X, I^j/I^{j+1}). \end{array}$$

With this fiber product at hand, one defines:

Definition 4.2.2. Let X be a separated, regular k -scheme of finite type. We define the n -th Chow–Witt group of X as

$$\widetilde{\text{CH}}^n(X) := H^n(C(X, G^n)) = \widetilde{Z}^n(X)/\widetilde{B}^n(X)$$

and the Chow–Witt groups twisted by L for a fixed line bundle L

$$\widetilde{\text{CH}}^n(X, L) := H^n(C(X, G^n, L)) = \widetilde{Z}^n(X, L)/\widetilde{B}^n(X, L).$$

We denote the differentials of $C(X, G^n)$ and $C(X, G^n, L)$ by ∂^n and will talk about these later.

Here, $\widetilde{Z}^n(X)$ (and $\widetilde{Z}^n(X, L)$) are called the (*twisted*) *oriented cocycles* and defined as the kernel of the complex at n . Similarly, $\widetilde{B}^n(X)$ (and $\widetilde{B}^n(X, L)$) denote the (*twisted*) *oriented coboundaries* and define the image of the complex at n .

To understand the complex $C(X, K_n^M)$, we recall that Milnor K -theory $K_*^M(F)$ of a field F is given as the graded tensor algebra

$$T(F^\times) = \mathbb{Z} \oplus F^\times \oplus (F^\times \otimes F^\times) \oplus \dots$$

divided by the equivalence relation $a \otimes (1 - a) = 0$. The Milnor K -groups $K_n^M(F)$ are then given by the n -th homogeneous components, see [Mil70].

The Milnor- K -theory complex is then given as follows. Note that the differentials are slightly complicated and can be found in detail, for example, in [Fas08, Chapter 2].

Definition 4.2.3. ([Fas08, Théorème 2.2.13]) For a noetherian scheme X over a field k and any $n \in \mathbb{Z}$, we define the *Milnor- K -theory complex* $C(X, K_n^M)$ as

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_{x \in X^{(0)}} K_n^M(k(x)) & \xrightarrow{d} & \bigoplus_{x \in X^{(1)}} K_{n-1}^M(k(x)) & \longrightarrow & \dots \\ & & \bigoplus_{x \in X^{(m)}} K_{n-m}^M(k(x)) & \xrightarrow{d} & \bigoplus_{x \in X^{(m+1)}} K_{n-m-1}^M(k(x)) & \longrightarrow & \dots \end{array}$$

Remark 4.2.4. Grading this complex by the codimension of points in X , one obtains isomorphism for all n

$$H^n(C(X, K_n^M)) \cong \text{CH}^n(X),$$

where the right-hand side denotes the Chow groups defined as cocycles modulo rational equivalence as in [Ful98, Section 1.3].

The other complexes involved in the definition of $C(X, G^n, L)$ come from the Gersten–Witt complex $C(X, W, L)$ by restriction to powers of the *fundamental ideals* or quotients of these. First, let us take a look at the construction of the Gersten–Witt complex.

Let $P(X)$ be the exact category of locally free \mathcal{O}_X -modules of finite type over X . Then, by [Bal99, Proposition 2.3], $(D^b(P(X)), R\mathrm{Hom}_{\mathcal{O}_X}(-, L))$ is a triangulated category with duality, and since X is regular, there is an equivalence $D^b(P(X)) \simeq D^b(\mathrm{Coh}(X))$ of triangulated categories with duality when considering the duality $R\mathrm{Hom}(-, L)$ on $D^b(\mathrm{Coh}(X))$.

Now, we can define the following subcategories analogously to Example 3.2.11.

Definition 4.2.5. The *homological support* $\mathrm{supph}(P_\bullet)$ of $P_\bullet \in D^b(P(X))$ is defined as

$$\mathrm{supph}(P_\bullet) := \bigcup_i \mathrm{supp}(H_i(P_\bullet)).$$

Definition 4.2.6. We define for each $n \in \mathbb{N}$

$$D^b(P(X))^{(n)} = \{P_\bullet \in D^b(P(X)) \mid \mathrm{codim}(\mathrm{supph}(P_\bullet)) \geq n\}.$$

This way, we obtain a filtration for $\dim(X) = d$

$$0 = D^b(P(X))^{(d+1)} \subset D^b(P(X))^{(d)} \subset \dots \subset D^b(P(X))^{(0)} = D^b(P(X)).$$

As in Section 4.1, the indices of the filtration decrease with each step. Note that the filtrations of triangulated categories as in Section 2.3 have *increasing* indices, which has as a consequence that the signs swap when comparing tensor triangular Chow(–Witt) groups and Chow(–Witt) groups of a scheme. More details will be given in the proof of Theorem 4.4.5.

Since $D^b(P(X))^{(n)}$ is a thick subcategory of $D^b(P(X))$ for all $n \in \mathbb{N}$, we also have quotients of the filtration components and exact sequences of triangulated categories with duality

$$D^b(P(X))^{(n+1)} \xrightarrow{I^n} D^b(P(X))^{(n)} \xrightarrow{Q^n} D^b(P(X))^{(n)} / D^b(P(X))^{(n+1)},$$

where all the functors are duality-preserving when considering the dualities induced by $R\mathrm{Hom}_{\mathcal{O}_X}(-, L)$ for a fixed line bundle L .

In particular, I^n and Q^n induce maps on Witt groups and Grothendieck–Witt groups for $i \in \mathbb{Z}$

$$\begin{aligned} W^{[i]}(D^b(P(X))^{(n+1)}) &\xrightarrow{i^{n,W}} W^{[i]}(D^b(P(X))^{(n)}) \\ &\xrightarrow{q^{n,W}} W^{[i]}(D^b(P(X))^{(n)} / D^b(P(X))^{(n+1)}) \end{aligned}$$

and

$$\begin{aligned} \mathrm{GW}^{[i]}(D^b(P(X))^{(n+1)}) &\xrightarrow{i^n} \mathrm{GW}^{[i]}(D^b(P(X))^{(n)}) \\ &\xrightarrow{q^n} \mathrm{GW}^{[i]}(D^b(P(X))^{(n)} / D^b(P(X))^{(n+1)}), \end{aligned}$$

respectively. One also obtains a boundary morphism by [Bal00, Theorem 6.2/Corollary and Definition 5.16] which is exactly the one from Theorem 3.3.3,

$$d_n^i : W^{[i]}(\mathbf{D}^b(P(X))^{(n)} / \mathbf{D}^b(P(X))^{(n+1)}) \rightarrow W^{[i+1]}(\mathbf{D}^b(P(X))^{(n+1)})$$

and, hence, a homomorphism

$$\begin{aligned} \tilde{\partial}_n^i &= q^{i+1, W} \circ d_n^i : W^{[i]}(\mathbf{D}^b(P(X))^{(n)} / \mathbf{D}^b(P(X))^{(n+1)}) \\ &\rightarrow W^{[i+1]}(\mathbf{D}^b(P(X))^{(n+1)} / \mathbf{D}^b(P(X))^{(n+2)}). \end{aligned}$$

If $i = n$, we write $d_n^i = d^n$ and $\tilde{\partial}_n^i = \tilde{\partial}^n$.

We can already define a complex with these components and differentials, but to get to the original definition of the Gersten–Witt complex, we consider the following.

Definition 4.2.7. (i) Let A be a regular local ring of dimension d and $A\text{-mod}_{fl}$ be the category of A -modules of finite length. Then, the derived Hom functor

$$\text{Ext}_A^d(-, A) : (A\text{-mod}_{fl})^{\text{op}} \rightarrow A\text{-mod}_{fl}$$

yields a duality on $A\text{-mod}_{fl}$, turning $(A\text{-mod}_{fl}, \text{Ext}_A^d(-, A))$ into an exact category with duality, see [Fas08, Section 3.3].

(ii) We define the *Witt group of finite length modules* as

$$W^{fl}(A) := W(A\text{-mod}_{fl}, \text{Ext}_A^d(-, A)).$$

(iii) The category of A -modules of finite type shall be denoted by $A\text{-mod}_{ft}$.

Theorem 4.2.8. ([Bal01, Theorem 4.3], [BW02, Lemma 6.4]) *We have an isomorphism*

$$W^{fl}(A) \xrightarrow{\cong} W^{[0]}(\mathbf{D}^b(A\text{-mod}_{fl}), \text{Ext}_A^d(-, A))$$

and duality-preserving categorical equivalences

$$\mathbf{D}_{fl}^b(P(A)) \xrightarrow{\cong} \mathbf{D}_{fl}^b(A\text{-mod}_{ft}) \xleftarrow{\cong} \mathbf{D}^b(A\text{-mod}_{fl}),$$

where $\mathbf{D}_{fl}^b(-)$ is the subcategory of complexes with finite length homology yielding an isomorphism

$$W^{[d]}(\mathbf{D}_{fl}^b(P(A))) \xrightarrow{\cong} W^{[0]}(\mathbf{D}^b(A\text{-mod}_{fl}, \text{Ext}_A^d(-, A))).$$

Here, the duality of the bounded derived category of complexes with finite length homology is induced by the duality on the respective original bounded derived category.

We will now use a concrete form of Theorem 2.3.9/Corollary 2.3.10, namely [BW02, Proposition 7.1], to obtain isomorphisms for all n and i after applying W

$$W^{[i]}(\mathbb{D}^b(P(X))^{(n)}/\mathbb{D}^b(P(X))^{(n+1)}) \xrightarrow{\cong} \bigoplus_{x \in X^{(n)}} W^{[i]}(\mathbb{D}_{fl}^b(P(\mathcal{O}_{X,x}))).$$

Now, we can finally define the Gersten–Witt complex, using the above considerations.

Definition 4.2.9. For a regular scheme X of finite dimension, we define the *Gersten–Witt complex* as

$$\dots \longrightarrow \bigoplus_{x \in X^{(n)}} W^{fl}(\mathcal{O}_{X,x}) \xrightarrow{d} \bigoplus_{y \in X^{(n+1)}} W^{fl}(\mathcal{O}_{X,y}) \longrightarrow \dots$$

and denote it by $C(X, W)$.

Remark 4.2.10. One can instead equip $\mathbb{D}^b(X)$ with the duality derived from $\mathrm{Hom}(-, L)$ for an invertible \mathcal{O}_X -module L and obtains analogously the twisted versions $W^{fl}(A, L)$ and $C(X, W, L)$. Note that the complexes $C^n(X, W)$ and $C^n(X, W, L)$ are isomorphic (but not necessarily canonically).

We will briefly recall the pullback square of the defining complex of Chow–Witt groups.

$$\begin{array}{ccc} C^i(X, G^j, L) & \longrightarrow & C^i(X, I^j, L) \\ \downarrow & & \downarrow \\ C^i(X, K_j^M) & \longrightarrow & C^i(X, I^j/I^{j+1}) \end{array}$$

We are now familiar with the Milnor– K -theory complex and the Gersten–Witt complex. To understand the complexes $C^i(X, I^j, L)$ and $C^i(X, I^j/I^{j+1})$, we will need the following lemma.

Lemma 4.2.11. ([MH73, Chapter III, §3, Lemma 3.3]) *Let F be a field. There exists a unique ideal I in $W(F)$ such that $W(F)/I \cong \mathbb{F}_2$. This ideal (consisting of all Witt classes of even dimension) is called the *fundamental ideal* of $W(F)$.*

We want to restrict the Gersten–Witt complex to a generalized version of this fundamental ideal.

Definition 4.2.12. Let B be a regular local ring of dimension n with residue field F . For any generator β of $\mathrm{Ext}_B^n(F, B)$, we define the m -th *fundamental ideal* $I^m(B) = I_m^{fl}(B)$ as

$$I_m^{fl}(B) := \phi_\beta(I^m(F)),$$

where $\phi_\beta : W(F) \rightarrow W^{fl}(B)$ is an isomorphism induced by β and $\phi_\beta(I^m(F))$ is independent of the choice of the generator β , compare [Fas08, Section 9.2]

To restrict the Gersten–Witt complex to the fundamental ideals, we need to check if the differentials behave well with these ideals, which is answered by the following theorem.

Proposition 4.2.13. ([Fas08, Theorem 9.2.4]) *The differentials of the Gersten–Witt complex preserve the fundamental ideal.*

Definition 4.2.14. We denote by $C(X, I^j)$ the complex

$$\dots \rightarrow \bigoplus_{x \in X^{(n)}} I^{j-n}(\mathcal{O}_{X,x}) \rightarrow \bigoplus_{x \in X^{(n+1)}} I^{j-(n+1)}(\mathcal{O}_{X,x}) \rightarrow \dots$$

graded by codimension of support. Analogously, for a fixed invertible \mathcal{O}_X -module L we denote by $C(X, I^j, L)$ the following complex

$$\dots \rightarrow \bigoplus_{x \in X^{(n)}} I^{j-n}(\mathcal{O}_{X,x}, L_x) \rightarrow \bigoplus_{x \in X^{(n+1)}} I^{j-(n+1)}(\mathcal{O}_{X,x}, L_x) \rightarrow \dots$$

The maps induced by the inclusion of powers of the fundamental ideals

$$C(X, I^{j+1}) \rightarrow C(X, I^j) \text{ and } C(X, I^{j+1}, L) \rightarrow C(X, I^j, L)$$

have canonically isomorphic cokernels (see [Fas08, Corollary E.1.3]). Thus, we can define:

Definition 4.2.15. We denote by $C(X, I^j/I^{j+1})$ the cokernel of the above morphisms graded by codimension of support.

We now have defined the components of the pullback square

$$\begin{array}{ccc} C^i(X, G^j, L) & \longrightarrow & C^i(X, I^j, L) \\ \downarrow & & \downarrow \\ C^i(X, K_j^M) & \longrightarrow & C^i(X, I^j/I^{j+1}). \end{array}$$

The arrow on the right is induced by the canonical projection. For the arrow on the bottom of the diagram, the reader is referred to [Fas08, Section 10.2]; the differentials of $C^i(X, G^j, L)$ are induced by the differentials on $C^i(X, K_j^M)$ and $C^i(X, I^j, L)$, see [Fas08, Lemma 10.2.12].

Remark 4.2.16. The complex $C^i(X, G^j, L)$ is called the *Rost–Schmid complex* and can also be defined directly as follows (in the for us important case $n := j = i$). Here, $\det(\Omega_{k(x)/k})$ is the determinant bundle of the differential sheaf, see [Fas20, Section 1.4], and L is a fixed invertible \mathcal{O}_X -module.

$$\begin{array}{c} \bigoplus_{x \in X^{(n-1)}} K_1^{\text{MW}}(k(x), \det(\Omega_{k(x)/k}) \otimes L) \\ \xrightarrow{\partial_1} \bigoplus_{x \in X^{(n)}} K_0^{\text{MW}}(k(x), \det(\Omega_{k(x)/k}) \otimes L) \\ \xrightarrow{\partial_0} \bigoplus_{x \in X^{(n+1)}} K_{-1}^{\text{MW}}(k(x), \det(\Omega_{k(x)/k}) \otimes L) \end{array}$$

Note that the differentials are complicated, see for example [Fas20] for more details, and in their explicit form not necessary for our purposes.

The definition of Milnor–Witt K -theory of fields F is due to Hopkins and Morel and can be found in [Mor12, Chapter 3], where it is defined as a quotient of the \mathbb{Z} -graded ring $A(F)$, generated in degree 1 by $[a]$ for all $a \in F^\times$ and by one additional generator η in degree -1 , defined by concrete relations. The basic idea we will make use of here is that we have isomorphisms $\mathrm{GW}(F) \cong K_0^{\mathrm{MW}}(F)$ and $\mathrm{W}(F) \cong K_{-i}^{\mathrm{MW}}(F)$ for all $i > 0$, see *loc.cit.*

Remark 4.2.17. It will be important for us to know that, by using [Fas07, Remark 3.23], there are isomorphisms

$$C^n(X, G^n, L) \cong \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n)} / \mathrm{D}^b(P(X))^{(n+1)}, R\mathrm{Hom}_{\mathcal{O}_X}(-, L)).$$

More details can be found in [Fas08, Remark 10.2.9].

Hence, we can rewrite the complex from Definition 4.2.2 as follows

$$\begin{aligned} \dots \rightarrow C^{n-1}(X, G^n, L) &\xrightarrow{\partial^{n-1}} \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n)} / \mathrm{D}^b(P(X))^{(n+1)}, R\mathrm{Hom}_{\mathcal{O}_X}(-, L)) \\ &\xrightarrow{\tilde{\partial}^n} \mathrm{W}^{[n+1]}(\mathrm{D}^b(P(X))^{(n+1)} / \mathrm{D}^b(P(X))^{(n+2)}, R\mathrm{Hom}_{\mathcal{O}_X}(-, L)) \rightarrow \dots \end{aligned}$$

Here, $\tilde{\partial}^n$ is defined as $q^{n+1, \mathrm{W}} \circ d^n$, where d^n is the boundary map from Theorem 3.3.3.

Remark 4.2.18. Let us recall some results about the connection between Chow groups and Chow–Witt groups.

- (i) Let X be a scheme of dimension d over an algebraically closed field k , then there is a natural isomorphism between $\widetilde{\mathrm{CH}}^d(X)$ and $\mathrm{CH}^d(X)$, see [Fas08, Remarque 10.2.16].
- (ii) If we replace K^{MW} in the Rost–Schmid complex as in the description before Remark 4.2.17 by K^{M} , we basically divide out the ideal generated by η . Since η commutes with the differentials, we obtain differentials on the level of Milnor K -theory. The cohomology at $i = -n$ now yields Chow groups instead of Chow–Witt groups, hence we get a homomorphism

$$\widetilde{\mathrm{CH}}^n(X, L) \rightarrow \mathrm{CH}^n(X),$$

which is, in general, neither injective nor surjective. For more details, compare [Fas13, HW19].

This section has served to recall the theory of Chow–Witt groups of a scheme. Together with the construction of tensor triangular Chow groups from Section 4.1, we can now glue these concepts together to sketch the idea, give the concrete construction, and prove the agreement theorem in the following two sections.

4.3 Tensor Triangular Chow–Witt Groups

We now come to the central section of the first part of this work, where we construct Chow–Witt groups of tensor triangulated categories. Before coming to the definition in Subsection 4.3.2, we give an overview of the idea in Subsection 4.3.1. This way, we prepare the main concepts of Section 4.4, where we prove the agreement of our definition to the classical definition of Chow–Witt groups of a scheme for the derived category of perfect complexes $D^{\text{perf}}(X)$.

Convention 4.3.1. Let \mathcal{T} be an essentially small rigid tensor triangulated category such that $\text{Spc}(\mathcal{T})$ is noetherian and $\frac{1}{2} \in \mathcal{T}$. We fix a \otimes -invertible object L and a dimension function \dim . When considered a triangulated category with duality, \mathcal{T} will be equipped with the duality $D_L := \text{hom}(-, L)$ as in Example 2.2.6 (ii). If not mentioned explicitly otherwise, restrictions of \mathcal{T} will always carry the restriction of D_L .

Moreover, we will assume that all quotients of the form $\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}$ are idempotent complete. This is true for a very wide range of examples, for example for $T = D^{\text{perf}}(X)$ when X is a regular scheme. We will comment on the case that the quotient is not idempotent complete for some n later in Section 4.6, see also [Kl16a, Remark 3.2.7].

4.3.1 Idea

Let X be a regular, separated scheme of finite type over a base field k of characteristic $\neq 2$. The natural idea to generalize Chow–Witt groups of a scheme is to generalize the oriented cocycles $\tilde{Z}^n(X)$ and coboundaries $\tilde{B}^n(X)$, hence, the kernel of the differential ∂^n and the image of the differential ∂^{n-1} from Definition 4.2.2.

The problem of this approach is that the differentials of the Rost–Schmid complex constructed in Section 4.2 are quite concrete and do not suggest a categorical generalization at first sight. However, we did see in Remark 4.2.17 that there is a way to rewrite at least a part of it in terms of triangulated categories with duality, namely by viewing Chow–Witt groups as the cohomology of the following complex:

$$\begin{aligned} \dots \rightarrow C^{n-1}(X, G^n) \xrightarrow{\partial^{n-1}} \text{GW}^{[n]}(D^b(P(X))^{(n)}/D^b(P(X))^{(n+1)}, R\text{Hom}_{\mathcal{O}_X}(-, L)) \\ \xrightarrow{\tilde{\partial}^n} \text{W}^{[n+1]}(D^b(P(X))^{(n+1)}/D^b(P(X))^{(n+2)}, R\text{Hom}_{\mathcal{O}_X}(-, L)) \rightarrow \dots \end{aligned}$$

Now, the map $\tilde{\partial}^n$ can be generalized directly to the tensor triangular setting since it already uses Grothendieck–Witt groups of triangulated categories with duality. Hence, we may construct a triangular form of $\tilde{\partial}^n$ when replacing $D^b(P(X))^{(n)}$ by filtration components $\mathcal{T}_{(n')}$ of suitable tensor triangulated categories.

We deliberately chose different indices here, since it is not clear at first sight whether $n' = n$ or else and, in fact, we will see later that for $\mathcal{T} = D^{\text{perf}}(X)$, we have $n' = -n$.

We briefly recall that, for an essentially small strongly tensor triangulated category equipped with a dimension function and a fixed \otimes -invertible object L , we have a filtration of triangulated categories with duality

$$\mathcal{T}_{(-\infty)} \subset \dots \subset (\mathcal{T}_{(n-1)}, D_L|_{\mathcal{T}_{(n-1)}}) \subset (\mathcal{T}_{(n)}, D_L|_{\mathcal{T}_{(n)}}) \subset \dots \subset \mathcal{T}_{(\infty)} = (\mathcal{T}, D_L).$$

We will omit the duality functor from the notation for better readability if we consider the duality D_L or its restrictions.

We obtain exact sequences of triangulated categories with duality and duality-preserving functors for each $n \in \mathbb{Z} \cup \{\pm\infty\}$

$$\mathcal{T}_{(n-1)} \xrightarrow{I_{\mathcal{T}}^n} \mathcal{T}_{(n)} \xrightarrow{Q_{\mathcal{T}}^n} \mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}$$

which induce exact sequences by Theorem 3.3.3

$$\mathrm{GW}^{[n]}(\mathcal{T}_{(n-1)}) \xrightarrow{i_{\mathcal{T}}^n} \mathrm{GW}^{[n]}(\mathcal{T}_{(n)}) \xrightarrow{q_{\mathcal{T}}^n} \mathrm{GW}^{[n]}(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})$$

and by Theorem 3.3.2

$$\mathrm{W}^{[n]}(\mathcal{T}_{(n-1)}) \xrightarrow{i_{\mathcal{T}}^{n,\mathrm{W}}} \mathrm{W}^{[n]}(\mathcal{T}_{(n)}) \xrightarrow{q_{\mathcal{T}}^{n,\mathrm{W}}} \mathrm{W}^{[n]}(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}).$$

The idea of generalizing the cocycles, i.e. the kernel of ∂^n , is now as follows: We know that $\ker(\partial^n) = \ker(\tilde{\partial}^n)$, and we can write $\tilde{\partial}^n$ in terms of maps coming from functors of the underlying triangulated categories with duality such as $q_{\mathcal{T}}^n$ or $i_{\mathcal{T}}^n$. Of course, we need to take care of (lower and upper!) indices, but, for the details, the reader is referred to Subsection 4.3.2.

The coboundaries, i.e. the image of ∂^{n-1} , however, are not that easily generalized since the term $C^{n-1}(X, G^n)$ above of the Rost–Schmid complex is still in its concrete algebro-geometric form. We prove in Theorem 4.4.3, that, even though we might not be able to write ∂^{n-1} in terms of maps coming from functors of triangulated categories with duality, we can still do so for its image. Concretely, we show that the image of ∂^{n-1} can be written only in terms of maps $q_{\mathcal{T}}^n$ and $i_{\mathcal{T}}^n$ coming from the inclusion and quotient functor. The passage to complicial exact categories with weak equivalences and duality is essential here since there are no higher Grothendieck–Witt groups of triangulated categories with duality.

Then again, we can generalize the coboundaries and, hence, the whole concept of Chow–Witt groups.

Remark 4.3.2. The attentive reader might observe that the idea works mainly with the underlying triangulated categories with duality and the tensor structure does not appear too often in the sketch of the idea. The tensor structure and the additional information of a dimension function only serve as a basis to filtrate the category by dimension of support.

However, one can easily generalize the definition for essentially small triangulated categories \mathcal{C} with duality containing $\frac{1}{2}$ that allow a filtration $\mathcal{C}_{(-\infty)} \dots \subset \mathcal{C}_{(n-1)} \subset \mathcal{C}_{(n)} \subset \dots \subset \mathcal{C}_{(\infty)} = \mathcal{C}$ of \mathcal{C} , where all the filtration components are triangulated categories with duality and the inclusion functors are duality-preserving.

Since the leading concept of this work is tensor triangular geometry, we specialize on filtrations coming from rigid tensor triangulated categories introduced in Section 2.3.

4.3.2 Construction and Definition

Finally, we come to the construction of tensor triangular Chow–Witt groups. We still follow Convention 4.3.1.

Recall the following, for us central localization sequence from Theorem 3.3.3

$$\begin{aligned} & \mathrm{GW}^{[n]}(\mathcal{T}_{(n-1)}) \xrightarrow{i_{\mathcal{T}}^n} \mathrm{GW}^{[n]}(\mathcal{T}_{(n)}) \xrightarrow{q_{\mathcal{T}}^n} \mathrm{GW}^{[n]}(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}) \\ & \xrightarrow{d_{\mathcal{T}}^n} \mathrm{W}^{[n+1]}(\mathcal{T}_{(n-1)}) \xrightarrow{i_{\mathcal{T}}^{n,W}} \mathrm{W}^{[n+1]}(\mathcal{T}_{(n)}) \xrightarrow{q_{\mathcal{T}}^{n,W}} \mathrm{W}^{[n+1]}(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}) \rightarrow \dots \end{aligned}$$

To define tensor triangular Chow–Witt groups in terms of these maps, we consider the following defining diagram.

$$\begin{array}{ccc} \mathrm{W}^{[n+1]}(\mathcal{T}_{(n-2)}) \xrightarrow{i_{\mathcal{T}}^{n-1,W}} \mathrm{W}^{[n+1]}(\mathcal{T}_{(n-1)}) & & \mathrm{GW}^{[n]}(\mathcal{T}_{(n)}) \xrightarrow{i_{\mathcal{T}}^{n+1}} \mathrm{GW}^{[n]}(\mathcal{T}_{(n+1)}) \\ & \swarrow d_{\mathcal{T}}^n & \downarrow q_{\mathcal{T}}^n \\ & & \mathrm{GW}^{[n]}(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}) \end{array} \quad (4.3)$$

Then, we can give the definition of *tensor triangular oriented cocycles* and *coboundaries*.

Definition 4.3.3. For \mathcal{T} as in Convention 4.3.1, we define its n -th *tensor triangular oriented cocycles (twisted by L)* for $n \in \mathbb{Z}$ as

$$\tilde{Z}_{\Delta}^n(\mathcal{T}, L) := (d_{\mathcal{T}}^n)^{-1}(\mathrm{Im}(i_{\mathcal{T}}^{n-1,W}))$$

and its n -th *tensor triangular oriented coboundaries (twisted by L)* as

$$\tilde{B}_{\Delta}^n(\mathcal{T}, L) := q_{\mathcal{T}}^n(\ker(i_{\mathcal{T}}^{n+1})),$$

where the maps are the ones from the defining diagram Diagram 4.3.

Proposition 4.3.4. *In the setting of Definition 4.3.3, $\tilde{B}_{\Delta}^n(\mathcal{T})$ is a subgroup of $\tilde{Z}_{\Delta}^n(\mathcal{T})$ and $\tilde{Z}_{\Delta}^n(\mathcal{T})$ is a subgroup of $\mathrm{GW}^{[n]}(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})$.*

Proof. What we need to check is whether $q_{\mathcal{T}}^n(\ker(i_{\mathcal{T}}^{n+1}))$ is a subgroup of $\tilde{Z}_{\Delta}^n(\mathcal{T}) := (d_{\mathcal{T}}^n)^{-1}(\mathrm{Im}(i_{\mathcal{T}}^{n-1,W}))$ and whether the latter is itself a subgroup of $\mathrm{GW}^{[n]}(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})$. We will omit the subscript \mathcal{T} from the notation here for better readability. However, be careful not to confuse it with the maps from the algebro-geometric setting in Definition 4.2.6.

The last statement follows from the fact that $(d^n)^{-1}(\mathrm{Im}(i^{n-1,W})) = \ker(q^{n-1,W} \circ d^n)$ by the exactness of the sequence from Theorem 3.3.3. Moreover, by construction, $q^n(\ker(i^{n+1}))$ is a subgroup of $\mathrm{GW}^{[n]}(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})$. It is left to check that

$$q^n(\ker(i^{n+1})) \subset (d^n)^{-1}(\mathrm{Im}(i^{n-1,W})).$$

Since $\text{Im}(q^n) = \ker(d^n)$ by exactness of the localization sequence (Theorem 3.3.3)

$$\begin{aligned} \text{GW}^{[n]}(\mathcal{T}_{(n-1)}) &\xrightarrow{i^n} \text{GW}^{[n]}(\mathcal{T}_{(n)}) \xrightarrow{q^n} \text{GW}^{[n]}(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}) \\ \xrightarrow{d^n} \text{W}^{[n+1]}(\mathcal{T}_{(n-1)}) &\xrightarrow{i^{n,W}} \text{W}^{[n+1]}(\mathcal{T}_{(n)}) \xrightarrow{q^{n,W}} \text{W}^{[n+1]}(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}) \longrightarrow \dots, \end{aligned}$$

we have, in particular, $\text{Im}(q^n) \subset \ker(d^n) = (d^n)^{-1}(\{0\})$. Since $0 \in \text{Im}(i^{n-1,W})$, we obtain $\text{Im}(q^n) \subset (d^n)^{-1}(\{0\}) \subset (d^n)^{-1}(\text{Im}(i^{n-1,W}))$, hence, $q^n(\ker(i^{n+1})) \subset (d^n)^{-1}(\text{Im}(i^{n-1,W}))$ as stated. \square

This guarantees that the following, for this work central new notion of *tensor triangular Chow–Witt groups* is well-defined.

Definition 4.3.5. For \mathcal{T} as in Convention 4.3.1, we define its n -th *tensor triangular Chow–Witt groups twisted by L* for $n \in \mathbb{Z}$ as

$$\widetilde{\text{CH}}_{\Delta}^n(\mathcal{T}, L) := \widetilde{Z}_{\Delta}^n(\mathcal{T}, L) / \widetilde{B}_{\Delta}^n(\mathcal{T}, L),$$

where $\widetilde{Z}_{\Delta}^n(\mathcal{T}, L)$ and $\widetilde{B}_{\Delta}^n(\mathcal{T}, L)$ are defined in Definition 4.3.3.

Remark 4.3.6. If we are given a dimension function \dim , we can always *shift* it by some $m \in \mathbb{Z}$ to obtain a dimension function \dim' . Then, we have isomorphisms $\widetilde{\text{CH}}_{\Delta}^n(\mathcal{T}, \dim) \cong \widetilde{\text{CH}}_{\Delta}^{n+m}(\mathcal{T}, \dim')$ and $\text{CH}_{\Delta}^n(\mathcal{T}, \dim) \cong \text{CH}_{\Delta}^{n+m}(\mathcal{T}, \dim')$ for all n . However, since the constructions depend on the choice of the dimension function a priori, it is not clear if the change of tensor triangular Chow(–Witt) groups can be controlled when changing the dimension function in arbitrary ways.

Remark 4.3.7. If we replace GW with K_0 in the construction, it coincides with Balmer’s tensor triangular Chow groups from [Bal13]. The assumption that all quotients of filtration components are idempotent complete is essential in this comparison.

We have given a construction of tensor triangular Chow–Witt groups in this section. However, the definition comes at the price of an agreement theorem that justifies the name, i.e. it shows that the definition agrees with the classical algebro-geometric case for regular X .

4.4 Agreement with Algebraic Geometry

Convention 4.4.1. Let X be a regular, separated scheme of finite type over a field k with $\text{char}(k) \neq 2$.

This section will justify the name of Definition 4.3.5. We will show that, for regular X , the definition for the tensor triangulated category $\mathcal{T} = \text{D}^{\text{perf}}(X)$ recovers the usual Definition 4.2.2 for X . Note that, in this case, \mathcal{T} satisfies all the conditions from Convention 4.3.1.

In Subsection 4.4.1, we reformulate the classical definition of oriented cocycles to a form that we can refine to the tensor triangular setting using the filtration by dimension of support. In Subsection 4.4.2, we do the same for classical oriented coboundaries. Finally, we merge these alternative definitions in Subsection 4.4.3 to show the agreements of tensor triangular and classical Chow–Witt groups and calculate an easy example by hand.

4.4.1 Reformulation of Classical Oriented Cocycles

As for Chow groups, we first aim to reformulate the definition of classical oriented cocycles in terms of maps coming from localization that we can directly generalize to the triangular setting.

Recall that $P(X)$ is the exact category of locally free \mathcal{O}_X -modules of finite type over a noetherian scheme X of dimension d . Keep in mind that, if in addition X is regular, the bounded derived category of $P(X)$ is equivalent to $D^b(X) := D^b(\text{Coh}(X))$ as a rigid tensor triangulated category.

The filtration by codimension of homological support from Definition 4.2.6

$$0 = D^b(P(X))^{(d+1)} \subset D^b(P(X))^{(d)} \subset \dots \subset D^b(P(X))^{(0)} = D^b(P(X))$$

yields short exact sequences of triangulated categories with duality for $0 \leq n \leq d$

$$D^b(P(X))^{(n+1)} \xrightarrow{I^n} D^b(P(X))^{(n)} \xrightarrow{Q^n} D^b(P(X))^{(n)} / D^b(P(X))^{(n+1)}.$$

We will need Walter’s localization sequence from Theorem 3.3.3 for triangulated categories with duality

$$\begin{aligned} & \text{GW}^{[n]}(D^b(P(X))^{(n+1)}) \xrightarrow{i^n} \text{GW}^{[n]}(D^b(P(X))^{(n)}) \\ & \xrightarrow{q^n} \text{GW}^{[n]}(D^b(P(X))^{(n)} / D^b(P(X))^{(n+1)}) \\ \xrightarrow{d^n} & \text{W}^{[n+1]}(D^b(P(X))^{(n+1)}) \xrightarrow{i^{n,W}} \text{W}^{[n+1]}(D^b(P(X))^{(n)}) \\ & \xrightarrow{q^{n,W}} \text{W}^{[n+1]}(D^b(P(X))^{(n)} / D^b(P(X))^{(n+1)}) \rightarrow \dots \end{aligned}$$

to reformulate the classical oriented cocycles for a scheme X .

Theorem 4.4.2. *Let X be a separated, regular scheme of finite type over a field k of characteristic $\neq 2$. With the above notation, there are isomorphisms for all n*

$$\tilde{Z}^{-n}(X, L) \cong (d^{-n})^{-1}(\text{Im}(i^{-n+1,W})),$$

where we are in the algebro-geometric setting as in Definition 4.2.2.

Proof. By Remark 4.2.17, we know that $\tilde{Z}^{-n}(X, L) := \ker(\partial^{-n}) \cong \ker(\tilde{\partial}^{-n})$, where

$$\begin{aligned} \tilde{\partial}^{-n} & := q^{-n+1,W} \circ d^{-n} : \text{GW}^{[-n]}(D^b(P(X))^{(-n)} / D^b(P(X))^{(-n+1)}, \text{Hom}_{\mathcal{O}_X}(-, L)) \\ & \rightarrow \text{W}^{[-n+1]}(D^b(P(X))^{(-n+1)}, \text{Hom}_{\mathcal{O}_X}(-, L)) \\ & \rightarrow \text{W}^{[-n+1]}(D^b(P(X))^{(-n+1)} / D^b(P(X))^{(-n+2)}, \text{Hom}_{\mathcal{O}_X}(-, L)). \end{aligned}$$

Since d^{-n} is exactly the boundary map from the localization sequence Theorem 3.3.3 for (Grothendieck–)Witt groups of triangulated categories with duality, we can use the exactness of that sequence to write $\tilde{Z}^{-n}(X, L)$ as

$$\begin{aligned}\tilde{Z}^{-n}(X, L) &:= \ker(\partial^{-n}) \cong \ker(\tilde{\partial}^{-n}) = \ker(q^{-n+1, \mathbb{W}} \circ d^{-n}) \\ &= (d^{-n})^{-1}(\ker(q^{-n+1, \mathbb{W}})) = (d^{-n})^{-1}(\operatorname{Im}(i^{-n+1, \mathbb{W}})).\end{aligned}$$

□

After having found an alternative description of algebraic oriented cocycles in terms of maps coming from the localization sequence for (Grothendieck–)Witt groups, we do the same for oriented coboundaries in the next subsection.

4.4.2 Reformulation of Classical Oriented Coboundaries

Reformulating classical oriented coboundaries in terms of maps coming from localization is trickier, as mentioned in Subsection 4.3.1. We recall from Example 3.2.11 that

$$(\operatorname{Ch}^b(P(X)), \operatorname{quis}, \operatorname{Hom}_{\mathcal{O}_X}(-, L))$$

is a complicial exact category with weak equivalences and duality by fixing the quasi-isomorphisms quis . We have subcategories

$$\operatorname{Ch}^b(P(X))^{(n)} := \{P_{\bullet} \in \operatorname{Ch}^b(P(X)) \mid \operatorname{codim}(\operatorname{supph}(P_{\bullet})) \geq n\}$$

and denote by $\operatorname{quis}^{n'}$ the morphisms in $\operatorname{Ch}^b(P(X))^{(n)}$ with cone in $\operatorname{Ch}^b(P(X))^{(n')}$ for $n' \geq n$, which again yield complicial exact categories with weak equivalences and duality

$$(\operatorname{Ch}^b(P(X))^{(n)}, \operatorname{quis}^{n'}, \operatorname{Hom}_{\mathcal{O}_X}(-, L)).$$

Recall that we denote the triangulated categories with duality obtained by inverting the quasi-isomorphisms $\operatorname{quis}^{n'}$ or quis by $\operatorname{Ch}^b(P(X))^{(n)}[(\operatorname{quis}^{n'})^{-1}]$ and $\operatorname{Ch}^b(P(X))^{(n)}[(\operatorname{quis})^{-1}]$, respectively.

By Theorem 3.3.4, we obtain for $n \in \mathbb{Z}$ a localization sequence for Grothendieck–Witt groups of complicial exact categories with weak equivalences and duality

$$\begin{aligned}\dots \rightarrow \operatorname{GW}_m^{[n]}(\operatorname{Ch}^b(P(X))^{(n+1)}, \operatorname{quis}) &\xrightarrow{i_{\operatorname{Ch}}^{n,m}} \operatorname{GW}_m^{[n]}(\operatorname{Ch}^b(P(X))^{(n)}, \operatorname{quis}) \\ &\xrightarrow{q_{\operatorname{Ch}}^{n,m}} \operatorname{GW}_m^{[n]}(\operatorname{Ch}^b(P(X))^{(n)}, \operatorname{quis}^{n+1}) \\ \xrightarrow{d_{\operatorname{Ch}}^{n,m}} \operatorname{GW}_{m-1}^{[n]}(\operatorname{Ch}^b(P(X))^{(n+1)}, \operatorname{quis}) &\xrightarrow{i_{\operatorname{Ch}}^{n,m-1}} \operatorname{GW}_{m-1}^{[n]}(\operatorname{Ch}^b(P(X))^{(n)}, \operatorname{quis}) \rightarrow \dots\end{aligned}$$

With these tools, we can prove a hermitian analog of [Qui73, Theorem 5.14], which has already been proven in [Hor08, Theorem 1.7] and [FS09, Theorem 33]. An important part of its proof will be the comparison of the localization sequences

in the different settings introduced in Section 3.3, and the comparison between localization sequence of the triangulated categories $\mathrm{Ch}^b(P(X))^{(n)}[(\mathrm{quis}^{n+1})^{-1}]$ and $\mathrm{D}^b(P(X))^{(n)}/\mathrm{D}^b(P(X))^{(n+1)}$. Recall that the notation for the latter is

$$\begin{aligned} & \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n+1)}) \xrightarrow{i^n} \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n)}) \\ \xrightarrow{q^n} & \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n)}/\mathrm{D}^b(P(X))^{(n+1)}) \xrightarrow{d^n} \mathrm{W}^{[n+1]}(\mathrm{D}^b(P(X))^{(n+1)}) \rightarrow \dots \end{aligned}$$

Now, we can formulate:

Theorem 4.4.3. *Let X be a separated, regular scheme of finite type over a field k of characteristic $\neq 2$. With the notation from above, there are isomorphisms for all n*

$$\tilde{B}^n(X, L) \cong q^n(\ker(i^{n-1})),$$

i.e. the image of

$$\begin{aligned} \partial^{n-1} : C^{n-1}(X, G^n, L) & \rightarrow C^n(X, G^n, L) \\ & \xrightarrow{\cong} \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n)}/\mathrm{D}^b(P(X))^{(n+1)}, R\mathrm{Hom}_{\mathcal{O}_X}(-, L)) \end{aligned}$$

of Definition 4.2.2 equals $q^n(\ker(i^{n-1}))$.

Here, the isomorphism to $\mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n)}/\mathrm{D}^b(P(X))^{(n+1)}, R\mathrm{Hom}_{\mathcal{O}_X}(-, L))$ is the one from Remark 4.2.17.

Proof of Theorem 4.4.3. We do the proof in three steps. First, we reformulate the oriented coboundaries in terms of maps coming from the localization sequence of exact categories with weak equivalences and duality from Theorem 3.3.4. Second, we move this description to the triangular setting after inverting the respective quasi-isomorphisms of the considered exact categories with weak equivalences and duality. At last, we compare the localization sequences for triangulated categories with duality as in Theorem 3.3.3 for the categories $\mathrm{Ch}^b(P(X))^{(n)}[(\mathrm{quis}^{n+1})^{-1}]$ and $\mathrm{D}^b(P(X))^{(n)}/\mathrm{D}^b(P(X))^{(n+1)}$, i.e. for the case of first inverting the quasi-isomorphisms and then filtrating the category, and vice versa.

By [FS09, Theorem 33], the n -th Chow–Witt group can also be defined as the cohomology of the following sequence of (Grothendieck–)Witt groups of complicial exact categories with weak equivalences and duality, all dualities again being derived from $\mathrm{Hom}_{\mathcal{O}_X}(-, L)$,

$$\begin{aligned} & \mathrm{GW}_1^{[n]}(\mathrm{Ch}^b(P(X))^{(n-1)}, \mathrm{quis}^n) \\ & \xrightarrow{\tilde{\partial}_{\mathrm{Ch}}^{1,n}} \mathrm{GW}^{[n]}(\mathrm{Ch}^b(P(X))^{(n)}, \mathrm{quis}^{n+1}) \\ & \xrightarrow{\tilde{\partial}_{\mathrm{Ch}}^{0,n}} \mathrm{W}^{[n+1]}(\mathrm{Ch}^b(P(X))^{n+1}, \mathrm{quis}^{n+2}). \end{aligned}$$

Hence, we have an isomorphism

$$H^n(C(X, G^n, L)) = \tilde{Z}^n(X, L)/\tilde{B}^n(X, L) \cong \ker(\tilde{\partial}_{\mathrm{Ch}}^{0,n})/\mathrm{Im}(\tilde{\partial}_{\mathrm{Ch}}^{1,n}),$$

where $C(X, G^n, L)$ is the complex from Definition 4.2.2. The sequence arises from the *Gersten–Grothendieck–Witt* spectral sequence from [FS09, Definition 26] and, thus, it follows by construction that, in particular, $\mathrm{Im}(\tilde{\partial}_{\mathrm{Ch}}^{1,n}) \cong \tilde{B}^n(X, L)$.

Now, Theorem 3.3.4 yields the long exact localization sequence for Grothendieck–Witt groups of exact categories with weak equivalences and duality

$$\begin{aligned} \dots &\rightarrow \mathrm{GW}_m^{[n]}(\mathrm{Ch}^b(P(X))^{(n+1)}, \mathrm{quis}) \xrightarrow{i_{\mathrm{Ch}}^{n,m}} \mathrm{GW}_m^{[n]}(\mathrm{Ch}^b(P(X))^{(n)}, \mathrm{quis}) \\ &\xrightarrow{q_{\mathrm{Ch}}^{n,m}} \mathrm{GW}_m^{[n]}(\mathrm{Ch}^b(P(X))^{(n)}, \mathrm{quis}^{n+1}) \\ \dots &\xrightarrow{d_{\mathrm{Ch}}^{n,m}} \mathrm{GW}_{m-1}^{[n]}(\mathrm{Ch}^b(P(X))^{(n+1)}, \mathrm{quis}) \xrightarrow{i_{\mathrm{Ch}}^{n,m-1}} \mathrm{GW}_{m-1}^{[n]}(\mathrm{Ch}^b(P(X))^{(n)}, \mathrm{quis}) \rightarrow \dots, \end{aligned}$$

and we know that $\tilde{\partial}_{\mathrm{Ch}}^{1,n}$ is the composition

$$\begin{aligned} \tilde{\partial}_{\mathrm{Ch}}^{1,n} &:= q_{\mathrm{Ch}}^{n,0} \circ d_{\mathrm{Ch}}^{m-1,1} : \mathrm{GW}_1^{[n]}(\mathrm{Ch}^b(P(X))^{(n-1)}, \mathrm{quis}^n) \\ &\rightarrow \mathrm{GW}_0^{[n]}(\mathrm{Ch}^b(P(X))^{(n)}, \mathrm{quis}) \\ &\rightarrow \mathrm{GW}_0^{[n]}(\mathrm{Ch}^b(P(X))^{(n)}, \mathrm{quis}^{n+1}). \end{aligned}$$

We can use the exactness of the localization sequence from Theorem 3.3.4 to conclude

$$\mathrm{Im}(\tilde{\partial}_{\mathrm{Ch}}^{1,n}) = \mathrm{Im}(q_{\mathrm{Ch}}^{n,0} \circ d_{\mathrm{Ch}}^{m-1,1}) = q_{\mathrm{Ch}}^{n,0}(\mathrm{Im}(d_{\mathrm{Ch}}^{m-1,1})) = q_{\mathrm{Ch}}^{n,0}(\ker(i_{\mathrm{Ch}}^{n-1,0})).$$

After having rewritten the algebro-geometric coboundaries in terms of maps in the localization sequence of exact categories with weak equivalences and duality, we will now pass to the triangular setting.

For this, we show that the horizontal arrows in the diagram

$$\begin{array}{ccc} \mathrm{GW}_0^{[n]}(\mathrm{Ch}^b(P(X))^{(n-1)}, \mathrm{quis}) & \longrightarrow & \mathrm{GW}^{[n]}(\mathrm{Ch}^b(P(X))^{(n-1)}[(\mathrm{quis})^{-1}]) \\ \uparrow i_{\mathrm{Ch}}^{n-1,0} & & \uparrow \\ \mathrm{GW}_0^{[n]}(\mathrm{Ch}^b(P(X))^{(n)}, \mathrm{quis}) & \longrightarrow & \mathrm{GW}^{[n]}(\mathrm{Ch}^b(P(X))^{(n)}[(\mathrm{quis})^{-1}]) \\ \downarrow q_{\mathrm{Ch}}^{n,0} & & \downarrow \\ \mathrm{GW}_0^{[n]}(\mathrm{Ch}^b(P(X))^{(n)}, \mathrm{quis}^{n+1}) & \longrightarrow & \mathrm{GW}^{[n]}(\mathrm{Ch}^b(P(X))^{(n)}[(\mathrm{quis}^{n+1})^{-1}]) \end{array}$$

are isomorphisms. The left column treats Grothendieck–Witt groups of exact categories with weak equivalences and duality, whereas the right column contains Grothendieck–Witt groups of triangulated categories with duality.

The isomorphisms follow from Proposition 3.2.14 and the fact that, by [FS09, Theorem 8], the part of the localization sequence from above (Theorem 3.3.4) for complicial exact categories with weak equivalences and duality starting in GW_0 coincides with the tensor triangular localization sequence Theorem 3.3.2 for the short exact sequence

$$\mathrm{Ch}^b(P(X))^{(n-1)}[(\mathrm{quis})^{-1}] \rightarrow \mathrm{Ch}^b(P(X))^{(n)}[(\mathrm{quis})^{-1}] \rightarrow \mathrm{Ch}^b(P(X))^{(n)}[(\mathrm{quis}^{n+1})^{-1}].$$

Lastly, we want to show that the horizontal morphisms in the diagram

$$\begin{array}{ccc}
 \mathrm{GW}^{[n]}(\mathrm{Ch}^b(P(X))^{(n-1)}[(\mathrm{quis})^{-1}]) & \xrightarrow{f} & \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n-1)}) \\
 \uparrow & & \uparrow i^{n-1} \\
 \mathrm{GW}^{[n]}(\mathrm{Ch}^b(P(X))^{(n)}[(\mathrm{quis})^{-1}]) & \xrightarrow{g} & \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n)}) \\
 \downarrow & & \downarrow q^n \\
 \mathrm{GW}^{[n]}(\mathrm{Ch}^b(P(X))^{(n)}[(\mathrm{quis}^{n+1})^{-1}]) & \xrightarrow{h} & \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n)}/\mathrm{D}^b(P(X))^{(n+1)})
 \end{array}$$

are isomorphism.

Both columns involve Grothendieck–Witt groups of triangulated categories with duality. In the left column, we first filtrate the exact category with weak equivalences and duality $\mathrm{Ch}^b(P(X))$ by codimension of homological support and then invert the given quasi-isomorphisms; in the column on the right, we first invert all quasi-isomorphisms in $\mathrm{Ch}^b(P(X))$ before filtrating.

The underlying functors of f and g are categorical equivalences since inverting quasi-isomorphisms commutes with restricting objectwise to the subcategory $(-)^{(k)}$ for $k \in \{n-1, n, n+1\}$. Moreover, we note that the underlying triangulated categories in the domain and codomain of h are equivalent by universality of the Verdier quotient functor (compare e.g. [Nee05, Proposition 2.1.24] or [Wei95, Exercise 10.3.2] for abelian categories). The isomorphisms on the level of GW now follow from Proposition 3.2.14.

Hence, we obtain a commutative diagram with the horizontal maps being isomorphisms

$$\begin{array}{ccc}
 \mathrm{GW}_0^{[n]}(\mathrm{Ch}^b(P(X))^{(n-1)}, \mathrm{quis}) & \xrightarrow{\cong} & \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n-1)}) \\
 \uparrow i_{\mathrm{Ch}}^{n-1,0} & & \uparrow i^{n-1} \\
 \mathrm{GW}_0^{[n]}(\mathrm{Ch}^b(P(X))^{(n)}, \mathrm{quis}) & \xrightarrow{\cong} & \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n)}) \\
 \downarrow q_{\mathrm{Ch}}^{n,0} & & \downarrow q^n \\
 \mathrm{GW}_0^{[n]}(\mathrm{Ch}^b(P(X))^{(n)}, \mathrm{quis}^{n+1}) & \xrightarrow{\cong} & \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n)}/\mathrm{D}^b(P(X))^{(n+1)}).
 \end{array}$$

Eventually, this yields $\tilde{B}^n(X, L) \cong q_{\mathrm{Ch}}^{n,0}(\ker(i_{\mathrm{Ch}}^{n-1,0})) \cong q^n(\ker(i^{n-1}))$ as claimed. \square

Remark 4.4.4. Note that, until now, we have only considered filtrations by *codimension of homological support* as in Definition 4.2.6, which is why there has not appeared a change of sign yet as announced before. We have not yet considered tensor triangular oriented cocycles/coboundaries; their definition requires the filtration of tensor triangulated categories by *codimension of support* as in Definition 2.3.4. Passing between these filtrations will involve change of signs.

In this and the last section, we have rewritten the algebro-geometric oriented cocycles $\tilde{Z}^n(X, L)$ and coboundaries $\tilde{B}^n(X, L)$ in terms of maps coming from the localization sequence

$$\begin{aligned} & \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n+1)}) \xrightarrow{i^n} \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n)}) \\ \xrightarrow{q^n} & \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n)} / \mathrm{D}^b(P(X))^{(n+1)}) \xrightarrow{d^n} \mathrm{W}^{[n+1]}(\mathrm{D}^b(P(X))^{(n+1)}) \xrightarrow{i^{n,W}} \dots \end{aligned}$$

for triangulated categories with duality. Hence, we have lifted the definition of Chow–Witt groups already to a triangular setting. However, to show the agreement between the definition of tensor triangular Chow–Witt groups and the classical Chow–Witt groups, we need to check whether these rewritten forms coincide with Definition 4.3.5 in the tensor triangular setting. We will do so in the following section.

4.4.3 Agreement of Chow–Witt Groups

Above, we have brought oriented cocycles and coboundaries to a form that we can generalize directly. In this sense, Theorem 4.4.3 and Theorem 4.4.2 pave the path for the following central agreement theorem this section aims to prove. It is a refinement of [Kl16a, Theorem 3.2.6].

Theorem 4.4.5. *Let X be a separated, regular scheme of finite type over a field k with $\mathrm{char}(k) \neq 2$, L a line bundle over X , and $\mathrm{D}^{\mathrm{perf}}(X)$ be equipped with the dimension function $-\mathrm{codim}_{\mathrm{Kroll}}$. Then, there are isomorphisms for all $n \in \mathbb{Z}$*

$$\begin{aligned} \tilde{Z}_{\Delta}^n(\mathrm{D}^{\mathrm{perf}}(X), \mathrm{hom}(-, L)) &\cong \tilde{Z}^{-n}(X, L), \\ \tilde{B}_{\Delta}^n(\mathrm{D}^{\mathrm{perf}}(X), \mathrm{hom}(-, L)) &\cong \tilde{B}^{-n}(X, L) \end{aligned}$$

and

$$\widetilde{\mathrm{CH}}_{\Delta}^n(\mathrm{D}^{\mathrm{perf}}(X), \mathrm{hom}(-, L)) \cong \widetilde{\mathrm{CH}}^{-n}(X, L).$$

Proof. We start in the algebro-geometric setting as in Definition 4.2.2. Then, Theorem 4.4.2 and Theorem 4.4.3 tell us that we have isomorphisms

$$\widetilde{\mathrm{CH}}^{-n}(X) \cong (d^{-n})^{-1}(\mathrm{Im}(i^{-n+1,W})) / q^{-n}(\mathrm{ker}(i^{-n-1})),$$

where the maps are the ones from the localization sequence of triangulated categories with duality

$$\begin{aligned} & \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n+1)}) \xrightarrow{i^n} \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n)}) \\ \xrightarrow{q^n} & \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n)} / \mathrm{D}^b(P(X))^{(n+1)}) \xrightarrow{d^n} \mathrm{W}^{[n+1]}(\mathrm{D}^b(P(X))^{(n+1)}) \xrightarrow{i^{n,W}} \dots \end{aligned}$$

Now, let $\mathcal{T} := \mathrm{D}^{\mathrm{perf}}(X)$ be the rigid tensor triangulated category from Example 2.2.6 (iii) equipped with $-\mathrm{codim}_{\mathrm{Kroll}}$, and L a fixed \otimes -invertible object; note

that here $\text{hom} := R\text{Hom}$. We know that, by our assumptions, there is a categorical equivalence $D^{\text{perf}}(X) \simeq D^b(P(X))$.

We recall the definition of the filtration components in the tensor-triangular setting as in Section 2.3

$$D^{\text{perf}}(X)_{(n)} := \{P \in D^{\text{perf}}(X) \mid -\text{codim}_{\text{Krull}}(\text{supp}(P)) \leq n\}$$

with

$$\text{supp}(P) := \{I \in \text{Spc}(D^{\text{perf}}(X)) \mid P \notin I\} \subset \text{Spc}(D^{\text{perf}}(X))$$

and the filtration components in the algebro-geometric setting as in Definition 4.2.6

$$\begin{aligned} D^b(P(X))^{(-n)} &:= \{P \in D^b(P(X)) \mid \text{codim}(\text{supph}(P)) \geq -n\} \\ &= \{P \in D^b(P(X)) \mid -\text{codim}(\text{supph}(P)) \leq n\} \end{aligned}$$

with

$$\text{supph}(P) := \bigcup_i \text{supp}(H_i(P)) \subset X.$$

We know by [Bal05b, Corollary 5.6] that, under the isomorphism $\text{Spec}(D^{\text{perf}}(X)) \cong X$ from Example 2.1.20, $\text{supph}(A) \subset X$ and $\text{supp}(A) \subset \text{Spc}(D^{\text{perf}}(X))$ coincide for $A \in D^{\text{perf}}(X)$. Thus, using $D^{\text{perf}}(X) \simeq D^b(P(X))$, we obtain equivalences of triangulated categories with duality

$$(D^{\text{perf}}(X)_{(n)}, \text{hom}(-, L)) \simeq (D^b(P(X))^{(-n)}, \text{Hom}(-, L))$$

and

$$\begin{aligned} &(D^{\text{perf}}(X)_{(n)} / D^{\text{perf}}(X)_{(n-1)}, \text{hom}(-, L)) \\ &\simeq (D^b(P(X))^{(-n)} / D^b(P(X))^{(-n+1)}, \text{Hom}(-, L)). \end{aligned}$$

It follows that the horizontal arrows in the commutative diagrams (where the dualities are omitted from the notation)

$$\begin{array}{ccc} \text{GW}^{[n]}(\mathcal{T}_{(n+1)}) & \longrightarrow & \text{GW}^{[n]}(D^b(P(X))^{(-n-1)}) \\ \uparrow i_{\mathcal{T}}^{n+1} & & \uparrow i^{-n-1} \\ \text{GW}^{[n]}(\mathcal{T}_{(n)}) & \longrightarrow & \text{GW}^{[n]}(D^b(P(X))^{(-n)}) \\ \downarrow q_{\mathcal{T}}^n & & \downarrow q^{-n} \\ \text{GW}^{[n]}(\mathcal{T}_{(n)} / \mathcal{T}_{(n-1)}) & \longrightarrow & \text{GW}^{[n]}(D^b(P(X))^{(-n)} / D^b(P(X))^{(-n+1)}) \end{array}$$

and

$$\begin{array}{ccc}
 \mathrm{GW}^{[n]}(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}) & \longrightarrow & \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(-n)}/\mathrm{D}^b(P(X))^{(-n+1)}) \\
 \downarrow d_{\mathcal{T}}^n & & \downarrow d^{-n} \\
 \mathrm{W}^{[n+1]}(\mathcal{T}_{(n-1)}) & \longrightarrow & \mathrm{W}^{[n+1]}(\mathrm{D}^b(P(X))^{(-n+1)}) \\
 \uparrow i_{\mathcal{T}}^{n-1, \mathrm{W}} & & \uparrow i^{-n+1, \mathrm{W}} \\
 \mathrm{W}^{[n+1]}(\mathcal{T}_{(n-2)}) & \longrightarrow & \mathrm{W}^{[n+1]}(\mathrm{D}^b(P(X))^{(-n+2)})
 \end{array}$$

coming from the above equivalences are isomorphisms. As a consequence, we have

$$\begin{aligned}
 \tilde{B}^{-n}(X, L) &:= \mathrm{Im}(\partial^{-n+1}) \cong q^{-n}(\ker(i^{-n-1})) \\
 &\cong q_{\mathcal{T}}^n(\ker(i_{\mathcal{T}}^{n+1})) =: \tilde{B}_{\Delta}^n(\mathrm{D}^{\mathrm{perf}}(X), \mathrm{hom}(-, L))
 \end{aligned}$$

and, moreover,

$$\begin{aligned}
 \tilde{Z}^{-n}(X, L) &:= \ker(\partial^{-n}) \cong \ker(\tilde{\partial}^{-n}) \cong (d^{-n})^{-1}(\mathrm{Im}(i^{-n+1, \mathrm{W}})) \\
 &\cong (d_{\mathcal{T}}^n)^{-1}(\mathrm{Im}(i_{\mathcal{T}}^{n-1, \mathrm{W}})) =: \tilde{Z}_{\Delta}^n(\mathrm{D}^{\mathrm{perf}}(X), \mathrm{hom}(-, L)).
 \end{aligned}$$

□

Remark 4.4.6. Fasel and Srinivas present an alternative definition of Chow–Witt groups of a scheme in terms of Grothendieck–Witt groups of triangulated categories with duality. One could also have started the attempt of generalizing Chow–Witt groups of a scheme at this point.

Consider the exact sequences of triangulated categories with duality

$$\begin{aligned}
 \mathrm{D}^b(P(X))^{(n+1)}[(\mathrm{quis}^{n+2})^{-1}] &\rightarrow \mathrm{D}^b(P(X))^{(n)}[(\mathrm{quis}^{n+2})^{-1}] \\
 &\rightarrow \mathrm{D}^b(P(X))^{(n)}[(\mathrm{quis}^{n+1})^{-1}]
 \end{aligned}$$

and

$$\begin{aligned}
 \mathrm{D}^b(P(X))^{(n)}[(\mathrm{quis}^{n+1})^{-1}] &\rightarrow \mathrm{D}^b(P(X))^{(n-1)}[(\mathrm{quis}^{n+1})^{-1}] \\
 &\rightarrow \mathrm{D}^b(P(X))^{(n-1)}[(\mathrm{quis}^n)^{-1}]
 \end{aligned}$$

They yield exact sequences by localization

$$\begin{aligned}
 \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n)}[(\mathrm{quis}^{n+2})^{-1}]) &\xrightarrow{\alpha} \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n)}[(\mathrm{quis}^{n+1})^{-1}]) \\
 &\xrightarrow{q^{n-1, \mathrm{W}} \circ d^n} \mathrm{W}^{[n+1]}(\mathrm{D}^b(P(X))^{(n+1)}[(\mathrm{quis}^{n+2})^{-1}])
 \end{aligned}$$

and

$$\begin{aligned}
 \mathrm{GW}_1^{[n]}(\mathrm{D}^b(P(X))^{(n-1)}[(\mathrm{quis}^n)^{-1}]) &\xrightarrow{q^n \circ d^{n-1}} \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n)}[(\mathrm{quis}^{n+1})^{-1}]) \\
 &\xrightarrow{\beta} \mathrm{GW}^{[n]}(\mathrm{D}^b(P(X))^{(n-1)}[(\mathrm{quis}^{n+1})^{-1}]),
 \end{aligned}$$

where $\mathrm{GW}_1^{[n]}(\mathrm{D}^b(P(X))^{(n-1)}[(\mathrm{quis}^n)^{-1}])$ is defined as the higher Grothendieck–Witt group $\mathrm{GW}_1^{[n]}(\mathrm{Ch}^b(P(X))^{(n-1)}, \mathrm{quis}^n)$ of the underlying complicial exact category with weak equivalences and duality, compare Remark 3.2.15. The localization sequences of triangulated categories with duality from Theorem 3.3.3 and of complicial exact categories with weak equivalences and duality from Theorem 3.3.4 are combined here, using that they coincide from Grothendieck–Witt groups of degree ≤ 0 on by Theorem 3.3.4.

In [FS09, Theorem 34], Fasel and Srinivas prove that

$$\widetilde{\mathrm{CH}}^n(X, L) = \mathrm{Im}(\alpha) / \ker(\beta)$$

for a regular $\mathbb{Z}[\frac{1}{2}]$ -linear scheme X and a line bundle L . Using the tools of this proof, one can show Theorem 4.4.2 and Theorem 4.4.3 in the following way.

By exactness and [FS09, Theorem 34], it follows that

$$\begin{aligned} \widetilde{Z}^n(X, L) &= \mathrm{Im}(\alpha) = \ker(q^{n-1, \mathrm{W}} \circ d^n) = (d^n)^{-1}(\ker(q^{n-1, \mathrm{W}})) \\ &= (d^n)^{-1}(\mathrm{Im}(i^{n-1, \mathrm{W}})). \end{aligned}$$

and

$$\widetilde{B}^n(X, L) = \ker(\beta) = \mathrm{Im}(q^n \circ d^{n-1}) = q(\mathrm{Im}(d^{n-1})) = q^n(\ker(i^{n-1})).$$

One of the main reasons why we did not go down the route of [FS09, Theorem 34] was that we wanted to distinguish more clearly the localization sequences in the setting of triangulated categories with duality and of complicial exact categories with weak equivalences and duality. On the other hand, we aimed to stay close to the definition of tensor triangular Chow groups introduced in [Bal13] and the corresponding agreement theorem [Kl16a, Theorem 3.2.6].

Remark 4.4.7. In Theorem 4.4.5, we assume X to be separated, regular, and of finite type over a field k with characteristic $\neq 2$. These assumptions originate in the definition of Chow–Witt groups we use, compare [Fas20, Section 2.1]. If one uses a more general definition of Chow–Witt groups for regular $\mathbb{Z}[\frac{1}{2}]$ -linear schemes as for example in [FS09, Definition 32], the proof of Theorem 4.4.5 also holds true for this slightly more general case (so do the proofs from Remark 4.4.6).

To get an intuitive understanding of why Theorem 4.4.5 is true, we compare easy Chow–Witt groups “by hand”.

Example 4.4.8. Let k be a field of characteristic $\neq 2$. We know that

$$\widetilde{\mathrm{CH}}^n(\mathrm{Spec}(k)) = \begin{cases} \mathrm{GW}(k) & \text{if } n = 0 \\ 0 & \text{else.} \end{cases}$$

We want to show the same for $\widetilde{\mathrm{CH}}_{\Delta}^n(\mathrm{D}^{\mathrm{perf}}(\mathrm{Spec}(k)))$.

Consider $\mathcal{T} := \mathrm{D}^{\mathrm{perf}}(\mathrm{Spec}(k))$ equipped with the dimension function $\dim := -\mathrm{codim}_{\mathrm{Krull}}$, i.e. the category of perfect complexes of $\mathcal{O}_{\mathrm{Spec}(k)}$ -modules. An $\mathcal{O}_{\mathrm{Spec}(k)}$ -module is a sheaf that sends open sets in $\mathrm{Spec}(k)$ to $\mathcal{O}_{\mathrm{Spec}(k)} = k$ -modules. Since

$\mathrm{Spec}(k)$ contains only one point, $\mathcal{O}_{\mathrm{Spec}(k)}$ -modules are completely determined by the k -vector space the point is sent to, i.e. we can identify $\mathcal{O}_{\mathrm{Spec}(k)}$ -modules with k -vector spaces. It follows that $D^{\mathrm{perf}}(\mathrm{Spec}(k))$ is given by perfect complexes of k -vector spaces.

We now want to understand what $\mathcal{T}_{(n)}$ looks like for $n \in \mathbb{Z}$. For this, we check that for all objects $A \neq 0$ in \mathcal{T} we have $\mathrm{supp}(A) = \{0\}$, hence $\dim(\mathrm{supp}(A)) = 0$, i.e. $-1 < \dim(\mathrm{supp}(A)) \leq 0$. For $0 \in \mathcal{T}$, we have $\mathrm{supp}(0) = \emptyset$ by [Bal07, Proposition 1.3 (i)]. Hence, we obtain

$$\mathcal{T}_{(n)} = \begin{cases} \mathcal{T} & \text{for } n \geq 0 \\ 0 & \text{for } n \leq -1 \end{cases}$$

and for the quotients

$$\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)} = \begin{cases} 0 & \text{for } n \geq 1 \\ \mathcal{T} & \text{for } n = 0 \\ 0 & \text{for } n \leq -1. \end{cases}$$

For $n = 0$, the defining diagram Diagram 4.3 becomes

$$\begin{array}{ccc} W^{[1]}(\mathcal{T}_{(-2)}) \xrightarrow{i^{-1,W}} W^{[1]}(\mathcal{T}_{(-1)}) & & \mathrm{GW}^{[0]}(\mathcal{T}_{(0)}) \xrightarrow{i^1} \mathrm{GW}^{[1]}(\mathcal{T}_{(1)}) \\ & \swarrow d^0 & \downarrow q^0 \\ & & \mathrm{GW}^{[0]}(\mathcal{T}_{(0)}/\mathcal{T}_{(-1)}) \end{array}$$

and thus

$$\begin{array}{ccc} W^{[1]}(0) \xrightarrow{0_i} W^{[1]}(0) & & \mathrm{GW}^{[0]}(\mathcal{T}) \xrightarrow{\mathrm{id}} \mathrm{GW}^{[1]}(\mathcal{T}). \\ & \swarrow 0_d & \downarrow \mathrm{id} \\ & & \mathrm{GW}^{[0]}(\mathcal{T}) \end{array}$$

It follows that

$$\widetilde{\mathrm{CH}}_{\Delta}^0(D^{\mathrm{perf}}(\mathrm{Spec}(k))) = 0_d^{-1}(\mathrm{Im}(0_i)) \big/_{\mathrm{id}(\ker(\mathrm{id}))} \mathrm{GW}(\mathcal{T}) \big/_0 = \mathrm{GW}(\mathcal{T}) \big/_0 = \mathrm{GW}(\mathcal{T}).$$

Using Proposition 3.2.7, we obtain

$$\widetilde{\mathrm{CH}}_{\Delta}^0(D^{\mathrm{perf}}(\mathrm{Spec}(k))) = \mathrm{GW}(\mathcal{T}) \cong \mathrm{GW}(k).$$

The cases $n \neq 0$ can be seen directly.

This subsection has proven that the construction of tensor triangular Chow–Witt groups we gave in Chapter 4 agrees with the classical definition of Chow–Witt groups of a scheme X for the derived category of perfect complexes $D^{\mathrm{perf}}(X)$, X being regular. Moreover, we have seen an easy example by hand. The following section discusses functorial properties of $\widetilde{\mathrm{CH}}_{\Delta}^n$.

4.5 Functoriality

We are in the situation of Convention 4.3.1. After having seen the agreement theorem between classical and tensor triangular Chow–Witt groups, we now show the functoriality of the construction from Chapter 4. We will see that we obtain, in particular, flat pullbacks and proper pushforwards.

First, we need to specify what kind of functors we allow.

Definition 4.5.1. Let \mathcal{T}, \mathcal{L} be tensor triangulated categories equipped with dimension functions, and let $F : \mathcal{T} \rightarrow \mathcal{L}$ be a triangulated functor. We say that F is of *relative dimension* m if there exists a smallest $m \in \mathbb{N}$ such that $F(\mathcal{T}_{(n)}) \subset \mathcal{L}_{(n+m)}$ for all n .

Remark 4.5.2. We do not require F to be a tensor functor following the idea of [Kl16a, Remark 4.1.2]. There are various examples of functors of relative dimension that are neither tensor functors nor duality-preserving, e.g. $a \otimes -$ when $\dim(\text{supp}(a)) \neq \pm\infty$, see [Kl16a, Proposition 4.1.6].

Remark 4.5.3. Suppose F is a functor of relative dimension m . By shifting the dimension function, we can always obtain $m = 0$. Recall that shifting the dimension function on a rigid tensor triangulated category \mathcal{T} by $m' \in \mathbb{Z}$ yields isomorphisms $\widetilde{\text{CH}}_{\Delta}^n(\mathcal{T}) \cong \widetilde{\text{CH}}_{\Delta}^{n+m'}(\mathcal{T})$ for all n , compare Remark 4.3.6.

We can generalize [Kl16a, Theorem 4.1.3] in the following way.

Theorem 4.5.4. Let \mathcal{T}, \mathcal{L} be rigid tensor triangulated categories equipped with dimension functions and let K and L be \otimes -invertible objects in \mathcal{T} and \mathcal{L} , respectively. Moreover, let $F : \mathcal{T} \rightarrow \mathcal{L}$ be a functor of relative dimension 0 that is duality-preserving when considering \mathcal{T} and \mathcal{L} triangulated categories with duality $(\mathcal{T}, \text{hom}(-, K))$ and $(\mathcal{L}, \text{hom}(-, L))$ as in Example 2.2.6. Then, F induces group homomorphisms for all $n \in \mathbb{Z}$

$$\begin{aligned} \tilde{z}^n : \widetilde{Z}_{\Delta}^n(\mathcal{T}, K) &\rightarrow \widetilde{Z}_{\Delta}^n(\mathcal{L}, L) \text{ and} \\ \tilde{c}^n : \widetilde{\text{CH}}_{\Delta}^n(\mathcal{T}, K) &\rightarrow \widetilde{\text{CH}}_{\Delta}^n(\mathcal{L}, L). \end{aligned}$$

Proof. We will omit the dualities from the notation for better readability. The functor F restricts to functors on the filtration components $F_i : \mathcal{T}_{(i)} \rightarrow \mathcal{L}_{(i)}$ for all i . Consider the following commutative diagram, which is the underlying right part of the defining Diagram 4.3,

$$\begin{array}{ccccc} \mathcal{T}_{(n)} & \xrightarrow{I_{\mathcal{T}}^{n+1}} & \mathcal{T}_{(n+1)} & & \\ \downarrow Q_{\mathcal{T}}^n & \searrow F_n & \searrow F_{n+1} & & \\ \mathcal{T}_{(n)}/\mathcal{T}_{(n-1)} & & \mathcal{L}_{(n)} & \xrightarrow{I_{\mathcal{L}}^{n+1}} & \mathcal{L}_{(n+1)} \\ & \searrow \bar{F}_n & \downarrow Q_{\mathcal{L}}^n & & \\ & & \mathcal{L}_{(n)}/\mathcal{L}_{(n-1)} & & \end{array}$$

Here, \overline{F}_n exists since $F(\mathcal{T}_{(n-1)}) \subset \mathcal{L}_{(n-1)}$ by definition, hence, F induces a morphism on quotients.

Since F is duality-preserving, so are F_i and \overline{F}_i for all i . Consequently, we obtain the commutative diagram

$$\begin{array}{ccccc}
 \mathrm{GW}^{[n]}(\mathcal{T}_{(n)}) & \xrightarrow{i_{\mathcal{T}}^{n+1}} & \mathrm{GW}^{[n]}(\mathcal{T}_{(n+1)}) & & \\
 \downarrow q_{\mathcal{T}}^n & \searrow f_n & & \searrow f_{n+1} & \\
 \mathrm{GW}^{[n]}(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}) & & \mathrm{GW}^{[n]}(\mathcal{L}_{(n)}) & \xrightarrow{i_{\mathcal{L}}^{n+1}} & \mathrm{GW}^{[n]}(\mathcal{L}_{(n+1)}) \\
 & \searrow \overline{f} & \downarrow q_{\mathcal{L}}^n & & \\
 & & \mathrm{GW}^{[n]}(\mathcal{L}_{(n)}/\mathcal{L}_{(n-1)}) & &
 \end{array} \tag{4.4}$$

by applying $\mathrm{GW}^{[n]}$. For the left part of the defining Diagram 4.3, note that the diagram

$$\begin{array}{ccc}
 \mathcal{T}_{(n-2)} & \xrightarrow{I_{\mathcal{T}}^{n-1}} & \mathcal{T}_{(n-1)} \\
 F_{n-2} \downarrow & & \downarrow F_{n-1} \\
 \mathcal{L}_{(n-2)} & \xrightarrow{I_{\mathcal{L}}^{n-1}} & \mathcal{L}_{(n-1)}
 \end{array}$$

commutes, and, as above, we can apply $\mathrm{W}^{[n+1]}$ and obtain a commutative diagram

$$\begin{array}{ccc}
 \mathrm{W}^{[n+1]}(\mathcal{T}_{(n-2)}) & \xrightarrow{i_{\mathcal{T}}^{n-1, \mathrm{W}}} & \mathrm{W}^{[n+1]}(\mathcal{T}_{(n-1)}) \\
 f'_{n-2} \downarrow & & \downarrow f'_{n-1} \\
 \mathrm{W}^{[n+1]}(\mathcal{L}_{(n-2)}) & \xrightarrow{i_{\mathcal{L}}^{n-1, \mathrm{W}}} & \mathrm{W}^{[n+1]}(\mathcal{L}_{(n-1)}).
 \end{array} \tag{4.5}$$

It is left to show that the diagram

$$\begin{array}{ccc}
 \mathrm{W}^{[n+1]}(\mathcal{T}_{(n-1)}) & \xleftarrow{d_{\mathcal{T}}^n} & \mathrm{GW}^{[n]}(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}) \\
 f'_{n-1} \downarrow & & \downarrow \overline{f}_n \\
 \mathrm{W}^{[n+1]}(\mathcal{L}_{(n-1)}) & \xleftarrow{d_{\mathcal{L}}^n} & \mathrm{GW}^{[n]}(\mathcal{L}_{(n)}/\mathcal{L}_{(n-1)})
 \end{array}$$

also commutes, but this is Lemma 4.5.5 below, and, thus, we can glue together Diagram 4.4 and Diagram 4.5. Hence, we have a homomorphism

$$\overline{f}_n : \mathrm{GW}^{[n]}(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}) \rightarrow \mathrm{GW}^{[n]}(\mathcal{L}_{(n)}/\mathcal{L}_{(n-1)})$$

which restricts to a homomorphism

$$\overline{f}_n : \widetilde{Z}_{\Delta}^n(\mathcal{T}) \rightarrow \mathrm{GW}^{[n]}(\mathcal{L}_{(n)}/\mathcal{L}_{(n-1)})$$

since $\tilde{Z}_\Delta^n(\mathcal{T})$ is a subgroup of $\mathrm{GW}^{[n]}(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})$ by Proposition 4.3.4. Now, the glued diagram yields

$$\bar{f}_n(\tilde{Z}_\Delta^n(\mathcal{T})) \subset \tilde{Z}_\Delta^n(\mathcal{L}),$$

so we set $\bar{f}_n =: \tilde{z}^n$.

To show that \tilde{z}_n induces a homomorphism on Chow–Witt groups, we observe directly from Diagram 4.4 and Diagram 4.5 that

$$\tilde{z}^n(\tilde{B}_\Delta^n(\mathcal{T})) \subset \tilde{B}_\Delta^n(\mathcal{L}).$$

□

Lemma 4.5.5. *The diagram*

$$\begin{array}{ccc} \mathrm{W}^{[n+1]}(\mathcal{T}_{(n-1)}) & \xleftarrow{d_{\mathcal{T}}^n} & \mathrm{GW}^{[n]}(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}) \\ f'_{n-1} \downarrow & & \downarrow \bar{f}_n \\ \mathrm{W}^{[n+1]}(\mathcal{L}_{(n-1)}) & \xleftarrow{d_{\mathcal{L}}^n} & \mathrm{GW}^{[n]}(\mathcal{L}_{(n)}/\mathcal{L}_{(n-1)}) \end{array}$$

from above commutes.

Proof. To see that this diagram commutes, we recall that it comes from the localization sequence Theorem 3.3.3. We have exact sequences (the horizontal lines) of triangulated categories with duality and (duality-preserving) maps between these as above such that the following diagram commutes.

$$\begin{array}{ccccc} \mathcal{T}_{n-1} & \xrightarrow{I_{\mathcal{T}}^n} & \mathcal{T}_{(n)} & \xrightarrow{Q_{\mathcal{T}}^n} & \mathcal{T}_{(n)}/\mathcal{T}_{(n-1)} \\ \downarrow F_{n-1} & & \downarrow F_n & & \downarrow \bar{F}_n \\ \mathcal{L}_{n-1} & \xrightarrow{I_{\mathcal{L}}^n} & \mathcal{L}_{(n)} & \xrightarrow{Q_{\mathcal{L}}^n} & \mathcal{L}_{(n)}/\mathcal{L}_{(n-1)} \end{array}$$

The horizontal sequences now give rise to long exact localization sequences (by Theorem 3.3.3) with induced maps between them. The square from the statement can now be found as a part of the lower diagram and, hence, commutes by construction of the differential in the localization sequences.

$$\begin{array}{ccccccc} \mathrm{GW}^{[n]}(\mathcal{T}_{(n)}) & \longrightarrow & \mathrm{GW}^{[n]}(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}) & \xrightarrow{d_{\mathcal{T}}^n} & \mathrm{W}^{[n+1]}(\mathcal{T}_{(n-1)}) & \longrightarrow & \mathrm{W}^{[n+1]}(\mathcal{T}_{(n)}) \\ \downarrow f_n & & \downarrow \bar{f}_n & & \downarrow f'_{n-1} & & \downarrow f'_n \\ \mathrm{GW}^{[n]}(\mathcal{L}_{(n)}) & \longrightarrow & \mathrm{GW}^{[n]}(\mathcal{L}_{(n)}/\mathcal{L}_{(n-1)}) & \xrightarrow{d_{\mathcal{L}}^n} & \mathrm{W}^{[n+1]}(\mathcal{L}_{(n-1)}) & \longrightarrow & \mathrm{W}^{[n+1]}(\mathcal{L}_{(n)}). \end{array}$$

□

As an example, let us consider flat pullbacks and proper pushforwards.

Example 4.5.6. (i) Let $f : X \rightarrow Y$ be a faithfully flat morphism between separated schemes of finite type over a field and consider the rigid tensor triangulated categories $D^{\text{perf}}(X)$ and $D^{\text{perf}}(Y)$ equipped with the dimension function $-\text{codim}_{K_{\text{rull}}}$. Then, by [Kl16a, Lemma 4.3.2], the functor $Lf^* : D^{\text{perf}}(Y) \rightarrow D^{\text{perf}}(X)$ has relative dimension 0. Hence, it induces homomorphisms on the respective tensor triangular Chow groups that coincide with the flat pullbacks of the usual Chow groups (see [Kl16a, Section 4.3]). For tensor triangular Chow–Witt groups, let K be a line bundle on Y . Assume that X and Y are regular. The functor

$$Lf^* : (D^{\text{perf}}(Y), \text{hom}(-, K)) \rightarrow (D^{\text{perf}}(X), \text{hom}(-, Lf^*(K)))$$

is duality-preserving by [CH11, Theorem 4.1] and, hence, it induces a morphism for all n between the respective Chow–Witt groups

$$Lf^* : \widetilde{\text{CH}}_{\Delta}^n(D^{\text{perf}}(Y), K) \rightarrow \widetilde{\text{CH}}_{\Delta}^n(D^{\text{perf}}(X), Lf^*(K)).$$

The author assumes that one can show that it coincides with the flat pullback for usual Chow–Witt groups. For a more general case of pullbacks, see [CH09, Theorem 4.1.2 and Corollary 4.1.3].

(ii) Let $f : X \rightarrow Y$ be a proper morphism between integral, regular, separated schemes of finite type over an algebraically closed field and consider the rigid tensor triangulated categories $D^{\text{perf}}(X)$ and $D^{\text{perf}}(Y)$ equipped with the dimension function $\dim_{K_{\text{rull}}}$. Then, by [Kl16a, Lemma 4.4.1], the functor $Rf_* : D^{\text{perf}}(X) \rightarrow D^{\text{perf}}(Y)$ has relative dimension 0. Hence, it induces homomorphisms on the respective tensor triangular Chow groups that coincide with the proper pushforwards of the usual Chow groups (see [Kl16a, Section 4.4]).

For tensor triangular Chow–Witt groups, let K be a line bundle on Y . The functor

$$Rf_* : (D^{\text{perf}}(X), \text{hom}(-, f^!(K))) \rightarrow (D^{\text{perf}}(Y), \text{hom}(-, K))$$

is duality-preserving by [CH11, Theorem 4.4] and, hence, it induces a morphism for all n between the respective Chow–Witt groups

$$Rf_* : \widetilde{\text{CH}}_{\Delta}^n(D^{\text{perf}}(X), f^!(K)) \rightarrow \widetilde{\text{CH}}_{\Delta}^n(D^{\text{perf}}(Y), K).$$

Here, $f^!$ is the right adjoint of Rf_* from [CH11, Theorem 3.8]. The author assumes that one can show that Rf_* coincides with the flat pullback for usual Chow–Witt groups. For a more general case of pushforwards, see [CH09, Corollary 4.3.3].

Summing up, this chapter has given a concrete definition of tensor triangular Chow–Witt groups in Section 4.3 after recalling tensor triangular Chow groups in Section 4.1 and classical Chow–Witt groups in Section 4.2. After this, we have seen the agreement with the algebro-geometric case in Section 4.4 and the functoriality above.

4.6 Chow(–Witt) Groups for Tensor Triangular Categories with Models

Convention 4.6.1. Let \mathcal{T} be as in Convention 4.3.1, but now we allow the existence of quotients of filtration components $\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}$ that are not idempotent complete. We also assume that \mathcal{T} is idempotent complete. This implies that all filtration components $\mathcal{T}_{(n)}$ are also idempotent complete. We will fix $L = \mathbf{1}$ and, hence, omit L from the notation.

In this section, ideas are merely sketched and can be generalized to arbitrary \otimes -invertible L . We investigate the case of tensor triangular Chow groups at first, although their Definition 4.1.2 does not require quotients of filtration components to be idempotent complete, whereas we do require it for the construction of tensor triangular Chow–Witt groups as in Definition 4.3.5.

We will mainly use the notation from [Sch06].

We want to redefine Chow groups of tensor triangulated categories as introduced in Definition 4.1.2, in particular, when quotients of filtration components are not necessarily idempotent complete. Later, we want to generalize tensor triangular Chow–Witt groups Definition 4.3.5 to the above case.

The underlying motivation is the following: By [Sch06, Remark 1], we know that if $\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}$ is not idempotent complete, $\mathbb{K}_{-1}(\mathcal{T}_{(n-1)})$ is nonzero. Consider the case of a *nice enough* singular scheme X , $\mathcal{T} = \mathrm{D}^{\mathrm{perf}}(X)$, still satisfying Bloch’s formula but such that $\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}$ is not idempotent complete. If Bloch’s formula holds, we would like to use this cohomological description of its Chow groups to translate the definition to the tensor triangular setting as seen in Section 4.1 for regular schemes.

If X is regular and negative K -groups vanish, the tensor triangular n -cocycles are everything, i.e. $K_0((\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})^{\natural})$; if not, negative K -groups appear and hence the tensor triangular n -cocycles may be a proper subgroup of $K_0((\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})^{\natural})$. We thus want to redefine tensor triangular n -cocycles.

What changes in the non-singular case is that we do not have an equivalence of tensor triangulated categories $\mathrm{D}^{\mathrm{perf}}(X) \simeq \mathrm{D}^b(\mathrm{Coh}(X))$ and, hence, the quotients of the filtration components do not have a good description, but the idempotent completion can be described as

$$(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})^{\natural} \simeq \coprod_{x \in X_{(n)}} \mathrm{D}_{\{x\}}^{\mathrm{perf}}(\mathcal{O}_{X,x}),$$

compare [Bal09, Proof of Theorem 2]. To adapt the construction of tensor triangular cocycles to this case, we thus need a localization sequence of the form

$$\begin{aligned} K_0(\mathcal{T}_{(n-1)}) &\xrightarrow{i_{\mathcal{T}}^n} K_0(\mathcal{T}_{(n)}) \xrightarrow{q_n'} K_0((\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})^{\natural}) \\ \xrightarrow{d_n^l} \mathbb{K}_{-1}(\mathcal{T}_{(n-1)}) &\xrightarrow{i_{\mathcal{T}}^n} \mathbb{K}_{-1}(\mathcal{T}_{(n)}) \xrightarrow{q_n'} \mathbb{K}_{-1}((\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})^{\natural}) \\ \xrightarrow{d_n^l} \mathbb{K}_{-2}(\mathcal{T}_{(n-1)}) &\rightarrow \dots, \end{aligned}$$

which we obtain from [Sch06, Theorem 1].

Since $\mathcal{T}_{(n)}$ is idempotent complete for all n , the maps $i_{\mathcal{T}}^n$ coincide in degree 0 with the maps $i_{\mathcal{T}}^n$ from Definition 4.1.2 and the maps q'_n in degree 0 are the composition of the maps $j_{\mathcal{T}}^n : K_0(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}) \rightarrow K_0((\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})^\natural)$, induced by idempotent completion, and $q_{\mathcal{T}}^n$ from Definition 4.1.2.

However, the price one has to pay for this is the introduction of *models* in order to obtain localization sequences, which means that we need our tensor triangulated category \mathcal{T} to *admit a model*. The above localization sequence is defined on the level of models, hence, there is a slight abuse of notation here. For more details on models in general, the reader is referred to [Sch06, Section 2.2]. The models Schlichting uses are *Frobenius models*, compare [Sch06, Sections 4–6], which are the same kind of models Klein uses in [Kl16b] to introduce a ring structure on tensor triangular Chow groups, but other models categories are possible such as the category of small exact categories.

In particular, our most prominent examples $D^{\text{perf}}(X)$ for a noetherian k -scheme X and the stable module category $kG\text{-stab}$ for G a finite group and k a field with characteristic dividing the order of G are such tensor triangulated categories which admit for instance Frobenius models; more examples of Frobenius models can be found in [Sch06, Chapter 6]. To see why admitting models might not be too strong a restriction the reader is referred to the introduction of [Sch06].

Hence, we can introduce the following variant of Definition 4.1.2, where the idea is to extend the Gersten complex to negative degrees.

Definition 4.6.2. Let \mathcal{T} be an idempotent complete, essentially small tensor triangulated category equipped with a dimension function \dim that admits a model and such that $\text{Spc}(\mathcal{T})$ is noetherian. When considering \mathcal{T} a triangulated category with duality, we fix the duality $\text{hom}(-, \mathbf{1})$ and restrictions of it on subcategories, quotients and idempotent completions (by Lemma 2.3.7, Lemma 2.3.8). Then, we define

$$Z_{\Delta'}^n(\mathcal{T}) := \ker(q'_{n-1} \circ d'_n) \subset K_0((\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})^\natural)$$

which, by exactness of the above localization sequence, equals

$$(d'_n)^{-1}(\text{Im}(i_{\mathcal{T}}^{n-1})),$$

and

$$\begin{aligned} \text{CH}_{\Delta'}^n(\mathcal{T}) &:= Z_{\Delta'}^n(\mathcal{T})/q'_n(\ker(i_{\mathcal{T}}^{n+1})) \\ &= Z_{\Delta'}^n(\mathcal{T})/j_{\mathcal{T}}^n \circ q_{\mathcal{T}}^n(\ker(i_{\mathcal{T}}^{n+1})). \end{aligned}$$

The algebraic n -cocycles from Definition 4.6.2 and Balmer’s Definition 4.1.2 differ a priori if the quotient $\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}$ is not idempotent complete, i.e. if $j_{\mathcal{T}}^n$ is not the identity, but the coboundaries coincide.

It follows from the construction that for $\mathcal{T} = D^{\text{perf}}(X)$, X non-singular (so, in particular, for $j_{\mathcal{T}}^n$ the identity), $\text{CH}_{\Delta'}^n(\mathcal{T})$ and $\text{CH}_{\Delta}^n(\mathcal{T})$, and thus by Lemma 4.1.3 also $\text{CH}^{-n}(X)$, coincide.

Remark 4.6.3. For noetherian schemes which may be singular, the idea of adapting Balmer’s definition of tensor triangular cocycles for the derived category of perfect complexes already appears in the unpublished work [Ya16, Definition 3.6 and Section 3.2]. We sketched the generalized idea for general idempotent complete, essentially small tensor triangulated categories with a dimension function that admit models, hence, in particular, for $\mathcal{T} = kG\text{-stab}$, where G is a finite group and k a field such that $\text{char}(k)$ divided $|G|$.

Let X be a connected quasi-projective variety of dimension $d \geq 2$ whose singular locus $\text{Sing}(X)$ is contained in a finite closed set Y . If X is *nice enough*, i.e. it in particular satisfy some kind of Bloch’s formula (see [Ya16, Question 3.19]), Yang’s definition for $\mathcal{T} = \text{D}^{\text{perf}}(X)$ coincides with the relative Chow groups $\text{CH}(X, Y)$ of Pedrini and Weibel from [PW86] by [Ya16, Corollary 3.21] after tensoring with \mathbb{Q} , hence so they do with Definition 4.6.2 after tensoring with \mathbb{Q} .

Balmer’s tensor triangular n -cocycles from Definition 4.1.2 differ a priori from $Z_{\Delta}^n(\mathcal{T})$ and for this reason in the special case mentioned above also from the relative Chow groups from Pedrini and Weibel.

In the case of tensor triangular Chow–Witt groups, we observe something similar. As usual, we now assume \mathcal{T} to be rigid, $\mathbb{Z}[\frac{1}{2}]$ -linear, and dualities as in Convention 4.3.1 when considering \mathcal{T} a triangulated category with duality.

The Witt group $W^{[n+1]}(\mathcal{T}_{(n-1)})$ may be nonzero even when $\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}$ is idempotent complete. As a consequence, the oriented n -cocycles may be a proper subgroup of $\text{GW}^{[n]}((\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})^{\natural})$, which follows from Definition 4.3.3. If the quotient is idempotent complete, the localization sequence from Theorem 3.3.3 still suffices to define tensor triangular Chow–Witt groups as we did in Definition 4.3.5.

If the quotient is not idempotent complete, as above we need a sequence of the form

$$\begin{aligned} & \text{GW}^{[n]}(\mathcal{T}_{(n-1)}) \xrightarrow{\tilde{i}_{\mathcal{T}}^n} \text{GW}^{[n]}(\mathcal{T}_{(n)}) \xrightarrow{\tilde{q}'_n} \text{GW}^{[n]}((\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})^{\natural}) \\ & \xrightarrow{\tilde{q}'_n} W^{[n+1]}(\mathcal{T}_{(n-1)}) \xrightarrow{\tilde{i}_{\mathcal{T}}^{n,W}} W^{[n+1]}(\mathcal{T}_{(n)}) \xrightarrow{\tilde{q}'_{n,W}} W^{[n+1]}((\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})^{\natural}) \\ & \xrightarrow{\tilde{q}'_n} W^{[n+2]}(\mathcal{T}_{(n-1)}) \rightarrow \dots, \end{aligned}$$

which we have after tensoring with $\mathbb{Z}[\frac{1}{2}]$ essentially by [Bal09, Remark 3]. Note that here, models have to carry some notion of *duality* such as complicial BiWaldhausen categories, compare [Bal09] and [Sch06, Section 6.5].

Again, the maps $\tilde{i}_{\mathcal{T}}^{n,(W)}$ coincide with the maps $i_{\mathcal{T}}^{n,(W)}$ from Definition 4.3.5 and the maps $\tilde{q}'_{n,(W)}$ are the composition of the map $\tilde{j}_{\mathcal{T}}^n : \text{GW}^{[n]}(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}) \rightarrow \text{GW}^{[n]}((\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})^{\natural})$ and $\tilde{j}_{\mathcal{T}}^{n,W} : W^{[n]}(\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}) \rightarrow W^{[n]}((\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)})^{\natural})$, respectively, induced by idempotent completion, and $q_{\mathcal{T}}^{n,(W)}$ from Definition 4.3.5.

We thus have a variant of Definition 4.3.5 for tensor triangulated categories admitting models:

Definition 4.6.4. Let \mathcal{T} be a $\mathbb{Z}[\frac{1}{2}]$ -linear, essentially small, idempotent complete, rigid tensor triangulated category equipped with a dimension function admitting

a model and such that $\mathrm{Spc}(\mathcal{T})$ is noetherian. When considering \mathcal{T} a triangulated category with duality, we fix the duality $\mathrm{hom}(-, \mathbb{1})$ and restrictions of it on subcategories. Moreover, we consider all groups being tensored by $\mathbb{Z}[\frac{1}{2}]$ here. We define

$$\begin{aligned}\tilde{Z}_{\Delta'}^n(\mathcal{T}) &:= (\tilde{d}'_n)^{-1}(\mathrm{Im}(\tilde{i}_{\mathcal{T}}^{n-1, \mathrm{W}})) \\ &= \ker(\tilde{q}'_{n-1, \mathrm{W}} \circ \tilde{d}'_n)\end{aligned}$$

and

$$\tilde{\mathrm{CH}}_{\Delta'}^n(\mathcal{T}) := \tilde{Z}_{\Delta'}^n(\mathcal{T}) / \tilde{q}'_n(\ker(\tilde{i}_{\mathcal{T}}^{n+1})) = \tilde{Z}_{\Delta'}^n(\mathcal{T}) / \tilde{j}_{\mathcal{T}}^n \circ q_{\mathcal{T}}^n(\ker(i_{\mathcal{T}}^{n+1})).$$

When $\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}$ is idempotent complete for all n , for example for $\mathcal{T} = \mathrm{D}^{\mathrm{perf}}(X)$ and X smooth, Definition 4.3.5 and Definition 4.6.4 coincide since $\tilde{j}_{\mathcal{T}}^n$ is the identity in this case. Moreover, when replacing GW by K_0 in the construction, Definition 4.6.4 agrees with Definition 4.6.2 and thus with Definition 4.1.2 when all quotients $\mathcal{T}_{(n)}/\mathcal{T}_{(n-1)}$ are idempotent complete.

In this section, we have redefined our definition of tensor triangular Chow–Witt groups for cases when quotients of the filtration components $\mathcal{T}_{(n)}$ are not idempotent complete. For this, we sketched the idea of introducing models for (rigid) tensor triangulated categories.

The following chapter is dedicated to concrete calculations of tensor triangular Chow–Witt groups in the setting of modular representation theory.

Chapter 5

Computations in Modular Representation Theory

In this chapter, we want to compute examples of tensor triangular Chow–Witt groups. Since, for the derived category of perfect complexes over a regular scheme, the definition coincides with the algebro-geometric one for regular schemes (Theorem 4.4.5), the field of algebraic geometry already provides us with many known examples. It is therefore interesting to consider Chow–Witt groups in new areas Definition 4.3.5 and Definition 4.6.4 have opened, for example in modular representation theory.

To do so, we will recall some basic concepts on group rings and the stable module category in Section 5.1. In Section 5.2, we will collect helpful theorems and propositions that will serve as tools for the computation of the Chow–Witt group of the stable module category $kG\text{-stab}$ – first for $G = \mathbb{Z}/p^n\mathbb{Z}$ and then we will give partial results for $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ with $p \neq 2$.

5.1 The Stable Module Category

Our input candidate for tensor triangular Chow–Witt groups in a non-algebro-geometric setting will be the tensor triangulated category $kG\text{-stab}$, known as the *stable module category*, for a field k whose characteristic divides the group order. We will focus on two examples, namely $G = \mathbb{Z}/p\mathbb{Z}$ and $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ for $p \neq 2$. Already for the latter, obstacles will occur so that we will only give partial results here.

In this section, we first collect basic notions and results on group rings and then on the stable module category for the reader’s convenience. As an introductory lecture, we recommend [Car96, BIK12, Ben98a]; we mainly follow these references in this section.

5.1.1 Recollection on Group Rings

We recall some essential notations and results concerning group rings.

Convention 5.1.1. Let G be a finite group and k a field of positive characteristic p .

Definition 5.1.2. We define the *group ring* kG as the set of all formal k -linear combinations $\sum_{g \in G} \alpha_g g$, where $\alpha_g \in k$ and $g \in G$. It becomes a ring and even a k -algebra via the compositions

$$\sum_{g \in G} \alpha_g g + \sum_{g \in G} \beta_g g := \sum_{g \in G} (\alpha_g + \beta_g) g$$

and

$$\left(\sum_{g \in G} \alpha_g g \right) \cdot \left(\sum_{h \in G} \beta_h h \right) := \sum_{g, h \in G} (\alpha_g \beta_h) gh.$$

As a k -module, kG is free with basis G .

Example 5.1.3. Let $G = \mathbb{Z}/p^n\mathbb{Z} = \langle g \mid g^{p^n} = 1 \rangle$, with g generating G , and $\text{char } k = p$. Then, the group ring is given by $kG \cong k[y]/(y^{p^n} - 1)$ which is isomorphic to $k[x]/(x^{p^n})$ via $x \mapsto y - 1$.

For later calculations, it will be important for us to have conditions whether a group ring is local and to then determine its unique maximal ideal as well as its residue field.

Lemma 5.1.4. ([Car96, Corollary 1.4]) *If G is a finite p -group, the group ring kG is local and its unique maximal ideal is the augmentation ideal (see [Car96, Chapter 1]).*

Lemma 5.1.5. ([Car96, Proposition 7.4]) *For groups A and B , we have $k(A \times B) \cong kA \otimes_k kB$.*

Example 5.1.6. (i) For $G = \mathbb{Z}/p^n\mathbb{Z}$, the unique maximal ideal of the local ring $kG \cong k[x]/(x^{p^n})$ is given by (x) , hence, its residue field $kG/(x)$ simply is k .

(ii) For $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, by Lemma 5.1.5 and (i), we have that

$$kG \cong k[x]/(x^p) \otimes_k k[y]/(y^p) \cong k[x, y]/(x^p, y^p)$$

with maximal ideal (x, y) , hence the residue field $kG/(x, y)$ again is k . More generally, the residue field of kE for any finite elementary abelian p -group E is k by the same argument.

(iii) The group ring $k\mathbb{Z}$ equals the ring of Laurent polynomials and is not local.

Remark 5.1.7. Note that since k is a field (more general a (left) Artinian ring), kG is left Artinian by [BIK12, Remark 1.24]. It follows that, for modules over kG , being finitely generated and having finite length is equivalent. This will be important for us in the computation of the Grothendieck–Witt group of kG -stab, the first step in the computation of its Chow–Witt group.

The following results on the semisimplicity of kG are well-known.

Theorem 5.1.8. (Maschke) *If the characteristic of k does not divide the order of G , then kG is a semisimple k -algebra, i.e. every nonzero kG -module is semisimple (if moreover kG is finite-dimensional).*

Theorem 5.1.9. (Converse of Maschke's theorem) *If kG is a semisimple k -algebra, then the characteristic of k does not divide the order of G .*

For the notion of a *Frobenius algebra* we follow [Hap88].

Definition 5.1.10. A k -algebra R is called *Frobenius algebra* if indecomposable projective R -modules and indecomposable injective R -modules coincide and if moreover it is *locally bounded* (see [Hap88, p.25]).

Example 5.1.11. Let G be a finite group and k a field, then kG is a finite-dimensional Frobenius k -algebra by [Ben98a, Proposition 3.1.2].

For our purposes, this is all we need to know about group rings. In the following, we recall the construction and basic results concerning the stable module category $kG\text{-stab}$.

5.1.2 Construction of $kG\text{-stab}$

Convention 5.1.12. If not stated otherwise, modules in this chapter we consider will always be finitely generated left modules. For the rest of this chapter, let G be a finite group and k a field of characteristic p dividing the order of G .

We denote by $kG\text{-mod}$ the abelian category of finitely generated left kG -modules. Recall that since k is a field, being finitely generated and having finite length is equivalent due to Remark 5.1.7. The category $kG\text{-mod}$ has the following structural properties.

Remark 5.1.13. (i) The tensor product $M \otimes_k N$ of finitely generated left kG -modules is again an object of $kG\text{-mod}$ when considering the linear extension of the diagonal action

$$g(m \otimes n) := gm \otimes gn$$

for $g \in G$, $m \in M$, and $n \in N$. Moreover, the set $\text{Hom}_k(M, N)$ of k -linear maps from M to N is an object of $kG\text{-mod}$ via linear extension of the action

$$(gf)(m) := f(g^{-1}m)$$

for $g \in G$, $m \in M$ and $f \in \text{Hom}_k(M, N)$.

(ii) The category $kG\text{-mod}$ becomes an abelian category with duality when equipping it with the duality $\text{Hom}_k(-, k)$, where k carries the trivial kG -module structure.

The category theoretic analogon of Definition 5.1.10 is the following.

Definition 5.1.14. A *Frobenius category* is an exact category \mathcal{E} which has enough injectives, enough projectives, and projective and injective objects coincide. A *functor of Frobenius categories* is a functor preserving projective-injective objects.

Example 5.1.15. For G a finite group and k a field, the abelian (hence, exact) category $kG\text{-mod}$ is a Frobenius category. This follows from Example 5.1.11 and [Ben98a, Proposition 1.6.2].

Now, we can define the stable module category via the category $kG\text{-mod}$ in the following way.

Definition 5.1.16. Let G be a finite group and k a field whose characteristic divides the group order. For the abelian Frobenius category $kG\text{-mod}$, we denote by $kG\text{-stab}$ its associated *stable category* with the same objects as $kG\text{-mod}$. The morphism group between two kG -modules M and N is given as the quotient

$$\mathrm{Hom}_k(M, N)/J,$$

where J is the subgroup of morphisms factoring through projective-injective objects of $kG\text{-mod}$. This construction is called the *stable module category*.

More generally, we can define the stable category of any Frobenius category in an analog manner.

Remark 5.1.17. A construction like in Definition 5.1.16 is called a *quotient category*. Note that the construction of a *localization of a category* is different in general since the amount of morphisms may increase by adding formal inverses to create isomorphisms, whereas, in a quotient category, the number of morphisms decreases. In both cases though, objects can become isomorphic that have not been isomorphic before. The construction of a *Serre quotient* has parallels to the construction of a quotient category, but behaves like a localization of a category in many cases.

Together with the definition of the stable module category come Hellers inverse loop space functors Ω and Ω^{-1} that encode the information of kernels of projective covers and of cokernels of injective hulls, respectively. Keep in mind that in our case, $kG\text{-mod}$ is a Frobenius category, hence, injective and projective modules coincide.

Definition-Lemma 5.1.18. ([He60]) *Let $\pi : P \rightarrow M$ be a projective cover of a kG -module M . Then, we define $\Omega(M) := \ker(\pi)$. Moreover, for an injective hull $i : M \rightarrow I$ of M , we define $\Omega^{-1}(M) := \mathrm{coker}(i)$.*

Two projective covers (or injective hulls) are isomorphic to each other in $kG\text{-stab}$. We obtain well-defined functors $\Omega, \Omega^{-1} : kG\text{-stab} \rightarrow kG\text{-stab}$ which are inverse to each other, yielding an autoequivalence of categories.

For our purposes, we of course are interested in the triangulated structure of $kG\text{-stab}$, which turns out to be even a rigid tensor triangulated category.

Proposition 5.1.19. ([Car96, Theorem 5.6]) *The category $kG\text{-stab}$ is a triangulated category with shift functor $T := \Omega^{-1}$ and exact triangles as follows. For $f \in \text{Hom}_{kG\text{-stab}}(X, Y)$, $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ is a standard triangle in $kG\text{-stab}$ if Z is the pushout in $kG\text{-mod}$ of f and $i_X : X \rightarrow I(X)$, where $I(X)$ is an injective hull, and if g and h appear in the following commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{i_X} & I(X) & \longrightarrow & T(X) & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow & & \downarrow = & & \\ 0 & \longrightarrow & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T(X) & \longrightarrow & 0. \end{array}$$

The exact triangles are the ones isomorphic to such a standard triangle.

Proposition 5.1.20. ([Bal07, Proposition 4.2]) *Let G be a finite group and k a field of characteristic dividing the group order. The stable module category $kG\text{-stab}$ is a rigid tensor triangulated category with tensor product \otimes_k and internal Hom $\text{hom} := \text{Hom}_k$.*

Remark 5.1.21. (i) Combining Proposition 5.1.20 and Example 2.2.6 (i), we can consider $kG\text{-stab}$ a triangulated category with duality $\text{Hom}_k(-, k)$.

We can see by hand that the duality is well-defined: Let A be in $kG\text{-mod}$. For any projective P , we have

$$\text{Hom}_k(A \oplus P, k) \cong \text{Hom}_k(A, k) \oplus \text{Hom}_k(P, k)$$

and since P is projective and $kG\text{-mod}$ Frobenius, $P^* = \text{Hom}_k(P, k)$ is also projective. It follows that $\text{Hom}(A \oplus P, k) \cong \text{Hom}(A, k)$ in $kG\text{-stab}$ by Theorem 5.2.2. If $f \sim g : A \rightarrow B$ in $kG\text{-stab}$, then $b \circ f \sim b \circ g$ for all $b \in \text{Hom}_k(B, k) = B^*$, hence, $f^* \sim g^*$.

(ii) As already mentioned in [Kl16a, Remark 6.2.3], we deliberately consider \otimes_k and consequently $\text{Hom}_k(-, k)$ as opposed to \otimes_{kG} . This is because the tensor product $M \otimes_{kG} N$ of two left kG -modules M, N does not have a natural left module structure.

We can view $kG\text{-stab}$ as an Verdier localization of the derived category of kG -modules $D^b(kG\text{-mod})$ as the following result by Rickard shows.

Theorem 5.1.22. ([Ric89, Theorem 2.1]) *Let G be a finite group and k a field dividing the group order. The natural functor $kG\text{-mod} \rightarrow D^b(kG\text{-mod})$ induces an equivalence of tensor triangulated categories*

$$F : kG\text{-stab} \xrightarrow{\cong} D^b(kG\text{-mod}) / D^b(kG\text{-proj}),$$

where $kG\text{-proj}$ denotes the subcategory of $kG\text{-mod}$ generated by projective kG -modules.

In our case, we need to make sure that the above equivalence preserves duality. The rigid tensor triangulated category $kG\text{-stab}$ becomes a triangulated category with duality when fixing the duality $\text{Hom}_k(-, k)$, as Remark 5.1.21 shows.

Lemma 5.1.23. *The above equivalence F is duality-preserving.*

Proof. The inclusion $kG\text{-mod} \rightarrow D^b(kG\text{-mod})$ sending a module M to the complex $\dots \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow \dots$ concentrated in degree 0 is duality-preserving and, thus, also $kG\text{-mod} \rightarrow D^b(kG\text{-mod})/D^b(kG\text{-proj})$. Since $kG\text{-mod}$ and $kG\text{-stab}$ have the same objects, the equivalence is duality-preserving on the level of objects. On the level of morphisms, note that $kG\text{-stab}$ and $D^b(kG\text{-mod})/D^b(kG\text{-stab})$ are well-defined with respect to taking duals. The statement now again follows from the fact that $kG\text{-mod} \rightarrow D^b(kG\text{-mod})/D^b(kG\text{-proj})$ is duality-preserving and duality in $D^b(kG\text{-mod})$ is defined degreewise: Let $f : M \rightarrow N$ be a morphism in $kG\text{-mod}$ and \bar{f} its equivalence class in $kG\text{-stab}$. Then,

$$F(\bar{f}^*) = F(\overline{f^*}) = [f^*] = [f]^*,$$

where $[-]$ denotes the class of a morphism in $D^b(kG\text{-mod})/D^b(kG\text{-proj})$. \square

Apart from the rigid tensor triangular structure, we were moreover interested in dimension functions Section 2.3.

Definition 5.1.24. Let $\mathcal{P} \in \text{Spc}(kG\text{-stab})$. We define its *Krull dimension* $\dim_{\text{Krull}}(\mathcal{P})$ as the maximal length n of a chain of irreducible closed subsets

$$\emptyset \subsetneq C_0 \subsetneq C_1 \subsetneq \dots \subsetneq C_n = \overline{\{\mathcal{P}\}}.$$

It gives rise to a dimension function in the sense of Definition 2.3.1 by [Bal07, Example 3.2].

We will now see that the *cohomology ring* of a group algebra coming with its *projective support variety* already encodes all the information on the Balmer spectrum of $kG\text{-stab}$.

Definition 5.1.25. We define the *cohomology ring of kG* as the graded ring

$$H^\bullet(G, k) := \bigoplus_{i \geq 0} \text{Ext}_{kG}^i(k, k)$$

if $p = 2$, and only as its even part if the characteristic p is odd.

Definition 5.1.26. The *projective support variety of kG* is given by

$$\mathcal{V}_G(k) := \text{Proj}(H^\bullet(G, k)).$$

The *projective support variety* $\mathcal{V}_G(M)$ of a kG -module M is defined as the subvariety of $\mathcal{V}_G(k)$ associated to the annihilator ideal $J(M)$ of $\text{Ext}_k^*(M, M)$ in $H^\bullet(G, k)$.

Proposition 5.1.27. ([Bal05b, Corollary 5.10]) *Let G be a finite group and k a field with characteristic dividing the group order. There is a homeomorphism*

$$\begin{aligned} \mathcal{V}_G(k) &\xrightarrow{\cong} \text{Spc}(kG\text{-stab}) \\ x &\mapsto \{M \in kG\text{-stab} \mid x \notin \mathcal{V}_G(M)\} \end{aligned}$$

Under this homeomorphism, $\mathcal{V}_G(M)$ corresponds to $\text{supp}(M) \subset \text{Spc}(kG\text{-stab})$ for M an object of $kG\text{-stab}$.

Moreover, we can compute the dimension of the support variety of a module via its *complexity*, an invariant depending on the growth of the dimension of components in projective resolutions.

Theorem 5.1.28. ([Car96, Theorem 4.3]) *Any object M of kG -mod has a minimal projective resolution, that is, a projective resolution $P_\bullet \rightarrow M$ such that for any other projective resolution $Q_\bullet \rightarrow M$ there exists an injective chain map $(P_\bullet \rightarrow M) \rightarrow (Q_\bullet \rightarrow M)$ and a surjective chain map $(Q_\bullet \rightarrow M) \rightarrow (P_\bullet \rightarrow M)$ that lift the identity on M .*

Definition 5.1.29. For a minimal projective resolution $P_\bullet \rightarrow M$ of an object M in kG -mod its *complexity* $c_G(M)$ is defined as the smallest $s \in \mathbb{Z}$ such that there is a constant $\kappa > 0$ with

$$\dim_k(P_n) \leq \kappa \cdot n^{s-1} \text{ for } n > 0.$$

We obtain the following correspondence between the dimension of $\mathcal{V}_G(M)$ and the complexity of M .

Proposition 5.1.30. ([Ben98b, Proposition 5.7.2]) *Let G be a finite group and k a field of characteristic dividing the group order. For an object M of kG -mod, we have*

$$\dim(\mathcal{V}_G(M)) = c_G(M) - 1.$$

Remark 5.1.31. The original statement of [Ben98b, Proposition 5.7.2] is that the complexity of a kG -module M equals the dimension of $\mathcal{V}_G(M)$. However, in [Ben98b, Section 5.1], $\mathcal{V}_G(M)$ is defined as the affine variant of the projective support variety we defined in Definition 5.1.26 and carries the name *cohomological variety*. Their dimensions differ by 1, hence, so do the statements of [Ben98b, Proposition 5.7.2] and Proposition 5.1.30.

In this section, we have recalled foundations on the group ring kG and the stable module category kG -stab. Next, we discuss in more detail the relation between the Frobenius category kG -mod and its stable category kG -stab. The (Grothendieck–)Witt group of the latter is harder to calculate, so we give a statement on how to compute it in terms of the (Grothendieck–)Witt group of kG -mod.

5.2 The (Grothendieck–)Witt Group of kG -stab

Convention 5.2.1. Let G be a finite group and k a field of characteristic $\neq 2$ dividing the group order. When considering kG -stab a triangulated category with duality, we fix the duality $\text{Hom}_k(-, k)$.

As mentioned before, this section intends to simplify the computations of the groups $\text{GW}^{[n]}(kG\text{-stab})$ and $\text{W}^{[n]}(kG\text{-stab})$, $n \in \mathbb{Z}$, by writing it in terms of $\text{GW}^{[n]}(kG\text{-mod})$ and $\text{W}^{[n]}(kG\text{-mod})$, respectively. Calculations like these will

turn out to be the first step in the determination of tensor triangular Chow–Witt groups of kG -stab in Section 5.3 and Section 5.5. Since the (Grothendieck–)Witt groups (and, for that matter, also the K_0 -groups) of kG -stab are harder to calculate by hand than the respective groups of kG -mod, we will use a variation of [TaWa91, Proposition 1] to break it down to a computation involving merely the Frobenius category kG -mod.

For this, we first need to understand how equivalences translate between kG -stab and kG -mod. The category kG -mod is a category *with cancellation*, compare [He58, p.486/487], which is why we can apply [He60, Theorem 2.2] to our case to obtain the following result.

Theorem 5.2.2. ([He60, Theorem 2.2]) *Let $f : M \rightarrow N$ be a morphism in kG -mod. Then, the following are equivalent:*

- (i) *[f] is an isomorphism in kG -stab*
- (ii) *There are projective modules M', N' and an isomorphism $f' : M \oplus M' \rightarrow N \oplus N'$ in kG -mod making the following diagram in kG -mod commute*

$$\begin{array}{ccc} M \oplus M' & \xrightarrow{f'} & N \oplus N' \\ \uparrow & & \downarrow \\ M & \xrightarrow{f} & N. \end{array}$$

With these tools at hand, we can prove the following theorem. It will be central in the computation of tensor triangular Chow–Witt groups of kG -stab since the first step is usually to determine its (shifted) Grothendieck–Witt group. The theorem is a generalization of a similar statement for K_0 , namely [TaWa91, Proposition 1], that Klein uses to calculate tensor triangular Chow groups of kG -stab in [Kl16a].

Theorem 5.2.3. *Let G be a finite group and k a field of characteristic $\neq 2$ dividing the order of G ; all dualities here are given by $\mathrm{Hom}_k(-, k)$.*

- (i) *The quotient map kG -mod $\rightarrow kG$ -stab from Definition 5.1.16 induces isomorphisms for even n*

$$\mathrm{GW}^{[n]}(kG\text{-stab}) \cong \mathrm{GW}^{[n]}(\mathrm{D}^b(kG\text{-mod}))/\mathrm{Im}(\alpha),$$

where $\alpha : \mathrm{GW}^{[n]}(\mathrm{D}^b(kG\text{-proj})) \rightarrow \mathrm{GW}^{[n]}(\mathrm{D}^b(kG\text{-mod}))$ is induced by the inclusion kG -proj $\rightarrow kG$ -mod.

In particular,

$$\mathrm{GW}(kG\text{-stab}) \cong \mathrm{GW}(kG\text{-mod})/\mathrm{Im}(\alpha')$$

for $\alpha' : \mathrm{GW}(kG\text{-proj}) \rightarrow \mathrm{GW}(kG\text{-mod})$ induced by the inclusion kG -proj $\rightarrow kG$ -mod.

(ii) The quotient map $kG\text{-mod} \rightarrow kG\text{-stab}$ from Definition 5.1.16 induces isomorphisms for all n

$$W^{[n]}(kG\text{-stab}) \cong W^{[n]}(D^b(kG\text{-mod}))/W^{[n]}(D^b(kG\text{-proj})).$$

In particular,

$$W(kG\text{-stab}) \cong W(kG\text{-mod})/W(kG\text{-proj})$$

and $W^{[n]}(kG\text{-stab}) = 0$ for n odd.

Proof. We prove (ii) first. Let m be odd. Then, by [BW02, Proposition 5.2], $W^{[m]}(D^b(kG\text{-mod})) = 0 = W^{[m]}(D^b(kG\text{-proj}))$. Moreover, by Theorem 5.1.22 and Theorem 4.1.4, which is applicable by Lemma 5.2.4 below, we have

$$\begin{aligned} W^{[m]}(kG\text{-stab}) &\cong W^{[m]}(D^b(kG\text{-mod})/D^b(kG\text{-proj})) \\ &\cong W^{[m]}(D^b(kG\text{-mod}/kG\text{-proj})) = 0, \end{aligned}$$

again using [BW02, Proposition 5.2]. Now, let n be even. We have Balmer's localization sequence Theorem 3.3.2

$$\begin{aligned} \dots \rightarrow W^{[n-1]}(D^b(kG\text{-mod})/D^b(kG\text{-proj})) &\rightarrow W^{[n]}(D^b(kG\text{-proj})) \rightarrow W^{[n]}(D^b(kG\text{-mod})) \\ &\rightarrow W^{[n]}(D^b(kG\text{-mod})/D^b(kG\text{-proj})) \rightarrow W^{[n+1]}(D^b(kG\text{-proj})) \rightarrow \dots \end{aligned}$$

By the above observations, we have that $W^{[n-1]}(D^b(kG\text{-mod})/D^b(kG\text{-proj})) = 0 = W^{[n+1]}(D^b(kG\text{-proj}))$. Consequently, the statement follows from the isomorphism theorem of abelian groups.

Using Proposition 3.2.13, Proposition 3.2.14, and Proposition 3.2.16, we observe that, for an abelian category \mathcal{A} , the Grothendieck–Witt group of \mathcal{A} coincides with the Grothendieck–Witt group of $D^b(\mathcal{A})$. It follows directly that $W(kG\text{-stab}) \cong W(kG\text{-mod})/W(kG\text{-proj})$.

For (i), consider Walter's localization sequence Theorem 3.3.3

$$\begin{aligned} \text{GW}^{[n]}(D^b(kG\text{-proj})) &\xrightarrow{\alpha} \text{GW}^{[n]}(D^b(kG\text{-mod})) \\ &\rightarrow \text{GW}^{[n]}(D^b(kG\text{-mod})/D^b(kG\text{-proj})) \rightarrow W^{[n+1]}(D^b(kG\text{-proj})) \rightarrow \dots \end{aligned}$$

For even n , $W^{[n+1]}(D^b(kG\text{-proj})) = 0$ by [BW02, Proposition 5.2]. It follows that the morphism $\text{GW}^{[n]}(D^b(kG\text{-mod})) \rightarrow \text{GW}^{[n]}(D^b(kG\text{-mod})/D^b(kG\text{-proj})) \cong \text{GW}^{[n]}(kG\text{-stab})$ is an epimorphism, where the latter isomorphism again follows from Theorem 5.1.22 and Theorem 4.1.4. The first statement now follows from the exactness of the sequence and the isomorphism theorem of abelian groups.

The second statement follows as in the case of Witt groups using that $\text{GW}(\mathcal{A}) \cong \text{GW}(D^b(\mathcal{A}))$ for an abelian category \mathcal{A} . \square

Lemma 5.2.4. *The condition for Theorem 4.1.4 is satisfied for the inclusion $kG\text{-proj} \rightarrow kG\text{-mod}$. Moreover, the equivalence*

$$D^b(kG\text{-mod}/kG\text{-proj}) \simeq D^b(kG\text{-mod})/D^b(kG\text{-proj})$$

is duality-preserving.

Proof. Let $0 \rightarrow P \rightarrow B \rightarrow C \rightarrow 0$ be a short exact sequence in $kG\text{-mod}$ with P in $kG\text{-proj}$. The category $kG\text{-mod}$ is Frobenius, hence, we have an injective hull $i_C : C \rightarrow I(C)$ with $I(C)$ in $kG\text{-proj}$. Note that $I(C)$ and P are both injective and projective. Consequently, the given sequence splits via an isomorphism $f : B \xrightarrow{\cong} P \oplus C$. Then, the following diagram commutes and the rows are exact:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & P & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \text{id} \downarrow & & f \downarrow \cong & & \text{id} \downarrow & & \\
 0 & \longrightarrow & P & \xrightarrow{i_1} & P \oplus C & \xrightarrow{p_2} & C & \longrightarrow & 0 \\
 & & \text{id} \downarrow & & (\text{id}, i_C) \downarrow & & i_C \downarrow & & \\
 0 & \longrightarrow & P & \xrightarrow{i_1} & P \oplus I(C) & \xrightarrow{p_2} & I(C) & \longrightarrow & 0.
 \end{array}$$

The second statement follows from the fact that the equivalence is induced by the duality-preserving short exact sequence

$$D^b(kG\text{-proj}) \rightarrow D^b(kG\text{-mod}) \rightarrow D^b(kG\text{-mod}/kG\text{-proj}).$$

□

Remark 5.2.5. In general, $\alpha : \text{GW}^{[n]}(kG\text{-proj}) \rightarrow \text{GW}^{[n]}(kG\text{-mod})$ from Theorem 5.2.3 above might not be injective and, hence, $\text{Im}(\alpha)$ might not be isomorphic to $\text{GW}^{[n]}(kG\text{-proj})$, similar to the K -theoretic case in [TaWa91, Proposition 1]. For odd n , the map $\text{GW}^{[n]}(kG\text{-mod}) \rightarrow \text{GW}^{[n]}(D^b(kG\text{-mod})/D^b(kG\text{-proj}))$ is not even surjective in general, so other techniques are required to investigate this case. We will not pursue this issue further here.

In order to do further calculations, we will moreover need the following version of the dévissage theorem, for GW and W , respectively. For readers not familiar with the theory of (Grothendieck-)Witt groups of fields, [Scha85, Lam05, MH73] are recommended as an introductory reference.

Proposition 5.2.6. *For k a field of characteristic $p \neq 2$ and G a finite p -group, the forgetful functor $U : kG\text{-mod} \rightarrow k\text{-mod}$ induces isomorphisms*

$$\text{W}(kG\text{-mod}, \text{Hom}_k(-, k)) \cong \text{W}(k)$$

and

$$\text{GW}(kG\text{-mod}, \text{Hom}_k(-, k)) \cong \text{GW}(k),$$

where the left-hand side denotes the shifted (Grothendieck-)Witt group of an exact category with duality as in Definition 3.2.5.

Proof. We know from Lemma 5.1.4 that kG is local with residue field k , hence, all simple modules are isomorphic (as kG -modules) to k , endowed with trivial G -action. It follows that the semisimple modules are isomorphic to k^n for some

$n \in \mathbb{N}$ since all modules are of finite length in our setting, i.e. they correspond to finite-dimensional k -modules.

Since the subcategory of kG -mod of semisimple modules is equivalent to k -mod, the statement follows directly from [QSS79, Theorem 6.10] (and [QSS79, Theorem 6.9] for Witt groups). \square

After having seen how to compute the (Grothendieck–)Witt group of kG -stab in terms of the (Grothendieck–)Witt group of kG -mod, we have enough tools at hand to determine the first non-algebro-geometric example of a Chow–Witt group, namely of the tensor triangulated category kG -stab for $G = \mathbb{Z}/p^n\mathbb{Z}$.

5.3 Tensor Triangular Chow–Witt Groups for $G = \mathbb{Z}/p^n\mathbb{Z}$

In order to compute the Chow–Witt group of the rigid tensor triangulated category kG -stab, consider the following setup.

Convention 5.3.1. In this section, we fix $n \in \mathbb{N}$ and $p \neq 2$ prime. Let k be a field of characteristic p and $G = \mathbb{Z}/p^n\mathbb{Z}$. When considering the tensor triangulated category kG -stab a triangulated category with duality, we fix the duality $\mathrm{Hom}_k(-, k)$ and restrictions of it on subcategories. Moreover, we equip kG -stab with the dimension function \dim_{Krull} (Definition 5.1.24).

We begin with determining the filtration components $kG\text{-stab}_{(i)}$ of kG -stab introduced in Definition 2.3.4. Recall that Proposition 5.1.27 tells us that it suffices to look at support varieties.

By [Car96, Theorem 7.3] we know that $\mathcal{V}_G(k)$ is a point, hence, so is $\mathcal{V}_G(M)$ for any non-projective kG -module M , i.e. $\dim_{\mathrm{Krull}}(\mathcal{V}_G(M)) = 0$ for all $M \neq 0$ in kG -stab. Under the isomorphism $\mathcal{V}_G(k) \cong \mathrm{Spc}(kG\text{-stab})$ from Proposition 5.1.27, the support of an object in kG -stab corresponds to its projective support variety. In particular, they have equal dimension. It follows that

$$kG\text{-stab}_{(i)} = \begin{cases} kG\text{-stab} & i \geq 0, \\ 0 & i < 0, \end{cases}$$

and thus

$$kG\text{-stab}_{(i)}/kG\text{-stab}_{(n-1)} = \begin{cases} kG\text{-stab} & i = 0, \\ 0 & \text{else,} \end{cases}$$

compare also [Kl16a, Section 6.3]. In particular, all quotients of filtration components are idempotent complete, which is why we can apply Definition 4.3.5. The only non-trivial degree of $\widetilde{\mathrm{CH}}_i^\Delta(kG\text{-stab})$ is $i = 0$. For this case, we obtain the

following defining diagram for $\mathcal{T} := kG\text{-stab}$.

$$\begin{array}{ccccc}
 W^{[1]}(\mathcal{T}_{(-2)}) & \xrightarrow{\tilde{i}_{\mathcal{T}}^{-1, W}} & W^{[1]}(\mathcal{T}_{(-1)}) & & \text{GW}(\mathcal{T}_{(0)}) & \xrightarrow{\tilde{i}_{\mathcal{T}}^1} & \text{GW}(\mathcal{T}_{(1)}) \\
 & & \swarrow \bar{d}_{\mathcal{T}}^0 & & \downarrow \bar{q}_{\mathcal{T}}^0 & & \\
 & & & & \text{GW}(\mathcal{T}_{(0)}/\mathcal{T}_{(-1)}) & &
 \end{array}$$

which, by the above considerations, equals

$$\begin{array}{ccccc}
 0 & \xrightarrow{0} & 0 & & \text{GW}(kG\text{-stab}) & \xrightarrow{\text{id}} & \text{GW}(kG\text{-stab}) \\
 & & \swarrow 0_a & & \downarrow \text{id} & & \\
 & & & & \text{GW}(kG\text{-stab}) & &
 \end{array}$$

So, we obtain

$$\begin{aligned}
 \widetilde{\text{CH}}_{\Delta}^0(kG\text{-stab}) &= (0_a)^{-1}(\text{Im}(0))/\text{id}(\ker(\text{id})) \\
 &= \text{GW}(kG\text{-stab})/0 \\
 &= \text{GW}(kG\text{-stab}).
 \end{aligned}$$

As a consequence, the computation of $\widetilde{\text{CH}}_{\Delta}^0(kG\text{-stab})$ amounts to the computation of $\text{GW}(kG\text{-stab})$. For this, we observe that kG is local with residue field k by Example 5.1.6. So, we use Theorem 5.2.3 and Proposition 5.2.6 to obtain

$$\begin{aligned}
 \text{GW}(kG\text{-stab}) &\cong \text{GW}(kG\text{-mod})/\text{Im}(\alpha) \\
 &\cong \text{GW}(k)/U(\text{Im}(\alpha))
 \end{aligned}$$

where $U : \text{GW}(kG\text{-mod}) \rightarrow \text{GW}(k\text{-mod})$ is the isomorphism from Proposition 5.2.6 and $\alpha : \text{GW}(kG\text{-proj}) \rightarrow \text{GW}(kG\text{-mod})$ is induced by the inclusion. Abusing notation, we also write U for the composition $(\dim, \det) \circ U$.

We have proven the following.

Proposition 5.3.2. *Let $G = \mathbb{Z}/p^n\mathbb{Z}$, k a field of prime characteristic $p \neq 2$ and $n \in \mathbb{N}$ and U, α as above. Then, the tensor triangular Chow–Witt groups of $kG\text{-stab}$ are given as*

$$\widetilde{\text{CH}}_{\Delta}^i(kG\text{-stab}) \cong \begin{cases} \text{GW}(k)/U(\text{Im}(\alpha)) & i = 0, \\ 0 & i \neq 0. \end{cases}$$

We can even be more concrete than this general result. If k is algebraically closed, Chow–Witt groups and Chow groups coincide as expected.

Example 5.3.3. For algebraically closed fields k , the map $\dim : \text{GW}(k) \rightarrow \mathbb{Z}$ is an isomorphism by [Lam05, Proposition II.3.1], hence

$$\widetilde{\text{CH}}_{\Delta}^i(kG\text{-stab}) \cong \begin{cases} \mathbb{Z}/p^n\mathbb{Z} & i = 0, \\ 0 & i \neq 0 \end{cases}$$

since kG is local, i.e. projective and free modules coincide. Thus, symmetric forms in $\text{Im}(\alpha)$ have k -dimension a multiple of p^n . It follows from [Kl16a, Proposition 6.3.2] that

$$\widetilde{\text{CH}}_{\Delta}^i(kG\text{-stab}) \cong \text{CH}_{\Delta}^i(kG\text{-stab})$$

for all i .

For the next example, we restrict to the case of k being finite, since $\text{GW}(k)$ is known for this special case. Again, it turns out that Chow–Witt and Chow groups coincide here.

Example 5.3.4. Let k be finite. We know by Proposition 5.2.6 that

$$\text{GW}(kG\text{-mod}) \cong \text{GW}(k).$$

By [Scha85, Chapter 2, Theorem 3.3 and Lemma 3.7] (and the considerations of [Scha85, Chapter 2, §2,]) we have the isomorphism

$$(\dim, \det) : \text{GW}(k) \xrightarrow{\cong} \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

given by the dimension and the determinant. Note that [Scha85, Chapter 2, Theorem 3.3] can be applied since, by [Scha85, Chapter 2, Theorem 3.8], every 3-dimensional quadratic space is isotropic (in the sense of [Scha85]). By Theorem 5.2.3 and Proposition 5.2.6 we hence have

$$\begin{aligned} \text{GW}(kG\text{-stab}) &\cong \text{GW}(kG\text{-mod})/\text{Im}(\alpha) \\ &\cong \text{GW}(k)/U(\text{Im}(\alpha)) \\ &\cong (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})/U(\text{Im}(\alpha)) \end{aligned}$$

for $U : \text{GW}(kG\text{-mod}) \rightarrow \text{GW}(k\text{-mod})$ and $\alpha : \text{GW}(kG\text{-proj}) \rightarrow \text{GW}(kG\text{-mod})$ as before.

The subgroup $\text{Im}(\alpha)$ is generated by equivalence classes of symmetric objects (P, ϕ) , where P is projective. Since kG is local by Lemma 5.1.4, all projective modules are free, and hence have k -dimension a multiple of p^n . Moreover, classes of objects of the form (P, ϕ) can have arbitrary determinant in each dimension $m \cdot p^n$, $m \in \mathbb{Z}$, hence, the map (\dim, \det) induces a group isomorphism

$$U(\text{Im}(\alpha)) \cong p^n\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

In the end, we have

$$\begin{aligned} \text{GW}(kG\text{-stab}) &\cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}/U(\text{Im}(\alpha)) \\ &\cong (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})/(p^n\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \\ &\cong \mathbb{Z}/p^n\mathbb{Z}. \end{aligned}$$

Hence,

$$\widetilde{\text{CH}}_{\Delta}^i(kG\text{-stab}) \cong \begin{cases} \mathbb{Z}/p^n\mathbb{Z} & i = 0, \\ 0 & i \neq 0. \end{cases}$$

Again, it follows from [Kl16a, Proposition 6.3.2] that

$$\widetilde{\text{CH}}_{\Delta}^i(kG\text{-stab}) \cong \text{CH}_{\Delta}^i(kG\text{-stab})$$

for all i .

The reader might find it surprising that $\widetilde{\text{CH}}_{\Delta}^i(kG\text{-stab}) \cong \text{CH}_{\Delta}^i(kG\text{-stab})$ not only for k algebraically closed but also for finite k as we have seen above. What actually happens when dividing out $\text{Im}(\alpha)$ is that we divide out such a big subgroup that symmetric objects of the same dimension become equivalent although they have different determinants, i.e. in particular are not isometric in $\text{GW}(k)$. To get a better intuition of this circumstance, we consider the case $k = \mathbb{F}_3$ and $G = \mathbb{Z}/3\mathbb{Z}$ in the following.

Example 5.3.5. Let $k = \mathbb{F}_3$ and $G = \mathbb{Z}/3\mathbb{Z}$, i.e. $p = 3$ and $n = 1$. Since $\text{GW}(k) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, each element in $\text{GW}(k)$ is uniquely determined (up to isometry) by its dimension together with its determinant, where the determinant is an element of $k^{\times}/(k^{\times})^2 = \{[1], [-1]\}$ since -1 is not a square in \mathbb{F}_3 .

Dividing out $U(\text{Im}(\alpha))$, symmetric objects of equal dimension now are in the same equivalence class. We check dimensions 1 and 2:

We have $\langle 1 \rangle \sim \langle -1 \rangle$ since

$$\langle 1 \rangle + \langle -1, 1, 1 \rangle = \langle -1, 1, 1, 1 \rangle = \langle -1 \rangle + \langle 1, 1, 1 \rangle$$

with $\langle 1, 1, 1 \rangle, \langle -1, 1, 1 \rangle$ in $U(\text{Im}(\alpha))$.

In dimension 2, we have

$$\langle 1, 1 \rangle + \langle -1, 1, 1 \rangle = \langle -1, 1, 1, 1, 1 \rangle = \langle -1, 1 \rangle + \langle 1, 1, 1 \rangle$$

with $\langle -1, 1, 1 \rangle, \langle 1, 1, 1 \rangle \in U(\text{Im}(\alpha))$. Analogous considerations can be done for different values of p^n for $p \neq 2$.

Summarizing these results, we have the following.

Proposition 5.3.6. *If $G = \mathbb{Z}/p^n\mathbb{Z}$, and k is either algebraically closed or finite of characteristic $p \neq 2$, we have that*

$$\widetilde{\text{CH}}_{\Delta}^i(kG\text{-stab}) \cong \text{CH}_{\Delta}^i(kG\text{-stab}) \cong \begin{cases} \mathbb{Z}/p^n\mathbb{Z} & i = 0, \\ 0 & i \neq 0. \end{cases}$$

In particular, $\text{GW}(kG\text{-stab}) \cong K_0(kG\text{-stab}) \cong \mathbb{Z}/p^n\mathbb{Z}$.

Proof. If $G = \mathbb{Z}/p^n\mathbb{Z}$, the first statement follows directly from Example 5.3.3 and Example 5.3.4. The second statement follows from

$$\mathrm{GW}(kG\text{-stab}) = \widetilde{\mathrm{CH}}_{\Delta}^0(kG\text{-stab}) \cong \mathrm{CH}_{\Delta}^0(kG\text{-stab}) = K_0(kG\text{-stab})$$

by Proposition 5.3.2 and [Kl16a, Section 6.3]. \square

We have just seen examples of tensor triangular Chow–Witt groups in the setting of representation theory. For $G = \mathbb{Z}/p^n\mathbb{Z}$, we obtained a general result, but also witnessed that, for k algebraically closed or k finite, Chow–Witt and Chow groups of $kG\text{-stab}$ coincide.

In the next section, we will consider the “next complicated” case, namely $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

5.4 Tensor Triangular Chow Groups for $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$

We are in the situation of Convention 5.3.1, but with $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

In [Kl16a], Klein computes the Chow groups for $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{char}(k) = 2$, but since we work with symmetric bilinear forms, characteristic 2 is usually a problem as mentioned in Remark 3.1.4. For this reason, we consider fields k of characteristic $p \neq 2$ and $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Remark 5.4.1. Note that there has been progress on the matter of characteristic 2 for Grothendieck–Witt and Witt groups. For example, works by Schlichting which we use here such as [Sch10a] do not make any assumptions on the characteristic. However, some of the tools we use do not have an analog for characteristic 2 yet. Since the focus of this work is merely to see some basic examples, we will not consider this case.

To get a sensible intuition of what the Chow–Witt groups of $kG\text{-stab}$ could possibly look like, we start with computing its tensor triangular Chow groups following [Kl16a] (here for $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, $p \neq 2$ as opposed to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$).

As for $G = \mathbb{Z}/p^n\mathbb{Z}$, we first determine the filtration components of the tensor triangulated category $kG\text{-stab}$. By [Car96, Thm 7.6] (compare [BBC09, Example 1.2]), it follows that $\mathcal{V}_G(k) = \mathbb{P}^1$, i.e. the modules of complexity ≤ 1 yield a proper subcategory $kG\text{-stab}_{(0)}$ of $kG\text{-stab}$. The filtration components of $kG\text{-stab}$ thus are

$$kG\text{-stab}_{(i)} := \begin{cases} kG\text{-stab} & i \geq 1, \\ kG\text{-stab}_{(0)} & i = 0, \\ 0 & i < 0, \end{cases}$$

and the only non-trivial quotients are

$$kG\text{-stab}_{(0)} / kG\text{-stab}_{(-1)} = kG\text{-stab}_{(0)}$$

and

$$kG\text{-stab}_{(1)}/kG\text{-stab}_{(0)} = kG\text{-stab}/kG\text{-stab}_{(0)}.$$

Hence, we distinguish the following non-trivial degrees i for $\text{CH}_i^\Delta(kG\text{-stab})$.

$i = 0$

First note that

$$\begin{aligned} Z_\Delta^\Delta(kG\text{-stab}) &:= K_0((kG\text{-stab}_{(0)}/kG\text{-stab}_{(-1)})^\natural) \\ &= K_0((kG\text{-stab}_{(0)})^\natural) = K_0(kG\text{-stab}_{(0)}) \end{aligned}$$

since $kG\text{-stab}_{(0)}$ is idempotent complete as a thick subcategory of an idempotent complete category. It follows that

$$\text{CH}_\Delta^0(kG\text{-stab}) := Z_\Delta^0(kG\text{-stab})/j^0 \circ q^0(\ker(i^1)) = K_0(kG\text{-stab}_{(0)})/\ker(i^1),$$

for $i^1 : K_0(kG\text{-stab}_{(0)}) \rightarrow K_0(kG\text{-stab})$ and by the isomorphism theorem of abelian groups

$$\text{CH}_\Delta^0(kG\text{-stab}) = K_0(kG\text{-stab}_{(0)})/\ker(i^1) \cong \text{Im}(i^1).$$

$i = 1$

We do not know whether the quotient $kG\text{-stab}/kG\text{-stab}_{(0)}$ is idempotent complete, which was a condition until now. So, we use the model-theoretic description of tensor triangular Chow groups introduced in Definition 4.6.2. It is applicable since $kG\text{-stab}$ admits a model by [Kl16b, Example 2.31].

The coboundaries (whose definition does not differ from Balmer's definition) are defined as

$$B_\Delta^1(kG\text{-stab}) = j^1 \circ q^1(\ker(i^2)),$$

where

$$i^2 : K_0(kG\text{-stab}_{(1)}) \rightarrow K_0(kG\text{-stab}_{(2)})$$

is the identity on $K_0(kG\text{-stab})$ since $kG\text{-stab}_{(i)} = kG\text{-stab}$ for $i \geq 1$. Hence,

$$B_\Delta^1(kG\text{-stab}) = j^1 \circ q^1(\ker(i^2)) = j^1 \circ q^1(0) = 0.$$

It follows that

$$\text{CH}_{\Delta'}^1(kG\text{-stab}) := Z_{\Delta'}^1(kG\text{-stab})/B_{\Delta'}^1(kG\text{-stab}) = Z_{\Delta'}^1(kG\text{-stab}).$$

By Definition 4.6.2,

$$Z_{\Delta'}^1(kG\text{-stab}) := \ker(q'_0 \circ d'_1) \subset K_0((kG\text{-stab}/kG\text{-stab}_{(0)})^\natural)$$

with

$$d'_1 : K_0((kG\text{-stab} / kG\text{-stab}_{(0)})^\natural) \rightarrow \mathbb{K}_{-1}(kG\text{-stab}_{(0)})$$

and

$$\begin{aligned} q'_1 : \mathbb{K}_{-1}(kG\text{-stab}_{(0)}) &\rightarrow \mathbb{K}_{-1}(kG\text{-stab}_{(0)} / kG\text{-stab}_{(-1)}) \\ &\rightarrow \mathbb{K}_{-1}((kG\text{-stab}_{(0)} / kG\text{-stab}_{(-1)})^\natural). \end{aligned}$$

Since $kG\text{-stab}_{(-1)} = 0$, the map q'_1 is the identity on $\mathbb{K}_{-1}(kG\text{-stab}_{(0)})$. Consequently,

$$\mathrm{CH}_{\Delta'}^1(kG\text{-stab}) = Z_{\Delta'}^1(kG\text{-stab}) = \ker(d'_1).$$

In conclusion, we have shown:

Proposition 5.4.2. *Let $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and k a field of prime characteristic $p \neq 2$ and $n \in \mathbb{N}$. Then, the tensor triangular Chow groups of $kG\text{-stab}$ are*

$$\mathrm{CH}_{\Delta'}^i(kG\text{-stab}) = \begin{cases} 0 & i \neq 0, 1, \\ \mathrm{Im}(i^1) & i = 0, \\ \ker(d'_1) & i = 1, \end{cases}$$

where $i^1 : K_0(kG\text{-stab}_{(0)}) \rightarrow K_0(kG\text{-stab})$ and $d'_1 : K_0((kG\text{-stab} / kG\text{-stab}_{(0)})^\natural) \rightarrow \mathbb{K}_{-1}(kG\text{-stab}_{(0)})$.

For $i = 0$, it amounts to computing the image of i^1 .

Remark 5.4.3. For the case $p = 2$, Klein does so in [Kl16a, Lemma 6.4.4]. He uses a complete characterization of the indecomposable modules of complexity 1 as the non-projective indecomposable modules of even dimension, see [Kl16a, Corollary 6.4.3]. At this point, concrete results about $p = 2$ are applied, which is why we use different arguments here.

Recall that, by Example 5.1.6, the group algebra of kG for $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ is given by

$$k(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}) \cong k[x, y]/(x^p, y^p).$$

We know from [Ben98b, Corollary 5.8.5] that if p does not divide the dimension of a module, then its support variety already is $\mathcal{V}_G(k)$, which in our case equals \mathbb{P}^1 . Hence, if p does not divide the dimension of a module, then its complexity is 2.

Example 5.4.4. Let $p = 3$. We want to give an example of an indecomposable 3-dimensional module of complexity 1. One can extend this example to $p > 3$ since this is a general procedure.

Let $J = (x, y)$ be the radical of kG , then $J^2 = (x^2, y^2, xy)$. We consider the indecomposable module

$$M = kG/J^2 = k[x, y]/(x^2, y^2, xy).$$

It is 3-dimensional with k -generators $1, x$ and y . To see that it has complexity 1, we construct a minimal resolution.

We define $f_0 : kG \rightarrow M$ as the canonical projection which has kernel $\ker(f_0) = (x^2, y^2, xy)$ as a kG -module. Next, we construct

$$\begin{aligned} f_1 : kG^3 &\rightarrow kG, \\ (1, 0, 0) &\mapsto x^2, \\ (0, 1, 0) &\mapsto y^2, \\ (0, 0, 1) &\mapsto xy, \end{aligned}$$

such that $\ker(f_0) = \text{Im}(f_1)$. We obtain the description of its kernel as $\ker(f_1) = ((x, 0, 0), (0, y, 0), (0, 0, x^2), (0, 0, y^2))$ and construct a map

$$\begin{aligned} f_2 : kG^4 &\rightarrow kG^3, \\ (1, 0, 0, 0) &\mapsto (x, 0, 0), \\ (0, 1, 0, 0) &\mapsto (0, y, 0), \\ (0, 0, 1, 0) &\mapsto (0, 0, x^2), \\ (0, 0, 0, 1) &\mapsto (0, 0, y^2), \end{aligned}$$

with $\text{Im}(f_2) = \ker(f_1)$ and $\ker(f_2) = ((x^2, 0, 0, 0), (0, y^2, 0, 0), (0, 0, x, 0), (0, 0, 0, y))$. We continue inductively by always sending the generators of kG^4 to the generators of the kernel from the step before. This way, we obtain that all f_i for $i \geq 3$ odd and for $i \geq 4$, respectively, are equal and we have a minimal resolution

$$\dots \rightarrow kG^4 \xrightarrow{f_3} kG^4 \xrightarrow{f_2} kG^3 \xrightarrow{f_1} kG \xrightarrow{f_0} M \rightarrow 0.$$

This procedure is described in [Car96, p.14]; the resolution is minimal by [Car96, Theorem 4.1].

Now, let $\lambda := 5$. We observe that $\dim(kG^3) = 3 \leq 5 \cdot 2^{1-1} = 5$ and $\dim(kG^4) = 4 \leq 5 \cdot i^{1-1}$ where $i \geq 3$ is the index of kG^4 in the free resolution. Hence, the complexity of M is indeed 1 whereas its dimension is 3, i.e. in particular divisible by 3.

On the other hand, the module $k \oplus k \oplus k$ is three-dimensional, but has complexity 2 by Proposition 5.1.30 and [Ben98b, Theorem 5.1.1 (ii)].

As we have just seen, there are modules of dimension 3 with different complexity.

However, for general p , such modules of the same dimension but different complexity are in the same equivalence class in $K_0(kG\text{-mod}) \stackrel{\dim}{\cong} \mathbb{Z}$. Consequently, we can still compute $\text{Im}(i^1)$. By the same arguments as in the case $G = \mathbb{Z}/p^n\mathbb{Z}$, we know that the dimension yields an isomorphism

$$\overline{\dim}_k : K_0(kG\text{-stab}) \xrightarrow{\cong} \mathbb{Z}/p^2\mathbb{Z}.$$

Restricting this isomorphism to $\mathrm{Im}(i^1) \subset K_0(kG\text{-stab})$, we obtain an injective group homomorphism

$$\overline{\mathrm{dim}}_k : \mathrm{Im}(i^1) \rightarrow \mathbb{Z}/p^2\mathbb{Z}.$$

Let A be the subgroup of $K_0(kG\text{-stab}) \cong \mathbb{Z}/p^2\mathbb{Z}$ generated by modules of dimension divisible by p . We have that

$$A \cong p(\mathbb{Z}/p^2\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$$

via dimension. Modules of complexity 1, i.e. modules in $kG\text{-mod}_{(0)}$, all have dimension divisible by p by contraposition of [Ben98b, Corollary 5.8.5] and the fact that $\mathcal{V}_G(k) = \mathbb{P}^1$. Thus, $\mathrm{Im}(i^1) \subset A$. On the other hand, there is a kG -module of dimension p and complexity 1, namely kG/J^2 for $J = (x, y)$ by Example 5.4.4. Since $1 = c_G(kG/J^2) = c_G((kG/J^2)^n)$ by [Ben98b, Theorem 5.1.1 (ii)], the composition

$$\mathrm{Im}(i^1) \hookrightarrow A \xrightarrow{\cong} \mathbb{Z}/p\mathbb{Z}$$

is also surjective. It follows that $\mathrm{Im}(i^1) \cong \mathbb{Z}/p\mathbb{Z}$.

Applying these computations to Proposition 5.4.2, we can now deduce the following.

Proposition 5.4.5. *Let $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and k a field with $\mathrm{char}(k) = p \neq 2$. Then,*

$$\mathrm{CH}_{\Delta}^i(kG\text{-stab}) = \begin{cases} 0 & i \neq 0, 1, \\ \mathbb{Z}/p\mathbb{Z} & i = 0, \\ \ker(d_1') & i = 1, \end{cases}$$

where $d_1' : K_0((kG\text{-stab}/kG\text{-stab}_{(0)})^{\natural}) \rightarrow \mathbb{K}_{-1}(kG\text{-stab}_{(0)})$.

For $i = 1$, we will not proceed further since it would exceed the extent of this work.

5.5 Tensor Triangular Chow–Witt Groups for $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$

Again, we are in the situation of Convention 5.3.1, but with $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

We would like to take a closer look at the defining diagram for Chow–Witt groups of $kG\text{-stab}$ for $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, at first for general k of characteristic $p \neq 2$.

We have seen the filtration components above, so we also consider the following cases where $\mathcal{T} := kG\text{-stab}$. Since we do not know in general whether $kG\text{-stab}/kG\text{-stab}_{(0)}$ is idempotent complete, the defining diagram is considered on the level of models as in Definition 4.6.4.

$i = 0$

Since $kG\text{-stab}_{(0)}$ is idempotent complete as a thick subcategory of an idempotent complete category, the defining diagram has the form

$$\begin{array}{ccc}
 0 & \xrightarrow{0} & 0 \\
 & \swarrow & \downarrow \text{id} \\
 & & \text{GW}(kG\text{-stab}_{(0)}) \\
 & \searrow & \downarrow \text{id} \\
 & & \text{GW}(kG\text{-stab})
 \end{array}$$

We obtain

$$\begin{aligned}
 \widetilde{\text{CH}}_{\Delta}^0(kG\text{-stab}) &= 0_d^{-1}(\text{Im}(0))/(\text{id}(\ker(\tilde{i}))) \\
 &= \text{GW}(kG\text{-stab}_{(0)})/\ker(\tilde{i}) \\
 &\cong \text{Im}(\tilde{i}),
 \end{aligned}$$

where the last isomorphism follows from the isomorphism theorem for abelian groups.

$i = 1$

As in the case of tensor triangular Chow groups, we use the model-theoretic version Definition 4.6.4 since we do not know whether $kG\text{-stab}/kG\text{-stab}_{(0)}$ is idempotent-complete.

The oriented coboundaries are defined as

$$\widetilde{B}_{\Delta'}^1(kG\text{-stab}) := \tilde{q}'_1(\ker(\tilde{i}^2)),$$

where

$$\tilde{i}^2 : \text{GW}^{[1]}(kG\text{-stab}_{(1)}) \rightarrow \text{GW}^{[1]}(kG\text{-stab}_{(2)})$$

is the identity on $\text{GW}^{[1]}(kG\text{-stab})$ since $kG\text{-stab}_{(i)} = kG\text{-stab}$ for $i \geq 1$. Hence,

$$\widetilde{B}_{\Delta'}^1(kG\text{-stab}) = \tilde{q}'_1(\ker(\tilde{i}^2)) = \tilde{q}'_1(0) = 0.$$

It follows that

$$\widetilde{\text{CH}}_{\Delta'}^1(kG\text{-stab}) := \widetilde{Z}_{\Delta'}^1(kG\text{-stab})/\widetilde{B}_{\Delta'}^1(kG\text{-stab}) = \widetilde{Z}_{\Delta'}^1(kG\text{-stab}).$$

By Definition 4.6.4,

$$\widetilde{Z}_{\Delta'}^1(kG\text{-stab}) = \ker(\tilde{q}'_{0,W} \circ \tilde{d}'_1) \subset \text{GW}^{[1]}((kG\text{-stab}/kG\text{-stab}_{(0)})^{\natural})$$

with

$$\tilde{d}'_1 : \text{GW}^{[1]}((kG\text{-stab}/kG\text{-stab}_{(0)})^{\natural}) \rightarrow \text{W}^{[2]}(kG\text{-stab}_{(0)})$$

and

$$\begin{aligned}
 \tilde{q}'_{0,W} : \text{W}^{[2]}(kG\text{-stab}_{(0)}) &\rightarrow \text{W}^{[2]}(kG\text{-stab}_{(0)}/kG\text{-stab}_{(-1)}) \\
 &\rightarrow \text{W}^{[2]}((kG\text{-stab}_{(0)}/kG\text{-stab}_{(-1)})^{\natural}).
 \end{aligned}$$

Since $kG\text{-stab}_{(-1)} = 0$, the map $\tilde{q}'_{0,W}$ is the identity on $W^{[2]}(kG\text{-stab}_{(0)})$. Consequently,

$$\widetilde{\text{CH}}_{\Delta'}^1(kG\text{-stab}) = \widetilde{Z}_{\Delta'}^1(kG\text{-stab}) = \ker(\tilde{d}'_1).$$

We have thus proven the following

Proposition 5.5.1. *For $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and k a field of characteristic $p \neq 2$, we have isomorphisms*

$$\widetilde{\text{CH}}_{\Delta'}^i(kG\text{-stab}) = \begin{cases} 0 & i \neq 0, 1, \\ \text{Im}(\tilde{i}) & i = 0, \\ \ker(\tilde{d}'_1) & i = 1 \end{cases}$$

for $\tilde{i} : \text{GW}(kG\text{-stab}_{(0)}) \rightarrow \text{GW}(kG\text{-stab})$ and $\tilde{d}'_1 : \text{GW}^{[1]}((kG\text{-stab}/kG\text{-stab}_{(0)})^\natural) \rightarrow W^{[2]}(kG\text{-stab}_{(0)})$.

For more concrete cases of k one can say a bit more, but we can only give partial results here.

Example 5.5.2. For k be algebraically closed, let

$$U : \text{GW}(kG\text{-mod}) \rightarrow \text{GW}(k\text{-mod})$$

be the morphism induced by the forgetful functor and

$$\alpha : \text{GW}(kG\text{-proj}) \rightarrow \text{GW}(kG\text{-mod})$$

the morphism induced by the inclusion. To compute the Chow–Witt group in degree 0, we consider the inclusion

$$i : \text{GW}(kG\text{-stab}_{(0)}) \rightarrow \text{GW}(kG\text{-stab})$$

and observe that

$$\text{GW}(kG\text{-stab}) \cong \text{GW}(kG\text{-mod}) / \text{Im}(\alpha) \cong \text{GW}(k) / U(\text{Im}(\alpha)) \cong \mathbb{Z}/p^2\mathbb{Z}$$

by Theorem 5.2.3 and Proposition 5.2.6; recall that by Example 5.1.6 the residue field of kG is k . The last isomorphism follows from the dimension isomorphism $\text{GW}(k) \cong \mathbb{Z}$ and the fact that kG is local (see Lemma 5.1.4), hence, free and projective modules coincide. So, their dimension is always a multiple of p^2 and a kG -module of dimension p^2 exists, namely kG itself.

As in the case of Chow groups, the image $\text{Im}(i)$ is now spanned by modules of complexity 1. These always have dimension divisible by p by contraposition of [Ben98b, Corollary 5.8.5] and support variety \mathbb{P}^1 . There is indeed a kG -module of dimension p , namely kG/J^2 from Example 5.4.4. Analogously, it follows that the restriction of the isomorphism $\overline{\dim}_k : \text{GW}(kG\text{-stab}) \rightarrow \mathbb{Z}/p^2\mathbb{Z}$ to $\text{Im}(\tilde{i})$

$$\overline{\dim}_k : \text{Im}(\tilde{i}) \rightarrow \mathbb{Z}/p^2\mathbb{Z}$$

is injective with image $\mathbb{Z}/p\mathbb{Z}$. It follows by Proposition 5.5.1, that

$$\widetilde{\text{CH}}_{\Delta}^0(kG\text{-stab})/\text{Im}(\tilde{i}) \cong p(\mathbb{Z}/p^2\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}.$$

For the computation of kernel of \tilde{d}'_1 , or even $\text{GW}^{[i]}(kG\text{-stab})$ as a first approach, we lack the necessary tools at the moment and will not pursue this further here. Therefore, we have that for k algebraically closed

$$\widetilde{\text{CH}}_{\Delta}^i(kG\text{-stab}) = \begin{cases} 0 & i \neq 0, 1, \\ \mathbb{Z}/p\mathbb{Z} & i = 0, \\ \ker(\tilde{d}'_1) & i = 1, \end{cases}$$

where $\tilde{d}'_1 : \text{GW}^{[1]}((kG\text{-stab}/kG\text{-stab}_{(0)})^{\natural}) \rightarrow \text{W}^{[2]}(kG\text{-stab}_{(0)})$.

In degree 0, our computation coincides with the case of Chow groups. To compare degree 1, further steps will need to be made.

After having treated algebraically closed fields, we now want to consider finite fields.

Example 5.5.3. Let k be finite and U, α as above. Analogously to Example 5.3.4, we consider the following isomorphism

$$(\dim, \det) : \text{GW}(k) \xrightarrow{\cong} \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

By Theorem 5.2.3 and Proposition 5.2.6 we, hence, have

$$\begin{aligned} \text{GW}(kG\text{-stab}) &\cong \text{GW}(kG\text{-mod})/\text{Im}(\alpha) \\ &\cong \text{GW}(k)/U(\text{Im}(\alpha)) \\ &\cong (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})/U(\text{Im}(\alpha)) \end{aligned}$$

Again, as in Example 5.3.4, kG is local by Lemma 5.1.4, all projective modules are free, hence, have a dimension a multiple of p^2 and arbitrary determinant in each dimension. It follows that $U(\text{Im}(\alpha)) \cong p^2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and, thus,

$$\begin{aligned} \text{GW}(kG\text{-stab}) &\cong (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})/(p^2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \\ &\cong \mathbb{Z}/p^2\mathbb{Z} \cong K_0(kG\text{-stab}). \end{aligned}$$

To determine the image of $\tilde{i} : \text{GW}(kG\text{-stab}_{(0)}) \rightarrow \text{GW}(kG\text{-stab})$, recall that, since $\mathcal{V}_G(k) = \mathbb{P}^1$ and by contraposition of [Ben98b, Corollary 5.8.5], modules in the image of \tilde{i} have dimension a multiple of p^2 but arbitrary determinants. Thus, as above, $\text{Im}(\tilde{i}) \cong p(\mathbb{Z}/p^2\mathbb{Z}) \cong p\mathbb{Z}/p^2\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z}$ and we obtain

$$\widetilde{\text{CH}}_{\Delta}^i(kG\text{-stab}) = \begin{cases} 0 & i \neq 0, 1, \\ \mathbb{Z}/p\mathbb{Z} & i = 0, \\ \ker(\tilde{d}'_1) & i = 1. \end{cases}$$

Again, Chow- and Chow–Witt groups coincide in degree 0 and further progress on degree 1 may be made in future work.

As before, we lack the necessary technical tools to compute the first shifted Grothendieck–Witt groups and kernels or images of maps between them.

However, from the examples, we can deduce the following results. The restriction to the cases where G is an elementary abelian p -group or $G = \mathbb{Z}/p^n\mathbb{Z}$ is due to the assumptions in Proposition 5.2.6.

Corollary 5.5.4. *If $G \cong (\mathbb{Z}/p\mathbb{Z})^r$ is a finite elementary abelian p -group and k is algebraically closed or finite of characteristic $p \neq 2$, we have that $\mathrm{GW}(kG\text{-stab}) \cong K_0(kG\text{-stab})$.*

Proof. Let U, α be as above. For G elementary abelian we have that

$$\mathrm{GW}(kG\text{-stab}) \cong \mathrm{GW}(kG\text{-mod})/\mathrm{Im}(\alpha) \cong \mathrm{GW}(k)/U(\mathrm{Im}(\alpha))$$

by Theorem 5.2.3 and Proposition 5.2.6.

For k algebraically closed, $\mathrm{GW}(k) \cong \mathbb{Z} \cong K_0(k)$ by dimension, and hence by [TaWa91, Proposition 1] $\mathrm{GW}(kG\text{-stab}) \cong K_0(kG\text{-stab})$.

If k is finite, $\mathrm{GW}(k) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ via dimension and determinant. Then, the subgroup $U(\mathrm{Im}(\alpha))$ is isomorphic to $p^r\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ analogously to Example 5.5.3. Hence, $\mathrm{GW}(kG\text{-stab}) \cong \mathbb{Z}/p^r\mathbb{Z} \cong K_0(kG\text{-stab})$ using again [TaWa91, Proposition 1], where both isomorphisms are given by the dimension. \square

In this chapter, we have made some very first computations concerning Chow–Witt groups of tensor triangulated categories outside the field of algebraic geometry. For $G = \mathbb{Z}/p^n\mathbb{Z}$, $p \neq 2$, we have seen that the only non-trivial Chow–Witt group is in degree 0 and isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$ and coincides with the non-oriented case. For $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and $p \neq 2$, the 0-th Chow–Witt group again coincides with the Chow group being $\mathbb{Z}/p\mathbb{Z}$. The other non-trivial degree is 1, but we lack technical tools such as an analogon of Proposition 5.2.6 to do further steps here. Our computations moreover show that the Chow–Witt groups of $kG\text{-stab}$ usually also heavily depends on the Grothendieck–Witt group of the base field k , which are hard to determine in general.

Chapter 6

The Spectrum of Artin Motives

After having seen how Chow–Witt groups of schemes can be generalized to the setting of tensor triangulated categories, we now want to investigate the role of Chow–Witt groups in the theory of motives. The first aim is to refine results of [BG23a] before computing some examples of Balmer spectra of Artin Milnor–Witt motives.

To begin with, this chapter recalls the construction of Voevodsky’s derived category of motives $\mathrm{DM}(k; R)$ in Section 6.1 and arguments and results of [BG23a] in Section 6.2 connecting Artin motives, permutation modules, and Mackey functors.

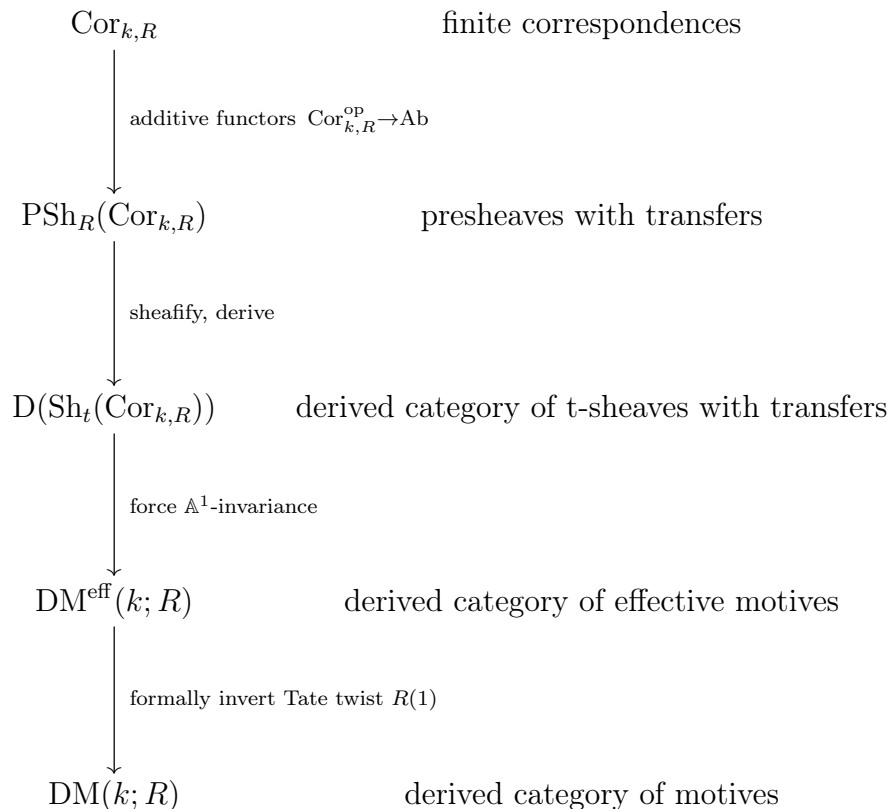
Convention 6.0.1. Let R be a commutative ring. In the rest of this work, let k be a perfect field with a fixed separable closure \bar{k} and absolute Galois group $\Gamma := \mathrm{Gal}(\bar{k}/k)$. If not mentioned otherwise, all schemes will be separated and of finite type over their respective base field (often k).

6.1 The Derived Category of Motives

We recall the construction of Voevodsky’s derived category of motives $\mathrm{DM}(k; R)$ over a field k with coefficients in a ring R . As an introductory lecture on this topic, we recommend [MVW06, Voe00], which we use as references for the contents of this section. The latter is the original work.

Consider the following diagram demonstrating the construction steps. We start with the category of finite correspondences discussed below and consider presheaves on this category. After that, one passes to the derived category of t -sheaves with transfers, usually for topologies such as Nisnevich, étale, or Zariski. In the next step, we force \mathbb{A}^1 -invariance to define the derived category of *effective motives*. Finally, we formally invert the Tate twist which is usually done by passing to spectrum objects, but we will not focus on this part here since we only

deal with effective motives in this work.



We start with the construction of the category of finite correspondences.

Definition 6.1.1. Let X be a smooth connected scheme over k and Y any separated scheme over k . An *elementary correspondence* from X to Y is defined as an irreducible closed subset $W \subset X \times Y$ whose associated integral subscheme is finite and surjective over X . If X is non-connected, an elementary correspondence from X to Y is an elementary correspondence from a connected component of X to Y . A *finite correspondence* is an element of the free abelian group $\text{Cor}_k(X, Y)$ generated by the elementary correspondences.

Definition 6.1.2. We define the category Cor_k of finite correspondences as follows. The objects are smooth separated schemes over k denoted by Sm_k . Its morphism sets are given by the abelian groups $\text{Cor}_k(X, Y)$. If X is not connected with connected components X_i , $i \in I$, we have $\text{Cor}_k(X, Y) = \bigoplus_{i \in I} \text{Cor}_k(X_i, Y)$. Composition of correspondences is illustrated in [MVW06, Figure 17A.1.]. For a ring R , we set the category $\text{Cor}_{k,R} := \text{Cor}_k \otimes_{\mathbb{Z}} R$, i.e., tensoring every morphism group between two objects with R .

Remark 6.1.3. Note that $\text{Cor}_{k,R}$ is a symmetric monoidal additive category with coproduct the disjoint union of k -schemes.

Example 6.1.4. Let $f : X \rightarrow Y$ be a morphism in Sm_k and X connected, then the graph Γ_f is an elementary correspondence from X to Y . If X is not connected, the sum of components of the graph yields a finite correspondence.

We obtain a functor

$$\gamma : \mathrm{Sm}_k \rightarrow \mathrm{Cor}_k,$$

being the identity on objects and sending a morphism to its graph.

In the next step of the construction of $\mathrm{DM}(k; R)$, we pass to (pre-)sheaves with transfers.

Definition 6.1.5. A *presheaf with transfers* is an additive, R -linear contravariant functor $\mathrm{Cor}_{k,R}^{\mathrm{op}} \rightarrow R\text{-Mod}$. The category of R -linear additive presheaves on $\mathrm{Cor}_{k,R}$ will be denoted by $\mathrm{PSh}_R(\mathrm{Cor}_{k,R})$.

A presheaf with transfers is a *t -sheaf with transfers* if it is a sheaf in the t -topology when restricted to Sm_k , where t can be Zariski (Zar), Nisnevich (Nis), or étale (et). We denote the corresponding category by $\mathrm{Sh}_t(\mathrm{Cor}_{k,R})$.

Example 6.1.6. Constant presheaves on Sm_k are presheaves with transfers. The classical Chow groups $\mathrm{CH}^i(-)$ are presheaves with transfers, see [MVW06, Example 2.5].

An important class of presheaves with transfers will be the representable objects. They are essential to define motivic complexes and the Tate twist we later wish to invert.

Definition 6.1.7. Let X be an object of Sm_k , then $R_{\mathrm{tr}}(X)$ is defined as the presheaf with transfers given by $R_{\mathrm{tr}}(X)(U) := \mathrm{Cor}_{k,R}(U, X)$, also denoted the *presheaf with transfers represented by X* .

Remark 6.1.8. By [Voe00, Lemma 3.1.2], $R_{\mathrm{tr}}(X)$ is a Nis-sheaf with transfers.

Definition 6.1.9. Let $q \in \mathbb{N}$. Then, we define the *motivic complex $R(q)$* as the complex of presheaves with transfers given by

$$R(q) := C_* R_{\mathrm{tr}}(\mathbb{G}_m^{\wedge q})[-q],$$

where \mathbb{G}_m is the multiplicative group and $C_* R_{\mathrm{tr}}(\mathbb{G}_m^{\wedge q}[-q])$ the complex obtained from the simplicial presheaf $U \mapsto R_{\mathrm{tr}}(\mathbb{G}_m^{\wedge q})[-q](U \times \Delta^\bullet)$ and Δ^\bullet is the cosimplicial scheme defined as

$$\Delta^n := \mathrm{Spec}(k[x_0, \dots, x_n]) / \left(\sum_{i=0}^n x_i = 1 \right).$$

Example 6.1.10. We consider the following examples of motivic complexes.

- (i) For a general ring R we have $R(q) = \mathbb{Z}(q) \otimes R$.
- (ii) If $q = 0$, then $\mathbb{Z}(0) = C_*(\mathbb{Z})$, which is quasi-isomorphic to \mathbb{Z} .
- (iii) For $q = 1$ we have $\mathbb{Z}(1) = C_* \mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m)[-1]$, which is quasi-isomorphic to $\mathcal{O}^*[-1]$ by [MVW06, Theorem 4.1].

In the next step, we force \mathbb{A}^1 -invariance to define effective motives.

Definition 6.1.11. We denote by $\mathcal{E}_{\mathbb{A}^1}$ the smallest thick (and closed under arbitrary direct sums) subcategory of $D(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{k,R}))$ containing maps of the form $R_{\mathrm{tr}}(X \times \mathbb{A}^1) \rightarrow R_{\mathrm{tr}}(X)$ and by $W_{\mathbb{A}^1}$ the class of maps with cone in $\mathcal{E}_{\mathbb{A}^1}$.

Then, we define the *triangulated category of effective motives* as the localization

$$\mathrm{DM}^{\mathrm{eff}}(k; R) := D(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{k,R}))[W_{\mathbb{A}^1}^{-1}],$$

which coincides with the Verdier quotient $D(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{k,R}))/\mathcal{E}_{\mathbb{A}^1}$, compare [Nee05, Proposition 2.1.24].

By the *motive* $M(X)$ of X in Sm_k we denote the object $R_{\mathrm{tr}}(X)$ in $\mathrm{DM}^{\mathrm{eff}}(k; R)$.

Moreover, we define the *triangulated category of geometric effective motives* $\mathrm{DM}^{\mathrm{eff},\mathrm{gm}}(k; R)$ as the localization of the category $\mathrm{K}_b(\mathrm{Cor}_{k,R}^{\natural})$ with respect to the thick subcategory \bar{T} from [Voe00, p.191], i.e., forcing \mathbb{A}^1 -invariance.

Finally, we pass to the triangulated category of motives in the last step. Since we will only consider effective motives in the rest of this work, we will not go into detail. For this, the reader is referred to [MVW06, Voe00].

If $t = \mathrm{Nis}$, we omit it from the notation.

Definition 6.1.12. We define the *triangulated category of motives* $\mathrm{DM}(k; R)$ as the category obtained from $\mathrm{DM}^{\mathrm{eff}}(k; R)$ after formally inverting the Tate twist operation $\otimes R(1)$.

Similarly, we set the *triangulated category of geometric motives* $\mathrm{DM}^{\mathrm{gm}}(k; R)$ as the category obtained from $\mathrm{DM}^{\mathrm{eff},\mathrm{gm}}(k; R)$ after formally inverting the Tate twist operation. By abuse of notation, we denote by the *motive* $M(X)$ of a smooth scheme X the class of $R_{\mathrm{tr}}(X)$ in $\mathrm{DM}(k; R)$.

Remark 6.1.13. The category $\mathrm{DM}^{\mathrm{eff}}(k; R)$ is a tensor triangulated category. It is compactly generated by motives $M(X)$ of smooth schemes. The category $\mathrm{DM}(k; R)$ is a tensor triangulated category in which every object is isomorphic to some $M(X)(-n)$ for a smooth scheme X and some $n \geq 0$.

The geometric motives form a rigid tensor triangulated category $\mathrm{DM}^{\mathrm{gm}}(k; R)$ by [Voe00, Theorem 4.3.7]. It is moreover essentially small and idempotent complete, hence, so is $\mathrm{DM}^{\mathrm{eff},\mathrm{gm}}(k; R)$. Moreover, the category of geometric (effective) motives coincides with the thick subcategory of compact objects of $\mathrm{DM}^{(\mathrm{eff})}(k; R)$.

Remark 6.1.14. The *triangulated category of Tate motives* is the localizing subcategory of $\mathrm{DM}(k; R)$ generated by Tate twists $R(n)$ for $n \in \mathbb{Z}$ and is a large tensor triangulated category. The *triangulated category of geometric Tate motives* is the thick subcategory of $\mathrm{DM}^{\mathrm{gm}}(k; R)$ generated by Tate twists $R(n)$ for $n \in \mathbb{Z}$. It is a small, rigid, idempotent complete tensor triangulated category.

After having recalled the construction of Voevodsky's derived category of motives, we will now recollect the most important results from [BG23a].

6.2 Artin Motives, Mackey Functors, and Permutation Modules

In this section, we recall constructions and main results from [BG23a] on the connection between Artin motives, permutation modules, and Mackey functors. The recollection is coarse and only included for the reader's convenience. For more details the reader is referred to the original work [BG23a], which we will follow here.

The aim of [BG23a] is to prove equivalences of tensor triangulated categories in the following diagram, where all the candidates will be recalled briefly.

$$\begin{array}{ccc}
 & \text{DAM}(k; R) & \\
 \swarrow \simeq & & \nwarrow \simeq \\
 \text{D Perm}(\Gamma; R) & \xleftarrow{\simeq} & \text{D}(\text{Mack}_R^{\text{coh}}(\Gamma))
 \end{array} \tag{6.1}$$

A part of these equivalences already appears in [Voe00, Proposition 3.4.1]. We begin with the upper part of the triangle.

Definition 6.2.1. The *derived category of Artin motives* $\text{DAM}(k; R)$ is defined as the localizing subcategory of $\text{DM}^{\text{eff}}(k; R)$ generated by the motives of 0-dimensional smooth k -schemes. Its compact part, the derived category of *geometric* Artin motives, is denoted by $\text{DAM}^{\text{gm}}(k; R)$. It can also be described as the thick subcategory of $\text{DM}^{\text{eff, gm}}(k; R)$ generated by motives $M(X)$ of zero-dimensional smooth k -schemes X .

Remark 6.2.2. The category $\text{DAM}(k; R)$ is a tensor triangulated category which is compactly generated by motives $M(X)$, where X is a 0-dimensional smooth scheme. Its compact part $\text{DAM}^{\text{gm}}(k; R)$ is an essentially small, idempotent complete, rigid tensor triangulated category by [HPS97, Theorem 2.1.3] (compare [Bal10b, Definition 44]). Moreover, one does not distinguish effective and non-effective (geometric) Artin motives by [BG23a, Remark 7.9].

Let $\text{Cor}_{k,R}^0$ be the full subcategory of $\text{Cor}_{k,R}$ spanned by the 0-dimensional smooth k -schemes. Balmer and Gallauer prove the following.

Proposition 6.2.3. ([BG23a, Corollary 7.10]) *The inclusion $\iota : \text{Cor}_{k,R}^0 \rightarrow \text{Cor}_{k,R}$ induces an equivalence of tensor triangulated categories*

$$\text{D}(\text{Sh}_{\text{Nis}}(\text{Cor}_{k,R}^0)) \xrightarrow[\iota]{\simeq} \text{DAM}(k; R)$$

that restricts to an equivalence on the level of compacts

$$\text{K}_b((\text{Cor}_{k,R}^0)^{\natural}) \xrightarrow{\simeq} \text{DAM}^{\text{gm}}(k; R),$$

where $(-)^{\natural}$ denotes the idempotent completion.

We now give the definition of permutation modules in the left corner of the triangle from Equation 6.1.

Let Γ be an arbitrary profinite group. Later, it will denote the absolute Galois group of k as in Convention 6.0.1.

In the category $\Gamma - \text{Sets}$, the objects are sets equipped with a continuous Γ -action. Morphisms are simply Γ -equivariant maps. By Γ -sets we denote the full subcategory consisting of *finite* sets with a continuous Γ -action.

Moreover, we define the category $\text{Mod}(\Gamma; R)$ with objects being R -modules endowed with the discrete topology and a continuous Γ -action. For X in $\Gamma - \text{Sets}$, we define $R(X)$ to be the free R -module on the basis X . The Γ -action on X can be R -linearly extended to $R(X)$. Hence, $R(X)$ is an object in $\text{Mod}(\Gamma; R)$. We obtain a functor

$$R(-) : \Gamma - \text{Sets} \rightarrow \text{Mod}(\Gamma; R)$$

that is symmetric monoidal, where the tensor product of two modules over R is equipped with the diagonal Γ -action.

Note that by [BG23a, Proposition 2.6] the category $\text{Mod}(\Gamma; R)$ is *Grothendieck abelian* with $\{R(\Gamma/H) \mid H \leq \Gamma \text{ open subgroup}\}$ as a set of finitely presented generators.

Definition 6.2.4. We call modules in the essential image of $R(-)$ *permutation modules*, the full subcategory of such is denoted by $\text{Perm}(\Gamma; R)$. Modules in the essential image of $R(-) : \Gamma - \text{sets} \rightarrow \text{Mod}(\Gamma; R)$ also yield a full subcategory which will be denoted by $\text{perm}(\Gamma; R)$, the *finitely generated permutation modules*.

For Balmer's and Gallauer's definition of the derived category of permutation modules $\text{D Perm}(\Gamma; R)$, the reader is referred to [BG23a, Definition 3.6]. Here, a Γ -acyclic complex is an object C in $\text{K Perm}(\Gamma; R)$ such that the complex C^H of H -fixed points is acyclic for every open subgroup $H \leq \Gamma$. By [BG23a, Corollary 3.13], $\text{D Perm}(\Gamma; R)$ is a tensor triangulated category.

For the right corner of the triangle from Equation 6.1, we need to introduce (cohomological) Mackey functors.

We will go into more detail when introducing the oriented counterpart in the following Construction 8.1.2. For a more concrete definition of $\text{span}(\Gamma)$, the reader is referred to [ThWe95, Section 2].

Definition 6.2.5. ([ThWe95, Section 2]) We denote by $\text{span}(\Gamma)$ the following category. Objects are Γ -sets and morphisms $X \rightarrow Y$ are given by isomorphisms classes of spans $X \leftarrow Z \rightarrow Y$, where $[X \leftarrow Z \rightarrow Y] = [X \leftarrow Z' \rightarrow Y]$ if there is an isomorphism $Z \cong Z'$ making the resulting diagrams commute. Composition is defined by pulling back.

Now, $\Omega(\Gamma)$ denotes the category obtained from $\text{span}(\Gamma)$ by group completing the Hom sets, where the monoidal structure is given by the disjoint union. The resulting category is additive. By extending the scalars formally to an arbitrary ring R , we obtain an additive category denoted by $\Omega_R(\Gamma)$.

Definition 6.2.6. An (R -linear) *Mackey functor* is an additive functor $M : \Omega(\Gamma)^{\text{op}} \rightarrow R\text{-Mod}$. Furthermore, if M sends spans $\Gamma/H \leftarrow \Gamma/K \rightarrow \Gamma/H$ for every open subgroup $K \leq H$ to multiplication by $[H : K]$, M is called a *cohomological Mackey functor*. We denote the resulting abelian categories by $\text{Mack}_R(\Gamma)$ and $\text{Mack}_R^{\text{coh}}(\Gamma)$, respectively.

If we extend scalars to R formally, Mackey functors are defined as R -linear additive functors $M : \Omega_R(\Gamma)^{\text{op}} \rightarrow R\text{-Mod}$, but we do not introduce a separate notation for this case.

Moreover, [BG23a, Proposition 4.17] tells us that we can view $\text{perm}(\Gamma; R)$ as a quotient

$$\text{perm}(\Gamma; R) \simeq \frac{\Omega_R(\Gamma)}{J_R(\Gamma)},$$

where $J_R(\Gamma)$ is the ideal generated by differences

$$(\Gamma/H \leftarrow \Gamma/K \rightarrow \Gamma/H) - [H : K] \cdot \text{id}_{\Gamma/H}$$

for $K \leq H \leq \Gamma$. We consequently have the following equivalent description of cohomological Mackey functors, which is cited in [BG23a, Corollary 4.22]).

Proposition 6.2.7. ([Yos83, Theorem 4.3]) *There is an equivalence of R -linear Grothendieck abelian categories*

$$\text{Mack}_R^{\text{coh}}(\Gamma) \simeq \text{PSh}_R(\text{perm}(\Gamma; R)),$$

where $\text{PSh}_R(-)$ denotes the category of R -linear additive presheaves with values in R -modules.

Now, let k and Γ be as in Convention 6.0.1.

To see how cohomological Mackey functors fit into Equation 6.1, we construct a functor

$$\begin{aligned} \Psi_{\bar{k}} : \text{Sh}_{\text{et}}(\text{Cor}_{k,R}) &\rightarrow \text{Mod}(\Gamma; R) \\ M &\mapsto \text{colim}_{k'} M(\text{Spec}(k')), \end{aligned}$$

where k' runs over finite field extensions of k contained in \bar{k} ; the colimit is taken in the category of R -modules and the $\Gamma = \text{Gal}(\bar{k}/k)$ -action is canonical. The functor $\Psi_{\bar{k}}$ is an equivalence of tensor categories by [BG23a, Lemma 6.13].

Proposition 6.2.8. ([BG23a, Proposition 6.14]) *Let k be a field with a fixed separable closure \bar{k} and absolute Galois group $\Gamma := \text{Gal}(\bar{k}/k)$. The composition $\Psi_{\bar{k}} \circ a_{\text{et}} : \text{Sh}_{\text{Nis}}(\text{Cor}_{k,R}) \rightarrow \text{Mod}(\Gamma; R)$, where a_{et} denotes the étale sheafification functor, restricts to an equivalence of tensor categories*

$$\text{Cor}_k^0 \otimes R \simeq \text{perm}(\Gamma; R).$$

Following [BG23a, Remark 6.16], we can extend $\Psi_{\bar{k}}$ to the Nisnevich-topology to get the following result.

Corollary 6.2.9. ([BG23a, Corollary 6.17]) *Let k be a field with a fixed separable closure \bar{k} and absolute Galois group $\Gamma := \text{Gal}(\bar{k}/k)$. The functor*

$$\Psi_{\bar{k}} : \text{Sh}_{\text{Nis}}(\text{Cor}_{k,R}^0) \xrightarrow{\simeq} \text{Mack}_R^{\text{coh}}(\Gamma)$$

is an equivalence of abelian categories.

Together with Proposition 6.2.3, this induces the equivalence of the right-hand side of the triangle in Equation 6.1.

In order to establish the equivalence at the bottom of the triangle in Equation 6.1, one defines a functor, called the *fixed-point functor*, via

$$\begin{aligned} \text{FP} : \text{Mod}(\Gamma; R) &\rightarrow \text{PSh}_R(\text{perm}(\Gamma; R)) \simeq \text{Mack}_R^{\text{coh}}(\Gamma) \\ M &\mapsto \text{Hom}_{\text{Mod}(\Gamma; R)}(-, M) \Big|_{\text{perm}(\Gamma; R)} \end{aligned}$$

which can be seen as a restricted version of Yoneda. It admits a left adjoint $\text{LP} : \text{PSh}_R(\text{perm}(\Gamma; R)) \simeq \text{Mack}_R^{\text{coh}}(\Gamma) \rightarrow \text{Mod}(\Gamma; R)$ defined by left Kan extension (see [BG23a, Remark 5.4]).

By [BG23a, Lemma 5.5], FP is R -linear, lax monoidal, fully faithful and preserves filtered colimits. To justify the name *fixed-point* functor, fix a Γ -module M . Then, the image of a generator $R(\Gamma/H)$ of $\text{perm}(\Gamma; R)$ under $\text{FP}(M)$ is isomorphic to M^H for any open subgroup $H \leq \Gamma$.

We want to understand compact objects better. Balmer and Gallauer prove the following using Neeman–Thomason localization, which can be transferred to different settings using Corollary 6.2.12 below.

Lemma 6.2.10. ([BG23a, Corollary 3.10]) *Let Γ be a profinite group. There is a canonical equivalence*

$$\text{D Perm}(\Gamma; R)^c \simeq \text{thick}(\text{perm}(\Gamma; R)) = \text{K}_b(\text{perm}(\Gamma; R)^\natural)$$

between the compact part of the triangulated category $\text{D Perm}(\Gamma; R)$ and the thick subcategory of $\text{K}(\text{Perm}(\Gamma; R))$ generated by finitely generated permutation modules.

Finally, [BG23a] deduces an equivalence between the derived category of permutation modules and the derived category of cohomological Mackey functors.

Proposition 6.2.11. ([BG23a, Corollary 5.7]) *Let Γ be a profinite group. The fixed-point functor FP induces an equivalence of tensor triangulated categories*

$$\text{FP} : \text{D Perm}(\Gamma; R) \xrightarrow{\simeq} \text{D}(\text{Mack}_R^{\text{coh}}(\Gamma))$$

The results are summarized in the following diagram.

Corollary 6.2.12. ([BG23a, Corollary 7.10]) *Let k be a field with a fixed separable closure \bar{k} and absolute Galois group $\Gamma := \text{Gal}(\bar{k}/k)$. There are equivalences of tensor triangulated categories*

$$\begin{array}{ccc} \text{DAM}(k; R) & \xleftarrow[\iota_1]{\simeq} & \text{D}(\text{Sh}_{\text{Nis}}(\text{Cor}_{k,R}^0)) \\ & & \uparrow \Psi_{\bar{k}}^{-1} \simeq \\ \text{D Perm}(\Gamma; R) & \xrightarrow[\text{FP}]{\simeq} & \text{D}(\text{Mack}_R^{\text{coh}}(\Gamma)) \end{array}$$

which restrict to equivalences on the level of compacts

$$\begin{array}{ccc} \text{DAM}^{\text{gm}}(k; R) & \xleftarrow{\simeq} & \text{K}_b((\text{Cor}_{k,R}^0)^{\natural}) \\ & \searrow \simeq & \nearrow \\ \text{K}_b(\text{perm}(\Gamma; R)^{\natural}) & & \end{array}$$

Proof. The statements follow from Proposition 6.2.3, Corollary 6.2.9 and Proposition 6.2.11. \square

Having seen the main results of [BG23a] and the construction of the derived category of motives $\text{DM}(k; R)$, the next chapter aims at refining the first step to an oriented setting. For this, the construction of the derived category of Milnor–Witt motives $\widetilde{\text{DM}}(k; R)$ is recalled in Section 7.1 before we generalize Proposition 6.2.3 to the Milnor–Witt setting in the following sections.

Chapter 7

The Derived Category of Artin Milnor–Witt Motives

This chapter aims at extending some results of [BG23a] to the category $\widetilde{\mathrm{DM}}(k; R)$ of Milnor–Witt motives. In the end, we want to construct the following categories and prove equivalences of tensor triangulated categories between them.

$$\begin{array}{ccc} \widetilde{\mathrm{DAM}}(k; R) & \xleftarrow{\cong} & \mathrm{D}(\mathrm{Sh}_{\mathrm{Nis}}(\widetilde{\mathrm{Cor}}_{k,R}^0)) \\ & & \downarrow \simeq \\ & & \mathrm{D}(\widetilde{\mathrm{Mack}}_R^{\mathrm{coh}}(k)). \end{array} \quad (7.1)$$

In Remark 8.2.7, we will comment on why we do not present an analog of the category $\mathrm{DPerm}(\Gamma; R)$ in our case.

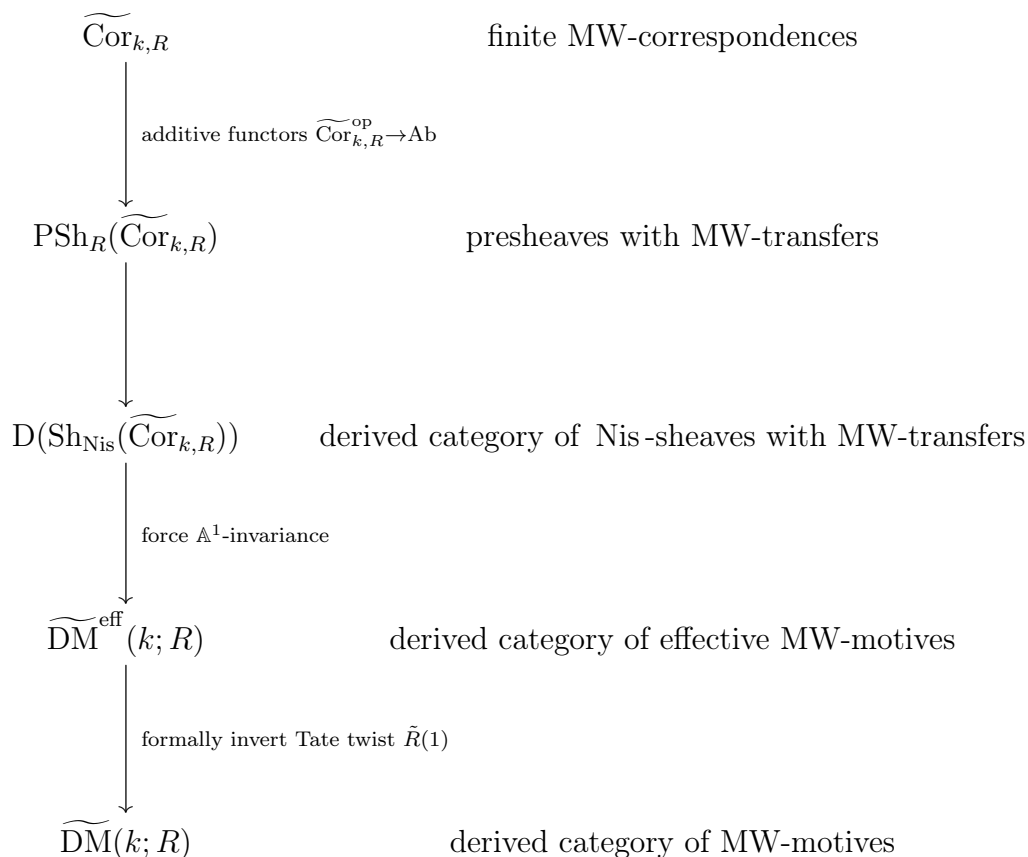
We begin with a recapitulation of the construction of the derived category of Milnor–Witt motives $\widetilde{\mathrm{DM}}(k; R)$ in the first section. Then, we discuss further details on the category $\widetilde{\mathrm{Cor}}_{k,R}$ of finite Milnor–Witt correspondences when restricting it to 0-dimensional schemes as in [BG23a]. In Section 7.3, we introduce Milnor–Witt Artin motives and prove the upper equivalence of the diagram above under the assumption that Conjecture 7.3.10 holds. For geometric Artin motives, we will prove the upper equivalence without assuming a conjecture to be true. The remaining equivalence will be treated in the Chapter 8.

Convention 7.0.1. For this chapter, let k be a perfect field of characteristic $\neq 2$ and R a ring.

7.1 Milnor–Witt Motives

In this section, we recall the theory of Milnor–Witt (=MW) motives following [CF22] and [DF22]. As for $\mathrm{DM}(k; R)$, we have similar steps to follow. The main difference here lies in the underlying category of finite correspondences, which refines to *finite MW-correspondences*. The other steps then follow in an analog

manner. Consider the following diagram illustrating the construction of $\widetilde{\mathrm{DM}}(k; R)$.



Because it will be a central object of study, we dedicate the first subsection to the construction of the category $\widetilde{\mathrm{Cor}}_{k,R}$ of finite MW-correspondences. The rest of the construction will follow in Subsection 7.1.2.

7.1.1 Milnor–Witt Correspondences

The idea why one would like to enlarge the category $\mathrm{Cor}_{k,R}$ of finite correspondences is that, for example, Chow–Witt groups (see Section 4.2) are not captured here. To understand why the concept of “decorating with \sim ” is interesting, the reader is, for example, referred to the introduction of [BC+22].

In order to create a category $\widetilde{\mathrm{Cor}}_{k,R}$ it therefore seems sensible to equip finite correspondences with symmetric bilinear forms, in this case over the function field of each irreducible component of the *support* of a finite correspondence. Together with some conditions on the symmetric bilinear form, this is exactly what the definition of MW-correspondences comes down to. However, due to technical reasons, we define the group $\widetilde{\mathrm{Cor}}_{k,R}(X, Y)$ in terms of Chow–Witt groups.

Remark 7.1.1. At first, we need to adapt our definition of Chow–Witt groups of schemes from Definition 4.2.2 slightly. Let X be a scheme and (i, L) a *graded line bundle* over X in the sense of [Fas20, Section 1.3]. One can define the Chow–Witt groups $\widetilde{\mathrm{CH}}^n(X, (i, L))$ twisted by a graded line bundle (i, L) . The addition of an

integer i to the line bundle L over X is only a technical convenience when treating products.

What we will be interested in now is the Chow–Witt group supported on a closed subset. For this, first recall that we can write Chow–Witt groups as the sheaf cohomology $\widetilde{\mathrm{CH}}^n(X, (i, L)) = H_X^n(X, \mathbf{K}_n^{\mathrm{MW}}(i, L))$, compare [AF16, Theorem 2.3.4]. Then, we define for a closed subset $Z \subset X$ the n -th Chow–Witt group supported on Z as $\widetilde{\mathrm{CH}}_Z^n(X, (i, L)) := H_Z^n(X, \mathbf{K}_n^{\mathrm{MW}}(i, L))$.

Having seen the definition of Chow–Witt groups with support, we want to define finite Milnor–Witt correspondences as the directed limit of such Chow–Witt groups.

Definition 7.1.2. Let X and Y be smooth schemes over a perfect field k of characteristic $\neq 2$ and Y equidimensional and let $T \subset X \times Y$ be a closed subset. We call T an *admissible subset* if the canonical morphism $T \rightarrow X$ is finite and maps each irreducible component of T surjectively onto an irreducible component of X . The poset of admissible subsets partially ordered by inclusion is denoted by $\mathcal{A}(X, Y)$.

Definition 7.1.3. Let X and Y be smooth schemes over a perfect field k of characteristic $\neq 2$ and Y equidimensional. Then, we define the group of *finite Milnor–Witt correspondences*

$$\widetilde{\mathrm{Cor}}_k(X, Y) := \varinjlim_{T \in \mathcal{A}(X, Y)} \widetilde{\mathrm{CH}}_T^{\dim(Y)}(X \times Y, \omega_Y),$$

where ω_Y is the graded line bundle $(\dim(Y), p_Y^* \omega_{Y/k})$, $p_Y : X \times Y \rightarrow Y$ being the canonical projection and $\omega_{Y/k} := \det(\Omega_{Y/k})$ is the determinant bundle of the differential sheaf as in Remark 4.2.16.

If Y is not equidimensional, i.e. $Y = \coprod_j Y_j$ with each Y_j equidimensional, we define

$$\widetilde{\mathrm{Cor}}_k(X, Y) := \prod_j \widetilde{\mathrm{Cor}}_{k, R}(X, Y_j).$$

For a ring R , we set $\widetilde{\mathrm{Cor}}_{k, R}(X, Y) := \widetilde{\mathrm{Cor}}_k(X, Y) \otimes_{\mathbb{Z}} R$.

We obtain a category $\widetilde{\mathrm{Cor}}_{k, R}$ with objects smooth k -schemes and morphisms groups from X to Y given by $\widetilde{\mathrm{Cor}}_{k, R}(X, Y)$. The composition is defined as in [CF22, Section 4.2].

Remark 7.1.4. When considering the set of elementary correspondences from Definition 6.1.1 a partially ordered set ordered by inclusion, it coincides with the poset of admissible subsets from Definition 7.1.2 since a scheme is integral if and only if it is reduced and irreducible.

Example 7.1.5. For arbitrary smooth schemes X , we have $\widetilde{\mathrm{Cor}}_k(X, \mathrm{Spec}(k)) = \widetilde{\mathrm{CH}}^0(X) = \mathbf{K}_0^{\mathrm{MW}}(X)$.

On the other hand, if Y has dimension d , we have

$$\widetilde{\mathrm{Cor}}_k(\mathrm{Spec}(k), Y) = \bigoplus_{y \in Y^{(d)}} \widetilde{\mathrm{CH}}_{\{y\}}^d(Y, \omega_Y) = \bigoplus_{y \in Y^{(d)}} \mathrm{GW}(k(y), \omega_{k(y)/k}).$$

For more details, see [CF22, Example 4.1.5].

As for the classical case, we can ask ourselves how to turn morphisms in Sm_k into finite MW-correspondences.

Example 7.1.6. Analogously to the functor $\gamma : \mathrm{Sm}_k \rightarrow \mathrm{Cor}_k$ from Example 6.1.4, we can assign to each morphism $f : X \rightarrow Y$ in Sm_k its *oriented graph* $\tilde{\gamma}_f$, which is constructed as follows.

Let Y be of dimension d_Y and $f : X \rightarrow Y$ a morphism in Sm_k . Further, let $\Gamma_f : X \rightarrow X \times Y$ be its graph. It follows that $\Gamma_f(X)$ is of codimension d_Y in $X \times Y$ and we obtain a finite pushforward $(\Gamma_f)_* : \mathbf{K}_0^{\mathrm{MW}}(X) \rightarrow \widetilde{\mathrm{CH}}_{\Gamma_f}^{d_Y}(X \times Y, \omega_Y)$. Eventually, we define the oriented graph of f as $\tilde{\gamma}_f := (\Gamma_f)_*(\langle 1 \rangle)$. For more details, see [CF22, Section 4.3].

In particular, we set $\tilde{\gamma}_{\mathrm{id}} := 1_X$ for $X = Y$. It is the identity of the composition of MW-correspondences defined in [CF22, Section 4.2].

This way, we obtain a functor

$$\tilde{\gamma} : \mathrm{Sm}_k \rightarrow \widetilde{\mathrm{Cor}}_k$$

being the identity on objects and sending a morphism f to its oriented graph $\tilde{\gamma}_f$.

In the beginning, we mentioned that finite Milnor–Witt correspondences can be considered finite correspondences in the sense of Voevodsky. To further comment on this, we introduce the notion of the *support* of a finite MW-correspondence.

Definition 7.1.7. For smooth X , equidimensional smooth Y , and $\alpha \in \widetilde{\mathrm{Cor}}_k(X, Y)$, we define the *support* of α as

$$\mathrm{supp}(\alpha) := \overline{\{x \in (X \times Y)^{(\dim(Y))} \mid \text{the component of } \alpha \text{ in } \overline{K_0^{\mathrm{MW}}(k(x), \Lambda(x) \otimes (\omega_Y)_x) \text{ is nonzero}\}}},$$

where $\Lambda(x)$ denotes the one-dimensional $k(x)$ -vector space $\wedge^n(\mathfrak{m}/\mathfrak{m}^2)^*$. Here, \wedge denotes the exterior power, $n = \dim(Y)$, $(-)^*$ denotes the dual, and \mathfrak{m}_x is the maximal ideal corresponding to x in $\mathcal{O}_{X,x}$.

Remark 7.1.8. As promised, we want to shed some light on the connection between finite Milnor–Witt correspondences and finite correspondences in the sense of Voevodsky mentioned above. For this, let $R = \mathbb{Z}$.

Moreover, let X and Y be smooth schemes and $\dim(Y) = d$. We describe finite Milnor–Witt correspondences alternatively (compare [CF22, Chapter 4]) as *admissible, unramified* elements in $G := \bigoplus_{x \in (X \times Y)^{(d)}} \mathrm{GW}(k(x), \omega_{k(x)/k})$.

Here, an element α of G is *unramified* if it is in the kernel of the residue map $G \rightarrow \bigoplus_{x \in (X \times Y)^{(d+1)}} W(k(x), \omega_{k(x)/k})$ (compare Section 4.2) and *admissible* if $\text{supp}(\alpha)$ is the union of the closure of admissible points. A point $x \in (X \times Y)^{(d)}$ is *admissible* if its closure is finite surjective over X .

Now, the support defined above can be interpreted as the union of the closure of points where the symmetric bilinear forms are non-trivial. The support of an admissible, unramified element in G is a finite correspondence in the sense of Voevodsky by [CF22, Lemma 4.1.8] using Remark 7.1.11 below. On the other hand, one can equip the support of a finite correspondence with a symmetric bilinear form that is moreover unramified and admissible and obtains a finite Milnor–Witt correspondence.

As usual, we are interested in the categorical properties of the construction. First, we see that $\widetilde{\text{Cor}}_{k,R}$ is an additive category with disjoint union as direct sum.

Moreover, we can define a tensor product for finite MW-correspondences that turns the category $\widetilde{\text{Cor}}_{k,R}$ into a symmetric monoidal (or tensor) category following [CF22, Section 4.4], where more details can be found.

Definition 7.1.9. We define the tensor product $X \otimes X'$ in $\widetilde{\text{Cor}}_{k,R}$ of smooth schemes X and X' as the underlying cartesian product $X \times X'$.

Let $\alpha \in \widetilde{\text{CH}}_{T_1}^{d_1}(X \times Y, \omega_Y) \otimes R$ and $\beta \in \widetilde{\text{CH}}_{T_2}^{d_2}(X' \times Y', \omega_{Y'}) \otimes R$ for some admissible subsets $T_1 \subset X \times Y$ and $T_2 \subset X' \times Y'$. The exterior product of Chow–Witt groups as defined in [Fas07, Chapter 4] yields a cycle

$$\alpha \times \beta \in \widetilde{\text{CH}}_{T_1 \times T_2}^{d_1+d_2}(X \times Y \times X' \times Y', p_{Y'}^* \omega_{Y'/k} \otimes p_Y^* \omega_{Y/k}) \otimes R.$$

We define the *tensor product* $\alpha \otimes \beta$ as

$$\alpha \otimes \beta := \sigma_*(\alpha \times \beta),$$

where

$$\sigma : X \times Y \times X' \times Y' \xrightarrow{\cong} X \times X' \times Y \times Y'$$

is the transpose isomorphism. It is not only a cycle but also a finite Milnor–Witt correspondence between $X \times X'$ and $Y \times Y'$, see [CF22, Section 4.4] for details. Hence, we can define for $\alpha \in \widetilde{\text{Cor}}_{k,R}^0(X, Y)$ and $\beta \in \widetilde{\text{Cor}}_{k,R}^0(X', Y')$ their tensor product as $\alpha \otimes \beta := \sigma_*(\alpha \times \beta)$.

Lemma 7.1.10. ([CF22, Lemma 4.4.2]) *The category $\widetilde{\text{Cor}}_{k,R}$ is symmetric monoidal by \otimes defined above.*

Remark 7.1.11. By replacing Chow–Witt groups with Chow groups in the construction of $\widetilde{\text{Cor}}_{k,R}$, one recovers classical finite correspondences $\text{Cor}_{k,R}$, see [CF22, Remark 4.3.3].

In particular, the forgetful homomorphisms

$$\widetilde{\text{CH}}_T^d(X \times Y, \omega_Y) \rightarrow \text{CH}_T^d(X \times Y)$$

yield an additive functor

$$\pi : \widetilde{\mathrm{Cor}}_{k,R} \rightarrow \mathrm{Cor}_{k,R}$$

such that $\gamma : \mathrm{Sm}_k \rightarrow \mathrm{Cor}_k$ is the composition $\pi \circ \tilde{\gamma} : \mathrm{Sm}_k \rightarrow \widetilde{\mathrm{Cor}}_{k,R} \rightarrow \mathrm{Cor}_{k,R}$.

On the other hand, the hyperbolic map $H_{X,Y} : \mathrm{CH}_T^d(X \times Y) \rightarrow \widetilde{\mathrm{CH}}_T^d(X \times Y, \omega_Y)$ does not yield a functor $\mathrm{Cor}_{k,R} \rightarrow \widetilde{\mathrm{Cor}}_{k,R}$ since identity and composition are not preserved by $H_{X,Y}$, compare [CF22, p.22].

In the next part, we proceed with the construction of the derived category of Milnor–Witt motives indicated at the beginning of this chapter.

7.1.2 Construction of the Derived Category of Milnor–Witt Motives

As in the construction of $\mathrm{DM}(k; R)$, the next step is the passage to presheaves and sheaves with transfers, in this case with MW-*transfers*. As in the classical case, sheaves with MW-transfers satisfy the sheaf property on the underlying category of smooth k -schemes.

Definition 7.1.12. A *presheaf with MW-transfers* is a contravariant, additive R -linear functor $\widetilde{\mathrm{Cor}}_k^{\mathrm{op}} \rightarrow R\text{-Mod}$. The corresponding category of presheaves with MW-transfers will be denoted by $\mathrm{PSh}_R(\widetilde{\mathrm{Cor}}_{k,R})$.

Let t be the Zariski, Nisnevich, or étale topology, denoted by Zar, Nis, or et, respectively. A *t -sheaf with MW-transfers* is a presheaf with MW-transfers whose restriction to Sm_k is a t -sheaf. We denote the corresponding category of t -sheaves with MW-transfers by $\mathrm{Sh}_t(\widetilde{\mathrm{Cor}}_{k,R})$.

Again, we will focus on one specific type of presheaves, namely the representable ones.

Definition 7.1.13. Let X be a smooth k -scheme. We denote the representable presheaf with MW-transfers $\widetilde{\mathrm{Cor}}_{k,R}(-, X)$ by $\tilde{c}_R(X)$. If $R = \mathbb{Z}$, we omit it from the notation.

Example 7.1.14. By [CF22, Lemma 5.0.3], the functor $X \mapsto \mathbf{K}_j^{\mathrm{MW}}(X) \otimes R$ is a presheaf with MW-transfers. We have that $\tilde{c}_R(\mathrm{Spec}(k)) = \mathbf{K}_0^{\mathrm{MW}} \otimes R$ for a field k .

Remark 7.1.15. The presheaf $\tilde{c}_R(X)$ is a Zar-sheaf with MW-transfers, but in general *not* a Nis-sheaf with transfers, see [CF22, Example 5.2.5], contrarily to the setting of (pre-)sheaves with transfers in the sense of Voevodsky as mentioned in Remark 6.1.8.

We have a forgetful functor $\tilde{\mathcal{O}} : \mathrm{Sh}_t(\widetilde{\mathrm{Cor}}_{k,R}) \rightarrow \mathrm{PSh}_R(\widetilde{\mathrm{Cor}}_{k,R})$ which has a left adjoint \tilde{a} by [DF22, Proposition 1.2.11 (1)], where \tilde{a} is the sheafification functor. We define the following.

Definition 7.1.16. Let t be Zar, Nis, or et. For a smooth scheme X , we set $\tilde{R}_t(X) := \tilde{a}(\tilde{c}_R(X))$. When $t = \mathrm{Nis}$, we sometimes omit it from the notation.

Remark 7.1.17. By definition, $\tilde{R}_t(X)$ is a t -sheaf with MW-transfers. It is called the t -sheaf associated to $\tilde{c}_R(X)$ and all t -sheaves of this form generate the abelian category $\mathrm{Sh}_t(\widetilde{\mathrm{Cor}}_{k,R})$. Moreover, we obtain a tensor product $\tilde{\otimes}$ on $\mathrm{Sh}_t(\widetilde{\mathrm{Cor}}_{k,R})$ uniquely characterized by the property $\tilde{R}_t(X) \tilde{\otimes} \tilde{R}_t(Y) = \tilde{R}_t(X \times Y)$ and commuting with colimits. For more details, see [DF22, 1.2.14].

Definition 7.1.18. Let $\mathcal{T}_{\mathbb{A}^1}$ denote the localizing triangulated subcategory of $\mathrm{D}(\mathrm{Sh}_t(\widetilde{\mathrm{Cor}}_{k,R}))$ generated by complexes

$$\dots \rightarrow 0 \rightarrow \tilde{R}_t(\mathbb{A}^1 \times X) \rightarrow \tilde{R}_t(X) \rightarrow 0 \rightarrow \dots$$

We define the *category of effective MW-motives* $\widetilde{\mathrm{DM}}_t^{\mathrm{eff}}(k; R)$ as the localization of $\mathrm{D}(\mathrm{Sh}_t(\widetilde{\mathrm{Cor}}_{k,R}))$ with respect to $\mathcal{T}_{\mathbb{A}^1}$. The category of *geometric effective MW-motives* $\widetilde{\mathrm{DM}}_t^{\mathrm{eff}, \mathrm{gm}}(k; R)$ is defined as the Verdier localization of $(\mathrm{K}_b(\widetilde{\mathrm{Cor}}_{k,R}))^{\natural}$ with respect to the thick triangulated subcategory containing complexes of the form (1) and (2) from [DF22, Definition 3.2.22]. Note that $(\mathrm{K}_b(\widetilde{\mathrm{Cor}}_{k,R}))^{\natural} \cong \mathrm{K}_b((\widetilde{\mathrm{Cor}}_{k,R})^{\natural})$ by [BS01, Corollary 2.12] (the same arguments hold for K instead of D).

Definition 7.1.19. Let X be a smooth scheme. Its MW-motive $\widetilde{M}(X)$ is defined as the class of $\tilde{R}_t(X)$ in $\widetilde{\mathrm{DM}}_t^{\mathrm{eff}}(k; R)$.

Remark 7.1.20. By [DF22, 3.2.4], $\widetilde{\mathrm{DM}}_t^{\mathrm{eff}}(k; R)$ is a tensor triangulated category with internal Hom. For $t = \mathrm{Nis}$, the motives $\widetilde{M}(X)$ for X a smooth k -scheme form a family of compact generators as seen in [DF22, Remark 3.2.24]. The category $\widetilde{\mathrm{DM}}_t^{\mathrm{eff}, \mathrm{gm}}(k; R)$ is also tensor triangulated with internal Hom by [BC+22, Introduction, (MW1)].

As in the case of Voevodsky’s derived category of motives, the next step would be to formally invert the Tate twist. In this setting, we have the following definition.

Definition 7.1.21. The object

$$\tilde{R}(1) := \widetilde{M}(\mathbb{P}^1) / \widetilde{M}(\{\infty\})[-2]$$

in $\widetilde{\mathrm{DM}}_t^{\mathrm{eff}}(k; R)$ is called the *Tate twist*.

Again, we will not go into detail about how to invert the Tate twist since we will only concentrate on effective Milnor–Witt motives in this work. The usual procedure of inverting the Tate twist operation, also called \mathbb{P}^1 -stabilization, is usually done by passage to spectrum objects, hence the name *MW-motivic spectra*. For more details, see [DF22, Section 3.3].

Definition 7.1.22. We define the *derived category of MW-motives* $\widetilde{\mathrm{DM}}(k; R)$ (also the *category of MW-motivic spectra*) as the category obtained from the derived category of effective motives $\widetilde{\mathrm{DM}}_{\mathrm{Nis}}^{\mathrm{eff}}(k; R)$ by formally inverting the Tate twist operation $\otimes \tilde{R}(1)$.

Remark 7.1.23. We know from [DF22, Proposition 3.3.4] that $\widetilde{\mathrm{DM}}(k; R)$ is a tensor triangulated category.

Having recalled the construction of the derived category of Milnor–Witt motives, we can now begin with formulating and proving Diagram 7.1 given at the beginning of this chapter.

7.2 Sheaves with Milnor–Witt Transfers on Zero-Dimensional Schemes

In this section, we will become more familiar with the category $\widetilde{\mathrm{Cor}}_{k,R}$ when allowing only 0-dimensional smooth k -schemes.

Definition 7.2.1. We denote by $\widetilde{\mathrm{Cor}}_{k,R}^0$ the full tensor subcategory of $\widetilde{\mathrm{Cor}}_{k,R}$ spanned by 0-dimensional smooth schemes.

For smooth k -schemes of dimension 0, the definition of finite Milnor–Witt correspondences simplifies in the following way.

Example 7.2.2. Let X and Y be smooth k -schemes. We know from Example 7.1.5 that $\widetilde{\mathrm{Cor}}_{k,R}(X, \mathrm{Spec}(k)) = \widetilde{\mathrm{CH}}^0(X)$. If Y is of dimension d , then $\widetilde{\mathrm{Cor}}_{k,R}(\mathrm{Spec}(k), Y) = \bigoplus_{y \in Y^{(d)}} \widetilde{\mathrm{CH}}_{\{y\}}^d(Y, \omega_Y) \otimes R$. We can go even further and show that for 0-dimensional X and Y

$$\widetilde{\mathrm{Cor}}_{k,R}^0(X, Y) = \widetilde{\mathrm{CH}}^0(X \times Y) \otimes R$$

and

$$\widetilde{\mathrm{Cor}}_{k,R}^0(X, Y) = \widetilde{\mathrm{CH}}^0(X \times Y) \otimes R = \widetilde{Z}^0(X \times Y) \otimes R.$$

It suffices to show that $X \times Y$ is an element in $\mathcal{A}(X, Y)$. In this case, $X \times Y$ is a final object in $\mathcal{A}(X, Y)$. We then have

$$\widetilde{\mathrm{Cor}}_{k,R}^0(X, Y) = \widetilde{\mathrm{CH}}_{X \times Y}^0(X \times Y) \otimes R = \widetilde{\mathrm{CH}}^0(X \times Y) \otimes R.$$

The canonical projection $X \times Y \rightarrow X$ is surjective and maps an irreducible component of $X \times Y$ onto an irreducible component of X , compare [CF22, Section 4.1]. For the finiteness of the projection, it suffices to show that $Y \rightarrow \mathrm{Spec}(k)$ is finite since base change of a finite morphism is finite. We assume $Y = \mathrm{Spec}(K)$ for a finite separable field extension K of k , because a morphism $\coprod_{i=1}^n \mathrm{Spec}(K_i) \rightarrow \mathrm{Spec}(k)$ is finite if each $\mathrm{Spec}(K_i) \rightarrow \mathrm{Spec}(k)$ is. But $\mathrm{Spec}(K) \rightarrow \mathrm{Spec}(k)$ is finite since K is a finite separable field extension of k , hence K is finitely generated as a k -vector space.

Remark 7.2.3. We did not include any twists in the above notion for the following reason. Let X and Y in $\widetilde{\mathrm{Cor}}_{k,R}^0$ be connected, i.e. $X = \mathrm{Spec}(K)$ and $Y = \mathrm{Spec}(L)$

for some finite separable field extensions $K, L \supset k$. Let $p_X : X \times Y \rightarrow X$ and $p_Y : X \times Y \rightarrow Y$ be the canonical projections.

In the original definition of $\widetilde{\text{Cor}}_{k,R}(X, Y)$, the Chow–Witt group of $X \times Y$ is twisted by $p_Y^* \omega_{Y/k}$. Since L is a separable field, $\omega_{Y/k}$ is canonically isomorphic to L (see [CF22, Lemma 2.2.1]). It follows that $p_Y^* \omega_{Y/k}$ is canonically isomorphic to $\omega_{X \times Y}$. By [Har77, Exercise II.8.3], $\omega_{X \times Y} \cong p_X^* \omega_X \otimes p_Y^* \omega_Y$ and since $X = \text{Spec}(K)$ and $Y = \text{Spec}(L)$, this is canonically equivalent to $\mathcal{O}_{X \times Y} \otimes \mathcal{O}_{X \times Y} \cong \mathcal{O}_{X \times Y}$ again by [CF22, Lemma 2.2.1]. For a smooth non-connected scheme X , ω_X is defined connected component by connected component, so, this observation linearly extends to non-connected X and Y .

Remark 7.2.4. The functor $\pi : \widetilde{\text{Cor}}_{k,R} \rightarrow \text{Cor}_{k,R}$, induced by the forgetful homomorphism from Chow–Witt to Chow groups, now restricts to a functor $\pi : \widetilde{\text{Cor}}_{k,R}^0 \rightarrow \text{Cor}_{k,R}^0$. It is the identity on objects and sends a morphism in $\widetilde{\text{CH}}^0(X \times Y) \otimes R$ to the respective element in $\text{CH}^0(X \times Y) \otimes R$.

Recall from [CF22, Section 4.3, p.21] that the hyperbolic homomorphism $H_{X,Y} : \text{Cor}_{k,R}(X, Y) \rightarrow \widetilde{\text{Cor}}_{k,R}(X, Y)$ does not yield a functor $\text{Cor}_{k,R} \rightarrow \widetilde{\text{Cor}}_{k,R}$ since identity morphisms and composition are not preserved:

The composition $\pi_{X,Y} \circ H_{X,Y}$ is the multiplication by 2 (see [CF22, Section 3, p.11]), where $\pi_{X,Y} : \widetilde{\text{Cor}}_{k,R}(X, Y) \rightarrow \text{Cor}_{k,R}(X, Y)$ is the forgetful homomorphism. This is also true when considering only 0-dimensional schemes.

The composition $H_{X,Y} \circ \pi_{X,Y}$ is not the identity either. For example, let $X = Y = \text{Spec}(k)$ for an algebraically closed field k and $R = \mathbb{Z}$. By [Lam05, Proposition II.3.1] and [Wei13, Lemma II.2.1], the dimension yields isomorphisms $\text{GW}(k) \cong \mathbb{Z}$ and $\text{K}_0(k) \cong \mathbb{Z}$. It follows that

$$\pi_{X,X} : \widetilde{\text{Cor}}_k(X, X) \cong \text{GW}(k) \cong \mathbb{Z} \rightarrow \mathbb{Z} \cong \text{K}_0(k) \cong \text{Cor}_k(X, X)$$

is the identity, whereas $H_{X,X}$ is multiplication by 2.

Remark 7.2.5. We recall from Remark 7.1.15 that $\tilde{c}_R(X) := \widetilde{\text{Cor}}_{k,R}(-, X)$ is a Zariski sheaf with Milnor–Witt transfers, but in general not a Nisnevich sheaf.

However, the Nisnevich and Zariski topology coincide on Sm_k^0 as mentioned in [BG23a, Remark 6.16]. Recall that t -sheaves with Milnor–Witt transfers and those with classical transfers are both defined to fulfill the sheaf property only on the underlying category Sm_k , so the same considerations apply to our case. We thus have

$$\text{PSh}_R(\widetilde{\text{Cor}}_{k,R}^0) = \text{Sh}_{\text{Zar}}(\widetilde{\text{Cor}}_{k,R}^0) = \text{Sh}_{\text{Nis}}(\widetilde{\text{Cor}}_{k,R}^0),$$

and, in particular, $\tilde{c}_R(X)$ is a sheaf in the Nisnevich topology for X in Sm_k^0 .

Even more, any presheaf on $\widetilde{\text{Cor}}_{k,R}^0$ is already a Zar-/Nis-sheaf with Milnor–Witt transfers: We can apply [BG23a, Remark 6.16] to the Milnor–Witt case since, in both cases, the sheaf condition needs to be satisfied only on the underlying presheaf on Sm_k^0 by definition.

However, for X not 0-dimensional, $\tilde{c}_R(X)$ is in general not a Nisnevich sheaf, see [CF22, Example 5.2.5].

Remark 7.2.6. By [DF22, Section 3.1, p.59], the derived category $D(\mathrm{Sh}_t(\widetilde{\mathrm{Cor}}_{k,R}^0))$ is a tensor triangulated category with internal Hom, hence so is $D(\mathrm{Sh}_t(\widetilde{\mathrm{Cor}}_{k,R}^0))$.

In this section, we have seen that the category $\widetilde{\mathrm{Cor}}_{k,R}^0$ of finite Milnor–Witt correspondences simplifies when restricting to 0-dimensional schemes; the morphism groups between X and Y are the Chow–Witt groups of the fiber product $X \times Y$. Moreover, although the representable presheaf is in general not a Nisnevich (nor an étale) sheaf with MW-transfers, it does enjoy this property when considering $\widetilde{\mathrm{Cor}}_{k,R}^0$.

In the next section, we will introduce Artin Milnor–Witt motives and establish the horizontal equivalence of Diagram 7.1 from the beginning of this chapter, assuming that Conjecture 7.3.10 is true. For geometric Artin Milnor–Witt motives, we will show the equivalence without assuming a conjecture to hold.

7.3 Artin Milnor–Witt Motives

In this section, we want to prove the upper equivalence of Diagram 7.1, which we recall here for convenience. We begin with introducing Milnor–Witt Artin motives in the upper left corner.

$$\begin{array}{ccc} \widetilde{\mathrm{DAM}}(k; R) & \xleftarrow{\simeq} & D(\mathrm{Sh}_{\mathrm{Nis}}(\widetilde{\mathrm{Cor}}_{k,R}^0)) \\ & & \downarrow \simeq \\ & & D(\widetilde{\mathrm{Mack}}_R^{\mathrm{coh}}(k)) \end{array}$$

Definition 7.3.1. Let k be a perfect field of characteristic $\neq 2$, R a commutative ring, and $t = \mathrm{Nis}, \mathrm{Zar}, \text{ or } \mathrm{et}$. We define the category $\widetilde{\mathrm{DAM}}_t(k; R) := \mathrm{Loc}(\widetilde{\mathrm{Cor}}_{k,R}^0)$ of *Artin MW-motives* as the localizing subcategory of $\widetilde{\mathrm{DM}}_t^{\mathrm{eff}}(k; R)$ generated by the motives of 0-dimensional smooth k -schemes. Moreover, we define the category of *geometric Artin MW-motives* as its compact part $\widetilde{\mathrm{DAM}}_t^{\mathrm{gm}}(k; R) := (\widetilde{\mathrm{DAM}}_t(k; R))^c$.

Convention 7.3.2. If omitted from the notation, $t = \mathrm{Nis}$.

Proposition 7.3.3. *Let k be an infinite perfect field of characteristic $\neq 2$ and R a ring. There is a canonical equivalence of tensor triangulated categories*

$$\widetilde{\mathrm{DAM}}^{\mathrm{gm}}(k; R) := (\widetilde{\mathrm{DAM}}(k; R))^c \simeq \mathrm{thick}(\widetilde{\mathrm{Cor}}_{k,R}^0) \simeq \mathrm{K}_b((\widetilde{\mathrm{Cor}}_{k,R}^0)^\natural)$$

between the compact part of the tensor triangulated category $\widetilde{\mathrm{DAM}}(k; R)$ and the thick triangulated subcategory of $\widetilde{\mathrm{DAM}}(k; R)$ generated by motives $\widetilde{M}(X)$ of zero-dimensional smooth k -schemes X , denoted $\mathrm{thick}(\widetilde{\mathrm{Cor}}_{k,R}^0)$.

Proof. Using [DF22, Remark 3.2.24], the first equivalence follows analogously to [BG23a, Notation 7.8] from Neeman–Thomason–Localization in [BG23a, Recollection 3.8(b)], compare [Nee92, Theorem 2.1].

For the second equivalence, it suffices to show that $\widetilde{\text{Cor}}_{k,R}^0(-, Y)$ considered a complex in degree 0 denoted by $\widetilde{M}(Y)$ is already Nisnevich-local and \mathbb{A}^1 -local in $\mathbf{K}_b((\widetilde{\text{Cor}}_{k,R})^{\mathbb{A}^1})$ for a smooth 0-dimensional k -scheme Y , i.e. local with respect to objects satisfying conditions (1) and (2) of [DF22, Definition 3.2.22]. By Remark 7.2.5, $\widetilde{\text{Cor}}_{k,R}^0(-, Y)$ is already a Nisnevich-sheaf with Milnor–Witt transfers, hence, Nisnevich-local. In particular, this shows that the map $i : \mathbf{K}_b(\widetilde{\text{Cor}}_{k,R}^0) \rightarrow \mathbf{D}(\text{Sh}_{\text{Nis}}(\widetilde{\text{Cor}}_{k,R}))$ is a fully faithful embedding.

It remains to show that $\widetilde{M}(Y)$ is \mathbb{A}^1 -local, but this follows from $\widetilde{M}(Y)$ being \mathbb{A}^1 -local in $\mathbf{D}(\text{Sh}_{\text{Nis}}(\widetilde{\text{Cor}}_{k,R}))$ by fully faithfulness of i , so we prove this statement instead. The proof is analog to the proof of the second statement of [BG23a, Proposition 7.4]. Since the category $\widetilde{\text{Cor}}_{k,R}^0$ is additive, it suffices to show the statement for schemes Y of the form $\text{Spec}(L)$ for a finite separable field extension L/k . This means that the map

$$\text{Hom}_{\mathbf{D}(\text{Sh}_{\text{Nis}}(\widetilde{\text{Cor}}_{k,R}^0))}(X, T^i(\text{Spec}(L))) \rightarrow \text{Hom}_{\mathbf{D}(\text{Sh}_{\text{Nis}}(\widetilde{\text{Cor}}_{k,R}^0))}(\mathbb{A}_X^1, T^i(\text{Spec}(L)))$$

is bijective for all $i \in \mathbb{Z}$ and for all smooth k -schemes X .

By Lemma 7.3.6, objects in $\widetilde{\text{Cor}}_{k,R}^0$ are their own duals. It follows that

$$\text{Hom}_{\mathbf{D}(\text{Sh}_{\text{Nis}}(\widetilde{\text{Cor}}_{k,R}^0))}(X, T^i(\text{Spec}(L))) \cong \text{Hom}_{\mathbf{D}(\text{Sh}_{\text{Nis}}(\widetilde{\text{Cor}}_{k,R}^0))}(X', T^i(\text{Spec}(k)))$$

and

$$\text{Hom}_{\mathbf{D}(\text{Sh}_{\text{Nis}}(\widetilde{\text{Cor}}_{k,R}^0))}(\mathbb{A}_X^1, T^i(\text{Spec}(L))) \cong \text{Hom}_{\mathbf{D}(\text{Sh}_{\text{Nis}}(\widetilde{\text{Cor}}_{k,R}^0))}(\mathbb{A}_{X'}^1, T^i(\text{Spec}(k)))$$

for $X' = X \otimes \text{Spec}(L)$. Hence, it amounts to show that

$$\text{Hom}_{\mathbf{D}(\text{Sh}_{\text{Nis}}(\widetilde{\text{Cor}}_{k,R}^0))}(X', T^i(\text{Spec}(k))) \rightarrow \text{Hom}_{\mathbf{D}(\text{Sh}_{\text{Nis}}(\widetilde{\text{Cor}}_{k,R}^0))}(\mathbb{A}_{X'}^1, T^i(\text{Spec}(k)))$$

is bijective. By [DF22, Corollary 3.1.8] (which is the Milnor–Witt analog of [Voe00, Proposition 3.1.8]), this is equivalent to showing that the map

$$H_{\text{Nis}}^i(X', \text{Spec}(k)) \rightarrow H_{\text{Nis}}^i(\mathbb{A}_{X'}^1, \text{Spec}(k))$$

is bijective, i.e. that $\widetilde{M}(\text{Spec}(k))$, which is \mathbf{K}_0^{MW} by Example 7.1.5, is strictly \mathbb{A}^1 -invariant. But this follows from [DF22, Theorem 3.2.10] and the fact that \mathbf{K}_0^{MW} is \mathbb{A}^1 -invariant by [Mor12, p.86]. \square

Remark 7.3.4. One could hope to drop the assumption of k being infinite in the above proposition.

Remark 7.3.5. The (Voevodsky) motive $M(X)$ of a 0-dimensional smooth scheme X is its own *tensor dual* in $\mathrm{DM}_t^{\mathrm{eff}}(k; R)$ and $\mathrm{DM}_t(k; R)$, i.e. $M(X)$ is isomorphic to $\mathrm{hom}(M(X), \mathbb{1})$. In particular, it is rigid in the sense of Definition 2.2.1, see [BG23a, Remark 7.9]. In this remark, Balmer and Gallauer use the equivalence between the derived category of permutation modules and $\mathrm{D}(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Cor}_{k,R}^0))$ and show that objects in $\mathrm{perm}(\Gamma; R)$ are already their own tensor duals. However, one can also see directly that objects in $\mathrm{Cor}_{k,R}^0$ enjoy this property and we generalize this idea to $\widetilde{\mathrm{Cor}}_{k,R}^0$ as follows.

Lemma 7.3.6. *The objects in $\widetilde{\mathrm{Cor}}_{k,R}^0$ are their own tensor duals.*

Proof. We will use that $\widetilde{\mathrm{Cor}}_{k,R}^0(X, Y) \cong \widetilde{\mathrm{Cor}}_{k,R}^0(Y, X)$ by Example 7.2.2. The category $\widetilde{\mathrm{Cor}}_{k,R}^0$ is additively generated by objects of the form $\mathrm{Spec}(K)$ for $K \supset k$ a separable finite field extension, so, it suffices to show that $\mathrm{Spec}(K)$ is rigid with dual $\mathrm{Spec}(K)$. Recall from Definition 7.1.9 that the tensor product in $\widetilde{\mathrm{Cor}}_{k,R}^0$ is given by the underlying cartesian product of schemes. To show that objects in $\widetilde{\mathrm{Cor}}_{k,R}^0$ are rigid, by Remark 2.2.2 we need to find (co-)evaluation maps

$$\begin{aligned} \eta : \mathrm{Spec}(k) &\rightarrow \mathrm{Spec}(K) \times \mathrm{Spec}(K) \in \mathrm{Hom}_{\widetilde{\mathrm{Cor}}_{k,R}^0}(\mathrm{Spec}(k), \mathrm{Spec}(K) \times \mathrm{Spec}(K)) \\ &\cong \mathrm{GW}(K \otimes K) \otimes R \end{aligned}$$

and

$$\begin{aligned} \varepsilon : \mathrm{Spec}(K) \times \mathrm{Spec}(K) &\rightarrow \mathrm{Spec}(k) \in \mathrm{Hom}_{\widetilde{\mathrm{Cor}}_{k,R}^0}(\mathrm{Spec}(K) \times \mathrm{Spec}(K), \mathrm{Spec}(k)) \\ &\cong \mathrm{GW}(K \otimes K) \otimes R \end{aligned}$$

fulfilling the triangle identities

$$(\mathrm{id}_{\mathrm{Spec}(K)} \otimes \varepsilon)(\eta \otimes \mathrm{id}_{\mathrm{Spec}(K)}) = \mathrm{id}_{\mathrm{Spec}(K)}$$

and

$$(\varepsilon \otimes \mathrm{id}_{\mathrm{Spec}(K)})(\mathrm{id}_{\mathrm{Spec}(K)} \otimes \eta) = \mathrm{id}_{\mathrm{Spec}(K)},$$

where $\mathrm{id}_{\mathrm{Spec}(K)} \in \mathrm{Hom}_{\widetilde{\mathrm{Cor}}_{k,R}^0}(\mathrm{Spec}(K), \mathrm{Spec}(K)) \cong \mathrm{GW}(K \otimes K) \otimes R$. We choose $\eta := \varepsilon := \langle 1 \rangle \in \mathrm{GW}(K \otimes K) \otimes R$. The triangle identities now easily follow from the fact that η, ε , and $\mathrm{id}_{\mathrm{Spec}(K)}$ all represent the identity element in $\mathrm{GW}(K \otimes K) \otimes R$.

We have shown that objects in the category of finite MW-correspondences are their own tensor duals and, in particular, rigid. \square

Remark 7.3.7. We can now generalize [BG23a, Remark 7.9]. By Voevodsky’s cancellation theorem [Voe00, Theorem 4.3.1], we know that the passage from geometric effective (Artin) motives to geometric non-effective (Artin) motives embodies turning each object rigid; for Milnor–Witt motives, this is shown by [DF22, Theorem 3.3.9]. But, by the above considerations, the Milnor–Witt motive $\widetilde{M}(X)$ of

a smooth 0-dimensional scheme X is its own tensor dual. In particular, it is rigid, so we do not distinguish effective and non-effective geometric Artin Milnor–Witt motives. Since the motives of the form $\widetilde{M}(X)$ with $\dim(X) = 0$ form a set of compact generators of $\widetilde{\text{DAM}}(k, R)$, it follows that we neither distinguish effective and non-effective Artin Milnor–Witt motives in general, just as in loc. cit.

We obtain the following structural properties for the category of (geometric) Artin motives.

Lemma 7.3.8. *The category $\widetilde{\text{DAM}}(k; R)$ is a tensor triangulated category with internal Hom and the category $\widetilde{\text{DAM}}^{\text{gm}}(k; R)$ is an essentially small, idempotent complete, rigid tensor triangulated category.*

Proof. The category $\widetilde{\text{DM}}^{\text{eff}}(k; R)$ is tensor triangulated and possesses an internal Hom adjoint to the tensor product by [DF22, Remark 3.2.4]. As a localizing subcategory, $\widetilde{\text{DAM}}(k; R)$ is again tensor triangulated with internal Hom. It is closed under \otimes since the tensor product is induced by the tensor product in $\text{Sh}_t(\widetilde{\text{Cor}}_{k,R}^0)$. It commutes with colimits and satisfies $\widetilde{R}_t(X) \otimes \widetilde{R}_t(Y) = \widetilde{R}_t(X \times Y)$ ([DF22, (1.2.14.a)]), where $X \times Y$ is again 0-dimensional, compare Remark 8.1.1. It follows from the \otimes -hom adjunction and Lemma 7.3.6 that $\widetilde{\text{DAM}}(k; R)$ is also closed under the internal Hom.

As the compact part of a compactly generated tensor triangulated category, the statement for $\widetilde{\text{DAM}}^{\text{gm}}(k; R)$ follows from [HPS97, Theorem 2.1.3] (compare [Bal10b, Definition 44]) under consideration of Proposition 7.3.3. \square

To show the equivalence between $\widetilde{\text{DAM}}^{\text{eff}}(k; R)$ and $\text{DSh}_{\text{Nis}}(\widetilde{\text{Cor}}_{k,R}^0)$, we construct the left adjoint functor $\tilde{!} : \text{D}(\text{Sh}_{\text{Nis}}(\widetilde{\text{Cor}}_{k,R}^0)) \rightarrow \text{D}(\text{Sh}_{\text{Nis}}(\widetilde{\text{Cor}}_{k,R}))$ and show that it is fully faithful and remains so under composition with the quotient functor to $\widetilde{\text{DM}}^{\text{eff}}(k)$ just as in the proof of [BG23a, Lemma 6.9]. It will follow that its essential image is precisely $\widetilde{\text{DAM}}^{\text{eff}}(k; R)$.

Construction 7.3.9. We consider the inclusion functor $\tilde{\iota} : \widetilde{\text{Cor}}_{k,R}^0 \rightarrow \widetilde{\text{Cor}}_{k,R}$. It induces a functor on the level of presheaves with values in R -modules

$$\tilde{\iota}^* : \text{PSh}_R(\widetilde{\text{Cor}}_{k,R}) \rightarrow \text{PSh}_R(\widetilde{\text{Cor}}_{k,R}^0),$$

defined as $\tilde{\iota}^*(F) = F \circ \tilde{\iota}$.

The category of R -modules is complete by [MacL71, p. 111] and $\widetilde{\text{Cor}}_{k,R}^0$ is locally small. More generally, it is small since its objects are the same as the objects in Sm_k^0 , which form a set. Hence, we can apply [KS06, Theorem 2.3.3] to see that the left adjoint

$$\tilde{!} : \text{PSh}_R(\widetilde{\text{Cor}}_{k,R}^0) \rightarrow \text{PSh}_R(\widetilde{\text{Cor}}_{k,R})$$

constructed as in [SGA4, (5.11)] exists.

Since $\tilde{\iota}$ is fully faithful, $\tilde{\iota}_!$ is fully faithful as well and there are isomorphisms

$$\mathrm{id}_{\mathrm{PSh}_R(\widetilde{\mathrm{Cor}}_{k,R}^0)} \cong \tilde{\iota}^* \tilde{\iota}_!$$

by [KS06, Theorem 2.3.3]. In particular, we have isomorphisms $\widetilde{\mathrm{Cor}}_{k,R}^0(-, X) \cong \tilde{\iota}^* \tilde{\iota}_!(\widetilde{\mathrm{Cor}}_{k,R}^0(-, X)) = \tilde{\iota}_!(\widetilde{\mathrm{Cor}}_{k,R}^0(-, X)) \circ \tilde{\iota}$ for each X in Sm_k^0 . Since $\tilde{\iota}$ is the inclusion, it follows that

$$\tilde{\iota}_!(\widetilde{\mathrm{Cor}}_{k,R}^0(-, X)) = \widetilde{\mathrm{Cor}}_{k,R}^0(-, X) = \widetilde{\mathrm{Cor}}_{k,R}(-, X)$$

for all X in Sm_k^0 .

Now, since $\tilde{\iota}$ is merely the inclusion, the functor $\tilde{\iota}^*$ preserves t -sheaves with MW-transfers and hence induces a functor

$$\mathrm{Sh}_t(\widetilde{\mathrm{Cor}}_{k,R}^0) \leftarrow \mathrm{Sh}_t(\widetilde{\mathrm{Cor}}_{k,R}) : \tilde{\iota}^*$$

which admits a left adjoint

$$\tilde{\iota}_! : \mathrm{Sh}_t(\widetilde{\mathrm{Cor}}_{k,R}^0) \rightarrow \mathrm{Sh}_t(\widetilde{\mathrm{Cor}}_{k,R})$$

defined as the composition $\mathrm{PSh}_R(\widetilde{\mathrm{Cor}}_{k,R}^0) \xrightarrow{\tilde{\iota}_!} \mathrm{PSh}_R(\widetilde{\mathrm{Cor}}_{k,R}) \xrightarrow{\tilde{a}_t} \mathrm{Sh}_t(\widetilde{\mathrm{Cor}}_{k,R})$ restricted to $\mathrm{Sh}_t(\widetilde{\mathrm{Cor}}_{k,R}^0)$, where \tilde{a}_t denotes the sheafification functor with respect to the topology t . Hence, we have constructed $\tilde{\iota}_! : \mathrm{Sh}_t(\widetilde{\mathrm{Cor}}_{k,R}^0) \rightarrow \mathrm{Sh}_t(\widetilde{\mathrm{Cor}}_{k,R})$ as a left adjoint functor.

It moreover induces a functor

$$\tilde{\iota}_! : \mathrm{D}(\mathrm{Sh}_t(\widetilde{\mathrm{Cor}}_{k,R}^0)) \rightarrow \mathrm{D}(\mathrm{Sh}_t(\widetilde{\mathrm{Cor}}_{k,R}))$$

on the level of derived categories which we will use later.

One might conjecture that [BG23a, Lemma 6.9] allows for a refinement to Chow–Witt correspondences. For $t = \mathrm{et}$, one can use that $\mathrm{Sh}_{\mathrm{et}}(\widetilde{\mathrm{Cor}}_{k,R}) \simeq \mathrm{Sh}_{\mathrm{et}}(\mathrm{Cor}_{k,R})$ by [DF22, Corollary 1.2.15 (3)].

Conjecture 7.3.10. *The inclusion $\tilde{\iota} : \widetilde{\mathrm{Cor}}_{k,R}^0 \rightarrow \widetilde{\mathrm{Cor}}_{k,R}$ and the étale sheafification functor \tilde{a}_{et} induce a commutative square of left adjoint tensor triangulated functors with the horizontal arrows being fully faithful.*

$$\begin{array}{ccc} \mathrm{Sh}_{\mathrm{Nis}}(\widetilde{\mathrm{Cor}}_{k,R}^0) & \xrightarrow{\tilde{\iota}_!} & \mathrm{Sh}_{\mathrm{Nis}}(\widetilde{\mathrm{Cor}}_{k,R}) \\ \tilde{a}_{\mathrm{et}} \downarrow & & \downarrow \tilde{a}_{\mathrm{et}} \\ \mathrm{Sh}_{\mathrm{et}}(\widetilde{\mathrm{Cor}}_{k,R}^0) & \xrightarrow{\tilde{\iota}_!} & \mathrm{Sh}_{\mathrm{et}}(\widetilde{\mathrm{Cor}}_{k,R}) \end{array}$$

The result motivates the following proposition, showing that $\tilde{\iota}_!$ remains fully faithful when passing to MW-motives, which is a refinement of [BG23a, Proposition 7.4].

For the rest of this subsection, we assume that Conjecture 7.3.10 is true.

Proposition 7.3.11. *Let k be a perfect field of characteristic $\neq 2$ and R a commutative ring. If Conjecture 7.3.10 is true, the functor*

$$\tilde{\iota}_! : D(\mathrm{Sh}_{\mathrm{Nis}}(\widetilde{\mathrm{Cor}}_{k,R}^0)) \rightarrow D(\mathrm{Sh}_{\mathrm{Nis}}(\widetilde{\mathrm{Cor}}_{k,R}))$$

induced by inclusion satisfies the following properties.

- (i) *It is tensor triangulated and fully faithful.*
- (ii) *The composition with the quotient functor*

$$\tilde{\iota}_! : D(\mathrm{Sh}_{\mathrm{Nis}}(\widetilde{\mathrm{Cor}}_{k,R}^0)) \rightarrow D(\mathrm{Sh}_{\mathrm{Nis}}(\widetilde{\mathrm{Cor}}_{k,R})) \rightarrow \widetilde{\mathrm{DM}}^{\mathrm{eff}}(k; R)$$

remains fully faithful.

Proof. Analogously to the proof of [BG23a, Proposition 7.4.(a)], we can deduce the first statement from Conjecture 7.3.10 since $\tilde{\iota}_!$ is tensor, fully faithful, triangulated, and admits a triangulated right adjoint $\tilde{\iota}^*$, so, the unit of the adjunction remains an isomorphism at the derived level.

To see that the composition with the quotient functor remains fully faithful, it suffices to show that the image of a 0-dimensional smooth scheme in $D(\mathrm{Sh}_{\mathrm{Nis}}(\widetilde{\mathrm{Cor}}_{k,R}))$ is \mathbb{A}^1 -local, but we have already shown this in the proof of Proposition 7.3.3. \square

Now, the following corollary refines the last equivalence of [BG23a, Corollary 7.10].

Corollary 7.3.12. *Let k be a perfect field of characteristic $\neq 2$ and R a commutative ring. If Conjecture 7.3.10 is true, there is an equivalence of tensor triangulated categories*

$$\tilde{\iota}_! : D(\mathrm{Sh}_{\mathrm{Nis}}(\widetilde{\mathrm{Cor}}_{k,R}^0)) \xrightarrow{\simeq} \widetilde{\mathrm{DAM}}(k; R).$$

Proof. By Construction 7.3.9, the fully faithful functor $\tilde{\iota}_! : D(\mathrm{Sh}_{\mathrm{Nis}}(\widetilde{\mathrm{Cor}}_{k,R}^0)) \rightarrow \widetilde{\mathrm{DM}}_{\mathrm{Nis}}^{\mathrm{eff}}(k; R)$ has image $\widetilde{\mathrm{DAM}}(k; R)$. \square

Chapter 8

Cohomological Milnor–Witt Mackey Functors

It might help to have a copy of [BG23a] at hand while reading this chapter.

After having proven the upper equivalence of Diagram 7.1 in the last chapter, we want to construct the vertical equivalence of

$$\begin{array}{ccc} \widetilde{\text{DAM}}(k; R) & \xleftarrow{\simeq} & \text{D}(\text{Sh}_{\text{Nis}}(\widetilde{\text{Cor}}_{k,R}^0)) \\ & & \downarrow \simeq \\ & & \text{D}(\widetilde{\text{Mack}}_R^{\text{coh}}(k)) \end{array}$$

refining the equivalence in [BG23a, Corollary 6.17].

We stay in the world of algebraic geometry; a Milnor–Witt analog of permutation modules will merely be commented in Remark 8.2.7. In Section 8.1, we introduce a category $\widetilde{\Omega}_R(k)$ closely related to classical span categories and investigate its relation to $\widetilde{\text{Cor}}_{k,R}^0$. The idea of Section 8.2 then is to construct a category of cohomological Milnor–Witt Mackey functors $\widetilde{\text{Mack}}_R^{\text{coh}}(k)$ and show that $\widetilde{\text{Mack}}_R^{\text{coh}}(k) \simeq \text{Sh}_{\text{Nis}}(\widetilde{\text{Cor}}_{k,R}^0)$.

Basics on symmetric bilinear forms and Grothendieck–Witt groups of fields and rings used here are taken from [Scha85, Lam05].

Convention 8.0.1. Let R be a commutative ring. In this chapter, let $\Gamma := \text{Gal}(\bar{k}/k)$ be the absolute Galois group of a perfect field k of characteristic $\neq 2$ for a fixed algebraic closure \bar{k} . For a subgroup H of Γ and an element $g \in \Gamma$, we set ${}^g H := gHg^{-1}$ and $H^g := g^{-1}Hg$. We will only consider open subgroups of Γ .

8.1 Span Categories

To begin with, we need an analog $\widetilde{\Omega}_R(k)$ of the additive category $\Omega_R(k)$ defined in [ThWe95, Section 2] and recalled in [BG23a, Remark 6.5]. The construction

is motivated by the idea of obtaining a functor $\widetilde{\Omega}_R(k) \rightarrow \widetilde{\text{Cor}}_{k,R}^0$ that sends a morphism to its oriented pushforward, analogously to the non-oriented case. After defining the category $\widetilde{\Omega}_R(k)$ we will go into more detail about how $\widetilde{\Omega}_R(k)$ and $\widetilde{\text{Cor}}_{k,R}^0$ relate.

We start with an observation on tensor products of 0-dimensional smooth k -schemes.

Remark 8.1.1. Recall that the fiber product of 0-dimensional smooth k -schemes is isomorphic to an object of Sm_k^0 . In the case $Z = \text{Spec}(K_Z)$, $Y = \text{Spec}(K_Y)$ and $V = \text{Spec}(K_V)$ we have that

$$Z \times_Y V = \text{Spec}(K_Z) \times_{\text{Spec}(K_Y)} \text{Spec}(K_V) = \text{Spec}(K_Z \otimes_{K_Y} K_V).$$

It suffices to restrict to this case since Sm_k^0 is additively generated by such schemes and pulling back is an additive functor. Now, the tensor product of fields is not always a field, but, in our case, it is isomorphic to a finite product of finite separable field extensions Ω_i of K_Y

$$K_Z \otimes_{K_Y} K_V \cong \prod_{i=1}^n \Omega_i,$$

see [Milne20, Theorem 1.18]. Hence,

$$\text{Spec}(K_Z \otimes_{K_Y} K_V) \cong \text{Spec}\left(\prod_{i=1}^n \Omega_i\right) = \prod_{i=1}^n \text{Spec}(\Omega_i)$$

is isomorphic to a finite disjoint union of spectra of finite separable field extensions over K_Y and, thus, over k , i.e. in particular, isomorphic to an object of Sm_k^0 .

Concretely, if $K_Z = K_Y(\alpha)$ for some $\alpha \in K_Z$, the minimal polynomial f of α has a decomposition in $K_V[X]$ into monic irreducible polynomials $f = f_1 \cdot \dots \cdot f_n$. Then, we have $\Omega_i = K_V[X]/(f_i)$ for all i by the Chinese Remainder Theorem since K_Z is separable over K_Y .

In the special case $K := K_Z = K_V$ and $L := K_Y$, let $L(\alpha) = K$ for some $\alpha \in K$ and let f be the minimal polynomial of α , hence, $K = L[X]/(f)$. Then, we have

$$K \otimes_L K \cong K \otimes_L L[X]/(f) \cong K[X]/(f).$$

In particular, since $\alpha \in K$, the linear polynomial $X - \alpha$ is a factor of f in $K[X]$, so,

$$K[X]/(X - \alpha) \cong K$$

is one of the Ω_i from above. It follows that the tensor product $K \otimes_L K$ contains at least one copy of K .

Let us now come to the definition of $\widetilde{\Omega}(k)$, which is a refinement of [ThWe95, Section 2].

Construction 8.1.2. Let k be a perfect field of characteristic $\neq 2$ and R a commutative ring. At first, we define the category $\widetilde{\text{span}}(k)$ as follows. The **objects** are the same as in Sm_k^0 , i.e., finite disjoint unions of spectra of finite separable field extensions of k .

Morphisms $f : X \rightarrow Y$ are given by isomorphism classes of so-called (*oriented*) *spans* consisting of pairs of morphisms

$$X \leftarrow Z := \coprod_{i=1}^n \text{Spec}(L_i) \rightarrow Y$$

in Sm_k^0 , where L_i is a finite separable field extension of k for each i , together with an equivalence class of symmetric bilinear forms

$$\langle \alpha_i \rangle \in \widetilde{Z}^0(\text{Spec}(L_i)) = \text{GW}(L_i)$$

for each i , where $\alpha_i \in L_i^\times$, i.e. the rank of $\langle \alpha_i \rangle$ is 1. We denote such a morphism by $X \leftarrow \coprod_{i=1}^n (\text{Spec}(L_i), \langle \alpha_i \rangle) \rightarrow Y$ or $\coprod_{i=1}^n (\text{Spec}(L_i), \langle \alpha_i \rangle) \rightarrow X \times Y$.

Two spans $X \leftarrow \coprod_{i=1}^n (\text{Spec}(L_i), \langle \alpha_i \rangle) \rightarrow Y$ and $X \leftarrow \coprod_{j=1}^m (\text{Spec}(L'_j), \langle \alpha'_j \rangle) \rightarrow Y$ are **isomorphic** if there is an isomorphism $f : Z = \coprod_{i=1}^n \text{Spec}(L_i) \rightarrow Z' = \coprod_{j=1}^m \text{Spec}(L'_j)$, so, in particular, $n = m$, making the diagram

$$\begin{array}{ccc} & Z & \\ & \swarrow & \searrow \\ X & & Y \\ & \nwarrow & \nearrow \\ & Z' & \end{array} \quad \begin{array}{c} \\ \cong \\ f \\ \\ \end{array}$$

commute and satisfying the following property. For all index pairs (i, j) with $f(\text{Spec}(L_i)) = \text{Spec}(L'_j)$, we have that $f'(\langle \alpha_i \rangle) = \langle \alpha'_j \rangle \in \widetilde{Z}^0(\text{Spec}(L'_j))$, where $f' : \widetilde{Z}^0(Z) \xrightarrow{\cong} \widetilde{Z}^0(Z')$ is the isomorphism on oriented 0-cocycles induced by f .

The **composition** in $\widetilde{\text{span}}(k)$ is defined as follows. Let $Z = \text{Spec}(L)$ and $V = \text{Spec}(K)$ for K, L finite separable field extensions of k . Consider morphisms $X \xleftarrow{r} (Z, \langle \alpha \rangle) \rightarrow Y$ and $Y \leftarrow (V, \langle \beta \rangle) \xrightarrow{t} W$ and let Y be connected w.l.o.g.

Consider the pullback square

$$\begin{array}{ccc} Z \times_Y V & \xrightarrow{pv} & V & \xrightarrow{t} & W \\ pz \downarrow & & \downarrow & & \\ Z & \longrightarrow & Y & & \\ r \downarrow & & & & \\ X & & & & \end{array} .$$

Since $\widetilde{Z}^0(Z) = \widetilde{\text{CH}}^0(Z)$ for all Z in Sm_k^0 , we can apply the pullback construction of Chow–Witt groups from [Fas08, Corollaire 10.4.3] in order to obtain elements

$$(pz)^*(\langle \alpha \rangle) \in \widetilde{Z}^0(Z \times_Y V)$$

and

$$(p_V)^*(\langle \beta \rangle) \in \widetilde{Z}^0(Z \times_Y V).$$

We fix an isomorphism $\varphi : Z \times_Y V \xrightarrow{\cong} \coprod_{i=1}^n \text{Spec}(\Omega_i)$ for finite separable field extensions Ω_i/k for all i , which exists by Remark 8.1.1. It induces an isomorphism φ' on oriented 0-cocycles. Then, $\varphi'((p_Z)^*(\langle \alpha \rangle))$ and $\varphi'((p_V)^*(\langle \beta \rangle))$ have the form

$$\langle \alpha'_1 \rangle \oplus \dots \oplus \langle \alpha'_n \rangle \in \bigoplus_{i=1}^n \widetilde{Z}^0(\text{Spec}(\Omega_i)) \cong \bigoplus_{i=1}^n \text{GW}(\Omega_i)$$

and

$$\langle \beta'_1 \rangle \oplus \dots \oplus \langle \beta'_n \rangle \in \bigoplus_{i=1}^n \widetilde{Z}^0(\text{Spec}(\Omega_i)) \cong \bigoplus_{i=1}^n \text{GW}(\Omega_i),$$

respectively, where $\alpha'_i, \beta'_i \in \Omega_i^\times$ for all i . Let

$$\langle \alpha'_i \rangle \cdot \langle \beta'_i \rangle \in \widetilde{Z}^0(\text{Spec}(\Omega_i)) = \widetilde{\text{CH}}^0(\text{Spec}(\Omega_i))$$

denote the intersection product of Chow–Witt groups (see [Fas07, Definition 6.1]), which coincides with the multiplication in $\text{GW}(\Omega_i)$ by [Fas07, Theorem 7.6]. The composition is then defined as the morphism

$$X \xleftarrow{r \circ p_Z} \prod_{i=1}^n (\text{Spec}(\Omega_i), \langle \alpha'_i \rangle \cdot \langle \beta'_i \rangle) \xrightarrow{\text{top}_V} W.$$

The construction is linearly extended to general Z, V, Y in Sm_k^0 . The composition is independent of the choice of φ , as Remark 8.1.3 shows.

One checks that the **identity element** in $\text{Hom}_{\widetilde{\text{span}}(k)}(Z, Z)$ for an object $Z = \coprod_{i=1}^n \text{Spec}(L_i)$ is the morphism

$$Z \xleftarrow{\text{id}} \prod_{i=1}^n (\text{Spec}(L_i), \langle 1 \rangle) \xrightarrow{\text{id}} Z,$$

where the L_i are finite separable field extensions of k . The composition is **associative** since the multiplication in Grothendieck–Witt groups is associative and the composition in the non-oriented counterpart $\text{span}(k)$ is as well.

We have a **monoid structure** on $\text{Hom}_{\widetilde{\text{span}}(k)}(X, Y)$ given by

$$\begin{aligned} & [X \leftarrow \prod_{i=1}^n (\text{Spec}(L_i), \langle \alpha_i \rangle) \rightarrow Y] + [X \leftarrow \prod_{j=1}^m (\text{Spec}(L'_j), \langle \alpha'_j \rangle) \rightarrow Y] \\ & := [X \leftarrow (\prod_{k=1}^{n+m} (\text{Spec}(L_k), \langle \alpha_k \rangle) \rightarrow Y), \end{aligned}$$

where $L_{n+j} := L'_j$ and $\langle \alpha_{n+j} \rangle := \langle \alpha'_j \rangle$ for all $j \in \{1, \dots, m\}$. One checks that the empty set gives rise to a zero element.

The **disjoint union** equips $\widetilde{\text{span}}(k)$ with finite products and coproducts, see Remark 8.1.4.

We denote by $\widetilde{\Omega}(k)$ the additive category after group completing the Hom-sets and imposing the following equivalence relation on Hom-sets. The morphism group $\text{Hom}_{\widetilde{\Omega}(k)}(X, Y)$ is then the group of isomorphism classes of oriented spans dividing out the equivalence relation

$$\begin{aligned} & [X \leftarrow (Z, \langle a \rangle) \rightarrow Y] + [X \leftarrow (Z, \langle b \rangle) \rightarrow Y] \\ &= [X \leftarrow (Z, \langle a + b \rangle) \rightarrow Y] + [X \leftarrow (Z, \langle ab(a + b) \rangle) \rightarrow Y], \end{aligned}$$

where X, Y are in Sm_k^0 , $Z = \text{Spec}(L)$ for some finite separable field extension L/k , $a, b, a + b \in L^\times$, and we proceed componentwise for general Z . This relation is inspired by the definition of a Grothendieck–Witt group of a field via relations, compare [Lam05, Theorem II.4.3]. Note that the relation

$$[X \leftarrow (Z, \langle a \rangle) \rightarrow Y] = [X \leftarrow (Z, \langle ab^2 \rangle) \rightarrow Y] \in \text{Hom}_{\widetilde{\Omega}(k)}(X, Y)$$

is already satisfied by construction.

We may **extend scalars** to a given commutative ring R in order to obtain a category $\widetilde{\Omega}_R(k)$. While objects are the same as in $\widetilde{\Omega}(k)$, oriented spans $f : (\coprod_{i=1}^n (\text{Spec}(L_i), \langle \alpha_i \rangle)) \rightarrow X \times Y$, $\alpha_i \in L_i^\times$, now satisfy $\langle \alpha_i \rangle \in \widetilde{Z}^0(\text{Spec}(L_i)) \otimes R = \text{GW}(L_i) \otimes R$.

Remark 8.1.3. Let the notation be as in Construction 8.1.2. The composition in $\widetilde{\text{span}}_R(k)$ (and therefore in $\widetilde{\Omega}(k)$) does not depend on the choice of the isomorphism $\varphi : Z \times_Y V \xrightarrow{\cong} \coprod_{i=1}^n \text{Spec}(\Omega_i)$. Take another isomorphism $\psi : Z \times_Y V \xrightarrow{\cong} \coprod_{i=1}^n \text{Spec}(\Omega'_i)$. We obtain isomorphisms

$$\bigoplus_{i=1}^n \widetilde{Z}^0(\text{Spec}(\Omega'_i)) \otimes R \xrightarrow{(\psi^{-1})'} \widetilde{Z}^0(Z \times_Y V) \otimes R \xrightarrow{\varphi'} \bigoplus_{i=1}^n \widetilde{Z}^0(\text{Spec}(\Omega_i)) \otimes R.$$

Then, the resulting compositions for the different choices φ and ψ are in the same isomorphism class in $\text{Hom}_{\widetilde{\text{span}}_R(k)}(X, W)$ via the isomorphism $\varphi \circ \psi^{-1}$ since it fulfills

$$(\varphi' \circ (\psi^{-1})')(\psi'((p_Z)^*(\langle \alpha \rangle))) = \varphi'((p_Z)^*(\langle \alpha \rangle))$$

and

$$(\varphi' \circ (\psi^{-1})')(\psi'((p_V)^*(\langle \beta \rangle))) = \varphi'((p_V)^*(\langle \beta \rangle)).$$

Remark 8.1.4. Let us briefly verify that the disjoint union is product and coproduct in $\widetilde{\text{span}}(k)$ and its non-oriented counterpart $\text{span}(k)$ from [ThWe95, Section 2].

First, we show that the disjoint union is a product. Let A, B, T, X, Y be connected 0-dimensional k -schemes. This observation extends to the non-connected

case. Let $T \longleftarrow A \xrightarrow{f_X} X$ and $T \longleftarrow B \xrightarrow{f_Y} Y$ be spans in $\text{span}(k)$. We construct a span

$$f = [T \longleftarrow A \coprod B \xrightarrow{(f_X, f_Y)} X \coprod Y].$$

Moreover, we set

$$\pi_X = [X \coprod Y \xleftarrow{i_X} X \xrightarrow{\text{id}} X]$$

and

$$\pi_Y = [X \coprod Y \xleftarrow{i_Y} Y \xrightarrow{\text{id}} Y].$$

Since the fiber product $(A \coprod B) \times_{X \coprod Y} X$ is simply A , the composition $\pi_X \circ f$ is the span

$$\begin{aligned} T &\longleftarrow (A \coprod B) \times_{X \coprod Y} X \rightarrow X \\ &= T \longleftarrow A \xrightarrow{f_X} X \end{aligned}$$

Similarly, $\pi_Y \circ f = f_Y$. One checks that for a different choice of f , at least one of the equations $\pi_X \circ f = f_X$ and $\pi_Y \circ f = f_Y$ is not satisfied. Hence, the choice of f is unique.

Now, consider spans $T \longleftarrow (A, \langle \alpha \rangle) \xrightarrow{f_X} X$ and $T \longleftarrow (B, \langle \beta \rangle) \xrightarrow{f_Y} Y$ in $\widetilde{\text{span}}(k)$. We equip the spans π_X , π_Y , and f from above with the forms $\langle 1 \rangle \in \widetilde{Z}^0(X)$, $\langle 1 \rangle \in \widetilde{Z}^0(Y)$, and $\langle \alpha \rangle \coprod \langle \beta \rangle$, respectively, where the latter notation means $\langle \alpha \rangle \in \widetilde{Z}^0(A)$ and $\langle \beta \rangle \in \widetilde{Z}^0(B)$. From the above description of the span $\pi_X \circ f = f_X$ it follows that the composition of $T \longleftarrow (A \coprod B, \langle \alpha \rangle \coprod \langle \beta \rangle) \xrightarrow{(f_X, f_Y)} X \coprod Y$ and $X \coprod Y \xleftarrow{i_X} X \xrightarrow{\text{id}} X$ is again $T \longleftarrow (A, \langle \alpha \rangle) \xrightarrow{f_X} X$. The same follows for the component Y . The choice of f in $\text{span}(k)$ has been unique and if we equip f with a different form than $\langle \alpha \rangle \coprod \langle \beta \rangle$, the respective diagram does not commute. Hence, the choice of f together with $\langle \alpha \rangle \coprod \langle \beta \rangle$ is unique as well.

This shows that the disjoint union is a product in both $\text{span}(k)$ and $\widetilde{\text{span}}(k)$. It can be seen directly that $\text{span}(k) \simeq \text{span}(k)^{\text{op}}$ and $\widetilde{\text{span}}(k) \simeq \widetilde{\text{span}}(k)^{\text{op}}$, so product and coproduct coincide.

We now turn back to the category $\widetilde{\Omega}_R(k)$. There is an additive forgetful functor

$$V : \widetilde{\Omega}_R(k) \rightarrow \Omega_R(k)$$

that is the identity on objects and is induced by the forgetful homomorphism $V : \widetilde{Z}^0(Z) \otimes R \rightarrow Z^0(Z) \otimes R$ on morphism groups.

We also obtain a functor from $\widetilde{\Omega}_R(k)$ to $\widetilde{\text{Cor}}_{k,R}^0$ induced by the pushforward of Chow–Witt groups. For an overview of the general construction of the pushforward of Chow–Witt groups, the reader is referred to [Fas20, Section 2.3]. Recall from Remark 7.2.3 that we may fix isomorphisms trivializing the twists in an canonical way in our case.

Lemma 8.1.5. *The functor $\tilde{\varepsilon} : \tilde{\Omega}_R(k) \rightarrow \widetilde{\text{Cor}}_{k,R}^0$ is well-defined with respect to the equivalence relation on Hom-sets.*

Proof. Let $f_\alpha := [(\text{Spec}(L), \langle \alpha \rangle) \xrightarrow{f} X \times Y]$ be in $\tilde{\Omega}_R(k)$, $\alpha, \beta, \alpha + \beta \in L^\times$, and $f_\beta, f_{\alpha+\beta}, f_{\alpha\beta(\alpha+\beta)}$ the same spans but equipped with the respective bilinear forms $\langle \beta \rangle, \langle \alpha + \beta \rangle$ and $\langle \alpha\beta(\alpha + \beta) \rangle$. We have to check that $\tilde{\varepsilon}(f_\alpha + f_\beta) = \tilde{\varepsilon}(f_{\alpha+\beta} + f_{\alpha\beta(\alpha+\beta)})$.

By additivity of $\tilde{\varepsilon}$, we have

$$\tilde{\varepsilon}(f_\alpha + f_\beta) = \tilde{\varepsilon}(f_\alpha) + \tilde{\varepsilon}(f_\beta) = f_*(\langle \alpha \rangle) + f_*(\langle \beta \rangle).$$

Since f_* is a group homomorphism ([Lam05, Corollary VII.1.5]) and because the relation $\langle \alpha \rangle + \langle \beta \rangle = \langle \alpha + \beta \rangle + \langle \alpha\beta(\alpha + \beta) \rangle$ holds in $\text{GW}(L) \otimes R$, it follows that

$$\begin{aligned} \tilde{\varepsilon}(f_\alpha + f_\beta) &= f_*(\langle \alpha \rangle + \langle \beta \rangle) = f_*(\langle \alpha + \beta \rangle + \langle \alpha\beta(\alpha + \beta) \rangle) \\ &= f_*(\langle \alpha + \beta \rangle) + f_*(\langle \alpha\beta(\alpha + \beta) \rangle) \\ &= \tilde{\varepsilon}(f_{\alpha+\beta}) + \tilde{\varepsilon}(f_{\alpha\beta(\alpha+\beta)}) = \tilde{\varepsilon}(f_{\alpha+\beta} + f_{\alpha\beta(\alpha+\beta)}). \end{aligned}$$

□

Lemma 8.1.6. *Let k be a perfect field of characteristic $\neq 2$ and R a ring. The pushforward of Chow–Witt groups induces an additive functor*

$$\tilde{\varepsilon} : \tilde{\Omega}(k) \rightarrow \widetilde{\text{Cor}}_k^0$$

being the identity on objects. It sends a span $f : (Z = \text{Spec}(L), \langle \alpha \rangle) \rightarrow X \times Y$ to the oriented pushforward $\tilde{\varepsilon}(f) := f_*(\langle \alpha \rangle) \in \widetilde{\text{CH}}^0(X \times Y) = \tilde{Z}^0(X \times Y)$ and we extend this definition linearly for general Z in Sm_k^0 . Extending scalars to a ring R , we obtain an additive functor $\tilde{\varepsilon} : \tilde{\Omega}_R(k) \rightarrow \widetilde{\text{Cor}}_{k,R}^0$.

Proof. Additivity is given by construction. To see that the identity is preserved, it thus suffices to check that $\text{Spec}(L) \leftarrow (\text{Spec}(L), \langle 1 \rangle) \rightarrow \text{Spec}(L)$ with $L \supset k$ a finite separable field extension and $1 \in L$ is preserved. We have by Example 8.1.9 below that

$$\begin{aligned} &\tilde{\varepsilon}(\text{Spec}(L) \leftarrow (\text{Spec}(L), \langle 1 \rangle) \rightarrow \text{Spec}(L)) \\ &= (\text{Tr}_L^L)_*(\langle 1 \rangle) = \text{id}_*(\langle 1 \rangle) = \langle 1 \rangle \in \tilde{Z}^0(\text{Spec}(L) \times \text{Spec}(L)). \end{aligned}$$

The fact that composition is preserved follows directly from the definition of the composition in $\widetilde{\text{Cor}}_k^0$ introduced in [CF22, Section 4.2]. The statement after extending scalars follows immediately. □

Let us consider examples and special cases in order to understand $\tilde{\varepsilon}$ better. Since, in our case, we only treat (finite coproducts of) zero-dimensional regular k -schemes, the definition simplifies and essentially breaks down to the Scharlau transfer of Grothendieck–Witt groups with respect to the transfer map, compare [Lam05, Chapter VII, Section 1].

Remark 8.1.7. Let $X \times Y \cong \operatorname{Spec}(\prod_{i=1}^n \Omega_i) = \coprod_{i=1}^n \operatorname{Spec}(\Omega_i)$ for X, Y in Sm_k^0 (by Remark 8.1.1), i.e. Ω_i is a finite separable field extension of k for every i . Recall from [BG23a, Remark 6.5] that a morphism $f : Z \rightarrow X \times Y$, $Z = \operatorname{Spec}(L)$, factors through one of the points $\operatorname{Spec}(\Omega_j)$ of $X \times Y$, and, thus, the pushforward $f_* : \operatorname{GW}(L) \otimes R = \widetilde{\operatorname{CH}}^0(Z) \otimes R \rightarrow \widetilde{\operatorname{CH}}^0(X \times Y) \otimes R = \bigoplus_{i=1}^n \operatorname{GW}(\Omega_i) \otimes R$ is given by the composition

$$f_* : \operatorname{GW}(L) \otimes R \xrightarrow{(\operatorname{Tr}_{\Omega_j}^L)_*} \operatorname{GW}(\Omega_j) \otimes R \hookrightarrow \bigoplus_{i=1}^n \operatorname{GW}(\Omega_i) \otimes R,$$

where $(\operatorname{Tr}_{\Omega_j}^L)_*$ is the Scharlau transfer of the field trace $\operatorname{Tr}_{\Omega_j}^L : L \rightarrow \Omega_j$, compare [Lam05, Chapter VII, Section 1]. It sends a one-dimensional form $\langle \alpha \rangle$, $\alpha \in L^\times$, to the scaled trace form

$$\begin{aligned} (\operatorname{Tr}_{\Omega_j}^L)_\alpha : L \times L &\rightarrow \Omega_j \\ (x, y) &\mapsto \operatorname{Tr}_{\Omega_j}^L(\alpha xy) \end{aligned}$$

of dimension $[L : \Omega_j]$.

To understand the functor $\tilde{\varepsilon}$ intuitively, we give some concrete examples for the case $k = \mathbb{R}$, hence, $\otimes = \otimes_{\mathbb{R}}$.

Example 8.1.8. Let $k = \mathbb{R}$ and $R = \mathbb{Z}$.

(i) Consider the span

$$\operatorname{Spec}(\mathbb{C}) \xleftarrow{\operatorname{id}} \operatorname{Spec}(\mathbb{C}) \xrightarrow{\overline{(-)}} \operatorname{Spec}(\mathbb{C})$$

corresponding to the ring homomorphism

$$\begin{aligned} \mathbb{C} \otimes \mathbb{C} &\cong \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C} \\ u \otimes v &\mapsto (uv, u\bar{v}) \mapsto u\bar{v}, \end{aligned}$$

hence, yielding the map

$$\begin{aligned} f : \operatorname{Spec}(\mathbb{C}) &\rightarrow \operatorname{Spec}(\mathbb{C} \times \mathbb{C}) \cong \operatorname{Spec}(\mathbb{C}) \amalg \operatorname{Spec}(\mathbb{C}) \\ (0) &\mapsto \mathbb{C} \times (0) \end{aligned}$$

factoring through the second component of $\operatorname{Spec}(\mathbb{C}) \amalg \operatorname{Spec}(\mathbb{C})$.

The pushforward $f_* : \operatorname{GW}(\mathbb{C}) \rightarrow \operatorname{GW}(\mathbb{C} \otimes \mathbb{C}) \cong \operatorname{GW}(\mathbb{C}) \oplus \operatorname{GW}(\mathbb{C})$ is now the composition of the identity $\operatorname{id} = (\operatorname{Tr}_{\mathbb{C}}^{\mathbb{C}})_*$ and the inclusion into the second component, hence the only equivalence class of one-dimensional forms $\langle 1 \rangle \in \operatorname{GW}(\mathbb{C}) \cong \mathbb{Z}$ is sent to itself in the second component of $\operatorname{GW}(\mathbb{C}) \oplus \operatorname{GW}(\mathbb{C})$ under $\tilde{\varepsilon}$.

(ii) Consider the span

$$\mathrm{Spec}(\mathbb{R}) \xleftarrow{\pi} \mathrm{Spec}(\mathbb{C}) \xrightarrow{\pi} \mathrm{Spec}(\mathbb{R})$$

corresponding to multiplication $m : \mathbb{R} \otimes \mathbb{R} \cong \mathbb{R} \rightarrow \mathbb{C}$ yielding

$$\begin{aligned} g : \mathrm{Spec}(\mathbb{C}) &\rightarrow \mathrm{Spec}(\mathbb{R}) \\ (0)_{\mathbb{C}} &\mapsto (0)_{\mathbb{R}}. \end{aligned}$$

Thus, g_* is the Scharlau transfer $(\mathrm{Tr}_{\mathbb{R}}^{\mathbb{C}})_* : \mathrm{GW}(\mathbb{C}) \rightarrow \mathrm{GW}(\mathbb{R})$ sending $\langle 1 \rangle \in \mathrm{GW}(\mathbb{C})$ to its trace form $(\mathrm{Tr}_{\mathbb{R}}^{\mathbb{C}})_1$.

To determine the matrix representation of $(\mathrm{Tr}_{\mathbb{R}}^{\mathbb{C}})_1$, we choose the \mathbb{R} -basis $e_1 = 1, e_2 = i$ of \mathbb{C} and determine

$$\mathrm{Tr}_{\mathbb{R}}^{\mathbb{C}}(e_i \cdot e_j)$$

for $i, j \in \{1, 2\}$. The multiplication by i is given by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so, for $e_i \neq e_j$, we have $\mathrm{Tr}_{\mathbb{R}}^{\mathbb{C}}(e_i e_j) = \mathrm{Tr}_{\mathbb{R}}^{\mathbb{C}}(i) = \mathrm{tr} \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) = 0 + 0 = 0$. Moreover, $\mathrm{Tr}_{\mathbb{R}}^{\mathbb{C}}(1) = 1 + 1 = 2$ and $\mathrm{Tr}_{\mathbb{R}}^{\mathbb{C}}(i^2) = -1 + (-1) = -2$, so the trace form $(\mathrm{Tr}_{\mathbb{R}}^{\mathbb{C}})_1$ is $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \langle 2, -2 \rangle \cong \langle 1, -1 \rangle = \mathbb{H} \in \mathrm{GW}(\mathbb{R})$.

Hence, the image of the span

$$\mathrm{Spec}(\mathbb{R}) \xleftarrow{i} (\mathrm{Spec}(\mathbb{C}), \langle 1 \rangle) \xrightarrow{i} \mathrm{Spec}(\mathbb{R})$$

under $\tilde{\varepsilon}$ in $\widetilde{\mathrm{Cor}}_k^0(\mathrm{Spec}(\mathbb{R}), \mathrm{Spec}(\mathbb{R}))$ is the hyperbolic form $\langle 1, -1 \rangle \in \mathrm{GW}(\mathbb{R})$.

For our purposes, it will be important to understand where spans of the following special type are sent to under $\tilde{\varepsilon}$, just as in [BG23a, Remark 6.5].

Example 8.1.9. Let $L/K/k$ be finite separable field extensions and consider the span

$$f = [\mathrm{Spec}(K) \xleftarrow{\pi} (\mathrm{Spec}(L), \langle \alpha \rangle) \xrightarrow{\pi} \mathrm{Spec}(K)].$$

We know from [BG23a, Remark 6.5] that the underlying map f without $\langle \alpha \rangle$ factors through $\mathrm{Spec}(K)$, hence its image in $\widetilde{\mathrm{Cor}}_{k,R}^0$ is given by

$$\tilde{\varepsilon}(f) = f_*(\langle \alpha \rangle) = (\mathrm{Tr}_K^L)_*(\langle \alpha \rangle) = (\mathrm{Tr}_K^L)_{\alpha} \in \mathrm{GW}(K) \otimes R \longrightarrow \mathrm{GW}(K \otimes K) \otimes R.$$

Forgetting the symmetric bilinear form $\langle \alpha \rangle$, the span f gets sent to $[L : K] \mathrm{id}_{\mathrm{Spec}(K)}$ under $\varepsilon : \Omega_R(k) \rightarrow \mathrm{Cor}_{k,R}^0$, see loc.cit.

More generally, the functor $\tilde{\varepsilon}$ is a generalization of the functor ε in the following sense.

Lemma 8.1.10. *Let k be a perfect field of characteristic $\neq 2$ and R a ring. The following diagram commutes, where the horizontal arrows are the forgetful functors.*

$$\begin{array}{ccc} \widetilde{\Omega}_R(k) & \longrightarrow & \Omega_R(k) \\ \widetilde{\varepsilon} \downarrow & & \downarrow \varepsilon \\ \widetilde{\text{Cor}}_{k,R}^0 & \longrightarrow & \text{Cor}_{k,R}^0. \end{array}$$

Proof. Let $R = \mathbb{Z}$ for better readability. All arrows are the identity on objects, hence we only have to check the statement for morphisms. Since all functors are additive, let $f : \text{Spec}(K_1) \leftarrow \text{Spec}(L) \rightarrow \text{Spec}(K_2)$ be a prototypical span in $\Omega(k)$ and $\langle \alpha \rangle \in \text{GW}(L)$, $\alpha \in L^\times$. We know that f factors through some point $P = \text{Spec}(k(P)) := f(\text{Spec}(L))$ of $\text{Spec}(K_1) \times \text{Spec}(K_2)$. Then, f_* factors as

$$\begin{aligned} f_* : Z^0(\text{Spec}(L)) \cong \mathbb{Z} &\rightarrow Z^0(P) \cong \mathbb{Z} \xrightarrow{\text{incl.}} Z^0(\text{Spec}(K_1) \times \text{Spec}(K_2)) \\ [\text{Spec}(L)] = 1 &\mapsto [L : k(P)] \cdot P \end{aligned}$$

and, in the oriented case, f_* factors as

$$\begin{aligned} f_* : \text{GW}(L) &\rightarrow \text{GW}(k(P)) \xrightarrow{\text{incl.}} \text{GW}(\text{Spec}(K_1) \times \text{Spec}(K_2)) \\ \langle \alpha \rangle &\mapsto (\text{Tr}_{k(P)}^L)_*(\langle \alpha \rangle). \end{aligned}$$

The forgetful functor $V : \text{GW}(k(P)) \rightarrow K_0(k(P)) \cong \mathbb{Z}$ is given by dimension and since $\dim((\text{Tr}_{k(P)}^L)_*(\langle \alpha \rangle)) = [L : k(P)]$ by Remark 8.1.7, the diagram commutes. The proof extends directly to general R . \square

We construct the following ideal of homomorphisms inspired by [BG23a, Notation 4.15].

Definition 8.1.11. We define the two-sided ideal $\widetilde{J}_R(k)$ of homomorphisms in $\widetilde{\Omega}_R(k)$ as the ideal generated, i.e. consisting of linear combinations (with respect to the sum as defined in Construction 8.1.2) with coefficients in R , by spans of the form

$$\begin{aligned} &[\text{Spec}(L') \xleftarrow{\pi_{L'}^{L'}} (\text{Spec}(L), \langle \alpha \rangle) \xrightarrow{\pi_{L'}^{L'}} \text{Spec}(L')] \\ &- \sum_{i=1}^{[L:L']} [\text{Spec}(L') \xleftarrow{\text{id}} (\text{Spec}(L'), \langle \alpha_i \rangle) \xrightarrow{\text{id}} \text{Spec}(L')] \end{aligned}$$

for $L/L'/k$ finite separable field extensions of k and $\langle \alpha \rangle \in \text{GW}(L) \otimes R$, where

$$\begin{aligned} \langle \alpha_1, \dots, \alpha_{[L:L']} \rangle &= \widetilde{\varepsilon}([\text{Spec}(L') \xleftarrow{\pi_{L'}^{L'}} (\text{Spec}(L), \langle \alpha \rangle) \xrightarrow{\pi_{L'}^{L'}} \text{Spec}(L')]) \\ &= (\text{Tr}_{L'}^L)_*(\alpha) \end{aligned}$$

is a matrix representation of the trace form of $\alpha \in L^\times$ in $\text{GW}(L') \otimes R$.

Note that two such matrix representations may differ by permutation of the entries or multiplication of the i -th entry by a square β_i^2 for $\beta_i \in (L')^\times$ and $i \in \{1, \dots, [L : L']\}$, see [Scha85, Lemma 3.5]. Since the addition of oriented spans is commutative, permutation of entries by some permutation π yields

$$\begin{aligned} & \sum_{i=1}^{[L:L']} [\text{Spec}(L') \xleftarrow{\text{id}} (\text{Spec}(L'), \langle \alpha_{\pi(i)} \rangle) \xrightarrow{\text{id}} \text{Spec}(L')] \\ &= \sum_{i=1}^{[L:L']} [\text{Spec}(L') \xleftarrow{\text{id}} (\text{Spec}(L'), \langle \alpha_i \rangle) \xrightarrow{\text{id}} \text{Spec}(L')]. \end{aligned}$$

On the other hand, multiplying α_i with a square yields isomorphic oriented spans by Construction 8.1.2, hence,

$$\begin{aligned} & \sum_{i=1}^{[L:L']} [\text{Spec}(L') \xleftarrow{\text{id}} (\text{Spec}(L'), \langle \alpha_i \beta_i^2 \rangle) \xrightarrow{\text{id}} \text{Spec}(L')] \\ &= \sum_{i=1}^{[L:L']} [\text{Spec}(L') \xleftarrow{\text{id}} (\text{Spec}(L'), \langle \alpha_i \rangle) \xrightarrow{\text{id}} \text{Spec}(L')]. \end{aligned}$$

Moreover, there is the Scharlau relation in $\text{GW}(L')$ for forms of dimension 2. Let w.l.o.g. $[L : L'] = 2$. Then,

$$\begin{aligned} & [\text{Spec}(L') \xleftarrow{\text{id}} (\text{Spec}(L'), \langle \alpha_1 \rangle) \xrightarrow{\text{id}} \text{Spec}(L')] \\ &+ [\text{Spec}(L') \xleftarrow{\text{id}} (\text{Spec}(L'), \langle \alpha_2 \rangle) \xrightarrow{\text{id}} \text{Spec}(L')] \\ &= [\text{Spec}(L') \xleftarrow{\text{id}} (\text{Spec}(L'), \langle \alpha_1 + \alpha_2 \rangle) \xrightarrow{\text{id}} \text{Spec}(L')] \\ &+ [\text{Spec}(L') \xleftarrow{\text{id}} (\text{Spec}(L'), \langle \alpha_1 \alpha_2 (\alpha_1 + \alpha_2) \rangle) \xrightarrow{\text{id}} \text{Spec}(L')] \end{aligned}$$

by definition of $\tilde{\Omega}_R(k)$. Consequently, the description of the ideal is independent of the choice of the α_i .

This yields an equivalence relation on each Hom-set of the form $\text{Hom}_{\tilde{\Omega}_R(k)}(\text{Spec}(H), \text{Spec}(H))$ in $\tilde{\Omega}_R(k)$ by defining the equivalence class of f in the quotient of abelian groups $\text{Hom}_{\tilde{\Omega}_R(k)}(\text{Spec}(H), \text{Spec}(H))/\tilde{J}_R(k)$ as $[f] := f + \tilde{J}_R(k)$.

We extend this equivalence relation via composition in the following way. Let $f = f_1 \circ f_2 \circ f_3$, $n \in \mathbb{N}$ be an arbitrary composition of some morphism $f \in \text{Hom}_{\tilde{\Omega}_R(k)}(\text{Spec}(K), \text{Spec}(L))$. If f_2 is an element of $\text{Hom}_{\tilde{\Omega}_R(k)}(\text{Spec}(H), \text{Spec}(H))$ for some finite separable field extension H/k , we say

$$f \sim f_1 \circ f_2' \circ f_3$$

for each $f_2' \in [f_2] = f_2 + \tilde{J}_R(k)$. We denote the quotient category defined this way by $\tilde{\Omega}_R(k)/\tilde{J}_R(k)$.

We need the following lemma that characterizes prototypical spans in $\tilde{\Omega}(k)$ just as in [BG23a, Remark 4.14], inspired by [ThWe95, Proposition 2.2].

Lemma 8.1.12. *Let K and H be open subgroups of $\Gamma := \text{Gal}(\bar{k}/k)$ for a fixed separable closure \bar{k} of a perfect field k with $\text{char}(k) \neq 2$. Every morphism in $\text{Hom}_{\tilde{\Omega}_R(k)}(\text{Spec}(\bar{k}^K), \text{Spec}(\bar{k}^H))$ is equivalent modulo $\tilde{J}_R(k)$ to a linear combination of spans of the form*

$$\text{Spec}(\bar{k}^K) \xleftarrow{{}^g\pi_{K^g \cap H}^K} \text{Spec}(\bar{k}^{K^g \cap H}, \langle \beta \rangle) \xrightarrow{\pi_{K^g \cap H}^H} \text{Spec}(\bar{k}^H),$$

where $[g] \in K \setminus \Gamma/H$.

Proof. Recall that there is a well-known equivalence $\text{Sm}_k^0 \simeq \Gamma$ -sets, by which every smooth connected 0-dimensional k -scheme is equivalent to $\text{Spec}(\bar{k}^H)$ for some open subgroup $H \leq \Gamma$.

Therefore, we can apply [ThWe95, Proposition 2.2] (compare [BG23a, Remark 4.14]) and conclude that we can rewrite a prototypical span in $\Omega_R(k)$

$$\text{Spec}(\bar{k}^K) \xleftarrow{{}^g\pi_L^K} \text{Spec}(\bar{k}^L) \xrightarrow{\pi_L^H} \text{Spec}(\bar{k}^H)$$

as the composition

$$\begin{aligned} & [\text{Spec}(\bar{k}^K) \xleftarrow{{}^g\pi_{K^g \cap H}^K} \text{Spec}(\bar{k}^{K^g \cap H}) = \text{Spec}(\bar{k}^{K^g \cap H})] \\ \circ & [\text{Spec}(\bar{k}^{K^g \cap H}) \xleftarrow{\pi_L^{K^g \cap H}} \text{Spec}(\bar{k}^L) \xrightarrow{\pi_L^{K^g \cap H}} \text{Spec}(\bar{k}^{K^g \cap H})] \\ \circ & [\text{Spec}(\bar{k}^{K^g \cap H}) = \text{Spec}(\bar{k}^{K^g \cap H}) \xrightarrow{\pi_{K^g \cap H}^H} \text{Spec}(\bar{k}^H)] \end{aligned}$$

and it follows that we can rewrite a prototypical span

$$\text{Spec}(\bar{k}^K) \xleftarrow{{}^g\pi_L^K} (\text{Spec}(\bar{k}^L), \langle \alpha \rangle) \xrightarrow{\pi_L^H} \text{Spec}(\bar{k}^H)$$

in $\tilde{\Omega}_R(k)$ as the composition

$$\begin{aligned} & [\text{Spec}(\bar{k}^K) \xleftarrow{{}^g\pi_{K^g \cap H}^K} (\text{Spec}(\bar{k}^{K^g \cap H}), \langle 1 \rangle) = \text{Spec}(\bar{k}^{K^g \cap H})] \\ \circ & [\text{Spec}(\bar{k}^{K^g \cap H}) \xleftarrow{\pi_L^{K^g \cap H}} (\text{Spec}(\bar{k}^L), \langle \alpha \rangle) \xrightarrow{\pi_L^{K^g \cap H}} \text{Spec}(\bar{k}^{K^g \cap H})] \\ \circ & [\text{Spec}(\bar{k}^{K^g \cap H}) = (\text{Spec}(\bar{k}^{K^g \cap H}), \langle 1 \rangle) \xrightarrow{\pi_{K^g \cap H}^H} \text{Spec}(\bar{k}^H)] \end{aligned}$$

since $\langle 1 \rangle \otimes \langle \alpha \rangle \otimes \langle 1 \rangle = \langle 1 \cdot \alpha \cdot 1 \rangle = \langle \alpha \rangle$ in $\text{GW}(\bar{k}^L) \otimes R$.

By definition of $\tilde{J}_R(k)$, the middle term is equivalent modulo $\tilde{J}_R(k)$ to

$$\sum_{i=1}^n [\text{Spec}(\bar{k}^{K^g \cap H}) \leftarrow (\text{Spec}(\bar{k}^{K^g \cap H}), \langle \alpha_i \rangle) \rightarrow \text{Spec}(\bar{k}^{K^g \cap H})]$$

for $n = [\bar{k}^L : \bar{k}^{K^g \cap H}]$, where $\langle \alpha_1 \rangle \perp \dots \perp \langle \alpha_n \rangle$ is the trace form of $\alpha \in L$, i.e. the image of $\langle \alpha \rangle$ under the pushforward

$$(\mathrm{Tr}_{\bar{k}^{K^g \cap H}}^{\bar{k}^L})_* : \mathrm{GW}(\bar{k}^L) \otimes R \rightarrow \mathrm{GW}(\bar{k}^{K^g \cap H}) \otimes R.$$

It follows that our original span is equivalent modulo $\widetilde{J}_R(k)$ to

$$\begin{aligned} & [\mathrm{Spec}(\bar{k}^K) \xleftarrow{g\pi_{K^g \cap H}^K} (\mathrm{Spec}(\bar{k}^{K^g \cap H}), \langle 1 \rangle) = \mathrm{Spec}(\bar{k}^{K^g \cap H})] \\ \circ \sum_{i=1}^n & ([\mathrm{Spec}(\bar{k}^{K^g \cap H}) \leftarrow (\mathrm{Spec}(\bar{k}^{K^g \cap H}), \langle \alpha_i \rangle) \rightarrow \mathrm{Spec}(\bar{k}^{K^g \cap H})]) \\ \circ & [\mathrm{Spec}(\bar{k}^{K^g \cap H}) = (\mathrm{Spec}(\bar{k}^{K^g \cap H}), \langle 1 \rangle) \xrightarrow{\pi_{K^g \cap H}^H} \mathrm{Spec}(\bar{k}^H)] \end{aligned}$$

which equals

$$\sum_{i=1}^n [\mathrm{Spec}(\bar{k}^K) \xleftarrow{g\pi_{K^g \cap H}^H} (\mathrm{Spec}(\bar{k}^{K^g \cap H}), \langle \alpha_i \rangle) \xrightarrow{\pi_{K^g \cap H}^H} \mathrm{Spec}(\bar{k}^H)].$$

□

Remark 8.1.13. At this point, we would have liked to have proven an analog of [BG23a, Proposition 4.17], namely that the functor $\tilde{\varepsilon} : \widetilde{\Omega}_R(k) \rightarrow \widetilde{\mathrm{Cor}}_{k,R}^0$ induces an equivalence of additive categories

$$\tilde{\varepsilon}' : \frac{\widetilde{\Omega}_R(k)}{\widetilde{J}_R(k)} \xrightarrow{\cong} \widetilde{\mathrm{Cor}}_{k,R}^0.$$

The idea was to construct an R -submodule \widetilde{M} of $\mathrm{Hom}_{\widetilde{\Omega}_R(k)}(\mathrm{Spec}(\bar{k}^K), \mathrm{Spec}(\bar{k}^H))$ generated by spans of the form

$$s_g^\alpha := [\mathrm{Spec}(\bar{k}^K) \xleftarrow{g\pi_{K^g \cap H}^K} (\mathrm{Spec}(\bar{k}^{K^g \cap H}), \langle \alpha \rangle) \xrightarrow{\pi_{K^g \cap H}^H} \mathrm{Spec}(\bar{k}^H)]$$

for $[g] \in K \backslash \Gamma / H$ and we wanted to show that

$$\mathrm{Hom}_{\widetilde{\mathrm{Cor}}_{k,R}^0}(\mathrm{Spec}(\bar{k}^K), \mathrm{Spec}(\bar{k}^H)) \cong \widetilde{M} \cong \frac{\mathrm{Hom}_{\widetilde{\Omega}_R(k)}(\mathrm{Spec}(\bar{k}^K), \mathrm{Spec}(\bar{k}^H))}{\widetilde{J}_R(k)},$$

where the composition of the isomorphisms is given by ε' . However, it is not clear whether

$$\tilde{\varepsilon} : \widetilde{M} \xrightarrow{\cong} \mathrm{Hom}_{\widetilde{\mathrm{Cor}}_{k,R}^0}(\mathrm{Spec}(\bar{k}^K), \mathrm{Spec}(\bar{k}^H)) \cong \mathrm{GW}(\bar{k}^K \otimes \bar{k}^H) \otimes R$$

is injective in general.

8.2 Milnor–Witt Mackey Functors

Having constructed the span category $\widetilde{\Omega}(k)$ and the passage to finite MW correspondences, we now define (cohomological) Mackey functors. We define Mackey functors in a different way than the direct analog of [BG23a, Definition 4.5] since its oriented version is not as concrete as its non-oriented counterpart. Our definition is inspired by [KY15, Proposition 2], which states an equivalent definition of cohomological Mackey functors via factorizations. One could show these definitions coincide in the Milnor–Witt case as well, but this would require some form of an equivalence as discussed in Remark 8.1.13.

Definition 8.2.1. Let k be a perfect field of characteristic $\neq 2$ and R a commutative ring. We define *MW Mackey functors* to be additive functors

$$(\widetilde{\Omega}(k))^{\text{op}} \rightarrow R\text{-Mod}.$$

A *cohomological* MW Mackey functor is a MW Mackey functor that factors through $\widetilde{\text{Cor}}_k^0$ via $\tilde{\varepsilon}$. We will denote the abelian categories of (cohomological) MW Mackey functors by $\widetilde{\text{Mack}}^{\text{coh}}(k) \subset \widetilde{\text{Mack}}(k)$.

As in the non-oriented case, we may extend scalars to R , denoting the resulting categories by $\widetilde{\text{Mack}}_R^{\text{coh}}(k) \subset \widetilde{\text{Mack}}_R(k)$. The (cohomological) MW Mackey functors in this case are R -linear additive functors $\widetilde{\Omega}_R(k)^{\text{op}} \rightarrow R\text{-Mod}$ (which factor through $\widetilde{\text{Cor}}_{k,R}^0$).

Conjecture 8.2.2. *The factorization via $\tilde{\varepsilon}$ is unique.*

The conjecture is true if the equivalence from Remark 8.1.13 holds, but we do not expect the latter to be true in this form. However, it might still be possible to show the uniqueness of the factorization independently.

For the rest of this work, we assume Conjecture 8.2.2 to be true.

The vertical equivalence of Diagram 7.1 is the following. After forgetting \sim , the result can be considered a refinement of [BG23a, Corollary 6.17].

Corollary 8.2.3. *We assume that Conjecture 8.2.2 holds. Let k be a perfect field of characteristic $\neq 2$ and R a commutative ring. Consider the functor*

$$\tilde{\Phi} : \widetilde{\text{Mack}}_R^{\text{coh}}(k) \rightarrow \text{PSh}_R(\widetilde{\text{Cor}}_{k,R}^0)$$

sending a cohomological MW Mackey functor M that factors as

$$\begin{array}{ccc} (\widetilde{\Omega}_R(k))^{\text{op}} & \xrightarrow{M} & R\text{-Mod} \\ & \searrow \tilde{\varepsilon} & \nearrow M' \\ & & (\widetilde{\text{Cor}}_{k,R}^0)^{\text{op}} \end{array}$$

to $\tilde{\Phi}(M) := M'$ and a natural transformation $\varphi : M_1 \Rightarrow M_2$, defined via $\varphi_X : M_1(X) \rightarrow M_2(X)$ for all objects X in $\tilde{\Omega}(k)$, to $\tilde{\Phi}(\varphi) := \varphi' : M'_1 \Rightarrow M'_2$ defined via $\varphi'_{\tilde{\varepsilon}(X)} : M'_1(\tilde{\varepsilon}(X)) \rightarrow M'_2(\tilde{\varepsilon}(X))$.

It is an equivalence of Grothendieck abelian categories.

Proof. The factorization is unique by Conjecture 8.2.2, so the statement follows directly. \square

Remark 8.2.4. Note that M' is, in particular, a Nisnevish sheaf with MW-transfers since every presheaf on $\widetilde{\text{Cor}}_{k,R}^0$ is automatically one by Remark 7.2.5. The above equivalence hence extends to an equivalence

$$\widetilde{\text{Mack}}_R^{\text{coh}}(k) \xrightarrow{\simeq} \text{Sh}_{\text{Nis}}(\widetilde{\text{Cor}}_{k,R}^0) = \text{PSh}_R(\widetilde{\text{Cor}}_{k,R}^0).$$

Let us say a few words about representation theoretical analogs.

We denote by $\Gamma := \text{Gal}(\bar{k}/k)$ the absolute Galois group of a perfect field k of characteristic $\neq 2$ for a fixed algebraic closure \bar{k} . The category $\tilde{\Omega}(k)$ from Construction 8.1.2 has a representation theoretical analog. For this, we recall the explicit description of the following equivalence (on the left-hand side determined by evaluation on additive generators)

$$\begin{aligned} \text{Sm}_k^0 &\xrightarrow{\simeq} \Gamma\text{-sets} \\ X &\mapsto X(\bar{k}) \\ \text{Spec}(\bar{k}^H) &\leftarrow \Gamma/H. \end{aligned}$$

Using this equivalence, we can directly transfer Construction 8.1.2 generalizing [BG23a, Definition 4.5/Recollection 4.2]. The definition is merely sketched here.

Definition 8.2.5. Let $\Gamma := \text{Gal}(\bar{k}/k)$ be the absolute Galois group of a perfect field k of characteristic $\neq 2$ for a fixed algebraic closure \bar{k} and R a commutative ring. Analogously to Construction 8.1.2, we define the category $\tilde{\Omega}(\Gamma)$ as the group completion of the category $\widetilde{\text{span}}(\Gamma)$ with objects finite Γ -sets, i.e. finite direct sums of finite permutation modules isomorphic to $R(\Gamma/H)$.

A morphism from X to Y is a span $X \leftarrow Z := \coprod_{i=1}^n R(\Gamma/H_i) \rightarrow Y$ together with a one-dimensional symmetric bilinear form $\langle \alpha_i \rangle \in \text{GW}(\bar{k}^{H_i})$ for each i . Composition and addition are defined analogously to Construction 8.1.2; extension of scalars yields an additive category $\tilde{\Omega}_R(\Gamma)$.

Remark 8.2.6. Let $\Gamma := \text{Gal}(\bar{k}/k)$ be the absolute Galois group of a perfect field k of characteristic $\neq 2$ for a fixed algebraic closure \bar{k} . The equivalence of categories $\Gamma\text{-sets} \simeq \text{Sm}_k^0$ induces equivalences of additive categories

$$\Omega(\Gamma) \simeq \Omega(k)$$

and

$$\tilde{\Omega}(\Gamma) \simeq \tilde{\Omega}(k)$$

and similarly after extending scalars to R .

Remark 8.2.7. let $\Gamma := \text{Gal}(\bar{k}/k)$ be the absolute Galois group of a perfect field k of characteristic $\neq 2$ for a fixed algebraic closure \bar{k} . One could (forcably) define the additive category $\widetilde{\text{perm}}(\Gamma; R)$ as follows. Objects are free R -modules of the form $R(X)$ for X in Γ -sets. The morphism group between additive generators $R(\Gamma/K)$ and $R(\Gamma/H)$ is defined as

$$\text{Hom}_{\widetilde{\text{perm}}(\Gamma; R)}(R(\Gamma/K), R(\Gamma/H)) := \text{GW}(\bar{k}^K \otimes \bar{k}^H) \otimes R.$$

Since $\text{Hom}_{\widetilde{\text{perm}}(\Gamma; R)}(R(\Gamma/K), R(\Gamma/H)) \cong \widetilde{\text{CH}}^0(\text{Spec}(\bar{k}^K) \times \text{Spec}(\bar{k}^H)) \otimes R$, we set the composition as the restriction of the usual composition in $\widetilde{\text{Cor}}_{k, R}^0$ introduced in [CF22, Section 4.2]. Immediately, one could generalize [BG23a, Proposition 6.14] to obtain an equivalence of R -linear tensor categories

$$\widetilde{\text{Cor}}_k^0 \otimes R \simeq \widetilde{\text{perm}}(\Gamma; R).$$

Then, we could define *cohomological MW Mackey functors* to be additive functors

$$(\widetilde{\Omega}_R(\Gamma))^{\text{op}} \rightarrow R\text{-Mod}.$$

factoring (conjecturally uniquely) through $\widetilde{\text{perm}}(\Gamma; R)$ and denote the resulting abelian categories of cohomological MW Mackey functors by $\widetilde{\text{Mack}}_R^{\text{coh}}(\Gamma)$. It would immediately follow that we have equivalences of abelian categories

$$\widetilde{\text{Mack}}_R^{\text{coh}}(\Gamma) \simeq \widetilde{\text{Mack}}_R^{\text{coh}}(k) \simeq \text{Sh}_{\text{Nis}}(\widetilde{\text{Cor}}_{k, R}^0).$$

However, the results only transfer directly from $\widetilde{\text{Cor}}_{k, R}^0$ and $\widetilde{\Omega}(k)$, and not much is won by this definition of $\widetilde{\text{perm}}(\Gamma; R)$.

We have not found a sensible way how to equip morphisms of R -modules with symmetric bilinear forms without using the equivalence $\text{Sm}_k^0 \simeq \Gamma$ -sets. For this reason, we do not have an intrinsic candidate for $\widetilde{\text{perm}}(\Gamma; R)$ and especially not for $\widetilde{\text{Perm}}(\Gamma; R)$ or $\widetilde{\text{Mod}}(\Gamma; R)$.

In the end, our computations in Chapter 9 do not rely on representation theoretical results as opposed to computations in [BG22a].

Chapter 9

The Balmer Spectrum of Artin Milnor–Witt Motives

We now want to compute Balmer spectra for initial examples. In Section 9.1, we determine, for example, the spectrum of (geometric) Artin Milnor–Witt motives over algebraically closed base fields. Then, Section 9.2 treats the base field \mathbb{R} with coefficients in $\mathbb{Z}/2$. We will assume all base fields k (and sometimes L) to be infinite. Following Remark 7.3.4, it may be possible to drop this assumption in the future.

9.1 Algebraically Closed Fields and First Computations

As expected, for algebraically closed fields with arbitrary coefficients, the derived category of Milnor–Witt Artin motives is equivalent to Voevodsky’s derived category of Artin motives.

Lemma 9.1.1. *Let k be an algebraically closed field of characteristic $\neq 2$ and R a commutative ring. Then, we have an equivalence of tensor triangulated categories $\widetilde{\text{DAM}}^{\text{gm}}(k; R) \simeq \widetilde{\text{DAM}}^{\text{gm}}(k; R)$. If Conjecture 7.3.10 holds, then $\text{DAM}(k; R) \simeq \widetilde{\text{DAM}}(k; R)$.*

Proof. First, we show that $\widetilde{\text{Cor}}_{k,R}^0 \simeq \text{Cor}_{k,R}^0$. Let $X, Y \in \text{Sm}_k^0$ and $T \subset X \times Y$ an admissible subset. Then, $X \times Y$ is zero-dimensional by Remark 8.1.1. It follows from [Fas08, Remarque 10.2.16] that

$$\widetilde{\text{CH}}_T^0(X \times Y) \otimes R \cong \text{CH}_T^0(X \times Y) \otimes R,$$

since k is algebraically closed. In particular, $\widetilde{\text{Cor}}_{k,R}^0 \simeq \text{Cor}_{k,R}^0$ by [CF22, Remark 4.3.3]. The result now follows from Proposition 7.3.3 and Corollary 7.3.12. \square

Corollary 9.1.2. *Let k be an algebraically closed field of characteristic $\neq 2$ and R a commutative ring. The Balmer spectra of $\widetilde{\text{DAM}}^{\text{gm}}(k; R)$ and $\text{DAM}^{\text{gm}}(k; R)$ are homeomorphic to $\text{Spec}(R)$.*

Proof. Follows from Lemma 9.1.1 and the observation from [BG22b, Corollary 2.9] that $\mathrm{Spc}(\mathrm{DAM}^{\mathrm{gm}}(k; R)) = \mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(R))$. \square

Moreover, we obtain a refinement of [BG22b, Proposition 2.8].

Proposition 9.1.3. *Let L be a finite separable field extension of k , both L and k of characteristic $\neq 2$, and K a perfect field such that the trace form $(\mathrm{Tr}_k^L)_1$ is invertible in $\mathrm{GW}(k) \otimes K$. Then, the map*

$$\mathrm{Spc}(\widetilde{\mathrm{DAM}}^{\mathrm{gm}}(L; K)) \rightarrow \mathrm{Spc}(\widetilde{\mathrm{DAM}}^{\mathrm{gm}}(k; K))$$

induced by the extension of scalars functor

$$\widetilde{\mathrm{Ext}}_k^L : \widetilde{\mathrm{Cor}}_{k,K}^0 \rightarrow \widetilde{\mathrm{Cor}}_{L,K}^0$$

is surjective.

Proof. We follow an argument similar to the standard argument in the proof of [BG22b, Proposition 2.8]. The extension functor $\widetilde{\mathrm{Ext}}_k^L$ admits a left adjoint $\widetilde{\mathrm{Res}}_k^L$ with unit η and counit ϵ which is also a right adjoint with unit $\tilde{\eta}$ and counit $\tilde{\epsilon}$, see [CF22, Section 6.2]. It is a tensor functor by [CF22, Section 6.2, p.32].

We obtain adjunctions after applying $\mathrm{K}_b((-)^{\natural})$ on both sides. Using [CF22, Lemma 6.2.2], we see that the composition

$$\epsilon \circ \tilde{\eta} : \mathrm{id}_{\mathrm{K}_b((\widetilde{\mathrm{Cor}}_{k,K}^0)^{\natural})} \rightarrow \widetilde{\mathrm{Res}}_k^L \circ \widetilde{\mathrm{Ext}}_k^L \rightarrow \mathrm{id}_{\mathrm{K}_b((\widetilde{\mathrm{Cor}}_{L,K}^0)^{\natural})}$$

is given by (degreewise) multiplication as a $(\mathrm{GW}(k) \otimes K)$ -module by the trace form $(\mathrm{Tr}_k^L)_1 \in \mathrm{GW}(k) \otimes K$. Since $(\mathrm{Tr}_k^L)_1$ is invertible in $\mathrm{GW}(k) \otimes K$ by assumption, $\epsilon \circ \tilde{\eta}$ is also invertible. Here, we use that composition in $\mathrm{Hom}_{\widetilde{\mathrm{Cor}}_{k,K}^0}(\mathrm{Spec}(k), \mathrm{Spec}(k))$ coincides with multiplication in $\mathrm{GW}(k) \otimes K$ by [Fas07, Theorem 7.6].

It follows that $\tilde{\eta}_A$ is injective for all objects A in $\mathrm{K}_b((\widetilde{\mathrm{Cor}}_{k,K}^0)^{\natural})$. Consequently, the functor $\widetilde{\mathrm{Ext}}_k^L : \mathrm{K}_b((\widetilde{\mathrm{Cor}}_{k,K}^0)^{\natural}) \rightarrow \mathrm{K}_b((\widetilde{\mathrm{Cor}}_{L,K}^0)^{\natural})$ is faithful. Moreover, it is a tensor triangulated functor since it is a tensor functor before applying $\mathrm{K}_b((-)^{\natural})$. The statement now follows from [Bal18, Theorem 1.3] and Proposition 7.3.3. \square

Remark 9.1.4. Unlike in the non-oriented case, it is not an immediate corollary of this proposition that the spectrum of $\widetilde{\mathrm{DAM}}^{\mathrm{gm}}(\mathbb{R}; K)$ is homeomorphic to a point when 2 is invertible in K (compare [BG22b, Corollary 2.9]). The reason for this is that the field extension \mathbb{C}/\mathbb{R} does not fulfill the assumption that the trace form is invertible.

Recall from Example 8.1.8 (ii) that $(\mathrm{Tr}_{\mathbb{R}}^{\mathbb{C}})_1 = \langle 1, -1 \rangle \in \mathrm{GW}(\mathbb{R}) \otimes K$, which is indeed not invertible. It is not even a monomorphism when considered an morphism in $\mathrm{Hom}_{\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0}(\mathrm{Spec}(\mathbb{R}), \mathrm{Spec}(\mathbb{R})) \cong \mathrm{GW}(\mathbb{R}) \otimes K$ since we have

$$\langle 1, -1 \rangle \langle 1 \rangle = \langle 1, -1 \rangle$$

and

$$\langle 1, -1 \rangle \langle -1 \rangle = \langle -1, 1 \rangle = \langle 1, -1 \rangle$$

by [Lam05, Proposition II.3.2] and [Scha85, Definition 1.4/1.7], but $\langle 1 \rangle \neq \langle -1 \rangle \in \mathrm{GW}(\mathbb{R}) \otimes K$. Here, we used that the composition in $\mathrm{Hom}_{\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0}(\mathrm{Spec}(\mathbb{R}), \mathrm{Spec}(\mathbb{R}))$ coincides with the multiplication in $\mathrm{GW}(\mathbb{R}) \otimes K$ by [Fas07, Theorem 7.6].

9.2 The Base Field \mathbb{R} and Coefficients in \mathbb{F}_2

Let $K = \mathbb{F}_2$ in the following. We want take initial steps to compute the Balmer spectrum of geometric Artin Milnor–Witt motives $\widetilde{\mathrm{DAM}}^{\mathrm{gm}}(\mathbb{R}; K)$ over the base field \mathbb{R} with coefficients in K .

Recall from Proposition 7.3.3 that we have a categorical equivalence

$$\widetilde{\mathrm{DAM}}^{\mathrm{gm}}(\mathbb{R}; K) \simeq \mathrm{K}_b((\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0)^\natural)$$

since \mathbb{R} is perfect infinite of characteristic $\neq 2$.

Remark 9.2.1. In the non-oriented case, these categories are equivalent to the bounded homotopy category of the abelian category $KC_2\text{-mod}$ of finitely generated KC_2 -modules by [BG23a, Proposition 6.14] since every KC_2 -module is already a permutation module. Moreover, [BG22a, Remark 10.5] recalls that there is a categorical equivalence between $KC_2\text{-mod}$ and the category of *Artin Chow motives* $\mathrm{AM}(\mathbb{R}, K)$. Its oriented analogon is precisely $\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0$, which is why $\widetilde{\mathrm{AM}}(\mathbb{R}, K)$ would also be a suitable notation for $\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0$.

Remark 9.2.2. An additive category is a *Krull-Schmidt category* if the endomorphism ring of every indecomposable object is a local ring, see [Rin84, p.52]. By definition, every object in $\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0$ and $\mathrm{Cor}_{\mathbb{R},K}^0$ is a finite direct sum of copies of $\mathrm{Spec}(\mathbb{R})$ and $\mathrm{Spec}(\mathbb{C})$. The latter is a Krull-Schmidt category, see [BG22b, Remark 3.1]. It follows that $\mathrm{End}_{\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0}(\mathrm{Spec}(\mathbb{C})) \cong \mathrm{End}_{\mathrm{Cor}_{\mathbb{R},K}^0}(\mathrm{Spec}(\mathbb{C}))$ is a local ring. Moreover, the endomorphism ring $\mathrm{End}_{\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0}(\mathrm{Spec}(\mathbb{R})) \cong K[C_2]$ is local as well. Consequently, $\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0$ is a Krull-Schmidt category with indecomposable objects $\mathrm{Spec}(\mathbb{R})$ and $\mathrm{Spec}(\mathbb{C})$.

The category $\mathrm{Cor}_{\mathbb{R},K}^0 \simeq KC_2\text{-mod}$ is abelian, but problems appear for its oriented counterpart.

Lemma 9.2.3. *The category $\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0$ is not abelian or the forgetful functor $\pi : \widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0 \rightarrow \mathrm{Cor}_{\mathbb{R},K}^0$ is not exact.*

Proof. Suppose $\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0$ is abelian and π is exact. We want to show that the morphism $H := \langle 1, -1 \rangle \in \mathrm{GW}(\mathbb{R}) \otimes K \cong \mathrm{Hom}_{\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0}(\mathrm{Spec}(\mathbb{R}), \mathrm{Spec}(\mathbb{R}))$ does not have a kernel.

Since π is exact by assumption, it commutes with kernels and we have

$$\begin{aligned} \pi(\ker(H)) &= \ker(\pi(H)) = \ker([\mathrm{Spec}(\mathbb{R}) \xrightarrow{0} \mathrm{Spec}(\mathbb{R})]) \\ &= [\mathrm{Spec}(\mathbb{R}) \xrightarrow{\mathrm{id}} \mathrm{Spec}(\mathbb{R})] \in \mathrm{Hom}_{\mathrm{Cor}_{\mathbb{R},K}^0}(\mathrm{Spec}(\mathbb{R}), \mathrm{Spec}(\mathbb{R})). \end{aligned}$$

Since $K = \mathbb{F}_2$, the identity morphism on $\mathrm{Spec}(\mathbb{R})$ has exactly two lifts under π in $\mathrm{Hom}_{\mathrm{Cor}_{\mathbb{R},K}^0}(\mathrm{Spec}(\mathbb{R}), \mathrm{Spec}(\mathbb{R})) \cong \mathrm{GW}(\mathbb{R}) \otimes K$, namely $\langle 1 \rangle$ and $\langle -1 \rangle$. Recall that composition in $\mathrm{Hom}_{\mathrm{Cor}_{\mathbb{R},K}^0}(\mathrm{Spec}(\mathbb{R}), \mathrm{Spec}(\mathbb{R}))$ coincides with multiplication in $\mathrm{GW}(\mathbb{R}) \otimes K$ by [Fas07, Theorem 7.6]. Since $\langle 1 \rangle \cdot \langle 1, -1 \rangle = \langle -1 \rangle \cdot \langle 1, -1 \rangle = \langle 1, -1 \rangle \neq 0$, the diagram

$$\begin{array}{ccc} & \mathrm{Spec}(\mathbb{R}) & \\ & \uparrow \kappa & \searrow H \\ \mathrm{Spec}(\mathbb{R}) & \xrightarrow{0} & \mathrm{Spec}(\mathbb{R}) \end{array}$$

does not commute for $\kappa \in \{\langle 1 \rangle, \langle -1 \rangle\}$, which contradicts the definition of a kernel in an abelian category, see [MacL71, Section VIII.1]. \square

Example 9.2.4. Let $\langle 1, -1 \rangle \in \mathrm{Hom}_{\mathrm{Cor}_{\mathbb{R},K}^0}(\mathrm{Spec}(\mathbb{R}), \mathrm{Spec}(\mathbb{R})) \cong \mathrm{GW}(\mathbb{R}) \otimes K$. We want to show that $\langle 1, -1 \rangle \circ \langle 1, -1 \rangle = 0 \in \mathrm{Hom}_{\mathrm{Cor}_{\mathbb{R},K}^0}(\mathrm{Spec}(\mathbb{R}), \mathrm{Spec}(\mathbb{R})) \cong \mathrm{GW}(\mathbb{R}) \otimes K$.

The composition in $\mathrm{Hom}_{\mathrm{Cor}_{\mathbb{R},K}^0}(\mathrm{Spec}(\mathbb{R}), \mathrm{Spec}(\mathbb{R}))$ coincides with the multiplication in $\mathrm{GW}(\mathbb{R}) \otimes K$ by [Fas07, Theorem 7.6]. By [Scha85, §2, Definition 1.4/1.7], the multiplication $\langle 1, -1 \rangle \cdot \langle 1, -1 \rangle$ in $\mathrm{GW}(\mathbb{R}) \otimes K$ is defined as

$$\langle 1, -1 \rangle \cdot \langle 1, -1 \rangle := \langle 1, -1, -1, 1 \rangle,$$

which equals $2\langle 1 \rangle + 2\langle -1 \rangle$. Since K is of characteristic 2, it follows that $\langle 1, -1 \rangle \cdot \langle 1, -1 \rangle = 0 \in \mathrm{GW}(\mathbb{R}) \otimes K$.

We want to take first steps in computing the Balmer spectrum of

$$\widetilde{\mathrm{DAM}}^{\mathrm{gm}}(\mathbb{R}; K) \simeq \mathrm{K}_b((\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0)^{\natural}).$$

It suffices to consider the tensor triangulated category $\mathrm{K}_b(\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0)$ since idempotent-completion does not change the Balmer spectrum by [Bal05b, Proposition 3.13].

In order to determine some points of $\mathrm{Spc}(\mathrm{K}_b(\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0))$, one can begin with computing preimages of tensor prime ideals under the tensor triangulated forgetful functor

$$\pi^* : \mathrm{K}_b(\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0) \rightarrow \mathrm{K}_b(\mathrm{Cor}_{\mathbb{R},K}^0),$$

which is defined by applying $\pi : \widetilde{\text{Cor}}_{\mathbb{R},K}^0 \rightarrow \text{Cor}_{\mathbb{R},K}^0$ at each stage by [BC+22, (MW1)]. It induces a continuous map

$$\begin{aligned} \text{Spc}(\pi^*) : \text{Spc}(\text{K}_b(\text{Cor}_{\mathbb{R},K}^0)) &\rightarrow \text{Spc}(\text{K}_b(\widetilde{\text{Cor}}_{\mathbb{R},K}^0)) \\ \mathcal{P} &\mapsto (\pi^*)^{-1}(\mathcal{P}) \end{aligned}$$

by [Bal05b, Proposition 3.6].

The left-hand side consists of the three tensor prime ideals $\langle M(\text{Spec}(\mathbb{C})) \rangle$, $\langle S \rangle$, and $\langle S, M(\text{Spec}(\mathbb{C})) \rangle$ as shown in [BG22a, Theorem 3.14]. Here, $M(\text{Spec}(\mathbb{C}))$ is $\text{Spec}(\mathbb{C})$ considered a complex concentrated in degree 0 and

$$S = \dots \rightarrow 0 \rightarrow \text{Spec}(\mathbb{R}) \xrightarrow{\eta} \text{Spec}(\mathbb{C}) \xrightarrow{\epsilon} \text{Spec}(\mathbb{R}) \rightarrow 0 \rightarrow \dots$$

is concentrated in degrees 0, 1, and 2, see [BG22a, (3.4)].

Notation 9.2.5. The notation $\langle C \rangle$ for a set of objects C in $\text{K}_b(\text{Cor}_{\mathbb{R},K}^0)$ stands for the thick tensor ideal generated by C , i.e., the smallest thick tensor ideal containing C .

We abuse notation here by writing $M(\text{Spec}(\mathbb{C}))$ for an object in $\text{K}_b(\text{Cor}_{\mathbb{R},K}^0)$, which originally denotes the motive of $\text{Spec}(\mathbb{C})$ in $\text{DM}(\mathbb{R}; K)$. Similarly, we denote by $\widetilde{M}(\text{Spec}(\mathbb{C}))$ not only the Milnor-Witt motive of $\text{Spec}(\mathbb{C})$ in $\widetilde{\text{DM}}(\mathbb{R}; K)$, but also the corresponding complex in $\text{K}_b(\widetilde{\text{Cor}}_{\mathbb{R},K}^0)$, i.e., the complex $\text{Spec}(\mathbb{C})$ concentrated in degree 0. Translating their notation from the setting of representation theory to algebraic geometry, some authors, e.g. in [BG22a, Section 3], use the notation $\text{Spec}(\mathbb{C})$ for the complex $\text{Spec}(\mathbb{C})$ concentrated in degree 0. We choose a different notation to better identify whether we mean an object in the oriented or non-oriented case.

As a consequence, in order to determine some points of $\text{Spc}(\text{K}_b(\widetilde{\text{Cor}}_{\mathbb{R},K}^0))$, we need to compute

$$(\pi^*)^{-1}(\langle M(\text{Spec}(\mathbb{C})) \rangle), (\pi^*)^{-1}(\langle S \rangle), \text{ and } (\pi^*)^{-1}(\langle S, M(\text{Spec}(\mathbb{C})) \rangle).$$

In the light of [BG22a, Corollary 3.15], this amounts to computing the kernels of (see loc.cit. for the notation in the third composition)

$$\text{K}_b(\widetilde{\text{Cor}}_{\mathbb{R},K}^0) \xrightarrow{\pi^*} \text{K}_b(\text{Cor}_{\mathbb{R},K}^0) \xrightarrow{\text{ext}_{\mathbb{C}}^{\mathbb{R}}} \text{K}_b(\text{Cor}_{\mathbb{C},K}^0) \simeq \text{K}_b(K\text{-mod}),$$

$$\text{K}_b(\widetilde{\text{Cor}}_{\mathbb{R},K}^0) \xrightarrow{\pi^*} \text{K}_b(\text{Cor}_{\mathbb{R},K}^0) \xrightarrow{q} \text{D}_b(\text{Cor}_{\mathbb{R},K}^0) \simeq \text{D}_b(KC_2\text{-mod}) \xrightarrow{\text{stab}} KC_2\text{-stab},$$

and

$$\text{K}_b(\widetilde{\text{Cor}}_{\mathbb{R},K}^0) \xrightarrow{\pi^*} \text{K}_b(\text{Cor}_{\mathbb{R},K}^0) \simeq \text{K}_b(KC_2\text{-mod}) \xrightarrow{\text{K}_b(\text{stab})} \text{K}_b(K\text{-mod}),$$

where $\text{ext}_{\mathbb{C}}^{\mathbb{R}}$ denotes the functor induced by extension of scalars.

Let us focus on some interesting objects in the category $\text{Ch}_b(\text{Cor}_{\mathbb{R},K}^0)$ that have multiple or no lifts in $\text{Ch}_b(\widetilde{\text{Cor}}_{\mathbb{R},K}^0)$.

Example 9.2.6. Let $K = \mathbb{F}_2$.

- (i) The composition $\mathrm{Spec}(\mathbb{R}) \xrightarrow{\eta} \mathrm{Spec}(\mathbb{C}) \xrightarrow{\epsilon} \mathrm{Spec}(\mathbb{R})$ in $\mathrm{Cor}_{\mathbb{R},K}^0$ appearing in the acyclic complex S has a unique lift

$$\mathrm{Spec}(\mathbb{R}) \xrightarrow{\langle 1 \rangle} \mathrm{Spec}(\mathbb{C}) \xrightarrow{\langle 1 \rangle} \mathrm{Spec}(\mathbb{R})$$

to $\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0$ since there are isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0}(\mathrm{Spec}(\mathbb{R}), \mathrm{Spec}(\mathbb{C})) &\cong \mathrm{GW}(\mathbb{C}) \otimes K \\ &\cong K \cong \mathrm{Hom}_{\mathrm{Cor}_{\mathbb{R},K}^0}(\mathrm{Spec}(\mathbb{R}), \mathrm{Spec}(\mathbb{C})) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Hom}_{\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0}(\mathrm{Spec}(\mathbb{C}), \mathrm{Spec}(\mathbb{R})) &\cong \mathrm{GW}(\mathbb{C}) \otimes K \\ &\cong K \cong \mathrm{Hom}_{\mathrm{Cor}_{\mathbb{R},K}^0}(\mathrm{Spec}(\mathbb{C}), \mathrm{Spec}(\mathbb{R})). \end{aligned}$$

However, we have seen in Example 8.1.8 (ii) that the composition is $\langle 1, -1 \rangle \neq 0 \in \mathrm{GW}(\mathbb{R})$. Consequently, the complex S does not give rise to a complex in $\mathrm{Ch}_b(\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0)$.

- (ii) On the other hand, the composition

$$\mathrm{Spec}(\mathbb{R}) \xrightarrow{\langle 1, -1 \rangle} \mathrm{Spec}(\mathbb{R}) \xrightarrow{\langle 1, -1 \rangle} \mathrm{Spec}(\mathbb{R})$$

in $\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0$ yields a complex (the composition is 0 by Example 9.2.4)

$$H^\bullet : \dots \rightarrow 0 \rightarrow \mathrm{Spec}(\mathbb{R}) \xrightarrow{\langle 1, -1 \rangle} \mathrm{Spec}(\mathbb{R}) \xrightarrow{\langle 1, -1 \rangle} \mathrm{Spec}(\mathbb{R}) \rightarrow 0 \rightarrow \dots,$$

which is not chain homotopy equivalent by the considerations below to

$$C^\bullet : \dots \rightarrow 0 \rightarrow \mathrm{Spec}(\mathbb{R}) \xrightarrow{0} \mathrm{Spec}(\mathbb{R}) \xrightarrow{0} \mathrm{Spec}(\mathbb{R}) \rightarrow 0 \rightarrow \dots$$

The forgetful functor $\pi : \widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0 \rightarrow \mathrm{Cor}_{\mathbb{R},K}^0$ sends both compositions $\langle 1, -1 \rangle \circ \langle 1, -1 \rangle$ and $0 \circ 0$ to

$$\mathrm{Spec}(\mathbb{R}) \xrightarrow{0} \mathrm{Spec}(\mathbb{R}) \xrightarrow{0} \mathrm{Spec}(\mathbb{R}),$$

which gives rise to a complex in $\mathrm{Ch}_b(\mathrm{Cor}_{\mathbb{R},K}^0)$.

Let us check that H^\bullet and C^\bullet are indeed not chain homotopy equivalent. If they were, we could find chain maps $f : H^\bullet \rightarrow C^\bullet$ and $g : C^\bullet \rightarrow H^\bullet$ such that $f \circ g \sim \mathrm{id}_{C^\bullet}$ and $g \circ f \sim \mathrm{id}_{H^\bullet}$. In order to make the diagrams mandatory for chain maps appearing in

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Spec}(\mathbb{R}) & \xrightarrow{H} & \mathrm{Spec}(\mathbb{R}) & \xrightarrow{H} & \mathrm{Spec}(\mathbb{R}) & \longrightarrow & 0 \\ & & \downarrow \uparrow f_1 & & \downarrow \uparrow f_2 & & \downarrow \uparrow f_3 & & \\ & & g_1 & & g_2 & & g_3 & & \\ & & \downarrow \uparrow & & \downarrow \uparrow & & \downarrow \uparrow & & \\ 0 & \longrightarrow & \mathrm{Spec}(\mathbb{R}) & \xrightarrow{0} & \mathrm{Spec}(\mathbb{R}) & \xrightarrow{0} & \mathrm{Spec}(\mathbb{R}) & \longrightarrow & 0 \end{array}$$

commutative, the concrete description of the Hom-sets in $\widetilde{\text{Cor}}_{\mathbb{R}, \mathbb{F}_2}^0$ implies that $g_1, g_2, f_2, f_3 \in \{0, \langle 1, -1 \rangle\}$. In particular, $g_2 \circ f_2 = f_2 \circ g_2 = 0$.

A chain homotopy equivalence between id_{C^\bullet} and $f \circ g$ is a collection of maps $h_i : C^i \rightarrow C^{i-1}$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Spec}(\mathbb{R}) & \xrightarrow{0} & \text{Spec}(\mathbb{R}) & \xrightarrow{0} & \text{Spec}(\mathbb{R}) \longrightarrow 0 \\ & & \downarrow \text{f}_1 \circ \text{g}_1 & \swarrow \text{h}_2 & \downarrow 0 & \swarrow \text{h}_3 & \downarrow \text{f}_3 \circ \text{g}_3 \\ 0 & \longrightarrow & \text{Spec}(\mathbb{R}) & \xrightarrow{0} & \text{Spec}(\mathbb{R}) & \xrightarrow{0} & \text{Spec}(\mathbb{R}) \longrightarrow 0 \end{array}$$

such that, in particular, $\langle 1 \rangle - 0 = 0 \circ h_2 + h_3 \circ 0 = 0$, which is a contradiction.

Lemma 9.2.7. *Let $K = \mathbb{F}_2$. The image of $\langle S \rangle$ under*

$$\text{Spc}(\pi^*) : \text{Spc}(\text{K}_b(\text{Cor}_{\mathbb{R}, K}^0)) \rightarrow \text{Spc}(\text{K}_b(\widetilde{\text{Cor}}_{\mathbb{R}, K}^0))$$

is contained in $\text{supp}(\widetilde{M}(\text{Spec}(\mathbb{C})))$.

Proof. It follows from [Bal05b, Proposition 3.6] and [BG22a, Theorem 3.14] that

$$\begin{aligned} (\text{Spc}(\pi^*))^{-1}(\text{supp}_{\text{K}_b(\widetilde{\text{Cor}}_{\mathbb{R}, K}^0)}(\widetilde{M}(\text{Spec}(\mathbb{C})))) &= \text{supp}_{\text{K}_b(\text{Cor}_{\mathbb{R}, K}^0)}(\pi^*(\widetilde{M}(\text{Spec}(\mathbb{C})))) \\ &= \text{supp}_{\text{K}_b(\text{Cor}_{\mathbb{R}, K}^0)}(M(\text{Spec}(\mathbb{C}))) = \langle S \rangle. \end{aligned}$$

Now, we can apply $\text{Spc}(\pi^*)$ on both sides and obtain

$$\begin{aligned} \text{Spc}(\pi^*)(\langle S \rangle) &= \text{Spc}(\pi^*)(\text{Spc}(\pi^*)^{-1}(\text{supp}_{\text{K}_b(\widetilde{\text{Cor}}_{\mathbb{R}, K}^0)}(\widetilde{M}(\text{Spec}(\mathbb{C})))) \\ &\subset \text{supp}_{\text{K}_b(\widetilde{\text{Cor}}_{\mathbb{R}, K}^0)}(\widetilde{M}(\text{Spec}(\mathbb{C}))). \end{aligned}$$

□

We can compute the preimage of $\langle M(\text{Spec}(\mathbb{C})) \rangle$ under π^* .

Lemma 9.2.8. *Let $K = \mathbb{F}_2$. The image of $\langle M(\text{Spec}(\mathbb{C})) \rangle$ under*

$$\text{Spc}(\pi^*) : \text{Spc}(\text{K}_b(\text{Cor}_{\mathbb{R}, K}^0)) \rightarrow \text{Spc}(\text{K}_b(\widetilde{\text{Cor}}_{\mathbb{R}, K}^0))$$

is the prime ideal $\langle \widetilde{M}(\text{Spec}(\mathbb{C})) \rangle$.

Proof. Since $\pi^*(\widetilde{M}(\text{Spec}(\mathbb{C}))) = M(\text{Spec}(\mathbb{C})) \in \langle M(\text{Spec}(\mathbb{C})) \rangle$, we have that $\widetilde{M}(\text{Spec}(\mathbb{C})) \in (\pi^*)^{-1}(\langle M(\text{Spec}(\mathbb{C})) \rangle)$. Consequently, we obtain $\langle \widetilde{M}(\text{Spec}(\mathbb{C})) \rangle \subset (\pi^*)^{-1}(\langle M(\text{Spec}(\mathbb{C})) \rangle) = \text{Spc}(\pi^*)(\langle M(\text{Spec}(\mathbb{C})) \rangle)$.

For the other inclusion, we recall that $\text{Cor}_{\mathbb{R}, K}^0$ and $\widetilde{\text{Cor}}_{\mathbb{R}, K}^0$ have the same objects and the additive generators satisfy $\text{Spec}(\mathbb{C}) \otimes_{\text{Spec}(\mathbb{R})} \text{Spec}(\mathbb{R}) \cong \text{Spec}(\mathbb{C})$ and $\text{Spec}(\mathbb{C}) \otimes_{\text{Spec}(\mathbb{R})} \text{Spec}(\mathbb{C}) \cong \text{Spec}(\mathbb{C}) \oplus \text{Spec}(\mathbb{C})$. Hence, complexes in $\langle M(\text{Spec}(\mathbb{C})) \rangle$ and $\langle \widetilde{M}(\text{Spec}(\mathbb{C})) \rangle$ have only (possibly 0) copies of $\text{Spec}(\mathbb{C})$ in each degree. Note

that the tensor unit $\widetilde{M}(\mathrm{Spec}(\mathbb{R}))$ considered a complex concentrated in degree 0 cannot appear as a direct summand of an object in $\langle \widetilde{M}(\mathrm{Spec}(\mathbb{C})) \rangle$. Otherwise, the thick tensor prime ideal $\mathrm{Spc}(\pi^*)(\langle M(\mathrm{Spec}(\mathbb{C})) \rangle)$ would not be proper, which is a contradiction. Analogously, $M(\mathrm{Spec}(\mathbb{R}))$ cannot appear as a direct summand of an object in $\langle M(\mathrm{Spec}(\mathbb{C})) \rangle$. The differentials are arbitrary morphisms in $\mathrm{Cor}_{\mathbb{R},K}^0$ and $\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0$, respectively, such that the composition of two consecutive differentials is 0.

Let $\widetilde{C} \in (\pi^*)^{-1}(\langle M(\mathrm{Spec}(\mathbb{C})) \rangle)$, i.e., $\pi^*(\widetilde{C}) = C \in \langle M(\mathrm{Spec}(\mathbb{C})) \rangle$. The functor π^* is given by applying the additive forgetful functor π degreewise, which is the identity on objects. Hence, $C_i = \widetilde{C}_i$ in each degree i . In particular, $\widetilde{C} \in \langle \widetilde{M}(\mathrm{Spec}(\mathbb{C})) \rangle$. \square

Remark 9.2.9. Applying Proposition 7.3.3 and using the fact that idempotent completion does not change the Balmer spectrum by [Bal05b, Proposition 3.13], we have computed one point of the Balmer spectrum $\mathrm{Spc}(\widetilde{\mathrm{DAM}}^{\mathrm{gm}}(\mathbb{R}; K))$ for K of characteristic 2. The next step is to compute the images under $\mathrm{Spc}(\pi^*)$ of $\langle S \rangle$ and $\langle S, M(\mathrm{Spec}(\mathbb{C})) \rangle$, which we conjecture to be $\langle 0 \rangle$ and $\langle \widetilde{M}(\mathrm{Spec}(\mathbb{C})) \rangle$, respectively. In future work, one could try to find tensor prime ideals in $\mathrm{Spc}(\mathrm{K}_b(\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0))$ that are not images under $\mathrm{Spc}(\pi^*)$.

However, one can show that the functor $\pi : \widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0 \rightarrow \mathrm{Cor}_{\mathbb{R},K}^0$ detects \otimes -nilpotence for morphisms. That is, if $\pi(f) = 0$ for a morphism f in $\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0$, then $f^{\otimes n} = 0$ for some $n \geq 1$. One can also show that this property transfers to the level of chain complexes, but it is not clear whether it transfers to the level of homotopy categories as well. If it does, one can apply [Bal18, Theorem 1.3] and conclude that the induced map on spectra $\mathrm{Spc}(\pi^*)$ is surjective.

In this case, computing the its image already yields a complete description of the Balmer spectrum $\mathrm{Spc}(\mathrm{K}_b(\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0))$. If the above conjecture of the computation of the other preimages proves to be true, the spectrum would actually consist of *less* points than $\mathrm{Spc}(\mathrm{K}_b(\mathrm{Cor}_{\mathbb{R},K}^0))$. This is unexpected, since “decorating with tilde” adds information and one would, in general, rather suspect a surjection in the opposite direction. However, we are in the special case of coefficients in a field K of characteristic 2. Therefore, unexpected results may appear. Still, the author expects $\mathrm{Spc}(\pi^*)$ not to be surjective in this case.

Remark 9.2.10. If the conjecture $\mathrm{Spc}(\pi^*)(\langle S \rangle) = \langle 0 \rangle$ proves to be true, it follows that $\langle 0 \rangle$ is a prime ideal in $\mathrm{K}_b(\widetilde{\mathrm{Cor}}_{\mathbb{R},K}^0)$. In $\mathrm{K}_b(\mathrm{Cor}_{\mathbb{R},K}^0)$, this is not the case. We repeat the argument in $\mathrm{K}_b(\mathrm{perm}(C_2, K)) \simeq \mathrm{K}_b(\mathrm{Cor}_{\mathbb{R},K}^0)$ after applying Corollary 6.2.12. By Frobenius reciprocity,

$$kC_2 \otimes M \cong \mathrm{Ind}_*^{C_2} \mathrm{Res}_*^{C_2} M = \mathrm{Ind}_*^{C_2} 0 = 0$$

for any acyclic complex $M \in \langle S \rangle = \mathrm{K}_{b,ac}(\mathrm{Cor}_{\mathbb{R},K}^0)$. Hence, in particular, $kC_2 \otimes S = 0$.

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