

Semigroup Approach to Boundary Feedback Systems

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Abstract Boundary

Feedback System



Abstract

Cauchy Problem

Consider an ***Abstract Boundary System***

$$\begin{cases} \dot{x}(t) = A_m x(t), & t \geq 0 \\ \dot{u}(t) = B u(t), & t \geq 0 \end{cases} \quad (\text{ABS})$$

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Consider an **Abstract Boundary Feedback System**

$$\left\{ \begin{array}{ll} \dot{x}(t) = A_m x(t), & t \geq 0 \\ \dot{u}(t) = Bu(t) + Cx(t), & t \geq 0 \\ Lx(t) = u(t), & t \geq 0 \\ x(0) = f_0, \quad u(0) = u_0, & \end{array} \right. \quad (\text{ABS})$$

where

- $x : [0, +\infty) \rightarrow X$, X - the state space,
- $u : [0, +\infty) \rightarrow \partial X$, ∂X - the boundary space,
- $f_0 \in X$, $u_0 \in \partial X$.

Abstract framework

Suppose that the linear operators A_m , B , C and L are related via the diagram:

$$\begin{array}{ccc} D(A_m) \subset X & \xrightarrow{A_m} & X \\ L \downarrow & & \downarrow C \\ D(B) \subset \partial X & \xrightarrow{B} & \partial X \end{array}$$

A_m -the *maximal operator*, describes internal dynamics on X ,

B -describes a dynamics on ∂X ,

L - the *boundary operator*,

C - the *feedback operator*.

Definition

We say that a function $x : [0, +\infty) \rightarrow X$ is a **classical solution** of (ABS) if

1. $x \in C^1([0, +\infty), X)$,
2. $x(t) \in D(A_m)$ for every $t \geq 0$,
3. $Lx \in C^1([0, +\infty), \partial X)$,
4. $u(t) \in D(B)$ for all $t \geq 0$,
5. x satisfies (ABS) .

Main assumptions

The following assumptions are crucial in the sequel:

- 1 The restriction $A_0 \subseteq A_m$, $D(A_0) := \ker L$ generates a strongly continuous semigroup $\{T_0(t)\}_{t \geq 0}$ on X .
- 2 The operator B generates a strongly continuous semigroup $\{S(t)\}_{t \geq 0}$ on ∂X .
- 3 The boundary operator $L : D(A_m) \rightarrow \partial X$ is surjective.
- 4 The operator $\begin{pmatrix} A_m \\ L \end{pmatrix} : D(A_m) \subset X \rightarrow X \times \partial X$ is closed.

Abstract framework

Let an operator $\mathcal{A}_0$ in $X \times \partial X$ be defined by

$$\begin{aligned} D(\mathcal{A}_0) &:= D(A_0) \times D(B) \\ &= \left\{ \begin{pmatrix} f \\ x \end{pmatrix} \in D(A_m) \times D(B) : Lf = 0 \right\}, \end{aligned}$$

$$\mathcal{A}_0 := \begin{pmatrix} A_0 & 0 \\ 0 & B \end{pmatrix}.$$

Then

$$\mathcal{T}_0(t) = \begin{pmatrix} T_0(t) & 0 \\ 0 & S(t) \end{pmatrix}.$$

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$$D(\mathcal{A}) := \left\{ \begin{pmatrix} f \\ x \end{pmatrix} \in D(A_m) \times D(B) : Lf = x \right\},$$

$$\mathcal{A} := \begin{pmatrix} A_m & 0 \\ 0 & B \end{pmatrix}.$$

Suppose now for simplicity, that the feedback operator $C \equiv 0$ and consider the ***Abstract Cauchy Problem***:

$$\begin{cases} \dot{\mathcal{X}}(t) = \mathcal{A}\mathcal{X}(t), & t \geq 0 \\ \mathcal{X}(0) = \mathcal{X}_0, \end{cases} \quad (\text{ACP})$$

Abstract framework

Theorem

The solution of Abstract Boundary Feedback System (ABS) corresponds to solution of the Abstract Cauchy Problem (ACP).

Lemma

Let $\lambda \in \rho(A_0)$. Then

$$D(A_m) = D(A_0) \oplus \ker(\lambda I - A_m).$$

Moreover, the restriction

$$L_\lambda := L|_{\ker(\lambda I - A_m)}$$

is invertible and its inverse, called the *Dirichlet operator*,

$$D_\lambda := (L|_{\ker(\lambda I - A_m)})^{-1}$$

is bounded.

Proof:

Let $\lambda \in \rho(A_0)$. Take $f \in D(A_0) \cap \ker(\lambda I - A_m)$.

Then $(\lambda I - A_0)f = 0$ and hence $f = 0$.

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$$\forall f \in D(A_m) \exists f_1 \in D(A_0) \quad (\lambda I - A_m)f = (\lambda I - A_0)f_1.$$

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Thus for any $f \in D(A_m)$ we have

$$f = f_1 + (f - f_1),$$

where $f_1 \in D(A_0)$ and $f - f_1 \in \ker(\lambda I - A_m)$.

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Since $D(A_0) = \ker L$ and L - surjective, thus

$$L_\lambda := L|_{\ker(\lambda I - A_m)} : \ker(\lambda I - A_m) \rightarrow \partial X \text{ - invertible.}$$

Let $f_n \in \ker(\lambda I - A_m)$ be such that $f_n \rightarrow f \in X$

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Then , denoting

$$\mathcal{V} := \begin{pmatrix} A_m \\ L \end{pmatrix} : D(A_m) \subset X \rightarrow X \times \partial X,$$

we obtain

$$\mathcal{V}f_n = \begin{pmatrix} \lambda f_n \\ Lf_n \end{pmatrix} \rightarrow \begin{pmatrix} \lambda f \\ x \end{pmatrix}.$$

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By the closedness of $\mathcal{V}$ we conclude

$$f \in D(A_m)$$

and

$$\mathcal{V}f = \begin{pmatrix} A_m f \\ Lf \end{pmatrix} = \begin{pmatrix} \lambda f \\ x \end{pmatrix}.$$

Lemma

Let $\lambda \in \rho(A_0)$. Then

$$(\lambda - \mathcal{A}) = (\lambda - \mathcal{A}_0) \begin{pmatrix} I & -D_\lambda \\ 0 & I \end{pmatrix} := \mathcal{W}_\lambda.$$

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i.e., if and only if $x \in D(B)$ and $(f - D_\lambda x) \in \ker(L) \subset D(A_m)$.

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$$D_\lambda x \in \ker(\lambda - A_m) \subset D(A_m) \implies$$

$$f \in D(A_m) \text{ and } L(f - D_\lambda x) = 0 \text{ i.e. } Lf = x.$$

Thus $D(\mathcal{A}) = D(\mathcal{W}_\lambda)$.

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Thus $D(\mathcal{A}) = D(\mathcal{W}_\lambda)$.

$$\begin{aligned} \mathcal{W}_\lambda \begin{pmatrix} f \\ x \end{pmatrix} &= \begin{pmatrix} \lambda - A_m & 0 \\ 0 & \lambda - B \end{pmatrix} \begin{pmatrix} f - D_\lambda x \\ x \end{pmatrix} \\ &= \begin{pmatrix} (\lambda - A_m)f \\ (\lambda - B)x \end{pmatrix} = (\lambda - \mathcal{A}) \begin{pmatrix} f \\ x \end{pmatrix}. \end{aligned}$$

Lemma

If $\lambda \in \rho(A_0)$, then

$$\lambda \in \rho(\mathcal{A}) \iff \lambda \in \rho(B)$$

Claim: $\lambda \in \rho(\mathcal{A}) \iff \lambda \in \rho(B)$

Proof.

Recall $\lambda - \mathcal{A} = (\lambda - \mathcal{A}_0) \begin{pmatrix} I & -D_\lambda \\ 0 & I \end{pmatrix} =: (\lambda - \mathcal{A}_0) \mathcal{R}_\lambda$

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$$R(\lambda, \mathcal{A}) = \mathcal{R}_\lambda^{-1} R(\lambda, \mathcal{A}_0) = \begin{pmatrix} R(\lambda, \mathcal{A}_0) & D_\lambda R(\lambda, B) \\ 0 & R(\lambda, B) \end{pmatrix}$$

Theorem (wellposedness)

If $0 \in \rho(A_0)$, then the following are equivalent

- $\mathcal{A}$ generates $C_0 - (\mathcal{T}(t))$
- (i) $Q(t) = -A_0 \int_0^t T_0(t-s)D_0S(s)ds \in \mathcal{L}(\partial X, X)$
- (ii) $\limsup_{t \rightarrow 0} \|Q(t)\| < \infty$

Claim: $\mathcal{A}$ generator $\iff Q(t) \in \mathcal{L}(\partial X, X)$ and $\limsup_{t \rightarrow 0} \|Q(t)\| < \infty$

Proof.

Consider $(\mathcal{T}(t))_{t \geq 0}$ defined as

$$\mathcal{T}(t) := \begin{pmatrix} T_0(t) & Q(t) \\ 0 & S(t) \end{pmatrix}$$

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$$\mathcal{T}(t) \in \mathcal{L}(X) \iff Q(t) \in \mathcal{L}(\partial X, X)$$

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$$\mathcal{T}(t) \in \mathcal{L}(X) \iff Q(t) \in \mathcal{L}(\partial X, X)$$

$$\mathcal{T}(t)\mathcal{T}(s) = \mathcal{T}(t+s) \iff T_0(t)Q(s) + Q(t)S(s) = Q(t+s)$$

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$$t \mapsto \mathcal{T}(t) \text{ strongly cont.} \iff t \mapsto Q(t) \text{ strongly cont. in } 0$$



Recall: $Q(t) = -A_0 \int_0^t T_0(t-s) D_0 S(s) ds$

- Integration by parts yields

$$Q(t) = D_0 S(t) - T_0(t) D_0 - \int_0^t T_0(t-s) D_0 S(s) B \text{ on } D(B)$$

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- $Q(t)$ strongly continuous on $D(B)$
- Since, $\limsup_{t \rightarrow 0} \|Q(t)\| < \infty$, $Q(t)$ strongly continuous on ∂X by uniform boundedness principle

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Yes, because

$$\mathcal{L}(\mathcal{T}(\cdot))(\lambda) = R(\lambda, \mathcal{A}) \quad \text{for big } \lambda$$

Corollary

If A_0, B generate analytic semigroups, then $\mathcal{A}$ generates analytic semigroup.

First example

Let $t \in [0, T]$,

$$\dot{x}(t) = A_m x(t),$$

$$\dot{u}(t) = Bu(t) + Cx(t),$$

$$Lx(t) = u(t),$$

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Let $t \in [0, T]$,

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$$\dot{x}_{|\partial\Omega}(t) = \Delta_{\partial\Omega} x_{|\partial\Omega}(t) - \mu(z) x_{|\partial\Omega}(t) + Cx(t),$$

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The operators

$$X = L^2(\Omega), \partial X = L^2(\partial\Omega).$$

$$\begin{aligned} A_m &= \Delta_\Omega, \\ D(A_m) &= D(\Delta_\Omega) \subset L^2(\Omega). \end{aligned}$$

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$$\begin{aligned} B &= \Delta_{\partial\Omega} - \mu(\cdot), \\ D(B) &\subset L^2(\partial\Omega) \quad \mu(z) \geq \varepsilon > 0. \end{aligned}$$

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$$C = 0.$$

The Cauchy-problem

Is the Cauchy-problem well-posed?

$$\dot{\mathcal{X}} = \begin{pmatrix} \dot{x} \\ \dot{x}|_{\partial\Omega} \end{pmatrix} = \begin{pmatrix} \Delta_{\Omega} & 0 \\ 0 & \Delta_{\partial\Omega} - \mu(\cdot) \end{pmatrix} \begin{pmatrix} x \\ x|_{\partial\Omega} \end{pmatrix} = \mathcal{A}\mathcal{X},$$

$$\mathcal{X}(0) = \begin{pmatrix} f_0 \\ u_0 \end{pmatrix},$$

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} f \\ u \end{pmatrix} \in D(\Delta_{\Omega}) \times D(B) \mid Lf = u \right\}.$$

Proof well-posedness

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Δ_Ω on $D(A_0)$ generates a analytic semigroup $(T_0(t))_{t \geq 0}$?

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What is A_0 ? $L : D(\Delta_\Omega) \rightarrow L^2(\partial\Omega)$

$$D(A_0) = \ker(L) = \{x \in D(\Delta_\Omega) \mid x|_{\partial\Omega} = 0\},$$

$$A_0 = \Delta_\Omega|_{D(A_0)}.$$

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2.

$\Delta_{\partial\Omega} - \mu(\cdot)$ on $D(B)$ generates an analytic semigroup
 $(S(t))_{t \geq 0}$?

Proof well-posedness

2.

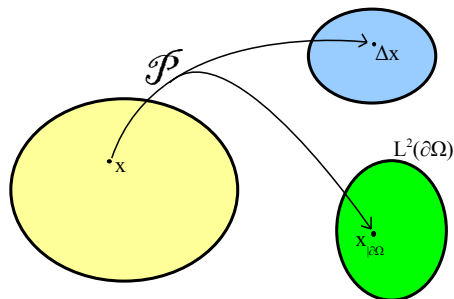
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Proof well-posedness

3.

$L : x \mapsto x|_{\partial\Omega}$ surjective?

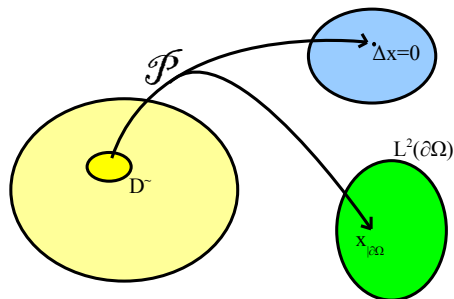


$\mathcal{P} : x \mapsto (\Delta_{\Omega} x, x|_{\partial\Omega})$
Isomorphism.

Proof well-posedness

3.

$L : x \mapsto x|_{\partial\Omega}$ surjective?



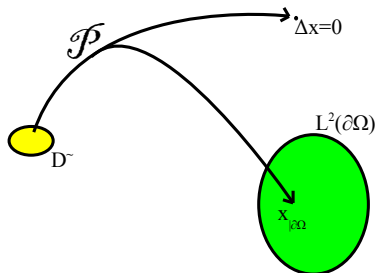
$$D^\sim = \{x \in D(\mathcal{P}) \mid \Delta x = 0\}$$

$\mathcal{P} : x \mapsto (\Delta_\Omega x, x|_{\partial\Omega})$
Isomorphism.

Proof well-posedness

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$L : x \mapsto x|_{\partial\Omega}$ surjective?



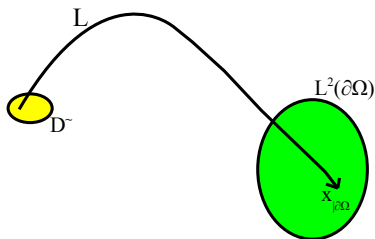
$$D^\sim = \{x \in D(\mathcal{P}) \mid \Delta x = 0\}$$

$\mathcal{P}|_{D^\sim} : x \mapsto (0, x|_{\partial\Omega})$
Isomorphism.

Proof well-posedness

3.

$L : x \mapsto x|_{\partial\Omega}$ surjective?

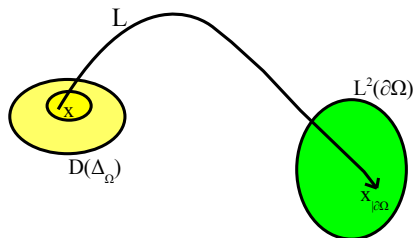


$L|_{D^-} : x \mapsto x|_{\partial\Omega}$
Isomorphism.

Proof well-posedness

3.

$L : x \mapsto x|_{\partial\Omega}$ surjective?



$$D^\sim \subset D(\Delta_\Omega) \Rightarrow$$

$$L|_{D(\Delta_\Omega)} : x \mapsto x|_{\partial\Omega} \text{ surjective.}$$

Proof well-posedness

4.

$$\begin{pmatrix} \Delta_\Omega \\ L \end{pmatrix} : D(\Delta_\Omega) \rightarrow L^2(\Omega) \times L^2(\partial\Omega) \text{ closed?}$$

For

$$x_n \rightarrow x \text{ in } L^2(\Omega)$$

$$\Delta_\Omega x_n \rightarrow g \text{ in } L^2(\Omega)$$

and

$$x_n|_{\partial\Omega} \rightarrow \varphi \text{ in } L^2(\partial\Omega)$$

Proof well-posedness

4.

$$\begin{pmatrix} \Delta_\Omega \\ L \end{pmatrix} : D(\Delta_\Omega) \rightarrow L^2(\Omega) \times L^2(\partial\Omega) \text{ closed?}$$

For

$$\begin{aligned} x_n &\rightarrow x \text{ in } L^2(\Omega) \\ \Delta_\Omega x_n &\rightarrow g \text{ in } L^2(\Omega) \\ &\text{and} \\ x_n|_{\partial\Omega} &\rightarrow \varphi \text{ in } L^2(\partial\Omega) \end{aligned}$$

must hold that

$$x \in D(\Delta_\Omega), \quad \Delta_\Omega x = g, \quad x|_{\partial\Omega} = \varphi.$$



Analytic semigroups

Zhandong:

A_0 generates $(T_0(t))_{t \geq 0}$ analytic

and

B generates $(S(t))_{t \geq 0}$ analytic

$\Rightarrow$

$\mathcal{A} = \begin{pmatrix} \Delta_\Omega & 0 \\ 0 & \Delta_{\partial\Omega} - \mu(\cdot) \end{pmatrix}$ generates $(\mathcal{T}(t))_{t \geq 0}$ analytic.

The Cauchy-problem

Now the Cauchy-problem is well-posed.

$$\dot{\mathcal{X}} = \begin{pmatrix} \dot{x} \\ \dot{x}|_{\partial\Omega} \end{pmatrix} = \begin{pmatrix} \Delta_{\Omega} & 0 \\ 0 & \Delta_{\partial\Omega} - \mu(\cdot) \end{pmatrix} \begin{pmatrix} x \\ x|_{\partial\Omega} \end{pmatrix} = \mathcal{A}\mathcal{X},$$

$$\mathcal{X}(0) = \begin{pmatrix} f_0 \\ u_0 \end{pmatrix},$$

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} f \\ u \end{pmatrix} \in D(\Delta_{\Omega}) \times D(B) \mid Lf = u \right\}.$$

$$\sigma(\tilde{\mathcal{A}})?$$

$$\sigma(\tilde{\mathcal{A}})?$$

Proposition

If $\lambda \in \rho(\mathbf{A}_0)$, then

$$\lambda \in \rho(\tilde{\mathcal{A}}) \iff \lambda \in \rho(\mathbf{B} + \mathbf{C}\mathbf{D}_\lambda)$$

$$R(\lambda, \tilde{\mathcal{A}})$$

$$=$$

$$\begin{pmatrix} R(\lambda, A_0) + D_\lambda R(\lambda, B + CD_\lambda)CR(\lambda, A_0) & D_\lambda R(\lambda, B + CD_\lambda) \\ R(\lambda, B + CD_\lambda)CR(\lambda, A_0) & R(\lambda, B + CD_\lambda) \end{pmatrix}$$

$$R(\lambda, \tilde{\mathcal{A}}) = \begin{pmatrix} R(\lambda, A_0) + \star & \star \\ \star & R(\lambda, B + CD_\lambda) \end{pmatrix}$$

$$s(A_0) < 0 \implies s(\tilde{\mathcal{A}}) < 0$$

$$s(A_0) < 0 \stackrel{C}{\implies} s(\tilde{\mathcal{A}}) < 0 \text{ ???}$$

Theorem

If $C, D_\lambda \geq 0$ for a $\lambda > s(A_0)$, then

$$s(\tilde{\mathcal{A}}) < 0 \iff s(A_0) < 0$$

Theorem

If $C, D_\lambda \geq 0$ for a $\lambda > s(A_0)$, then

$$s(\tilde{\mathcal{A}}) < 0 \iff s(A_0) < 0 \text{ \& } s(B + CD_0) < 0$$

$$R(\lambda, \tilde{\mathcal{A}}) = \begin{pmatrix} R(\lambda, A_0) + \star & \star \\ \star & R(\lambda, B + CD_\lambda) \end{pmatrix}$$

Proposition

If $\lambda \in \rho(A_0)$, then

$$\lambda \in \rho(\tilde{\mathcal{A}}) \iff \lambda \in \rho(B + CD_\lambda)$$

Second example

Let $t \in [0, T]$,

$$\dot{x}(t) = \Delta_{\Omega} x(t),$$

$$\dot{x}|_{\partial\Omega}(t) = \Delta_{\partial\Omega} x|_{\partial\Omega}(t) - \mu(z)x|_{\partial\Omega}(t) + Cx(t),$$

$$Lx(t) = x|_{\partial\Omega}(t),$$

$$x(0) = f_0, \quad u(0) = u_0.$$

Second example

Let $t \in [0, T]$,

$$\dot{x}(t) = \Delta_{\Omega} x(t),$$

$$\dot{x}|_{\partial\Omega}(t) = \Delta_{\partial\Omega} x|_{\partial\Omega}(t) - \mu(z)x|_{\partial\Omega}(t) + \int_{\Omega} k(y, \cdot) x(t, y) dy,$$

$$Lx(t) = x|_{\partial\Omega}(t),$$

$$x(0) = f_0, \quad u(0) = u_0.$$

The operators

$$X = L^2(\Omega), \partial X = L^2(\partial\Omega).$$

$$A_m = \Delta_\Omega,$$

$$D(A_m) = D(\Delta_\Omega) \subset L^2(\Omega).$$

$$B = \Delta_{\partial\Omega} - \mu(\cdot),$$

$$D(B) \subset L^2(\partial\Omega) \quad \mu(z) \geq \varepsilon > 0.$$

The operators

$$X = L^2(\Omega), \partial X = L^2(\partial\Omega).$$

$$\begin{aligned} A_m &= \Delta_\Omega, \\ D(A_m) &= D(\Delta_\Omega) \subset L^2(\Omega). \end{aligned}$$

$$\begin{aligned} B &= \Delta_{\partial\Omega} - \mu(\cdot), \\ D(B) &\subset L^2(\partial\Omega) \quad \mu(z) \geq \varepsilon > 0. \end{aligned}$$

$$\begin{aligned} (Cg)(z) &= \int_\Omega \underbrace{k(y,z)}_{\geq 0} g(y) dy, \quad g \in L^2(\Omega), \quad z \in \partial\Omega. \\ C &: L^2(\Omega) \rightarrow L^2(\partial\Omega). \end{aligned}$$

The Cauchy-problem

$$\dot{\mathcal{X}} = \begin{pmatrix} \dot{x} \\ \dot{x}|_{\partial\Omega} \end{pmatrix} = \begin{pmatrix} \Delta_{\Omega} & 0 \\ \int_{\Omega} k(y, \cdot) dy & \Delta_{\partial\Omega} - \mu(\cdot) \end{pmatrix} \begin{pmatrix} x \\ x|_{\partial\Omega} \end{pmatrix} = \tilde{\mathcal{A}}\mathcal{X},$$

$$\mathcal{X}(0) = \begin{pmatrix} f_0 \\ u_0 \end{pmatrix},$$

$$D(\tilde{\mathcal{A}}) = \left\{ \begin{pmatrix} f \\ u \end{pmatrix} \in D(\Delta_{\Omega}) \times D(B) \mid Lf = u \right\}.$$

$\tilde{\mathcal{A}}$ generates $(\tilde{T}(t))_{t \geq 0}$.

Question

Question:

If the semigroup $(T_0(t))_{t \geq 0}$ generated by A_0 is exp. stable,
for which boundary feedback operators

$C : L^2(\Omega) \rightarrow L^2(\partial\Omega)$ does the semigroup $(\tilde{T}(t))_{t \geq 0}$
remain exp. stable?

Theorem

If

- ① $L^2(\Omega)$, $L^2(\partial\Omega)$ are Banach lattices,
- ② A_0 and B generate positive semigroups $(T_0(t))_{t \geq 0}$,
 $(S(t))_{t \geq 0}$,
- ③ $C : L^2(\Omega) \rightarrow L^2(\partial\Omega)$ is bounded,
- ④ $C \geq 0$ and $D_\lambda \geq 0$ for all $\lambda \in \mathbb{R}$ large enough,

then

$$[s(A_0) < 0 \quad \text{and} \quad s(B + CD_0) < 0] \quad \Leftrightarrow \quad s(\tilde{\mathcal{A}}) < 0.$$



$(\tilde{T}(t))_{t \geq 0}$ is exp. stable

Theorem

If

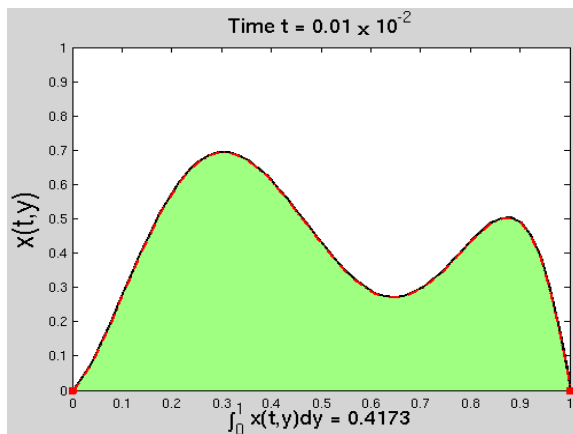
- 1 $L^2(\Omega), L^2(\partial\Omega)$ are Banach lattices,



Theorem

If

- 1 $L^2(\Omega)$, $L^2(\partial\Omega)$ are Banach lattices,
- 2 A_0 and B generate positive semigroups $(T_0(t))_{t \geq 0}$,
 $(S(t))_{t \geq 0}$,



Theorem

If

- 1 $L^2(\Omega)$, $L^2(\partial\Omega)$ are Banach lattices,
- 2 A_0 and B generate positive semigroups $(T_0(t))_{t \geq 0}$,
 $(S(t))_{t \geq 0}$,
- 3 $C : L^2(\Omega) \rightarrow L^2(\partial\Omega)$ is bounded,

$$C : L^2(\Omega) \rightarrow L^2(\partial\Omega)$$
$$\int_{\Omega} k(y, z) g(y) dy \leq \underbrace{\|k(\cdot, z)\|_{L^2(\Omega)}}_{< \infty} \|g\|_{L^2(\Omega)}.$$

Theorem

If

- ① $L^2(\Omega)$, $L^2(\partial\Omega)$ are Banach lattices,
- ② A_0 and B generate positive semigroups $(T_0(t))_{t \geq 0}$,
 $(S(t))_{t \geq 0}$,
- ③ $C : L^2(\Omega) \rightarrow L^2(\partial\Omega)$ is bounded,
- ④ $C \geq 0$ and $D_\lambda \geq 0$ for all $\lambda \in \mathbb{R}$ large enough,

$$(Cg)(z) = \int_{\Omega} \underbrace{k(y, z)}_{\geq 0} \underbrace{g(y)}_{\geq 0} dy \geq 0 \quad \forall g \geq 0$$

$D_\lambda \geq 0$ Hopf maximum principle.

Theorem

If

- ① $L^2(\Omega)$, $L^2(\partial\Omega)$ are Banach lattices,
- ② A_0 and B generate positive semigroups $(T_0(t))_{t \geq 0}$, $(S(t))_{t \geq 0}$,
- ③ $C : L^2(\Omega) \rightarrow L^2(\partial\Omega)$ is bounded,
- ④ $C \geq 0$ and $D_\lambda \geq 0$ for all $\lambda \in \mathbb{R}$ large enough,

then

$$s(A_0) < 0$$



$(T_0(t))_{t \geq 0}$ is exp. stable

Theorem

If

- ① $L^2(\Omega)$, $L^2(\partial\Omega)$ are Banach lattices,
- ② A_0 and B generate positive semigroups $(T_0(t))_{t \geq 0}$,
 $(S(t))_{t \geq 0}$,
- ③ $C : L^2(\Omega) \rightarrow L^2(\partial\Omega)$ is bounded,
- ④ $C \geq 0$ and $D_\lambda \geq 0$ for all $\lambda \in \mathbb{R}$ large enough,

then

$$[s(A_0) < 0 \quad \text{and} \quad s(B + CD_0) < 0] \quad \Leftrightarrow \quad s(\tilde{\mathcal{A}}) < 0.$$



$(\tilde{T}(t))_{t \geq 0}$ is exp. stable

Lets look at

$$s(B + CD_0).$$

Lets look at

$$s(B + CD_0).$$

$$D_0 = (L_{\ker A_m})^{-1} : L^2(\partial\Omega) \rightarrow \ker A_m.$$

$$\Rightarrow D_0 x \in \ker \Delta_\Omega \quad \Rightarrow \quad \Delta_\Omega(D_0 x) = 0$$

$$\Rightarrow D_0 x \text{ is a harmonic function!}$$

Our message

For our system is

stability independent of diffusion.