Invertible solutions of the Lyapunov equation

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1 Introduction

In this talk we study the implications of the existence of a bounded and boundedly invertible solution, $X$, to the Sylvester equation

$$\langle A_1 z_1, X z_2 \rangle + \langle z_1, X A_2 z_2 \rangle = 0,$$

(1)

where $z_1 \in D(A_1)$, $z_2 \in D(A_2)$. $A_1$ and $A_2$ are closed, densely defined linear operators on $Z$ with $Z$ a Hilbert space.
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$$A_1^* X = -X A_2.$$  

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We show that this does not hold in general. We begin by deriving an equivalent condition to (1) when $A_1$ and $A_2$ are both infinitesimal generators.
Lemma

Let $A_1$, $A_2$ be the infinitesimal generators of the $C_0$-semigroups $(T_1(t))_{t \geq 0}$ and $(T_2(t))_{t \geq 0}$, respectively. Then $X \in \mathcal{L}(Z)$ satisfies the Sylvester equation

$$\langle A_1 z_1, X z_2 \rangle + \langle z_1, X A_2 z_2 \rangle = 0, \quad z_1 \in D(A_1), z_2 \in D(A_2)$$

if and only if

$$T_1^*(t) X T_2(t) = X, \quad \text{for all } t \geq 0.$$
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T_1^*(t) X T_2(t) = X, \quad \text{for all } t \geq 0.
$$

If $X$ is (boundedly) invertible, then

$$
X^{-1} T_1^*(t) X T_2(t) = I, \quad \text{for all } t \geq 0.
$$

Thus $\left( X^{-1} T_1^*(t) X \right)_{t \geq 0}$ is the left-inverse of $(T_2(t))_{t \geq 0}$. □
Proof: Easy by differentiating (one direction) or (other implication) substituting $z_1 = T_1(t)z_{10}$ and $z_2 = T_2(t)z_{20}$, with $z_{10} \in D(A_1)$, $z_{20} \in D(A_2)$. □
2 Left-Invertibility

We begin with the definition of a left-invertible semigroup.

**Definition**

The $C_0$-semigroup $(T(t))_{t \geq 0}$ is left-invertible if there exists a function $t \mapsto m(t)$ such that $m(t) > 0$ and for all $z_0 \in Z$ there holds

$$m(t) \| z_0 \| \leq \| T(t) z_0 \|, \quad t \geq 0.$$  

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Louis and Wexler, 1983, showed the following equivalence.
Theorem

Let \((T(t))_{t \geq 0}\) be a \(C_0\)-semigroup on the Hilbert space \(Z\). Then the following are equivalent:

1. \((T(t))_{t \geq 0}\) is left-invertible;

2. There exists a \(C_0\)-semigroup \((S(t))_{t \geq 0}\) such that \(S(t)T(t) = I\) for all \(t \geq 0\).
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Hence if left-invertible, then the left-inverse can be chosen as a semigroup.
Theorem

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\[\square\]

Hence if left-invertible, then the left-inverse can be chosen as a semigroup. The proof of Louis and Wexler uses optimal control. We present a new/adjusted proof using an invertible solution of a Lyapunov equation.
Proof:

2. \( \Rightarrow 1. \) is trivial. So we concentrate on the other implication.

Let \( A \) be the infinitesimal generator of \( (T(t))_{t \geq 0} \). Choose \( \omega \in \mathbb{R} \) such that \( A - \omega I \) is exponentially stable.

Now for \( z \in Z \)

\[
m_0 \| z \|^2 = \int_0^\infty m(t)^2 e^{-2\omega t} \| z \|^2 dt
\leq \int_0^\infty e^{-2\omega t} \| T(t)z \|^2 dt \leq M \| z \|^2
\]

Define \( \langle z, Xz \rangle = \int_0^\infty e^{-2\omega t} \| T(t)z \|^2 \). Then
\begin{itemize}
  \item $m_0 I \leq X \leq MI$
  \item $X$ is the solution to the Lyapunov equation
    \[
    \langle (A - \omega I)z_1, Xz_2 \rangle + \langle z_1, X(A - \omega I)z_2 \rangle = -\langle z_1, z_2 \rangle,
    \]
    for $z_1, z_2 \in D(A)$.
\end{itemize}
• $m_0 I \leq X \leq MI$

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$$\langle (A - \omega I)z_1, Xz_2 \rangle + \langle z_1, X(A - \omega I)z_2 \rangle = -\langle z_1, z_2 \rangle,$$

for $z_1, z_2 \in D(A)$.

We rewrite this Lyapunov equation to the Sylvester equation

$$\langle (A - \omega I + X^{-1})z_1, Xz_2 \rangle + \langle z_1, X(A - \omega I)z_2 \rangle = 0.$$
The Sylvester equation

$$\langle (A - \omega I + X^{-1})z_1, Xz_2 \rangle + \langle z_1, X(A - \omega I)z_2 \rangle = 0.$$ 

is a special case of our general Sylvester equation. Thus by the previous lemma we know that the semigroup generated by $A - \omega I$ is left invertible, and

$$X^{-1}T_1(t)^*XT(t)e^{-\omega t} = I$$

where $(T_1(t))_{t \geq 0}$ is the semigroup generated by $A - \omega I + X^{-1}$. Thus $S(t) := X^{-1}T_1(t)^*Xe^{-\omega t}$ is the left-inverse of $T(t)$. \qed
Looking at the proof, the following is an easy consequence.

**Corollary**

If $C \in \mathcal{L}(Z, Y)$ is exactly observable, i.e. there exists an $m_0 > 0$ and $t_1 > 0$ such that

$$\int_0^{t_1} \|CT(t)z\|^2 dt \geq m_0 \|z\|^2$$

then $(T(t))_{t \geq 0}$ is left-invertible.
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then $(T(t))_{t \geq 0}$ is left-invertible.

The left-inverse semigroup is “generated” by $A - \omega I + X^{-1}C^*C$. \(\square\)
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**Example**

Consider the left-shift semigroup on $L^2(0, 1)$, i.e.

$$(T(t)f)(\eta) = \begin{cases} f(\eta + t) & \eta + t \in [0, 1] \\ 0 & \eta + t \geq 1 \end{cases}$$

with the observation at $\eta = 0$, i.e.,

$$Cf = f(0).$$

Then $(T(t))_{t \geq 0}$ is not left-invertible, but it is exactly observable

$$\int_0^1 |CT(t)f|^2 dt = \|f\|^2.$$
Remark

One could understand the difficulty as follows:

Is the operator $A + X^{-1}C^*C$ an infinitesimal generator?
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**Theorem**

Let \((T(t))_{t \geq 0}\) be a \(C_0\)-semigroup on the Hilbert space \(Z\) with generator \(A\). Then the following are equivalent:

1. \((T(t))_{t \geq 0}\) is left-invertible;

2. There exists a bounded operator \(Q\) and an equivalent inner product such that \(A + Q\) generates an isometric semigroup in the new norm.

□
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2. There exists a bounded operator \(Q\) and an equivalent inner product such that \(A + Q\) generates an isometric semigroup in the new norm.

Markus Haase has proved a similar result for generators of groups.
Proof: $1. \implies 2.$

By our previous proof we have the existence of a $X \in \mathcal{L}(Z)$, boundedly invertible, such that

$$\langle (A - \omega I)z_1, Xz_2 \rangle + \langle z_1, X(A - \omega I)z_2 \rangle = -\langle z_1, z_2 \rangle,$$

for $z_1, z_2 \in D(A)$. Now we write it as

$$\langle (A - \omega I + \frac{1}{2}X^{-1})z_1, Xz_2 \rangle + \langle z_1, X(A - \omega I + \frac{1}{2}X^{-1})z_2 \rangle = 0.$$

By defining $Q = -\omega I + \frac{1}{2}X^{-1}$, and taking as new inner product $\langle z_1, z_2 \rangle_{\text{new}} = \langle z_1, Xz_2 \rangle$, we obtain the desired result. \qed
3 When do we have that $A_1^*X = -XA_2$?

We return to our general Sylvester equation, see (1),

$$\langle A_1 z_1, Xz_2 \rangle + \langle z_1, XA_2 z_2 \rangle = 0,$$

and wonder when $A_1^*X = -XA_2$.

We have the following result:
Theorem

Assume that \( X \in \mathcal{L}(Z) \) is boundedly invertible and satisfies the Sylvester equation

\[
\langle A_1 z_1, X z_2 \rangle + \langle z_1, X A_2 z_2 \rangle = 0.
\]

If \( A_1, A_2 \) satisfy

- The intersection of \( \rho(A_2) \) with the complement of the point spectrum of \( -A_1^* \) is non-empty, or
- \( XD(A_2) = D(A_1^*) \),

then

\[
A_1^* = -X A_2 X^{-1}
\]
When do we have that $A_1^* X = -X A_2$?

Hence if $A_1, A_2$ generate a $C_0$-semigroup, then they generate a group and

$$T_1^*(t) = X T_2(-t) X^{-1}.$$
4 Riccati equations

It is clear that if $X$ is an invertible, self-adjoint solution to the Lyapunov equation

$$\langle Az_1, Xz_2 \rangle + \langle z_1, XAz_2 \rangle = -\langle z_1, z_2 \rangle,$$

then $X^{-1}$ satisfies the Riccati equation

$$\langle AX^{-1}z_1, z_2 \rangle + \langle z_1, AX^{-1}z_2 \rangle + \langle X^{-1}z_1, X^{-1}z_2 \rangle = 0.$$

However, there are other relations.
The ARE

\[ A^*X +XA -XBB^*X +C^*C = 0 \]

can be written as (weak form)

\[ -\langle Cz_1, Cz_2 \rangle = \langle Az_1, Xz_2 \rangle + \langle z_1, X(A - BB^*X)z_2 \rangle \]


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This last (Lyapunov) equation also holds when \( B \) is unbounded.
Theorem

Assume \( C \in \mathcal{L}(Z, Y) \) and let \( X \) be a self-adjoint, invertible solution of

\[
\langle Az_1, Xz_2 \rangle + \langle z_1, XA_{opt}z_2 \rangle = -\langle Cz_1, Cz_2 \rangle,
\]
and assume further that \( \rho(-A^* - X^{-1}C^*C) \cap \rho(A_{opt}) \neq \emptyset \). Then

1. \( D(A_{opt}) = X^{-1}D(A^*) \)

2. \( A_{opt} \) and \( A + X^{-1}C^*C \) generate a \( C_0 \)-group, \( (T_{opt}(t))_{t \geq 0} \), and
   \( (TX^{-1}C^*C(t))_{t \geq 0} \), respectively, and

\[
T_{opt}(t) = X^{-1}TX^{-1}C^*(-t)^*X.
\]