AG Düsseldorf-Wuppertal: Algebra & Geometrie

Seminar: Dessins d'enfants

Sommersemester 2017

Schedule

We meet Thursdays on the five dates indicated below.
Every meeting consists of two talks of 60 minutes each.
Düsseldorf: Time 16:30, Room 25.22.03.73 // **Wuppertal:** Time 16:00, Room HS7 (?)

Meeting	Date	Talk	Speaker
1 (HHU)	20.4.	1	Alejandra Garrido
	20.4.	2	Kevin Langlois
2 (BUW)	4.5.	3	Oihana Garayalde Ocaña
	4.5.	4	Tobias Hemmert
3 (HHU)	18.5.	5	Steffen Kionke
	18.5.	6	Leif Zimmermann
4 (BUW)	1.6.	7	Benno Kuckuck
	1.6.	8	
5 (HHU)	22.6.	9	Andrea Fanelli
	22.6.	10	Sean Tilson

Organizers: Halupczok, Hornbostel, Klopsch, Orlik, Reineke, Schröer, Späth, Zibrowius

Programme: Steffen Kionke

Programme

> *This discovery, which is technically so simple, made a very strong impression on me, and it represents a decisive turning point in the course of my reflections [...]. I do not believe that a mathematical fact has ever struck me quite so strongly as this one, nor had a comparable psychological impact.* – A. Grothendieck[1]

Grothendieck is refering to the discovery of (what he called) *dessins d'enfants*. A dessin d'enfant is a bicoloured graph (each vertex is colored black or white) drawn on a compact oriented topologial surface X such that removing the graph decomposes X into open cells. Such a dessin gives rise to the structure of a Riemann surface on X together with a map to the Riemann sphere which ramifies over the points 0,1 and ∞ (the white vertices are the inverse images of 0, the black vertices the inverse images of 1 and each cell contains an inverse image of ∞). However, the category of compact Riemann surfaces is equivalent to the category of smooth projective complex curves. Belyi's three point theorem states that such a curve admits a dominant morphism to $\mathbb{P}^1(\mathbb{C})$ which ramifies over at most three points if and only if the curve is defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers. The consequence is astonishing: The absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the set of dessins (which can be described as purely combinatorial objects). The aim of this seminar is to learn the basic theory of dessins d'enfants and to get an idea of the action of the absolute Galois group. For more information I recommend to look at the short article [Za03].

Structure of the seminar: Our main reference is the book "Introduction to Compact Riemann Surfaces and Dessins d'Enfants" of Girondo and González-Diez [GGD]. It is available as an eBook if you search for it via the website of the library in Düsseldorf. It is easy to read and contains a lot of examples. Our main interest lies in Chapter 4. However, I tried to include many other sources. The talks vary in length and difficulty. In case you have the impression that your talk is too long, please shorten it – there is no need to hurry (ask me if you are unsure which part to leave out). On the other hand, if your talk is too short: add examples! If you have any questions, please don't hesitate to ask:
`steffen.kionke@uni-duesseldorf.de`

Meeting 1: 20.04.17 in Düsseldorf

1. Introduction to Riemann surfaces

Ref.: parts of 1.1 and 1.2 in [GGD] and II.4 in [Mi] $\rightsquigarrow$ *Riemann surfaces*

The purpose of this talk is to give an introduction to Riemann surfaces. The focus lies on the properties of holomorphic maps between compact Riemann surfaces. Take the viewpoint that the audience is familiar with the general concept of a manifold (smooth or topologcial) and emphasize the specific features of Riemann surfaces.

Give the definition of a Riemann surface [GGD, Def.1.2]. Mention examples: the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ and the upper half plane $\mathbb{H}$ ([GGD, Ex.1.5,1.6]). Define the genus g of a compact Riemann surface S via the Euler characteristic and the formula $\chi(S) = 2 - 2g$.

Define holomorphic maps between Riemann surfaces and introduce the field $M(S)$ of meromorphic functions (1.15, 1.16 [GGD]). Give an example such as the Riemann sphere

[1] Grothendieck: *Esquisse d'un programme*; english translation from [SchL].

(see Prop. 1.26). Discuss the normal form theorem (II. Prop. 4.1 in [Mi]) and define the multiplicity of a holomorphic map at a point P ([Mi, II.Def.4.2]). Define branch point, ramification point, branch value as in [GGD, Def.1.31]. Explain II. Prop. 4.8 in [Mi] and define the degree of holomorphic maps between compact Riemann surfaces. State (and proof) the Riemann-Hurwitz formula (II. Thm. 4.16 in [Mi] or Thm.1.60 & Thm.1.76 in [GGD]). Discuss the local structure of morphisms of Riemann surfaces: After removing the branch points one obtains a finite covering of surfaces (1.2.6 especially Theorem 1.74 in [GGD]).

2. Riemann surfaces, function fields and algebraic curves

Ref.: 1.3 in [GGD] (and parts of I.6 in [Ha]) $\rightsquigarrow$ *algebraic geometry, Riemann surfaces*

The aim of this talk is to discuss the equivalence of the following four categories: (1) compact Riemann surfaces (with non-constant holomorphic maps), (2) function fields of transcendence degree 1, (3) algebraic curves over $\mathbb{C}$ (with dominant rational maps) and (4) smooth projective curves over $\mathbb{C}$ (with dominant morphisms). This is not an easy theorem, so the aim is to give an idea of the proof.

Discuss Sect. 1.3 in [GGD] up to Prop. 1.95 to explain the equivalence of (1), (2) and (3) (you have to leave out some technical details). Mention the correspondence of points on a compact Riemann surface and valuations of its function field (cf. Sect. 3.4: Thm. 3.23). Then clarify that the same description is available for the points of a complex smooth projective curve (see I. Thm. 6.9 in [Ha]). Indicate how this can be used to establish an equivalence of (2), (3) and (4) [Ha, Cor.6.12]. Translate the notions of the previous talk – degree, branch points, multiplicity at a point – to curves (see IV.2 (p.299) in [Ha]).

Meeting 2: 04.05.17 in Wuppertal

3. Definition of dessins d'enfants

Ref.: 4.1 and 4.2.1 (pp. 207–215) in [GGD] $\rightsquigarrow$ *topology*

Mention the concept of an orientation of a smooth manifold (cf. 1.2.1 [GGD]) and deduce that Riemann surfaces are orientable. Give the definition of a *dessin d'enfant* (Def. 4.1). Clarify that a dessin is not just an abstract graph (e.g. Fig. 4.1). Introduce *the permutation representation pair* of a dessin (Sect. 4.1.1). Give a small example and discuss Prop. 4.10 and Prop. 4.13. This is a good opportunity to mention the different definition of dessins given in [Za03]. Finally, explain the triangle decomposition in Sect. 4.2.1 and state Summary 4.15. You may describe the triangle decomposition using an example instead of going through the details.

4. The correspondence between dessins and Belyi pairs

Ref.: 4.2.2 and 4.3 (pp. 215–227) in [GGD] $\rightsquigarrow$ *topology, Riemann surfaces*

The goal of this talk is to explain Theorem 4.25: Equivalence classes of dessins correspond to equivalence classes of Belyi pairs. Discuss Sect. 4.2.2: a dessin on a surface X gives rise a complex structure on X and a Belyi map $X \to \widehat{\mathbb{C}}$ (Summary 4.16). Explain the underlying idea of Lemma 1.80 (cf. Thm. 4.6 and Thm. 8.4 in [Fo]). Prove Prop. 4.18 and give the definition of a Belyi pair (4.19). (Skip the long Example 4.21). Explain with Sect. 4.3 how a Belyi pair can be used to construct a dessin (Prop. 4.22). Combine this to prove Thm. 4.25. Explain how to find the Belyi function associated to a dessin on the Riemann sphere using some examples from Sect. 4.6.1.

Meeting 3: 18.05.17 in Düsseldorf

5. Belyi functions and the abc-conjecture

Ref.: pp. 1–4 in [Wo06], pp. 137–138 in [LZ] $\rightsquigarrow$ *number theory*

The aim of this talk is to give an idea of the connections between Belyi functions and the abc-conjecture. Introduce the abc-conjecture from number theory. Present the ABC-Theorem for polynomials and its relation to Belyi functions (Thm. 1, Prop. 1 and Prop. 2. in [Wo06], another source is 10.1 in [JW]). Then discuss Sect. 2.5.4 in [LZ] and indicate how to construct "high quality triples" via Belyi functions. Explain Examples 2.3.1 and 2.5.16 in [LZ] (how to find the Belyi function?). Attention: in this book the white vertices are not always drawn, they are supposed to lie in the middle of each edge. (Alternatively, if this is too boring for you, explain [El91] instead.)

6. Belyi's three point theorem

Ref.: [Kö04] $\rightsquigarrow$ *algebraic geometry*

The purpose of this talk is to explain Belyi's theorem: *a complex smooth projective curve X is defined over $\overline{\mathbb{Q}}$ if and only if there is a non-constant morphism to $\mathbb{P}^1(\mathbb{C})$ with at most 3 branching values.* The "only if" direction is due to Belyi and is rather elementary. The "if" part is called the "obvious" direction, but is only obvious modulo more difficult results of Weil. The plan is to prove "only if" and to sketch "if" following [Kö04].

Introduce the action of $\mathrm{Aut}(\mathbb{C})$ on complex curves, define the moduli field and explain what it means for a curve to be "defined over K" (1.1–1.4). Mention Ex. 1.7. and Thm. 1.8 without proof. State Belyi's Theorem (3.3) and proof the "only if" direction (various different options: 3.4 – 3.6 in [Kö04], the short proof in [Go14] or [GGD, 3.1.1]). If time permits, sketch the "if" direction.

Meeting 4: 01.06.17 in Wuppertal

7. Fuchsian triangle groups

Ref.: 2.4 in [GGD] $\rightsquigarrow$ *group theory*

State the classical Uniformization Theorem: *the simply connected Riemann surfaces are $\widehat{\mathbb{C}}$,$\mathbb{H}$ and $\mathbb{C}$ (p.81 in [GGD]).* As every connected Riemann surface is covered by one of these, it is of interest to construct groups acting properly discontinuously on the simply connected surfaces. Important examples are triangle groups. The aim is to discuss the hyperbolic triangle groups in detail.

Recall the definition of Fuchsian groups (for $\mathrm{Aut}(\mathbb{H})$ see Prop. 1.27). Mention facts from 2.4.1 and 2.4.2 if needed. Discuss 2.4.3 in detail (Thm. 2.27 can be skipped if there is not enough time). Describe the generators and relations of $\Gamma_{n,m,l}$. Mention the similar construction of spherical and euclidean triangle groups (Rem. 2.30 or [Jo, pp. 139–143]). Then explain 2.4.4: Introduce the groups $\Gamma(1)$ and $\Gamma(2)$ and show that they are triangle groups (Thm. 2.31, Thm. 2.34). Spell out why $\Gamma(2) \cong \Gamma_{\infty,\infty,\infty}$ (rank-2 free group!) and why $\mathbb{H}/\Gamma(2)$ is a sphere with three points removed.

8. The Uniformization Theorem

Ref.: 4.3.1, 4.4. in [GGD] $\rightsquigarrow$ *group theory, Riemann surfaces*

Introduce the monodromy homomorphism of a morphism of compact Riemann surfaces (see 2.7, pp. 148–150) in [GGD]). Discuss the monodromy of a Belyi function in 4.3.1 (Prop. 4.29). Explain Sect. 4.4. in detail. In particular, prove the Uniformization Theorem 4.31 (see also Cor. 1–Cor. 3 in [JS96]). Give an example.

Meeting 5: 22.06.17 in Düsseldorf

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9. The action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on dessins

Ref.: 3.5 and 4.5 in [GGD] $\rightsquigarrow$ *algebraic geometry/Riemann surfaces*

The aim is to define the Galois action on dessins d'enfants and to establish some basic properties. I suggest to use the language of algebraic curves instead of Riemann surfaces.

Remind the audience of the action of $\mathrm{Aut}(\mathbb{C})$ on complex curves (see 1.1. – 1.2 in [Kö04]). State Theorem 3.28 and prove some of the statements. (The proof is written up for Riemann surfaces but most parts seem to translate directly for curves; cf. [Ha, I.6] as in talk 2). Proceed to discuss Sect. 4.5. Use Belyi's theorem (Talk 6) to define an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on (equivalence classes of) dessins. Translate Thm. 3.28 into Thm. 4.46. State Thm. 4.48 on the faithfulness of the Galois action and prove it for genus $g = 0$ and/or $g = 1$. The case $g = 0$ (4.5.1) is elementary but a bit technical. Genus $g = 1$ (4.5.2) is short but needs some preparation: recall the Weierstraß equation of elliptic curves, the definition of the j-invariant and that the j-invariant determines the curve (see III.§1, Prop.1.4 in [Si]).

10. Regular dessins and the Grothendieck-Teichmüller group

Ref.: 4.4.2, 4.4.3 in [GGD] and 5.1–5.4 in [Gu14] $\rightsquigarrow$ *profinite groups*

The aim of this talk is to introduce regular dessins and to define the (coarse) Grothendieck-Teichmüller group following [Gu14]. This is probably too much, but both parts depend only mildly on each other. Decide freely which part you want to emphasize.

Discuss the automorphism group of a dessin [GGD, 4.4.2]. Define regular dessins and explain that they correspond to the Belyi pairs which are Galois coverings (Thm. 4.43 in [GGD]). Proceed to explain parts 5.1–5.4 in [Gu14]. It is not requested that you give all the details. Discuss Theorem 5.4. in [Gu14]: There is an injective homomorphism of profinite groups $\Gamma : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Out}(\widehat{F_2})$. Indicate how $\mathrm{Out}(\widehat{F_2})$ acts on dessins. The image of Γ is contained in a subgroup, the coarse Grothendieck-Teichmüller group. Give the definition of this group (p. 368). If time permits relate it to Drinfeld's definition.

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