AG DÜSSELDORF-WUPPERTAL
ALGEBRA UND GEOMETRIE

ALGEBRAIC GROUPS.

Facts

Where and when: Every second Thursday alternatively in Düsseldorf and Wuppertal.

Website: http://www2.math.uni-wuppertal.de/~hornbost/AGDW.html

The Book: Springer, T. A. *Linear algebraic groups.* Reprint of the 1998 second edition. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2009. xvi+334 pp.

Program

Meeting	Date	Where	Talk	Speaker
1	23.10.2014	Düsseldorf	1	B. Schilson
1	23.10.2014	Düsseldorf	2	C. Bärligea
2	06.11.2014	Wuppertal	3	S. Sagave
2	06.11.2014	Wuppertal	4	S.-T. Stahn
3	20.11.2014	Düsseldorf	5	R. Gonzales
3	20.11.2014	Düsseldorf	6	B. Klopsch
4	04.12.2014	Wuppertal	7	A. Dönmez
4	04.12.2014	Wuppertal	8	M. Bender
5	18.12.2014	Düsseldorf	9	A. Thillaisundaram
5	18.12.2014	Düsseldorf	10	A. Samokhin
6	08.01.2015	Wuppertal	11	M. Kuschkowitz
6	08.01.2015	Wuppertal	12	M. Reineke
7	22.01.2015	Düsseldorf	13	S. Schröer
7	22.01.2015	Düsseldorf	14	N. Perrin
8	05.02.2015	Wuppertal	15	S. Meinhardt
8	05.02.2015	Wuppertal	16	S. Meinhardt

Introduction

In this seminar we want to read the book *Linear Algebraic Groups* by T.A. Springer. The first aim of the seminar is that everyone can gain a basic knowledge on central objects for our research groups: linear algebraic groups.

Since this could be the subject of a full lecture, we will have to skip some details so the program of the talks may seem rather long but we do not want to present all the proofs in their full generality. In fact we want to get an idea of how the classification and the structure theorems for reductive groups work, what the techniques are, where difficulties can occur and where to look for the details if needed.

The topic being at the crossroad of several areas we will be forced to assume some knowledge in group theory and in (very) basic algebraic geometry. We shall recall what we mean by a variety in the first talk. Several other algebro-geometric concepts will be explained during the seminar. If you do not feel comfortable with algebraic geometry, you should read the first chapter of the book.

Most of the time the talks at the same day depend on each other so the speakers should agree on what they will need and discuss.

Day 1: Basics

Talk 1. Algebraic groups: Definitions. In this talk we want to present what we shall mean by *algebraic group* in the seminar. This is based on Chapter 2.

Chapter 1. Reminders on geometry: recall the definition of an affine variety (1.4.3) and of a variety (1.6.9).

2.1. First definitions: deal with Sections (2.1.1), (2.1.2) and (2.1.4). Here are defined algebraic groups and their corresponding algebraic equivalent: Hopf algebras. Examples are important.

2.2. First properties: discuss Sections (2.2.1) up to (2.2.8). The statements should be given without proof. For the exercise (2.2.2) one should simply give examples of connected and disconnected algebraic groups.

2.3. Action of groups: discuss Sections (2.3.1) up to 2.3.7) without the exercises. There should be no proof given except for statements in (2.3.6) and (2.3.7) which should be proven: in particular any affine algebraic group is a closed subgroup of some GL_n whence the name *linear algebraic group*.

2.4. Jordan decomposition: state (2.4.4) without proof and explain how to get (2.4.5). Discuss (2.4.7) and state (2.4.8) and (2.4.9) without proof. Discuss (2.4.11): for non algebraically closed fields the are some issues here. State (2.4.12) and (2.4.13) without proof and prove (if time permits) the last statement (2.4.14).

Talk 2. Commutative Algebraic groups. In this talk we study the structue of commutative algebraic groups. This is based on chapter 3.

3.1. Structure: Prove (3.1.1) and also Exercise (2.4.10.(2)). Discuss without proof (3.1.2) and (3.1.3). A proof of this last statement will be given later on in Talk ...ICI.

3.2. Tori: discuss Sections (3.2.1) up to (3.2.9) and prove (3.2.3), (3.2.6) and (3.2.8). This last statement: rigidity of tori enables to define the Weyl group (3.2.9). Discuss (3.2.11) up to (3.2.15). The example of $\lambda : \mathbb{G}_m \to \mathrm{GL}_n$ given by

$$\lambda(t) = \begin{pmatrix} tI_k & 0 \\ 0 & t^{-1}I_{n-k} \end{pmatrix}$$

is worth mentioning.

Day 2: Derivations and Lie algebras

In this second day we will introduce differential geometric concepts with an algebraic geometric point of view and apply this to groups in order to get Lie algebras.

Talk 3. Differential geometry. 4.1. Derivations and tangent spaces: Discuss Sections (4.1.1) up to (4.1.8) withouh proof. Give some examples out of (4.1.9).

4.2 Separability: we want to introduce the notion of separable morphisms in connection with differentials. Sections (4.2.1) up to (4.2.12) without proof except for (4.2.7) that should be proved: our main use will be Lemma (4.2.7). Sections (4.2.13) , (4.2.14) and (4.2.15) will be used later and should be explained.

4.3 Geometry: here we explain some geometric applications of what have been done before. Our main results will be Theorems (4.3.3), (4.3.6) and (4.3.7) and should be proved. Discuss what is needed for these proofs.

Talk 4. Lie algebras. We apply the previous techniques to groups. The whole Section from (4.4.1) up to (4.4.219 should be discussed with details. We want to understand properly how to go from algebraic groups to Lie algebras. Section (4.4.3) deals with positive characteristic explaining that the groups induced a richer structure that only a Lie algebra structure on its tangent space.

Day 3: Quotients

We go one step further in the interaction between groups and geometry and construct quotients of a groups by a closed subgroup. For this we need few more geometric techniques.

Talk 5. Morphisms. We will discuss several properties of morphisms.

5.1. Topological properties: discuss Sections (5.1.1) up to (5.1.7) only proving Theorem (5.1.6). The example of the twisted cubic (5.1.8.(2)) should be briefly recalled.

5.2. Normality: we prove a weak version of Zariski's Main Theorem. This is Theorem (5.2.8) which should be proved. The rest should be discussed without proof.

5.3. Homogeneous space: here comes the geometric application of what we did so far. Discuss Sections (5.3.1) up to (5.3.4) and prove Theorem (5.3.4).

5.4 Semisimple automorphisms: this will be needed later. Discuss Section (5.4.1) up to (5.4.8) proving Corollaries (5.4.5), (5.4.7) and (5.4.8).

Talk 6. Quotients. The full Section 4.1 should be discussed with details since the construction of quotients is a basic tool for dealing with algebraic groups. If we are lacking of time we could avoid proving Proposition (5.5.10).

Day 4: Borel subgroups and solvable groups

We define the very important class of parabolic subgroups and study the general structure of solvable groups.

Talk 7. Borel and parabolic subgroups. Parabolic subgroups P of G are defined using the properties of the quotient G/P.

6.1. Yet more geometry: complete varieties: discuss the full Section 6.1 proving only Lemma (6.1.5) and Theorem (6.1.6): complete algebraic groups are abelian whence the name *abelian varieties*.

6.2. Parabolic subgroups: discuss Sections (6.2.1) up to (6.2.10). One should prove (6.2.5), (6.2.6) and (6.2.7). Give examples of Borel and parabolic subgroups using (6.2.11).

Talk 8. Solvable groups. One should discuss in details both Sections 6.3 and 6.4 where the structure of solvable groups and the properties of maximal tori and Borel subgroups are discussed. Our main technical result is Theorem (6.3.5) while our main goals are Theorems (6.4.1), (6.4.5)(6.4.9).

Day 5: Root datum

We construct a root system for any reductive group

Talk 9. Semisimple rank 1. We define the root system of a reductive group. This is based on Chapter 7.

7.1. Weyl group: discuss with proof Sections (7.1.1) up to (7.1.5). Discuss the rest of the section 7.1 and prove Theorem (7.1.9).

7.2. Semisimple groups of rank 1: here only state Theorem (7.2.4).

7.3. Reductive groups of semisimple rank 1: prove Proposition (7.3.1) and discuss (7.3.2) and (7.3.3) without proof (explain that (7.3.3) more or less comes from Theorem (7.2.4)). Discuss (7.3.4) up to (7.3.6) without proof. Here again explain that the proof of (7.3.6) follows from the SL_2 or PGL_2 case via Theorem (7.2.4).

7.4. Root datum: explain what a root datum is and define the root datum of a reductive group. This is (7.4.1) up to (7.4.3). Prove (7.4.4): for algebraically closed fields, the root datum is reduced. Prove Proposition (7.4.6).

7.5. Here only briefly explain the results of this section without proof.

7.6. Unipotent radical: if time permits one should prove Theorem (7.6.3) and Corollary (7.6.4).

Talk 10. Structure of reductive groups. We describe the structure of reductive groups. This is based on Chapter 8.

8.1. Structure: this is an important section and should be fully discussed.

8.2. Positive roots: discuss Sections (8.2.1) up to (8.2.10) proving Propositions (8.2.1), (8.2.4) and (8.2.5), Theorem (8.2.8) and Corollary (8.2.10).

Day 6: Applications of the Structure Theorem

Talk 11. First applications of the Structure Theorem. We study the Bruhat decomposition, the structure of parablic subgroups and equivariant line bundles.

8.3. Bruhat decomposition: discuss without proof Section (8.3.1) up to (8.3.4). Prove the results in Sections (8.3.5) up to (8.3.11).

8.4. Parabolic subgroups: prove the Structure Theorem on parabolic subgroups (8.4.3) as well as (8.4.4) and (8.4.5).

8.5. Schubert varieties: discuss Section 8.5 proving only (8.5.2) and (8.5.3).

Equivariant line bundles: explain the construction (8.5.7) and prove Theorem (8.5.8). If there is no time left only prove the easy direction.

Talk 12. Quotients. The problem of constructing quotients for the algebraic action of a (reductive) algebraic group on a variety will be presented and several examples discussed.

Day 7: Chevalley's Theorem

We prove that any algebraic group G has a maximal connected affine (so a linear algebraic group) normal closed subgroup G_{aff} whose quotient is an abelian variety. The map $G \to G/G_{\text{aff}}$ is the Albanese map. This will be based on the first chapter of the book by Brion, Samuel and Uma, *Lectures on the structure of algebraic groups and geometric applications* available at
`http://www-fourier.ujf-grenoble.fr/~mbrion/chennai.pdf`

Talk 15. Chevalley's Theorem part I. Preparation results.
Sections 2.1 and 2.2 (admitting Theorem 2.2.2).

Talk 16. Chevalley's Theorem part II. Proof of the theorem.
Section 2.3.

Day 8: Tannakian Duality and Classification

Talk 17. Tannakian Duality. In this talk we study the close relationship between an affine group scheme and its category of finite dimensional representations. In fact, this category together with its tensor product structure can be used to recover the group entirely. Moreover, every tensor category satisfying some reasonable assumptions and possessing a tensor functor to the category of finite dimensional vector spaces can be realized as the category of representations of a certain affine group scheme. Applications arising from different branches of mathematics will be given. The talk will mostly follow Deligne's article `http://www.jmilne.org/math/xnotes/tc.pdf`.

Talk 18. Classification of semisimple algebraic groups. As an application of Tannakian duality we will give a classification of semisimple algebraic groups following Milne `http://arxiv.org/pdf/0705.1348`. After recalling the notion of a root system, we discuss the classification of semisimple Lie algebras in terms of root systems. In contrast to Springer's approach, we will construct the semisimple algebraic group associated to a given root system by studying a suitable subcategory of the category of representations of the Lie algebra given by the root system. Although this approach works only in characteristic zero, it is very conceptual and makes clear the relation between semisimple Lie algebras, semisimple algebraic groups, and tensor categories.