# Interacting particle systems with applications to infection problems[1]

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# 1 Introduction

The theory of interacting particle systems (= IPS) is a fast growing area in modern probability and infinite dimensional analysis with various applications in, e.g., mathematical physics, theoretical biology, ecology, social sciences and economy. The aim is to describe the time evolution of a huge collection of interacting entities. Such entities are called (microscopic) particles and are considered, depending on the particular choice of model, as molecules, cells, plants or animals, humans and agents of a market. The collection of all particles, which is typically of order at least $10^4 - 10^{23}$, is called microscopic state. Each particle from this state can, in principle, be described by a physical/ecological/biological mechanism. A detailed understanding of such mechanism yields the possibility to describe the time evolution of the microscopic state by solutions to certain systems of equations. Nevertheless, the complex structure of each particle makes it practically impossible to determine all parameters involved. Moreover, due to the huge number of particles it is hopeless to solve or even provide reasonable simulations for such large systems of equations. As a simplification each particle is therefore modelled as a random process. The parameters of such processes should be chosen in such a way that they fit with the experimental data. Moreover, the huge number of particles is described by statistical properties such as expectations, correlations and particle densities. A mathematical realization of this ideas leads, in the simplest case, to the description of a microscopic state in terms of a Markov process.

In this lecture notes we restrict our attention to IPS where particles may die or create new particles due to random influences. Models of this type are the so-called birth-and-death processes. Classical birth-and-death dynamics are described by a system of ordinary differential equations, also known as Kolmogorov's differential equations, and are usually studied by semigroup methods on (weighted) spaces of summable real-valued sequences, cf. Feller, Kato [Kat54, Fel68, Fel71, HP74]. More recent attempts study such equations on the spaces $\ell^p$ for $p \in [1, \infty)$, see Arlotti, Banasiak [BA06] and others [BLM06, TV06]. All these models have in common that they do not include the positions of the described particles. However, many models from applications are intrinsically based on the positions of particles described (see e.g. the BDLP-model [BP97, BP99, DL00, DL05] but also [Neu01, BCF$^+$14, KM66, SEW05, FFH$^+$15]).

The simplest possibility to include spatial structure is to assign to each particle a fixed site on a graph (e.g. from the lattice $\mathbb{Z}^d$). This are the so-called lattice models. For such models a rigorous study by semigroup methods is adequate and a detailed presentation can be found in the classical book of Liggett [Lig05] and references therein. Several models, such as the BDLP model, require that the positions of the particles are not a priori fixed. This means that $\mathbb{Z}^d$ should be replaced by a continuous location space, e.g. $\mathbb{R}^d$.

Birth-and-death processes in the continuum share several properties with their lattice analogues, but also include numerous unexpected features and require essentially different techniques for their mathematical treatment. Taking into account that they describe real-world particles it leads to the natural assumption that all particles are indistinguishable and any two particles cannot occupy the same position in the location space, say for simplicity $\mathbb{R}^d$. It is commonly used to model such microscopic states $\gamma$ as linear combination of point-masses $\delta_x$, where $x \in \mathbb{R}^d$ is the position of a particle in the system. Hence we may write

$$\gamma = \sum_{k \geq 1} \delta_{x_k}.$$

Here we encounter two different cases which have to be treated by different techniques.

- Finite population if $\gamma(\mathbb{R}^d) < \infty$
- Infinite population if $\gamma(\mathbb{R}^d) = \infty$.

The Markov dynamics corresponding to finite populations can be analysed by a measure-valued generalization of Kolmogorov's differential equations. This equations have been first analysed by Feller [Fel40] and have been afterwards further investigated in the next 60 years, cf. [FMS14] and many others. A summary with applications to interacting particle systems is provided in the book of Chen [Che04]. More recent results in this direction can be found in [EW03, Kol06, Bez15, Fri16] (see also the references therein).

In this lectures we focus on spatial birth-and-death processes for infinite populations. Our aim is to provide a comprehensive introduction to the construction of the corresponding processes and outline how particular properties can be studied. For this purpose we consider a particular example of a spatial branching process where particles may die at constant rate and, moreover, undergo a certain proliferation mechanism.

This lectures are organized as follows. A general description of spatial birth-and-death processes is given in the second section. In order to realize the general approach we introduce in section three the notion of correlation functions. Finally we apply this scheme in the last section.

# 2 Spatial birth-and-death processes

## 2.1 The configuration space

It will be convenient to identify a microscopic state $\gamma$ with a subset of $\mathbb{R}^d$, i.e. we consider the microscopic state as a collection of positions $x \in \mathbb{R}^d$. The state space (= configuration space) is, by definition, the collection of all microscopic states $\gamma$. For technical reasons it is assumed to be the space of all locally finite configurations

$$\Gamma = \{\gamma \subset \mathbb{R}^d \mid |\gamma \cap K| < \infty \quad \text{for all compacts } K \subset \mathbb{R}^d\}.$$

Here and in the following we write $|A|$ for the number of elements in $A \subset \mathbb{R}^d$. Note that each $\gamma \in \Gamma$ can be identified with the locally finite Radon measure $\sum_{x\in\gamma} \delta_x$. The configuration space is equipped with the smallest topology such that

$$\Gamma \ni \gamma \longmapsto \sum_{x\in\gamma} f(x) \in \mathbb{R}$$

is continuous for any continuous functions $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ having compact support.

**Theorem 2.1.** *(see [KK06]) The following assertions hold:*

1. $\Gamma$ *is a Polish space.*
2. $\Gamma$ *is not locally compact.*

Denote by $\mathcal{B}(\Gamma)$ the corresponding Borel-$\sigma$-algebra. It is the smallest $\sigma$-algebra such that

$$\Gamma \ni \gamma \longmapsto |\gamma \cap \Lambda| \in \mathbb{N}_0$$

is measurable for any compact $\Lambda \subset \mathbb{R}^d$.

## 2.2 The birth-and-death rates

Particles, in the framework of spatial birth-and-death processes, may randomly disappear and new particles may appear in the configuration $\gamma$. Death of a particle $x \in \gamma$ is described by the death rate $d(x,\gamma) \geq 0$. Similarly, $b(x,\gamma) \geq 0$ describes the birth rate and distribution of a new particle $x \in \mathbb{R}^d \backslash \gamma$. The following are our guiding examples in this theory.

**Example 2.2.** *Let $m(x) \geq 0$ be bounded and $\varphi \geq 0$ be integrable. Define the relative energy of the configuration $\gamma$ w.r.t. the point $x$ by*

$$E_{\varphi}(x,\gamma) = \sum_{y\in\gamma} \varphi(x-y).$$

*Note that $E_{\varphi}(x,\gamma)$ is not necessarily finite for all $\gamma$. This is, e.g., the case if $\varphi$ has compact support. Particular examples for the death rate are:*

*(i)* $d(x,\gamma) = m(x) + E_{\varphi}(x,\gamma)$.

*(ii)* $d(x,\gamma) = e^{E_{\varphi}(x,\gamma)}$

*Concerning the birth rates our main examples are:*

*(iii)* $b(x,\gamma) = ze^{-E_{\varphi}(x,\gamma)}$ *with* $z \geq 0$.

*(iv)* $b(x,\gamma) = E_{\varphi}(x,\gamma)$.

*New examples may be obtained by taking positive linear combinations of these rates.*

These examples are closely related to infinite populations in the following sense. Consider first (iii) with, for simplicity, $\varphi = 0$. The total birth rate is then

$$\int_{\mathbb{R}^d} b(x,\gamma)dx = \int_{\mathbb{R}^d} zdx = \infty, \quad \forall \gamma \in \Gamma.$$

Roughly speaking the birth mechanism creates infinitely many new particles in $\mathbb{R}^d$ in any (arbitrarily small) period of time $[0,\varepsilon]$, $\varepsilon > 0$. These new particles would be distributed according to a Poisson random measure with intensity measure $zdx$ on $\mathbb{R}^d$.

Example (iv) has an even more sophisticated structure. Suppose that we have given only finitely many particles at time zero, i.e. $|\gamma| < \infty$. Then

$$\int_{\mathbb{R}^d} b(x,\gamma)dx = \sum_{y\in\gamma} \int_{\mathbb{R}^d} \varphi(x-y)dx = \|\varphi\|_{L^1}|\gamma| < \infty,$$

i.e. only finitely many new particles may appear. As a consequence, the configuration of the birth-and-death Markov process (see next section) will have the property $|\gamma| < \infty$ for all moments of time. However, if the initial particle configuration is infinite, then, by the same reasoning, infinitely many particles will be created in any (arbitrarily small) period of time $[0,\varepsilon]$, $\varepsilon > 0$.

More generally, the following particular form of the birth-and-death rates is commonly used:

(A) For each $n \geq 0$ there exist measurable, symmetric functions $D_n, B_n : (\mathbb{R}^d)^n \longrightarrow \mathbb{R}$

$$d(x,\gamma) = \sum_{n=0}^{\infty} \sum_{\{y_1,\dots,y_n\}\subset\gamma} D_n(x;y_1,\dots,y_n), \tag{2.1}$$

$$b(x,\gamma) = \sum_{n=0}^{\infty} \sum_{\{y_1,\dots,y_n\}\subset\gamma} B_n(x;y_1,\dots,y_n). \tag{2.2}$$

Here $D_n(x,y_1,\dots,y_n)$ describes the interaction of a particle at position $x$ and other particles at positions $y_1,\dots,y_n$. The case $n=0$ corresponds to constnat birth-and-death rates (no interactions), whereas $n=1$ describes the important case of pair interactions (see examples (i), (iii), (iv)).

**Exercise 1.** *Let $g : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a function. Prove that for all $\gamma \in \Gamma$, with $|\gamma| < \infty$*

$$\sum_{\xi\subset\gamma} \prod_{x\in\xi} g(x) = \prod_{x\in\gamma}(1+g(x)).$$

**Exercise 2.** *Show that example (ii) satisfies condition (A) with the particular choice*

$$D_n(x;y_1,\dots,y_n) = \prod_{k=1}^{n} \left(e^{\varphi(x-y_k)} - 1\right).$$

*Likewise show that example (iii) satisfies condition (A) with the particular choice*

$$B_n(x;y_1,\dots,y_n) = z\prod_{k=1}^{n} \left(e^{-\varphi(x-y_k)} - 1\right).$$

Note that $d(x,\gamma), b(x,\gamma)$ defined in condition (A) are, in general, only well-defined for $\gamma \in \Gamma$ with $|\gamma| < \infty$. If $|\gamma| = \infty$, then these sums do not need to be absolutely convergent.

**Remark 2.3.** *One can show that*

$$\sum_{n=1}^{\infty} \frac{C^n}{n!} \int_{(\mathbb{R}^d)^n} \left(|D_n(x;y_1,\dots,y_n)| + |B_n(x;y_1,\dots,y_n)|\right) dy_1 \dots dy_n < \infty, \quad \forall x \in \mathbb{R}^d$$

*for some constant $C > 0$ implies that*

$$\int_{\Gamma} (d(x,\gamma) + b(x,\gamma)) d\mu(\gamma) < \infty, \quad \forall x \in \mathbb{R}^d$$

*holds for sufficiently many probability measures $\mu$ on $\Gamma$.*

In order to prove this remark we will need some additional results introduced in section 3. However, we do not want to go into details, all expressions given in the end of this section are therefore only formal.

## 2.3 The Markov operator

Based on the birth-and-death rates $d, b$ we want to study properties of a Markov process $(\gamma_t)_{t\geq 0}$ build by the following two elementary events:

- death of particles: $\gamma \longmapsto \gamma \backslash x$, where $x \in \gamma$.
- birth of particles $\gamma \longmapsto \gamma \cup x$ where $x \notin \gamma$.

This process should have (at least formally) the Markov generator given by the heuristic expression

$$(LF)(\gamma) = \sum_{x\in\gamma} d(x, \gamma\backslash x)(F(\gamma\backslash x) - F(\gamma)) + \int_{\mathbb{R}^d} b(x,\gamma)(F(\gamma \cup x) - F(\gamma))dx, \quad \gamma \in \Gamma. \tag{2.3}$$

For simplicity of notation we write $\gamma\backslash x$, $\gamma \cup x$ instead of $\gamma \cup \{x\}$ and $\gamma\backslash\{x\}$, respectively. In view of assumption (A) this expression is well-defined for any $\gamma$ with $|\gamma| < \infty$ and $F$ suitable chosen. However, similarly to $b$ and $d$, we cannot guarantee that such an expression is well-defined for all $\gamma \in \Gamma$. Additional restrictions on $\gamma$ and $F$ have to be made. Let us briefly describe to possible classes of functions commonly used.

(a) **Additive-type functions.** Motivated by the formulas (2.1) we consider functions $F$ of the form

$$F(\gamma) = \sum_{n=0}^{\infty} \sum_{\{x_1,\dots,x_n\}\subset\gamma} G^{(n)}(x_1, \dots, x_n), \tag{2.4}$$

where $G^{(n)} : (\mathbb{R}^d)^n \longrightarrow \mathbb{R}$ is a sequence of compactly supported, bounded, symmetric and measurable functions. In the case of $n = 0$ we identify $G^{(0)} \in \mathbb{R}$ with the constant function.

(b) **Multiplicative type functions.** Consider the particular case of functions

$$G^{(n)}(x_1, \dots, x_n) = \prod_{k=1}^{n} g(x_k)$$

where $g$ is continuous and has compact support. Then $F$ given by (2.4) satisfies

$$F(\gamma) = \sum_{n=0}^{\infty} \sum_{\{x_1,\dots,x_n\}\subset\gamma} \prod_{k=1}^{n} g(x_k) = \prod_{x\in\gamma} (1 + g(x)), \tag{2.5}$$

where we have used Exercise 1.

## 2.4 The Markov dynamics

### Markov process and Martingale problem

The martingale problem is a mathematical formulation what is meant by saying that a stochastic process $(\gamma_t)_{t\geq 0}$ on $\Gamma$ is a Markov process associated with the generator $L$. Here and below we let $D(L)$ be the collection of all functions either given by (2.4) or by (2.5), respectively.

**Definition 2.4.** *Let $\mu_0$ be a probability measure on $\Gamma$. A stochastic process $(\gamma_t)_{t\geq 0}$ on a stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is said to be a solution to the martingale problem posed by $L$ with initial distribution $\mu_0$ if the following properties hold:*

*(i)* $\gamma_0$ *has law* $\mu_0$.

*(ii) For each* $F \in D(L)$ *the process*

$$F(\gamma_t) - F(\gamma_0) - \int_0^t (LF)(\gamma_s)ds, \quad t \geq 0 \tag{2.6}$$

*is a* $(\mathcal{F}_t)_{t\geq 0}$ *martingale.*

The general theory on martingale problems can be found in [EK86]. Condition (ii) is equivalent to say that $\mathbb{E}(H) = 0$ holds for

$$H := \left( F(\gamma_t) - F(\gamma_s) - \int_s^t (LF)(\gamma_r)dr \right) h_1(\gamma_{t_1}) \cdots h_n(\gamma_{t_n})$$

and all $0 \leq t_1, \ldots, t_n \leq s < t$, $n \geq 1$, $h_1, \ldots, h_n$ continuous and bounded. Hence it is a condition on the finite dimensional distributions of the Markov process. Let us mention two cases where a Markov process has been constructed.

**Example 2.5.** 1. *Contact model, i.e.* $d(x, \gamma) = m > 0$ *constant and*

$$b(x, \gamma) = m + \sum_{y \in \gamma} a^+(x - y)$$

*was considered by Kondratiev, Skorokhod in [KS06]. Existence of a Markov process* $(\gamma_t)_{t\geq 0}$ *was shown. By construction it is localized in a proper subspace* $\Gamma' \subset \Gamma$, *i.e.* $\gamma_t \in \Gamma'$.

2. *A pure probabilistic approach by stochastic differential equations has been developed by Garcia, Kurtz [GK06]. Namely, for* $d(x, \gamma) = 1$ *and a birth intensity with*

$$|b(x, \gamma \cup y) - b(x, \gamma)| \leq a(x, y), \quad x, y \in \mathbb{R}^d$$

*such that* $a(x, y)$ *satisfies some additional continuity condition, existence and uniqueness has been established, and under additional conditions it was shown that this process is ergodic. Again the construction uses deeply that the process can be localized in a proper subspace* $\Gamma'' \subset \Gamma$.

Unfortunately several models from mathematical biology and ecology, see eg. [FFH$^+$15, KK16], are not covered by this results. Any reasonable extension of the techniques developed in [KS06, GK06] is still absent. The main difficulty is related with the possibility to control the number of particles in any bounded volume. It is worth to mention that any stochastic process with càdlàg paths, is necessarily contained in a proper subspace of $\Gamma$. The general theory of martingale problems requires that the state space is at least a locally compact Polish space. Since it is not the case for $\Gamma$, one necessarily has to restrict $\Gamma$ onto a proper subspace. Such sub-spaces may be obtained from proper Lyapunov functionals which, at least on the formal level, exist for many particular models.

**Fokker-Planck equation**

One possible way to construct (and study) a Markov process is related to its transition probabilities. Let us briefly describe how such transition probabilities can be obtained. Suppose that we have given a solution $(\gamma_t)_{t\geq 0}$ to the martingale problem with initial distribution $\mu_0$. Denote by $\mu_t$ the law of $\gamma_t$, e.g. defined by the relation

$$\mathbb{E}(F(\gamma_t)) = \int_\Gamma F(\gamma)d\mu_t(\gamma), \quad F \in D(L), \quad t \geq 0.$$

Taking the expectations in (2.6) and using the martingale property we get

$$\int_\Gamma F(\gamma)d\mu_t(\gamma) = \int_\Gamma F(\gamma)d\mu_0(\gamma) + \int_0^t \int_\Gamma (LF)(\gamma)d\mu_s(\gamma)ds, \quad F \in D(L),\ t \geq 0. \tag{2.7}$$

This is the so-called Fokker-Planck equation. Any solution $(\mu_t)_{t\geq 0}$ to this equation may be regarded as the one-dimensional distribution of a Markov process associated with $L$.

**Remark 2.6.** *The transition probabilities $p_t(\gamma, \cdot)$ of the Markov process $(\gamma_t)_{t\geq 0}$ satisfy the Fokker-Planck equation with initial distribution $\mu_0 = \delta_\gamma$.*

In this lectures we will mainly focus on the Fokker-Planck equation. Due to some technical reasons we also have also to study the (backward) Kolmogorov equation for functions

$$\frac{\partial F_t}{\partial t} = LF_t, \quad F_t|_{t=0} = F_0. \tag{2.8}$$

Until now there does not exist any technique applicable for the study of this equation on spaces of continuous functions. An alternative approach was described in [KKM08] and will be employed for a particular model in the last section.

# 3 Harmonic analysis on the configuration space

## 3.1 The K-transform

In this section we want to exhibit formula (2.4). Let $\Gamma_0 = \{\eta \subset \mathbb{R}^d \mid |\eta| < \infty\}$ be the space of finite configurations. It is equipped with the smallest $\sigma$-algebra such that

$$\Gamma_0 \ni \eta \longmapsto |\eta \cap \Lambda| \in \mathbb{N}_0$$

is measurable for any compact $\Lambda \subset \mathbb{R}^d$.

**Remark 3.1.** *Since $\Gamma_0$ admits the decomposition $\Gamma_0 = \bigsqcup_{n=0}^\infty \Gamma_0^{(n)}$ with $\Gamma_0^{(0)} = \{\varnothing\}$ and $\Gamma_0^{(n)} = \{\eta \in \Gamma_0 \mid |\eta| = n\}$ each function $G$ can be identified with a sequence of symmetric functions $G^{(n)} : (\mathbb{R}^d)^n \longrightarrow \mathbb{R}$ defined by*

$$G^{(n)}(x_1, \dots, x_n) = \begin{cases} G(\{x_1, \dots, x_n\}), & |\{x_1, \dots, x_n\}| = n \\ 0, & \textit{else} \end{cases}. \tag{3.1}$$

*Here we identify $G^{(0)}$ with the constant function. Then*

$$G(\eta) = G^{(n)}(x_1, \ldots, x_n), \quad \eta = \{x_1, \ldots, x_n\}.$$

Denote by $B_{bs}(\Gamma_0)$ the space of all bounded measurable functions $G : \Gamma_0 \longrightarrow \mathbb{R}$ for which there exists $N \in \mathbb{N}$ and a compact $\Lambda$ such that

$$G(\eta) = 0, \quad \text{if } |\eta| > N \text{ or } \eta \cap \Lambda^c \neq \varnothing.$$

Using the identification $G = (G^{(n)})_{n \geq 0}$ we see that $G \in B_{bs}(\Gamma_0)$ if and only if

- There exists $N \in \mathbb{N}$ such that $G^{(n)} = 0$ for $n > N$.
- $G^{(n)}$ is bounded, measurable and has compact support for all $n \geq 0$.

The next definition is motivated by (2.4).

**Definition 3.2.** *The K-transform is for $G \in B_{bs}(\Gamma_0)$ defined by*

$$(KG)(\gamma) = \sum_{\eta \Subset \gamma} G(\eta), \quad \gamma \in \Gamma. \tag{3.2}$$

*Here $\Subset$ means that the sum runs over all finite subsets $\eta$ of $\gamma$.*

**Exercise 3.** *Show that $(KG)(\gamma)$ is well-defined for any $G \in B_{bs}(\Gamma_0)$ and any $\gamma \in \Gamma$. Moreover, prove that*

$$(K^{-1}F)(\eta) = \sum_{\xi \subset \eta} (-1)^{|\eta \backslash \xi|} F(\xi)$$

*satisfies $K^{-1}KG = G$ for any $G \in B_{bs}(\Gamma_0)$.*
***Hint:*** *Use for this purpose the combinatorial relation*

$$\sum_{\xi \subset \eta} \sum_{\zeta \subset \xi} H(\zeta, \eta, \eta \backslash \xi) = \sum_{\zeta \subset \eta} \sum_{\xi \subset \eta \backslash \zeta} H(\zeta, \eta, \xi), \quad \eta \in \Gamma_0.$$

## 3.2 The correlation functions

Let us describe the space of probability measures on which the Fokker-Planck equation will be solved. Additional details in this section can be found in [KK02] (see also the references therein).

**Definition 3.3.** *The Lebesgue-Poisson measure $\lambda$ is defined by the relation*

$$\int_{\Gamma_0} G(\eta) d\lambda(\eta) = G(\varnothing) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} G(\{x_1, \ldots, x_n\}) dx_1 \ldots dx_n$$

*for all $G$ such that*

$$\|G\|_{L^1(\Gamma_0, d\lambda)} := \int_{\Gamma_0} |G(\eta)| d\lambda(\eta) = |G(\varnothing)| + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} |G(\{x_1, \ldots, x_n\})| dx_1 \ldots dx_n < \infty.$$

**Lemma 3.4.** *Let $L^1(\Gamma_0, d\lambda)$ be the Banach space of equivalence classes of functions integrable w.r.t. $\lambda$. Then $B_{bs}(\Gamma_0) \subset L^1(\Gamma_0, d\lambda)$ is dense w.r.t. $\|\cdot\|_{L^1(\Gamma_0,d\lambda)}$.*

*Beweis.* Take $G \in L^1(\Gamma_0, d\lambda)$ and let $(G^{(n)})_{n\geq 0}$ be symmetric functions defined by (3.1). Then

$$\frac{1}{n!}\|G^{(n)}\|_{L^1((\mathbb{R}^d)^n)} = \frac{1}{n!}\int\limits_{(\mathbb{R}^d)^n} |G(\{x_1,\ldots,x_n\})|dx_1\ldots dx_n$$

$$\leq |G(\varnothing)| + \sum_{n=1}^{\infty}\frac{1}{n!}\int\limits_{(\mathbb{R}^d)^n} |G(\{x_1,\ldots,x_n\})|dx_1\ldots dx_n = \|G\|_{L^1(\Gamma_0,d\lambda)} < \infty$$

Hence $G^{(n)} \in L^1((\mathbb{R}^d)^n)$ and we can find a sequence $G_k^{(n)}$ of bounded, symmetric functions with compact support such that $\|G_k^{(n)}\|_{L^1((\mathbb{R}^d)^n)} \leq \|G^{(n)}\|_{L^1((\mathbb{R}^d)^n)}$ and

$$\|G^{(n)} - G_k^{(n)}\|_{L^1((\mathbb{R}^d)^n)} \longrightarrow 0, \quad k \to \infty. \tag{3.3}$$

Define $G_k \in B_{bs}(\Gamma_0)$ by

$$G_k(\eta) = \begin{cases} G_k^{(n)}(x_1,\ldots,x_n), & \eta = \{x_1,\ldots,x_n\}, \quad n \leq k \\ 0, & \eta = \{x_1,\ldots,x_n\}, \quad n > k \end{cases}.$$

Hence we obtain for $N \leq k$

$$\|G - G_k\|_{L^1(\Gamma_0,d\lambda)} = \sum_{n=1}^{k}\frac{1}{n!}\|G^{(n)} - G_k^{(n)}\|_{L^1((\mathbb{R}^d)^n)} + \sum_{n=k+1}^{\infty}\frac{1}{n!}\|G^{(n)}\|_{L^1((\mathbb{R}^d)^n)}$$

$$= \sum_{n=1}^{N}\frac{1}{n!}\|G^{(n)} - G_k^{(n)}\|_{L^1((\mathbb{R}^d)^n)} + \sum_{n=N+1}^{k}\frac{1}{n!}\|G^{(n)} - G_k^{(n)}\|_{L^1((\mathbb{R}^d)^n)} + \sum_{n=k+1}^{\infty}\frac{1}{n!}\|G^{(n)}\|_{L^1((\mathbb{R}^d)^n)}$$

$$\leq \sum_{n=1}^{N}\frac{1}{n!}\|G^{(n)} - G_k^{(n)}\|_{L^1((\mathbb{R}^d)^n)} + 2\sum_{n=N+1}^{\infty}\frac{1}{n!}\|G^{(n)}\|_{L^1((\mathbb{R}^d)^n)}.$$

Since the second series is convergent we can make it arbitrary small by taking $N$ and hence $k$ sufficiently large. For each fixed $N$ the first term tends by (3.3) to zero. □

The Lebesgue-Poisson measure satisfies the following important integration by parts formula.

**Theorem 3.5.** *Let $G : \Gamma_0 \times \Gamma_0 \times \Gamma_0 \longrightarrow \mathbb{R}$ be measurable. Then*

$$\int\limits_{\Gamma_0}\sum_{\xi\subset\eta} G(\xi, \eta\backslash\xi, \eta)d\lambda(\eta) = \int\limits_{\Gamma_0}\int\limits_{\Gamma_0} G(\xi, \eta, \eta\cup\xi)d\lambda(\xi)d\lambda(\eta) \tag{3.4}$$

*whenever one side of the equality is finite for $|G|$.*

*Beweis.* Let $G \geq 0$, then using $\xi = \{x_1, \dots, x_n\}$ and $\eta = \{x_{n+1}, \dots x_{n+m}\}$ we get

$$\begin{aligned}\int_{\Gamma_0}\int_{\Gamma_0} G(\xi,\eta,\eta\cup\xi)d\lambda(\eta)d\lambda(\xi) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{n!}\frac{1}{m!}\int_{(\mathbb{R}^d)^n}\int_{(\mathbb{R}^d)^m} G(\{x\}_1^n, \{x\}_{n+1}^{n+m}, \{x\}_1^{n+m})dx_1^{n+m} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{m=0}^{n}\binom{n}{m}\int_{(\mathbb{R}^d)^l} G(\{x\}_1^m, \{x\}_{m+1}^n, \{x\}_1^n)dx_1^n \\ &= \int_{\Gamma_0}\sum_{\xi\subset\eta} G(\xi, \eta\backslash\xi, \eta)d\lambda(\eta).\end{aligned}$$

This shows the assertion in the case $G \geq 0$. For the general case, let $G = G^+ - G^-$ with $G^\pm \geq 0$. Then

$$\int_{\Gamma_0}\int_{\Gamma_0} G^\pm(\xi,\eta,\eta\cup\xi)d\lambda(\eta) = \int_{\Gamma_0}\sum_{\xi\subset\eta} G^\pm(\xi,\eta\backslash\xi,\eta)d\lambda(\eta).$$

In particular the left-hand side is finite if and only if the right-hand side is finite. (3.4) can be checked by using $G = G^+ - G^-$ and above equality. □

**Definition 3.6.** *Let $\mu$ be a probability measure on $\Gamma$ with finite local moments, i.e.*

$$\int_{\Gamma} |\gamma\cap\Lambda|^n d\mu(\gamma) < \infty, \quad n \geq 0$$

*for all compacts $\Lambda$. The correlation function $k_\mu$ is defined by*

$$\int_{\Gamma}(KG)(\gamma)d\mu(\gamma) = \int_{\Gamma_0} G(\eta)k_\mu(\eta)d\lambda(\eta), \quad G \in B_{bs}(\Gamma_0). \tag{3.5}$$

At this point it is worth to mention that not every probability measure $\mu$ on $\Gamma$ has a correlation function. However, it is possible to characterize the class of probability measures for which the correlation function exists (see [KK02]).

**Lemma 3.7.** *Let $\mu$ be with finite local moments and suppose that its correlation function $k_\mu$ exists. Then*

*(a) $k_\mu$ is uniquely determined by relation* (3.5).

*(b) $k_\mu(\emptyset) = 1$ and $k_\mu \geq 0$.*

*(c) $k_\mu$ satisfies the integrability condition*

$$\int_{\Lambda^n} k_\mu(\{x_1,\dots,x_n\})dx_1\dots dx_n < \infty, \quad \forall n \geq 1$$

*and all compacts $\Lambda$.*

*Beweis.* (a) Suppose that there exist two functions $k_\mu, h_\mu$ with property (3.5). Then

$$\int_{\Gamma_0} G(\eta)k_\mu(\eta)d\lambda(\eta) = \int_{\Gamma_0} G(\eta)h_\mu(\eta)d\lambda(\eta), \quad \forall G \in B_{bs}(\Gamma_0).$$

Since $B_{bs}(\Gamma_0)$ is dense in $L^1(\Gamma_0, d\lambda)$ this equality holds by approximation for all $G \in L^1(\Gamma_0, d\lambda)$ which proves the assertion.
(b) For each $G \in B_{bs}(\Gamma_0)$ with $G \geq 0$ we have by (3.5)

$$0 \leq \int_{\Gamma} (KG)(\gamma)d\mu(\gamma) = \int_{\Gamma_0} G(\eta)k_\mu(\eta)d\lambda(\eta).$$

This equality extends by standard density arguments to all $0 \leq G \in L^1(\Gamma_0, d\lambda)$. Taking $G(\eta) = 0^{|\eta|}$ gives with $KG(\gamma) = 1$

$$k_\mu(\varnothing) = \int_{\Gamma_0} 0^{|\eta|}k_\mu(\eta)d\lambda(\eta) = \mu(\Gamma) = 1.$$

(c) We have

$$\begin{aligned}
&\int_{\Gamma} |\gamma \cap \Lambda|^n d\mu(\gamma) = \int_{\Gamma} \left( \sum_{x \in \gamma} \mathbb{1}_\Lambda(x) \right)^n d\mu(\gamma) = \int_{\Gamma} \sum_{x_1, \ldots, x_n \in \gamma} \mathbb{1}_\Lambda(x_1) \cdots \mathbb{1}_\Lambda(x_n) d\mu(\gamma) \\
&\geq \int_{\Gamma} \sum_{\{x_1, \ldots, x_n\} \subset \gamma} \mathbb{1}_\Lambda(x_1) \cdots \mathbb{1}_\Lambda(x_n) d\mu(\gamma) = \int_{(\mathbb{R}^d)^n} \mathbb{1}_\Lambda(x_1) \cdots \mathbb{1}_\Lambda(x_n) k_\mu(\{x_1, \ldots, x_n\}) dx_1 \ldots dx_n \\
&= \int_{\Lambda^n} k_\mu(\{x_1, \ldots, x_n\}) dx_1 \ldots dx_n.
\end{aligned}$$

□

**Exercise 4.** *Let $\mu$ be a probability measure on $\Gamma$ such that its correlation $k_\mu$ exists and satisfies*

$$k_\mu(\eta) \leq AC^{|\eta|}, \quad \eta \in \Gamma_0 \tag{3.6}$$

*for some constants $A, C > 0$. Prove that $\int_\Gamma e^{\alpha|\gamma \cap \Lambda|}d\mu(\gamma)$ is finite for any compact $\Lambda$ and $\alpha \in \mathbb{R}$.*

**Theorem 3.8.** *Let $\mu$ be a probability measure with finite local moments and suppose that its correlation function exists. Then $K$ can be extended to a bounded linear operator $K : L^1(\Gamma_0, k_\mu d\lambda) \longrightarrow L^1(\Gamma, d\mu)$ with*

$$\|KG\|_{L^1(\Gamma, d\mu)} \leq \|K|G|\|_{L^1(\Gamma, d\mu)} \leq \|G\|_{L^1(\Gamma_0, d\lambda)}.$$

*Moreover, (3.2) holds $\mu$-a.e. for any $G \in L^1(\Gamma_0, d\lambda)$ where the series is absolutely convergent.*

This lemma can be used to prove the following.

**Exercise 5.** *Suppose that the birth-and-death rates $d, b$ satisfy (A) and assume that there exists $C > 0$ such that*

$$\sum_{n=1}^{\infty} \frac{C^n}{n!} \int\limits_{(\mathbb{R}^d)^n} \left(|D_n(x; y_1, \ldots, y_n)| + |B_n(x; y_1, \ldots, y_n)|\right) dy_1 \ldots dy_n < \infty, \quad x \in \mathbb{R}^d.$$

*Show that*

$$\int\limits_{\Gamma} (d(x,\gamma) + b(x,\gamma))\, d\mu(\gamma) < \infty, \quad x \in \mathbb{R}^d$$

*holds for all $\mu$ such that its correlation function exists and satisfies* (3.6)*.*

Let us describe below one example. Let $\rho \geq 0$ be a locally integrable function on $\mathbb{R}^d$. The Poisson measure $\pi_\rho$ on $\Gamma$ is the unique probability measure such that

$$\pi_\rho \left(\{\gamma \in \Gamma \mid |\gamma \cap \Lambda| = n\}\right) = \frac{1}{n!} \left( \int\limits_{\Lambda} \rho(x) dx \right)^n \exp\left( - \int\limits_{\Lambda} \rho(x) dx \right)$$

holds for all $n \geq 0$ and all compacts $\Lambda \subset \mathbb{R}^d$. At this point one should prove that such a measure exists and is uniquely determined (see [AKR98]). Moreover, one can show that

$$\int\limits_{\Gamma} \prod_{x \in \gamma} (1 + g(x))\, d\pi_\rho(\gamma) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \int\limits_{\mathbb{R}^d} g(x)\rho(x) dx \right)^n$$

$$= \int\limits_{\Gamma_0} \prod_{x \in \eta} g(x) \prod_{x \in \eta} \rho(x) d\lambda(\eta)$$

and hence the correlation function for $\pi_\rho$ is given by $k_{\pi_\rho}(\eta) = \prod\limits_{x \in \eta} \rho(x)$.

**Exercise 6.** *Show that for each compact $\Lambda \subset \mathbb{R}^d$*

$$\pi_\rho(\{\gamma \in \Gamma \mid |\gamma \cap \Lambda| > 0\}) = 1 - \exp\left( - \int\limits_{\Lambda} \rho(x) dx \right).$$

**Exercise 7.** *Suppose that $\rho$ is not integrable. Prove that $\Gamma_0 \in \mathcal{B}(\Gamma)$ and show that $\pi_\rho(\Gamma_0) = 0$.*

# 4 Free cell-proliferation

In this section we describe a model for the proliferation of cells. It is assumed that each cell has an exponential distributed lifetime with parameter $m > 0$. Moreover, each cell has another exponential distributed time, the so-called proliferation time, with parameter $\lambda > 0$. The corresponding elementary event is the splitting of a cell at position $x \in \gamma$ into two new cells. The position of the new cells is determined by the probability distribution

$$a(x - y_1, x - y_2) dy_1 dy_2$$

and $a \geq 0$ is assumed to be symmetric in both variable and satisfies $a(x_1, x_2) = a(x_2, x_1)$. The Markov generator is assumed to be given by

$$(LF)(\gamma) = m \sum_{x \in \gamma} (F(\gamma \backslash x) - F(\gamma)) \\ + \lambda \sum_{x \in \gamma} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x - y_1, x - y_2)(F(\gamma \backslash x \cup y_1 \cup y_2) - F(\gamma)) dy_1 dy_2$$

Note that due to the absence of interactions particles corresponding to this Markov dynamics evolve independently of each other. Additional details on this model, but also more general models, can be found in [FFH$^+$15].

## 4.1 The backward Kolmogorov equation

Our aim is to study the backward Kolmogorov equation. For this purpose we follow the general approach described in [KKM08]. We seek for a solution to (2.8) of the form

$$F_t(\gamma) = (KG_t)(\gamma) = \sum_{\eta \Subset \gamma} G_t(\eta).$$

Differentiating this equation (formally) gives

$$\frac{d}{dt} KG_t = LKG_t = KK^{-1}LKG_t$$

and hence $G_t$ should satisfy the Cauchy problem

$$\frac{dG_t}{dt} = \widehat{L}G_t, \quad G_t|_{t=0} = G_0, \tag{4.1}$$

where $\widehat{L} = K^{-1}LK$, i.e. $LK = K\widehat{L}$. In view of Theorem 3.8 and Exercise 5 it is reasonable to seek for solutions in the space of integrable functions.

**Lemma 4.1.** *Let $\widehat{L} := K^{-1}LK$ be defined on $B_{bs}(\Gamma_0)$. Then $\widehat{L} = \widehat{L}_0 + \widehat{L}_1$ is given by*

$$(\widehat{L}_0 G)(\eta) = -(m - \lambda)|\eta| G(\eta) + 2\lambda \sum_{x \in \eta} \int_{\mathbb{R}^d} b(x - y) \left( G(\eta \backslash x \cup y) - G(\eta) \right) dy \tag{4.2}$$

*and*

$$(\widehat{L}_1 G)(\eta) = \lambda \sum_{x \in \eta} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x - y_1, x - y_2) G(\eta \backslash x \cup y_1 \cup y_2) dy_1 dy_2. \tag{4.3}$$

*Here $b \geq 0$ describes the effective proliferation and is given by $b(x) = \int_{\mathbb{R}^d} a(x, y) dy$.*

*Beweis.* Using the $K-$transform we obtain for $x \in \gamma$

$$(KG)(\gamma\backslash x) - (KG)(\gamma) = -\sum_{\eta \Subset \gamma\backslash x} G(\eta \cup x)$$

and therefore for the first part

$$\begin{aligned} m\sum_{x\in\gamma}((KG)(\gamma\backslash x) - (KG)(\gamma)) &= -m\sum_{x\in\gamma}\sum_{\eta\Subset\gamma\backslash x} G(\eta\cup x) \\ &= -m\sum_{\eta\Subset\gamma}\sum_{x\in\eta} G(\eta) = -mK(|\cdot|G)(\gamma), \end{aligned}$$

where we have used the combinatorial relation

$$\sum_{x\in\gamma}\sum_{\eta\Subset\gamma\backslash x} H(x,\eta\cup x) = \sum_{\eta\Subset\gamma}\sum_{x\in\eta} H(x,\eta).$$

For the cell-division we first note that for $x \in \gamma$ and $y_1, y_1 \notin \gamma$

$$\begin{aligned} &(KG)(\gamma\backslash x \cup y_1 \cup y_1) - (KG)(\gamma) \\ &= \sum_{\eta\Subset\gamma\backslash x} \left(G(\eta\cup y_1) + G(\eta\cup y_2) + G(\eta\cup y_1\cup y_2) - G(\eta\cup x)\right). \end{aligned}$$

Therefore the birth-part is given by

$$\sum_{x\in\eta}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x-y_1, x-y_2)\left(G(\eta\backslash x\cup y_1) + G(\eta\backslash x\cup y_2) + G(\eta\backslash x\cup y_1\cup y_2) - G(\eta)\right) dy_1 dy_2.$$

In the first two terms of the second part the integration over $y_1$ and $y_2$ respectively can be carried out, which gives together with the substitution $y_1, y_2 \to y$

$$\begin{aligned} &\lambda\sum_{x\in\eta}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(x-y_1, x-y_2)\left(G(\eta\backslash x\cup y_1) + G(\eta\backslash x\cup y_2)\right) dy_1 dy_2 \\ &= \lambda\sum_{x\in\eta}\int_{\mathbb{R}^d} b(x-y)G(\eta\backslash x\cup y)dy. \end{aligned}$$

Altogether we obtain formulas (4.2) and (4.3). □

Our aim is to prove that the Cauchy problem (4.1) is well-posed in a suitable chosen Banach space. The choice of the Banach space, in general, depends on the particular model one has in mind. In our case it is reasonable to use the Banach space $\mathcal{L}_C$ of equivalence classes of functions with finite norm

$$\|G\|_{\mathcal{L}_C} = \int_{\Gamma_0} |G(\eta)||\eta|!^2 C^{|\eta|} d\lambda(\eta) = |G^{(0)}| + \sum_{n=1}^{\infty} C^n n! \int_{(\mathbb{R}^d)^n} |G^{(n)}(x_1,\ldots,x_n)| dx_1 \ldots dx_n.$$

**Theorem 4.2.** *Suppose that $m \leq \lambda$ and*

$$\kappa := \sup_{y_1, y_2 \in \mathbb{R}^d} \int_{\mathbb{R}^d} a(x - y_1, x - y_2) dx < \infty. \tag{4.4}$$

*Then for any $C > 0$ the operator $(\widehat{L}, B_{bs}(\Gamma_0))$ is closable on $\mathcal{L}_C$ and its closure $(\widehat{L}, D_C(\widehat{L}))$ satisfies the following assertions:*

*(i) $(\widehat{L}, D_C(\widehat{L}))$ is given by the same formulas as above with domain*

$$D_C(\widehat{L}) = \{G \in \mathcal{L}_\alpha \mid \|\widehat{L}_0 G\|_{\mathcal{L}_C} < \infty\}. \tag{4.5}$$

*(ii) $(\widehat{L}, D_C(\widehat{L}))$ is the generator of a strongly continuous, positivity preserving semigroup $\widehat{T}_C(t)$ such that*

$$\|\widehat{T}_C(t) G\|_{\mathcal{L}_C} \leq e^{\frac{\kappa\lambda}{C} t} \|G\|_{\mathcal{L}_C}, \quad t \geq 0. \tag{4.6}$$

*Beweis.* First observe that $\widehat{L}_V$ satisfies

$$(\widehat{L}_0 G)^{(n)} = \widehat{L}_0^{(n)} G^{(n)}, \quad n \in \mathbb{N}, \quad G = (G^{(n)})_{n=0}^{\infty}.$$

where we have $\widehat{L}_0^{(n)} = A_n + B_n$ for $n \in \mathbb{N}$ and

$$(A_n G^{(n)})(x_1, \dots, x_n) = -(m - \lambda) n G^{(n)}(x_1, \dots, x_n)$$

$$(B_n G)^{(n)}(x_1, \dots, x_n) = 2\lambda \sum_{k=1}^{n} \int_{\mathbb{R}^d} b(x_k - y) \Big( G^{(n)}(x_1, \dots, \hat{x}_k, y, \dots, x_n) - G^{(n)}(x_1, \dots, x_n) \Big) dy.$$

Here $\hat{x}_k$ means that integration over the variable $x_k$ should be omitted.
*Step 1.* The operator $\widehat{L}_0^{(n)}$ is bounded on $L^1((\mathbb{R}^d)^n)$ and $(e^{t\widehat{L}_0^{(n)}})_{t \geq 0}$ is a positive semigroup with

$$\|e^{t\widehat{L}_0^{(n)}} G^{(n)}\|_{L^1((\mathbb{R}^d)^n)} \leq e^{(\lambda - m)n} \|G^{(n)}\|_{L^1((\mathbb{R}^d)^n)}, \quad n \geq 0, \quad t \geq 0. \tag{4.7}$$

First, an easy computation shows that

$$\|B_n G^{(n)}\|_{L^1((\mathbb{R}^d)^n)} \leq 4\lambda n, \quad \|A_n G^{(n)}\|_{L^1((\mathbb{R}^d)^n)} = n|m - \lambda| \|G^{(n)}\|_{L^1((\mathbb{R}^d)^n)}$$

and hence $\widehat{L}_0^{(n)}$ is bounded on $L^1((\mathbb{R}^d)^n)$. Since $A_n$ and $B_n$ commute ( $A_n B_n = B_n A_n$ ) we conclude $e^{t\widehat{L}_0^{(n)}} = e^{tA_n} e^{tB_n} = e^{tB_n} e^{tA_n}$ where these exponentials are defined by their Taylor series. By $e^{tA_n} G^{(n)} = e^{-(m-\lambda)nt} G^{(n)}$ this semigroup preserves positivity. The second semigroup $e^{tB_n}$ perserves positivity since it describes the transition probabilities of a Random walk in continuous time on $\mathbb{R}^d$. Alternatively one could also consider another decomposition of $B_n$, argue as for $A_n$ and prove the assertion directly. It remains to prove that $e^{t\widehat{L}_0^{(n)}}$ satisfies (4.7). Take $G \in L^1(\Gamma_0, d\lambda)$ with $G \geq 0$. Then

$$\frac{d}{dt}\|e^{tD_n}G^{(n)}\|_{L^1((\mathbb{R}^d)^n)} = \frac{d}{dt}\int\limits_{(\mathbb{R}^d)} e^{tD_n}G^{(n)}dx_1\dots dx_n = \int\limits_{(\mathbb{R}^d)^n} D_n e^{tD_n}G^{(n)}dx_1\dots dx_n$$
$$= (\lambda - m)n\int\limits_{(\mathbb{R}^d)^n} G^{(n)}dx_1\dots dx_n = (\lambda - m)n\|G^{(n)}\|_{L^1((\mathbb{R}^d)^n)}$$

and hence (4.7) holds with equality. In the general case decompose $G^{(n)}$ into its positive and negative parts.

*Step 2.* Let $e^{t\widehat{L}_0}G := (e^{t\widehat{L}_0^{(n)}}G^{(n)})_{n=0}^{\infty}$ be defined on $\mathcal{L}_C$. Then $(e^{t\widehat{L}_0})_{t\geq 0}$ is a strongly continuous, positive semigroup of contractions on $\mathcal{L}_C$. Moreover, its generator $(\widehat{L}_0, D_C(\widehat{L}_0))$ is given by

$$D_C(\widehat{L}_0) = \left\{ G \in \mathcal{L}_C \mid \sum_{n=1}^{\infty} C^n n! \|\widehat{L}_0^{(n)}G^{(n)}\|_{L^1((\mathbb{R}^d)^n)} < \infty \right\}$$

with $\widehat{L}_0 G = (\widehat{L}_0^{(n)}G^{(n)})_{n=0}^{\infty}$. Finally $B_{bs}(\Gamma_0)$ is a core for this generator. The proof of this step is not difficult, but long. It is left as an exercise for the reader.

*Step 3.* The operator $\widehat{L}_1$ is bounded on $\mathcal{L}_C$.

Take $G \in \mathcal{L}_C$, then we apply (3.4) three times to deduce

$$\|\widehat{L}_1 G\|_{\mathcal{L}_C} = C\lambda \int\limits_{\Gamma_0}\int\limits_{(\mathbb{R}^d)^3} a(x-y_1, x-y_2)|G(\eta \cup y_1 \cup y_1)|C^{|\eta|}(|\eta|+1)!^2 dx dy_1 dy_2 d\lambda(\eta)$$
$$= \frac{\lambda}{C}\int\limits_{\Gamma_0} |G(\eta)|\frac{1}{|\eta|^2}\sum_{y_1\in\eta}\sum_{y_2\in\eta\backslash y_1}\int\limits_{\mathbb{R}^d} a(x-y_1, x-y_2)dx C^{|\eta|}|\eta|!^2 d\lambda(\eta) \leq \frac{\kappa\lambda}{C}\|G\|_{\mathcal{L}_C}.$$

*Step 4.* In this step we now prove the assertions. First using perturbation theory for bounded operators (see [EN00]) we immediately see that $\widehat{L} = \widehat{L}_0 + \widehat{L}_1$ with domain (4.5) is the generator of a strongly continuous semigroup $\widehat{T}_C(t)$ on $\mathcal{L}_C$ The Trotter-product formula gives

$$\widehat{T}_C(t)G = \lim_{n\to\infty}\left(e^{\frac{t}{n}\widehat{L}_0}e^{\frac{t}{n}\widehat{L}_1}\right)^n G, \quad t \geq 0, \ G \in \mathcal{L}_C$$

and hence (4.6) holds. Since $\widehat{L}_1$ preserves positivity, so does $e^{\frac{t}{n}\widehat{L}_1}$ and hence also $\widehat{T}_C(t)$ preserves positivity. It remains to show that $B_{bs}(\Gamma_0)$ is a core for the generator. This is left to the reader. □

As a consequence, the general theory of strongly continuous semigroups (see [EN00]) shows that

$$\frac{dG_t}{dt} = \widehat{L}G_t, \quad G_t|_{t=0} = G_0$$

has for each $G_0 \in D_C(\widehat{L})$ the unique classical solution $\widehat{T}_C(t)G_0$ in $\mathcal{L}_C$.

## 4.2 The evolution of correlation functions

In this section we want to study (2.7) in terms of correlation functions. Suppose that we have given a solution $(\mu_t)_{t\geq 0}$ to (2.7) such that its associated sequence of correlation functions $(k_t)_{t\geq 0}$ exists. By (2.7) together with the definition of $\widehat{L}$ and the $K$-transform we get

$$\int_{\Gamma_0} G(\eta)k_t(\eta)d\lambda(\eta) = \int_{\Gamma_0} G(\eta)k_t(\eta)d\lambda(\eta) + \int_0^t \int_{\Gamma_0} (\widehat{L}G)(\eta)k_s(\eta)d\lambda(\eta)ds$$
$$= \int_{\Gamma_0} G(\eta)k_t(\eta)d\lambda(\eta) + \int_0^t \int_{\Gamma_0} G(\eta)(L^{\Delta}k_s)(\eta)d\lambda(\eta)ds.$$

where the operator $L^{\Delta}$ is adjoint to $\widehat{L}$ defined by the relation

$$\int_{\Gamma_0} (\widehat{L}G)(\eta)k(\eta)d\lambda(\eta) = \int_{\Gamma_0} G(\eta)(L^{\Delta}k)(\eta)d\lambda(\eta), \quad G \in B_{bs}(\Gamma_0)$$

and $k \in \mathcal{K}_C = (\mathcal{L}_C)^*$. The latter space can be identified with the Banach space with norm

$$\|k\|_{\mathcal{K}_C} = \operatorname{ess\,sup}_{\eta\in\Gamma_0} \frac{|k(\eta)|}{|\eta|!^2 C^{|\eta|}}.$$

Above equation is a weak form of the Cauchy problem

$$\frac{dk_t}{dt} = L^{\Delta}k_t, \quad k_t|_{t=0} = k_0. \tag{4.8}$$

Its solutions describe the evolution of correlation functions and therefore determine a solution to (2.7). It is an Markov analogue of the BBGKY-hierarchy known from physics.

**Lemma 4.3.** *For $k : \Gamma_0 \longrightarrow \mathbb{R}$ such that $|k(\eta)| \leq |\eta|!^2 C^{|\eta|}$ for some constant $C > 0$ we have $L^{\Delta} = L_0^{\Delta} + L_1^{\Delta}$, where $L_0^{\Delta}$ is given by the same expression as $\widehat{L}_0$ and $L_1^{\Delta}$ by*

$$(L_1^{\Delta}k)(\eta) = \lambda \sum_{y_1\in\eta} \sum_{y_2\in\eta\setminus y_1} \int_{\mathbb{R}^d} a(x-y_1, x-y_2)k(\eta \cup x\setminus y_1\setminus y_2)dx. \tag{4.9}$$

*Beweis.* The negative multiplication part will not change and for the second part we get

$$\lambda \int_{\Gamma_0} \sum_{x\in\eta} \int_{\mathbb{R}^d} b(x-y)G(\eta\setminus x \cup y)dyk(\eta)d\lambda(\eta) = \lambda \int_{\Gamma_0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} b(x-y)G(\eta\cup y)k(\eta\cup x)dydxd\lambda(\eta)$$
$$= \lambda \int_{\Gamma_0} \sum_{y\in\eta} \int_{\mathbb{R}^d} b(x-y)k(\eta\cup x\setminus y)dxG(\eta)d\lambda(\eta).$$

Finally

$$\begin{aligned}
&\int_{\Gamma_0} (\widehat{L}_1 G)(\eta) k(\eta) d\lambda(\eta) \\
&= \lambda \int_{\Gamma_0} \sum_{x\in\eta} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x-y_1, x-y_2) G(\eta\backslash x \cup y_1 \cup y_2) dy_1 dy_2 k(\eta) d\lambda(\eta) \\
&= \lambda \int_{\Gamma_0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x-y_1, x-y_2) G(\eta \cup y_1 \cup y_2) k(\eta \cup x) dx dy_1 dy_2 d\lambda(\eta) \\
&= \lambda \int_{\Gamma_0} \sum_{y_1\in\eta} \sum_{y_2\in\eta\backslash y_1} \int_{\mathbb{R}^d} a(x-y_1, x-y_2) k(\eta \cup x\backslash y_1 \backslash y_2) dx G(\eta) d\lambda(\eta)
\end{aligned}$$

proves the assertion. □

The following is an immediate consequence of the general semigroup theory. A refined uniqueness statement follows from the general results in [Lem10].

**Corollary 4.4.** *Suppose that $\lambda \leq m$ and (4.4) holds. Then for each $k_0 \in \mathcal{K}_C$ there exists a unique solution to the weak formulation of (4.8). This solution is given by $\widehat{T}_C(t)^* k_0$.*

However, it is possible to solve this equation component-wise under less restrictive assumptions. First observe that the components of $\widehat{L}_0 = L_0^{\Delta}$ are also bounded on $L^{\infty}((\mathbb{R}^d)^n)$ for any $n \geq 0$. Let $k_0 = (k_0^{(n)})_{n=0}^{\infty}$ be non-negative and measurable such that $k_0^{(n)} \in L^{\infty}((\mathbb{R}^d)^n)$, then $k_t^{(n)} := e^{tL_0^{\Delta}} k^{(n)} = e^{(\lambda-m)tn} e^{tB_n} k^{(n)}$ preserves again positivity. Moreover it is the unique component-wise solution to

$$\frac{\partial k_t}{\partial t} = L_0^{\Delta} k_t, \quad k_t|_{t=0} = k_0.$$

Denote by $C_n^{\Delta}$ the operator given by (4.9) taking functions from $n$ variables to functions with $n+1$ variables, i.e.

$$(C_{n+1}^{\Delta} k^{(n)})(x_1, \dots, x_{n+1}) = \lambda \sum_{k=1}^{n+1} \sum_{\substack{j=1 \\ j\neq k}}^{n+1} \int_{\mathbb{R}^d} a(x-x_k, x-x_j) k^{(n)}(x_1, \dots, \hat{x}_k, \hat{x}_j, x, \dots, x_{n+1}) dx.$$

In view of $e^{tL_0^{\Delta}} k_0^{(n)} = e^{(\lambda-m)nt} e^{tB_n t} k^{(n)}$ the solution to (4.8) is given by

$$k_t^{(n+1)} = e^{-(m-\lambda)(n+1)t} e^{tB_{n+1}} k_0^{(n+1)} + \int_0^t e^{-(m-\lambda)(n+1)(t-s)} e^{(t-s)B_{n+1}} C_{n+1}^{\Delta} k_s^{(n)} ds. \tag{4.10}$$

The next statement establishes asymptotic clustering for the evolution of correlation functions constructed above.

**Theorem 4.5.** *For each $k_0 \geq 0$ measurable, such that $k_0^{(n)} \in L^{\infty}((\mathbb{R}^d)^n)$, there exist a unique solution $k_t \geq 0$, given recursively by formula (4.10). Moreover, this solution has the following properties:*

(a) *Suppose that $\kappa$ (given by (4.4)) is finite, then for each initial condition satisfying $k_0(\eta) \leq |\eta|!C^{|\eta|}$ for some constant $C > 0$, this solution obeys the bound*

$$k_t(\eta) \leq |\eta|!(C+t)^{|\eta|}(1+\theta)^{|\eta|}\theta(t)^{|\eta|}e^{-(m-\lambda)|\eta|t}$$

*with $\theta(t) = \max\{1, \lambda, \lambda e^{(m-\lambda)t}\}$.*

(b) *Suppose that there exists $\delta > 0$ such that $a(x,y) \geq \alpha > 0$ for some $\alpha > 0$ and all $|x|, |y| \leq \delta$. Then for each $k_0(\eta) = C^{|\eta|}$ the solution $k_t$ satisfies for any $\eta \in \Gamma_0$ with*

$$\forall x, y \in \eta,\ x \neq y :\ |x-y| < \frac{\delta}{2}$$

*the estimate*

$$k_t(\eta) \geq \beta^{|\eta|}e^{-(m-\lambda)|\eta|t}|\eta|! \quad t \geq 1, \tag{4.11}$$

*where $\beta = \min\{C, |B_\delta|\lambda\alpha\tau\}$ with $\tau = \begin{cases} \frac{1-e^{-(\lambda-m)}}{\lambda-m} & ,\lambda > m \\ 1 & ,\lambda \leq m \end{cases}$ and $|B_\delta|$ is the Lebesgue volume of the ball $B_\delta$ of radius $\delta$.*

(c) *Suppose the that $a(x,y) \geq \alpha > 0$ for $|x| \leq \delta$ and $y \in \mathbb{R}^d$. Then for each $k_0(\eta) = C^{|\eta|}$ the solution satisfies (4.11) for all $\eta \in \Gamma_0$.*

*Beweis.* For the bound from above we proceed by induction on the number of cells $|\eta|$. The first correlation function is given by

$$k_t^{(1)} = e^{-(m-\lambda)t}e^{tB_1}k_0^{(1)}$$

and hence by positivity of $(e^{tB_1})_{t\geq 0}$ and $e^{tB_1}C = C$

$$k_t^{(1)} \leq e^{-(m-\lambda)t}C \leq (C+t)(1+\kappa)\theta(t)e^{-(m-\lambda)t}.$$

For $n \to n+1$ we get with $|\eta| = n+1$

$$\begin{aligned}
k_t^{(n+1)} &\leq e^{-(m-\lambda)(n+1)t}(n+1)!C^{n+1} + \int_0^t e^{-(m-\lambda)(n+1)(t-s)}e^{(t-s)B_{n+1}}C_{n+1}^{\Delta}k_s^{(n)}ds \\
&\leq e^{-(m-\lambda)(n+1)t}(n+1)!C^{n+1} \\
&\quad + (1+\kappa)^{n+1}(n+1)!\lambda n\int_0^t e^{-(m-\lambda)(n+1)(t-s)}(C+s)^n\theta(s)^n e^{-(m-\lambda)ns}ds \\
&\leq e^{-(m-\lambda)(n+1)t}(n+1)!C^{n+1} \\
&\quad + (n+1)!\theta(t)^{n+1}(1+\kappa)^{n+1}\left((C+t)^{n+1} - C^{n+1}\right)e^{-(m-\lambda)(n+1)t} \\
&\leq (n+1)!(C+t)^{n+1}(1+\kappa)^{n+1}\theta(t)^{n+1}e^{-(m-\lambda)(n+1)t}.
\end{aligned}$$

Here we used the fact that for $s \leq t$ we have $\theta(s) \leq \theta(t)$. For the second part let $k_0^{(n)} = C^n$, then $e^{tB_n}k_0 = C^n$ and therefore $k_t^{(1)} = e^{-(m-\lambda)t}C \geq \beta e^{-(m-\lambda)t}$. For $n \to n+1$ and $t \geq 1$ we obtain

$$\begin{aligned} k_t^{(n+1)} &\geq e^{-(m-\lambda)(n+1)t}C^{n+1} + |B_\delta|\lambda\alpha\beta^n \int_0^t e^{-(m-\lambda)(n+1)(t-s)}(n+1)ne^{-(m-\lambda)ns}n!ds \\ &\geq e^{-(m-\lambda)(n+1)t} \int_0^t e^{(m-\lambda)s}ds \cdot (n+1)!|B_\delta|\lambda\alpha\beta^n \\ &\geq e^{-(m-\lambda)(n+1)t}\beta^{n+1}(n+1)! \end{aligned}$$

where we have used for $\lambda > m$

$$\int_0^t e^{-(\lambda-m)s}ds = \frac{1-e^{-(\lambda-m)t}}{\lambda-m} \geq \frac{1-e^{-(\lambda-m)}}{\lambda-m}, \quad t \geq 1$$

and for $\lambda \leq m$

$$\int_0^t e^{-(\lambda-m)s}ds \geq t \geq 1.$$

□

Above estimates show that if the probability distribution $a$ has no hard core, i.e. $a(0) > 0$ for continuous distributions, then the system will consist of clusters. Appearance of such clusters is caused by properties of the operator $L_1^\Delta$. The part $L_1^\Delta$ contains information about asymptotic behaviour, speed of propagation etc., whereas $L_1^\Delta$ contains information about correlations of the system. Assume for simplicity that in the cell-division the position of the new cells are independent of each other. Then we may write $a(x,y) = c(x)c(y)$ for some symmetric function $0 \leq c \in L^1(\mathbb{R}^d)$ normalized to 1. If for example $c$ is continuous and non-vanishing, then previous assumptions are satisfied and we get the bound

$$\beta^n n! e^{-(m-\lambda)nt} \leq k_t^{(n)}.$$

The same results have been shown in [KKP08] for the case $a(x,y) = c(x)\delta(y)$, where each cell creates a new cell and its location is described by the kernel $c$. In contrast to this model, the old cell will not die. Clearly such models should have the same qualitative properties.

**Open problem:** Is it possible to use the ideas developed in [KKP08] to prove that this process has infinitely many invariant measures, characterized as the solutions to the stationary equation

$$L^\Delta k_{\text{inv}} = 0, \quad k_{\text{inv}}(\emptyset) = 1, \quad k_{\text{inv}}(\{x\}) = \rho \in \mathbb{R}.$$

Moreover, does ergodicity holds in the sense that

$$\int_{\Gamma_0} G(\eta)k_t(\eta)d\lambda(\eta) \longrightarrow \int_{\Gamma_0} G(\eta)k_{\text{inv}}(\eta)d\lambda(\eta), \quad t \to \infty$$

for all $G \in B_{bs}(\Gamma_0)$? If yes, can this technique be extended to more complicated models?

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