Jonas Kremer WS 2017/2018

Probability theory

Christmas Exercise Sheet

Exercise 1 *(4 Points)*
Let $(X_n)_{n\in\mathbb{N}} \subset \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ be a sequence of independent, identically distributed random variables. Prove that

$$n\mathbb{P}\left(|X_1| \geq \varepsilon\sqrt{n}\right) \to 0 \text{ as } n \to \infty, \quad \text{for all } \varepsilon > 0.$$

Does $n^{1/2}\max\{|X_1|, \ldots, |X_n|\}$ converges in probability as $n \to \infty$?

Exercise 2 *(4 Points)*
Let X and $(X_n)_{n\in\mathbb{N}}$ be random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that

$$X_n \to X \text{ in probability } \iff \lim_{n\to\infty} \mathbb{E}\left[\frac{|X_n - X|}{1 + |X_n - X|}\right] = 0.$$

Exercise 3 *(4 Points)*
Let X, Y be random variables and suppose that the joint distribution of (X, Y) on $\mathbb{R}^2$ is given by

$$\mu_{(X,Y)}(dx, dy) = \exp\{-x - y\}\mathbb{1}_{\mathbb{R}^2_+}(x, y)dxdy.$$

Show that

(a) the random variables $X + Y$ and X/Y are independent.

(b) the random variables $X + Y$ and $X/(X + Y)$ are independent.

Exercise 4 *(4 Points)*
Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\mathcal{F}, \mathcal{G}$ two σ-algebras on Ω such that $\mathcal{F} \subset \mathcal{G} \subset \mathcal{A}$. Let X be a random variable with $\mathbb{E}[X^2] < \infty$. Show that

$$\mathbb{E}\left[(X - \mathbb{E}[X \mid \mathcal{G}])^2\right] + \mathbb{E}\left[(\mathbb{E}[X \mid \mathcal{G}] - \mathbb{E}[X \mid \mathcal{F}])^2\right] = \mathbb{E}\left[(X - \mathbb{E}[X \mid \mathcal{F}])^2\right].$$

Exercise 5 *(4 Points)*
Let X and Y be two independent $\exp(\lambda)$-distributed random variables $(\lambda > 0)$. Calculate

$$\mathbb{E}\left(X \,|\, X + Y\right) \quad \text{and} \quad \mathbb{P}\left(X \leqslant x \,|\, X + Y\right) \quad \text{for } x \geq 0.$$

Exercise 6 *(4 Points)*
For a $\alpha > 0$ let $(X_n)_{n\in\mathbb{N}}$ be an independent and identically $[0, \alpha]$-uniform distributed sequence of random variables. Let $Y_n := \max\{X_1, \dots, X_n\}$, $Z_n := n(\alpha - Y_n)$, and $\mu_n := \mathbb{P} \circ Z_n^{-1}$. Prove that μ_n converges weakly to μ^α as $n \to \infty$, where μ^α denotes the exponential distribution with parameter $1/\alpha$.

Exercise 7 *(4 Points)*
Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables with corresponding distribution functions

$$F_n(x) = \left(1 - \frac{1}{n}\right) \mathbb{1}_{[0,n)}(x) + \mathbb{1}_{[n,\infty)}(x).$$

Show that X_n converges in distribution to a random variable X with distribution function F. Further, prove that $\mathbb{E}[|X_n|^k]$ does not converge to $\mathbb{E}[|X|^k]$. Is this contradicting Portmanteau's Theorem?

Exercise 8 *(4 Points)*
For two distribution functions F and G let

$$d_L(F, G) := \inf\left\{\varepsilon > 0 \,:\, F(t - \varepsilon) - \varepsilon \leq G(t) \leq F(t + \varepsilon) + \varepsilon \,\forall t \in \mathbb{R}\right\}.$$

One can check that d_L defines a metric on the set of distribution functions. Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of distribution functions, and F be a distribution function.

(a) Suppose $F_n(t) \to F(t)$ as $n \to \infty$ for all continuity points t of F. Prove that

$$d_L(F_n, F) \to 0 \quad \text{as } n \to \infty.$$

(b) Conversely, suppose that $d_L(F_n, F) \to 0$ as $n \to \infty$. Prove that

$$F_n(t) \to F(t) \text{ as } n \to \infty \text{ for all continuity point } t \text{ of } F.$$