Quasi-hyperbolic semigroups

Yuri Tomilov (joint work with C. Batty (Oxford))

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Wuppertal, 18 July, 2011
Let $T$ be a bounded operator on a Banach space $X$. 
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The class of *contraction* (power bounded) operators $T$ (or operator semigroups) on $X$:

$$\|T\| \leq 1 \quad \text{(sup}_{n \geq 0} \|T^n\| = c < \infty.)$$

is **comparatively** well-understood.
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is comparatively well-understood.

Our aim is to try to understand an opposite class of *expansion* operators (operator semigroups) satisfying

$$
\|T^n x\| \geq c \|x\|
$$

at least in a certain sense to be made precise.
Hyperbolic operators

Definition A bounded linear operator $T$ on a Banach space $X$ is said to be \textit{hyperbolic} if

$$X = X_s \oplus X_u,$$

where $X_s$ and $X_u$ are closed $T$-inv. subspaces of $X$, $T \mid_{X_u}$ is invertible, and

$$\| (T \mid_{X_s})^n \| \leq \frac{1}{2}, \quad \| (T \mid_{X_u})^{-n} \| \leq \frac{1}{2} \quad \text{for some } n \in \mathbb{N}.$$
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In other words, for some $\alpha < 1$ and $\beta > 1$,

$$\| T^n x \| \leq C \alpha^n \| x \| \quad (x \in X_s, n \in \mathbb{N}), \quad \| T^n x \| \geq c \beta^n \| x \| \quad (x \in X_u, n \in \mathbb{N})$$

**Note:**

for non-zero $x \in X$ either $\| T^n x \| \geq c_x \beta^n$ or $\| T^n x \| \leq C_x \alpha^n, (n \in \mathbb{N})$.

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$T$ is hyperbolic if and only if $\sigma(T) \cap \Gamma = \emptyset \quad (\Gamma \text{ is the unit circle}).$
One of motivations

**Theorem [Krein]** A difference equation

\[ x_{n+1} = Tx_n + b_n, \quad n \in \mathbb{Z}, \]

admits a unique solution in \( l^\infty(\mathbb{Z}, X) \) for every \((b_n)_{n \in \mathbb{Z}} \in l^\infty(\mathbb{Z}, X)\) if and only if \( T \) is hyperbolic.

Here \( l^\infty(\mathbb{Z}, X) \) can be replaced by a variety of other spaces.
Observation If $T$ is hyperbolic and $Y$ is a closed $T$-invariant subspace of $X$, then $T \upharpoonright_Y$ may not be hyperbolic, but each non-zero orbit contracts exponentially or expands exponentially, with uniform exponent.
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Definition (Eisenberg, Hedlund (1970)): Assume $T$ is invertible.

- $T$ is *expansive* if for each $x$ there exists $n_x \in \mathbb{Z}$ such that
  \[ \| T^{n_x} x \| \geq 2 \| x \| ; \]

- $T$ is *uniformly expansive* if there exists $n \in \mathbb{N}$ (independent of $x$) such that
  \[ \max(\| T^{n} x \|, \| T^{-n} x \|) \geq 2 \| x \| \]
  for all $x$.  

Hedlund (1971):

$T$ is uniformly expansive $\iff$ $\sigma_{ap}(T) \cap \Gamma = \emptyset$ $\iff$ $\| (T - \lambda)^{n} x \| \geq c \| x \|$ ($x \in X, \lambda \in \Gamma$).
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$\iff \|(T - \lambda)x\| \geq c\|x\|$ (for $x \in X, \lambda \in \Gamma$)
Quasi-hyperbolic operators

$T$ is not necessarily invertible

**Definition** $T$ is *quasi-hyperbolic* if there exists $n \in \mathbb{N}$ (independent of $x$) such that

$$\max \left( \| T^{2n} x \|, \| x \| \right) \geq 2 \| T^n x \|$$

for all $x \in X$. 

Elementary properties

$T$ hyperbolic $\Rightarrow$ $T$ quasi-hyperbolic

$T$ is uniformly expansive $\iff$ $T$ is quasi-hyperbolic and invertible

$T$ quasi-hyperbolic $\Rightarrow$ $T \upharpoonright Y$ quasi-hyperbolic

$T$ quasi-hyperbolic $\Rightarrow$ $\sigma_{ap}(T) \cap \Gamma = \emptyset$
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**Elementary properties**

- $T$ hyperbolic $\implies$ $T$ quasi-hyperbolic
- $T$ is uniformly expansive $\iff$ $T$ is quasi-hyperbolic and invertible
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Theorem (Read 1986, 88; Müller 1988) Let $T$ be a bounded linear operator on $X$. There is a Banach space $Y$ and a bounded operator $S$ on $Y$ such that $X$ is isometrically embedded in $Y$, $S \restriction X = T$, $\|S\| = \|T\|$ and $\sigma(S) = \sigma_{ap}(T)$.

Corollary $T$ is the restriction of a hyperbolic operator to a closed invariant subspace.

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**Theorem (Read 1986,88; Müller 1988)** Let $T$ be a bounded linear operator on $X$. There is a Banach space $Y$ and a bounded operator $S$ on $Y$ such that $X$ is isometrically embedded in $Y$, $S|_X = T$, $\|S\| = \|T\|$ and $\sigma(S) = \sigma_{ap}(T)$.

**Corollary**

$T$ is the restriction of a hyperbolic operator to a closed invariant subspace.

$T$ is quasi-hyperbolic $\iff$ $\sigma_{ap}(T) \cap \Gamma = \emptyset$
Examples

1. Weighted shifts (Ridge, 1970)

\[ X = l^2(\mathbb{Z}), \quad S_w(x) = (w_n x_{n-1})_{n \in \mathbb{Z}}, \quad x = (x_n)_{n \in \mathbb{Z}} \in X \]

The spectrum of \( S_w \) is an annulus centered at 0.

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If

\[ w_n = \begin{cases} 2, & n \geq 0 \\ \frac{1}{2}, & n < 0 \end{cases} \]

then \( \sigma_{ap}(S_w) = \frac{1}{2}\Gamma \cup 2\Gamma \).
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If

\[ w_n = \begin{cases} \frac{1}{2}, & n \geq 0 \\ 2, & n < 0 \end{cases} \]

then \( \sigma_{\text{ap}}(S'_w) = \{ \lambda : \frac{1}{2} \leq |\lambda| \leq 2 \} \).
2. Wave equations (Cooper, Koch 1995)

The problem

\[
\Omega = \{(x, t) \in \mathbb{R}_+^2 : 0 < x < 1 + \frac{\sin(\pi t)}{2\pi}\}
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\[
u_{tt} - u_{xx} = 0 \text{ in } \Omega
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u = 0 \text{ on } \partial \Omega
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\[
u(\cdot, 0) = f \in W_0^{1,2}(0, 1)
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u_t(\cdot, 0) = g \in L^2(0, 1)
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is well-posed.

For the monodromy operator \( U(2, 0) : (f, g) \mapsto (u(\cdot, 2), u_t(\cdot, 2)) \) on \( X = W_0^{1,2} \times L^2 : \)

\[ \sigma(U(2, 0)) = \{ \lambda : \frac{1}{\sqrt{3}} \leq |\lambda| \leq \sqrt{3} \}, \quad \sigma_{\text{ap}}(U(2, 0)) \cap \Gamma = \emptyset; \]
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u_{tt} - u_{xx} &= 0 \text{ in } \Omega \\
u &= 0 \text{ on } \partial\Omega \\
u(\cdot, 0) &= f \in W_{0,2}^1(0, 1) \\
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is well-posed. For the monodromy operator \(U(2, 0) : (f, g) \mapsto (u(\cdot, 2), u_t(\cdot, 2))\) on \(X = W_{0,2}^1 \times L^2:\)

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\(U(2, 0)\) is quasi-hyperbolic and the energy \(\|U(t, 0)x\|^2, x \in X \setminus \{0\}\), grows exponentially in either forward or backward time.
3. Hyperbolic and quasi-hyperbolic operators appear naturally in the **smooth dynamics on manifolds** (operator-theoretical characterization of Anosov and quasi-Anosov maps; Mather-Maneñ theory)

Skip.
A question:

Is there a nice condition which characterises those operators $T$ on $X$ such that

- there exists a hyperbolic operator $S$ on a Banach space $Y$
- $X$ is continuously embedded in $Y$ and
- $T = S|_X$?
Hyperbolic Semigroups

**Definition** A $C_0$-semigroup $\mathcal{T} = \{ T(t) : t \geq 0 \}$ (with generator $A$) is hyperbolic if there is a splitting

$$X = X_s \oplus X_u,$$

where $X_s$ and $X_u$ are closed $\mathcal{T}$-invariant subspaces of $X$, $T(t) \upharpoonright X_u$ is invertible for some (or all) $t > 0$, and

$$\| T(t) \upharpoonright X_s \| < \frac{1}{2}, \quad \| (T(t) \upharpoonright X_u)^{-1} \| < \frac{1}{2}$$

for some (or all) $t > 0$. 

$T$ is hyperbolic if and only if $T(1)$ is hyperbolic.

The problem: spectral mapping theorem $\sigma(T(t))\{0\} = e^{t\sigma(A)}$ does not in general hold for $C_0$-semigroups.

For Hilbert spaces, $T$ is hyperbolic $\Leftrightarrow A - q$ is invertible for each $q \in \mathbb{R}$ and $\sup_{s \in \mathbb{R}} \| (A - q)^{-1} \| < \infty$ (Gearhart-Prüss).
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For Hilbert spaces, $\mathcal{T}$ is hyperbolic $\iff A - is$ is invertible for each $s \in \mathbb{R}$ and $\sup_{s \in \mathbb{R}} \| (A - is)^{-1} \| < \infty$ (Gearhart-Prüss).
A motivation for hyperbolicity

**Theorem** [Krein, Daletskii, Latushkin, Pruess, Schnaubelt, Zhikov, ...] If $A$ is the generator of a $C_0$-semigroup $\mathcal{T}$ then

$$x'(t) = Ax(t) + f(t), \quad t \in \mathbb{R},$$

admits the unique bounded (mild) continuous solution on $\mathbb{R}$ for every bounded continuous $f$ if and only if $\mathcal{T}$ is hyperbolic.
**Definition** A $C_0$-semigroup $\mathcal{T} = \{ T(t) : t \geq 0 \}$ is quasi-hyperbolic if there exists $t$ (independent of $x$) such that

$$\max (\| T(2t)x \|, \| x \|) \geq 2 \| T(t)x \|$$

for all $x \in X$. 

**Properties:**

- $\mathcal{T}$ is quasi-hyperbolic $\iff$ $\mathcal{T}(1)$ is quasi-hyperbolic $\iff$ $\mathcal{T}$ is a restriction of a hyperbolic semigroup $\iff$ $\sigma_{ap}(\mathcal{T}(1)) \cap \Gamma = \emptyset = \implies \| A - is \| \geq c \| x \|, s \in \mathbb{R}$.

**Basic examples:** weighted shift semigroups on $L^p(\mathbb{R})$. 

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**Basic examples:** weighted shift semigroups on $L^p(\mathbb{R})$
**Remark** There exist a semigroup $\mathcal{T} = (T(t))_{t \geq 0}$ such that $\mathcal{T}$ is not quasi-hyperbolic, but $A$ satisfies lower bounds

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$$\|(A - is)x\| \geq c\|x\|, \quad (s \in \mathbb{R}, x \in D(A)) :$$

Let $a > 2q/p, \quad 1 < p < 2 < q < \infty$

$$X := L_p(\mathbb{R}, e^{2x} \, dx) \cap L_q(\mathbb{R}, w(x) \, dx), \quad w(x) := \begin{cases} e^{ax} & (x \leq 0), \\ 1 & (x > 0), \end{cases}$$

$$\|f\|_X = \left\{ \int_{\mathbb{R}} |f(x)|^p e^{2x} \, dx \right\}^{1/p} + \left\{ \int_{\mathbb{R}} |f(x)|^q w(x) \, dx \right\}^{1/q}.$$
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Let $$(T(t)f)(s) = f(s + t) \ (s, t \in \mathbb{R}).$$

Then $\sigma(A) \cap i\mathbb{R} = \emptyset, \quad \sup_{s \in \mathbb{R}} \|(is - A)^{-1}\| < \infty.$
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Let $(T(t)f)(s) = f(s + t) \ (s, t \in \mathbb{R})$.
Then $\sigma(A) \cap i\mathbb{R} = \emptyset$, $\sup_{s \in \mathbb{R}} \| (is - A)^{-1} \| < \infty$. However,

$\forall t > 0 \ \exists f \in X, \| f \| = 1 : \quad \| T(-t)f \|_X < 2\| f \|_X, \quad \| T(t)f \|_X < 2\| f \|_X$. 

Yuri Tomilov (IM PAN, Warsaw and Nichola)

Quasi-hyperbolic semigroups
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Characterisations of quasi-hyperbolicity

**Theorem**  
a) Let $\mathcal{T}$ be a $C_0$-semigroup on a Hilbert space $X$ with generator $A$. Then $\mathcal{T}$ is quasi-hyperbolic if and only if $A$ satisfies lower bounds

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b) If $X$ is a Banach space then $\mathcal{T}$ is quasi-hyperbolic if and only if the multiplication operator

$$(M_{A-i}f)(s) = (A - is)f(s)$$

is a *lower Fourier multiplier* on $L^p(\mathbb{R}, X)$, $1 \leq p < \infty$, i.e.

$$\|\mathcal{F}^{-1}M_{A-i}\mathcal{F} f(s)\|_{L^p} \geq c\|f\|_{L^p}$$

for all Schwartz functions $f : \mathbb{R} \mapsto D(A)$, where $\mathcal{F}$ is the Fourier transform on $L^1(\mathbb{R}, X)$. 
What do lower bounds for $A$ imply?

For simplicity of statement, assume that $(T(t))_{t \in \mathbb{R}}$ is a $C_0$-group, i.e., each $T(t)$ is invertible.

**Theorem** Let $A$ be the generator of a $C_0$-group $(T(t))_{t \in \mathbb{R}}$ on a Banach space $X$, and assume that

$$\| (A - is)x \| \geq c \| x \| \quad (s \in \mathbb{R}, x \in D(A)).$$
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**Theorem** Let $A$ be the generator of a $C_0$-group $(T(t))_{t \in \mathbb{R}}$ on a Banach space $X$, and assume that

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\|(A - is)x\| \geq c\|x\| \quad (s \in \mathbb{R}, x \in D(A)).
$$

Then for each non-zero $x$,

(i) $\|T(t)x\|$ grows faster than polynomially either as $t \to \infty$ or as $t \to -\infty$, and
What do lower bounds for $A$ imply?

For simplicity of statement, assume that $(T(t))_{t \in \mathbb{R}}$ is a $C_0$-group, i.e., each $T(t)$ is invertible.

**Theorem** Let $A$ be the generator of a $C_0$-group $(T(t))_{t \in \mathbb{R}}$ on a Banach space $X$, and assume that

$$\| (A - is)x \| \geq c \| x \| \quad (s \in \mathbb{R}, x \in D(A)).$$

Then for each non-zero $x$,

(i) $\| T(t)x \|$ grows faster than polynomially either as $t \to \infty$ or as $t \to -\infty$, and

(ii) There exists $\epsilon_x > 0$ such that

$$\int_{-\infty}^{\infty} \| T(t)x \| e^{-\epsilon_x |t|} \, dt = \infty.$$
Continuous embedding?

If $A$ satisfies

$$\|(A - is)x\| \geq c\|x\| \quad (s \in \mathbb{R}, x \in D(A))$$

can $X$ be continuously embedded in a space $Y$ in such a way that there is a hyperbolic $C_0$-semigroup $\{S(t) : t \geq 0\}$ on $Y$ such that $T(t) = S(t) \mid_X$?
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A necessary condition for this:
each orbit nontrivial orbit $T(t)x$ should grow exponentially in forward or backward time, with uniform exponent.
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If $(T(t))_{t \geq 0}$ has growth bound 0 (the spectral radius of $T(t)$ is 1) and $A$ satisfies the condition above, then $T$ is not quasi-hyperbolic, but such growth does occur (in negative time).
THANK YOU FOR YOUR ATTENTION!
M compact Riemann manifold, with tangent bundle \( TM \), \( \varphi \) a diffeomorphism of \( M \).

**Definition** \( \varphi \) is Anosov if \( TM = TM_s \oplus TM_u \) where \( D\varphi \) contracts \( TM_s \) exponentially in positive time and contracts \( TM_u \) exponentially in negative time.

\( C(TM) \) Banach space of continuous sections of \( TM \) (with sup norm)

Define push-forward operator on \( C(TM) \) :

\[
(E_{\varphi}f)(\theta) = D\varphi(\varphi^{-1}\theta)f(\varphi^{-1}\theta) \quad (\theta \in M)
\]

Mather (1968): \( \varphi \) is Anosov if and only if \( E_{\varphi} \) is hyperbolic.
**Definition** \( \varphi \) is quasi-Anosov if, for all \( \theta \in M \) and all non-zero \( x \in TM_\theta \),

\[
\{ (D\varphi)^n(\theta)x : n \in \mathbb{Z} \}
\]

is unbounded.

Mané (1977): \( \varphi \) is quasi-Anosov if and only if \( M \) can be embedded in a manifold \( N \) on which \( \varphi \) can be extended to an Anosov diffeomorphism.

Moreover, \( \varphi \) is quasi-Anosov if and only if \( \sigma_{ap}(E\varphi) \cap \Gamma = \emptyset \), i.e.,

\( E\varphi \) is quasi-hyperbolic