Kernel and eigenfunction estimates for some second order elliptic operators

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(Joint work with E.M. Ouhabaz)

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1 Introduction

2 Heat kernel estimates
Consider

\[
Au = - \sum_{k,j=1}^{n} \partial_k (a_{kj} \partial_j u) + Vu, \\
Bu = - \Delta u + Vu.
\]

The associated quadratic forms

\[
a(u, u) := \sum_{k,j=1}^{n} \int_{\mathbb{R}^n} a_{kj} \partial_k u \partial_j u + \int_{\mathbb{R}^n} V |u|^2 \\
b(u, u) := \int_{\mathbb{R}^n} |\nabla u|^2 + \int_{\mathbb{R}^n} V |u|^2,
\]

\[u \in D(a) = D(b) = \{ u \in W^{1,2}(\mathbb{R}^n); \int_{\mathbb{R}^n} V |u|^2 < \infty \}.\]
Assumptions

\[
\begin{aligned}
(H) \quad & a_{kj} = a_{jk} \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}), \quad \partial_j a_{kj} = o(|x|^\frac{\alpha}{2}) \text{ as } |x| \to \infty, \\
& \eta |\xi|^2 \leq \sum_{j,k=1}^n a_{kj} (x) \xi_k \xi_j \leq \Lambda |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n, \\
& V \in L^1_{\text{loc}}(\mathbb{R}^n) \text{ such that } V(x) \geq |x|^\alpha, \quad \alpha > 2.
\end{aligned}
\]
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Spectrum

\[ \lim_{|x| \to +\infty} V(x) = +\infty \] implies that \( A \) and \( B \) have compact resolvents. Thus,

\[ \sigma(A) = \{ \mu_i; \ i = 0, 1, \ldots \} \]

\[ \sigma(B) = \{ \lambda_i; \ i = 0, 1, \ldots \} . \]

Let \( (\psi_i)_{i \geq 0} \) and \( (\varphi_i)_{i \geq 0} \) the corresponding normalized eigenfunctions of \( A \) and \( B \), respectively.
Gaussian estimates 1

It is known that $A$ and $B$ have heat kernels $k_t(x, y)$ and $p_t(x, y)$ satisfying

$$p_t(x, y) \leq \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}, \quad k_t(x, y) \leq Ct^{-n/2} e^{-c\frac{|x-y|^2}{t}}$$

for $t > 0$ and constants $c, C > 0$
Gaussian estimates 2


\[ p_t(x, y) \leq \frac{C}{t^{n/2}} e^{-\lambda_0 t} e^{-\frac{|x-y|^2}{4t}} \left[ 1 + \lambda_0 t + \frac{|x-y|^2}{2t} \right]^\frac{n}{2} \]

and

\[ k_t(x, y) \leq \frac{C}{t^{n/2}} e^{-\mu_0 t} e^{-\frac{\rho^2(x,y)}{4t}} \left[ 1 + \mu_0 t + \frac{\rho^2(x,y)}{2t} \right]^\frac{n}{2} , \quad t > 0, \]

where

\[ \rho(x, y) := \sup \{ \phi(x) - \phi(y) : \phi \in C_c^\infty(\mathbb{R}^n), \] \[
\sum_{k,j=1}^n a_{kj} \partial_k \phi \partial_j \phi \leq 1 \text{ a.e. on } \mathbb{R}^n \}. \]
E.B. Davies in 1984 showed

\[ p_t(x, y) \leq C e^{ct-b} \varphi_0(x) \varphi_0(y), \quad (1) \]

\( x, y \in \mathbb{R}^n, 0 < t \leq 1, \) where \( C, c \) are constants and \( b > \frac{\alpha+2}{\alpha-2} \).

Using Lyapunov functions techniques Metafune and Spina [JEE 7, 2007] obtained (1) with \( b = \frac{\alpha+2}{\alpha-2} \).
Intrinsic ultracontractivity for $B$

Davies showed also

$$c_1 |x|^{-\beta} e^{-\frac{|x|^\gamma}{\gamma}} \leq \varphi_0(x) \leq c_2 |x|^{-\beta} e^{-\frac{|x|^\gamma}{\gamma}}$$

(2)

for large $|x|$, $\beta = \frac{\alpha}{4} + \frac{n-1}{2}$, $\gamma = 1 + \frac{\alpha}{2}$. By (1),

$$p_t(x, y) \leq C e^{ct-b} (|x||y|)^{-\beta} e^{-\frac{|x|^\gamma}{\gamma}} e^{-\frac{|y|^\gamma}{\gamma}}$$

(3)

for large $|x|$, $|y|$, and $0 < t \leq 1$. 
The main theorem

**Theorem 1.** Assume \((H)\) with \(\Lambda < 1\). Then,

\[
k_t(x, y) \leq C e^{-\mu_0 t} e^{ct^{-b}} (|x||y|)^{-\beta} e^{-\frac{|x|^\gamma}{\gamma} - \frac{|y|^\gamma}{\gamma}}
\]

for large \(|x|, |y|\) and all \(t > 0\). Here \(C, c > 0, b > \frac{\alpha+2}{\alpha-2}, \beta = \frac{\alpha}{4} + \frac{n-1}{2}\) and \(\gamma = 1 + \frac{\alpha}{2}\). 
Outline

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Sketch of the proof

On $L^2_{\varphi} := L^2(\mathbb{R}^n, \varphi^2 dx)$ define

$$\tilde{a}(u, v) := a(\varphi u, \varphi v), \quad \tilde{b}(u, v) := b(\varphi u, \varphi v)$$

$$D(\tilde{a}) = D(\tilde{b}) = \{u \in L^2_{\varphi}; \varphi u \in D(a) = D(b)\}.$$ 

their associated kernels are

$$\tilde{k}_t(x, y) = \frac{k_t(x, y)}{\varphi(x)\varphi(y)}, \quad \tilde{p}_t(x, y) = \frac{p_t(x, y)}{\varphi(x)\varphi(y)}.$$ 

Using $\varphi \approx \varphi_0$, $|\nabla \varphi| \approx |\nabla \varphi_0|$ and the Beurling-Deny criterion for $(\varphi_0^{-1}e^{-tB}\varphi_0)$ on $L^2_{\varphi_0}$ we deduce

$$1 \wedge u \in D(\tilde{a}), \quad \forall 0 \leq u \in D(\tilde{a}).$$
Sketch of the proof

We have

\[ \tilde{a}(u, v) = \sum_{j,k=1}^{n} \int_{\mathbb{R}^n} a_{kj} \partial_k u \partial_j v \varphi^2 \, dx + \int_{\mathbb{R}^n} W_a u v \varphi^2 \, dx, \]

where \( W_a = V - \sum_{j,k=1}^{n} \partial_j a_{kj} \frac{\partial_k \varphi}{\varphi} \quad \text{and} \quad W_a(x) \geq -\lambda_a. \)
Sketch of the proof

Thus,

\[ \tilde{a}(1 \wedge u, (u - 1)^+) = \int_{\mathbb{R}^n} W_a(1 \wedge u)(u - 1)^+ \varphi^2 \, dx \]

\[ \geq -\lambda_a \int_{\mathbb{R}^n} (1 \wedge u)(u - 1)^+ \varphi^2 \, dx. \]

Applying Beurling-Deny

\[ \| e^{-t\tilde{A}} \|_{\mathcal{L}(L^\infty)} \leq e^{\lambda_a t}, \quad t \geq 0. \]
Sketch of the proof

Since $\tilde{b}$ satisfies a log-Sobolev inequality (see E.B. Davies) and by ellipticity, $\tilde{b}(u, u) \leq \max\{\frac{1}{\eta}, 1\} \tilde{a}(u, u)$, it follows that $\tilde{a}$ satisfies the same log-Sobolev inequality. Hence, with the $L^\infty$-contractivity, we deduce that $e^{-t\tilde{A}}$ is ultracontractive and

$$k_t(x, y) = \frac{k_t(x, y)}{\varphi(x)\varphi(y)} \leq C e^{ct-b}, \quad 0 < t \leq 1.$$
Sketch of the proof

For \( t \geq 1 \), by

\[
k_{s+r}(x, y) = \int_{\mathbb{R}^n} k_s(x, z)k_r(y, z) \, dz, \quad s, \, r > 0,
\]

\[
k_t(x, x) = \left\| e^{-\left(\frac{t}{2} - \frac{1}{2}\right)A}k_{\frac{1}{2}}(x, \cdot) \right\|_{L^2}^2 \leq e^{-2\mu_0\left(\frac{t}{2} - \frac{1}{2}\right)} \left\| k_{\frac{1}{2}}(x, \cdot) \right\|_{L^2}^2 \leq Me^{-\mu_0 t} \varphi^2(x).
\]

The result follows from

\[
k_t(x, y) \leq \sqrt{k_t(x, x)}\sqrt{k_t(y, y)}, \quad x, \, y \in \mathbb{R}^n.
\]
Estimates for the eigenfunctions

Corollary 2.

\[ |\psi_j(x)| \leq C|x|^{-\beta} e^{-\frac{|x| \gamma}{\gamma}} \]

for large \(|x|\) a \(C > 0\) with \(\beta = \frac{\alpha}{4} + \frac{n-1}{2}\), \(\gamma = 1 + \frac{\alpha}{2}\).
Proof

\[ |\psi_j(x)| e^{-\mu_j t} = |e^{-tA}\psi_j(x)| = |\int_{\mathbb{R}^n} k_t(x, y)\psi_j(y) \, dy| \leq \left( \int_{\mathbb{R}^n} k_t(x, y)^2 \, dy \right)^{1/2} \|\psi_j\|_2 \]

\[ = (k_{2t}(x, x))^{1/2}. \]
The general case

If $\Lambda \geq 1$, we study first

$$H := -\Delta + \theta |x|^\alpha$$

with $0 < \theta < 1$. One proves that its ground state $\phi_0$ satisfies

$$\phi_0(x) \approx \phi(x), \quad |\nabla \phi_0(x)| \approx |\nabla \phi(x)|,$$

where $\phi(x) = |x|^{-\beta} e^{-\frac{\sqrt{\theta}}{\gamma} |x|^\gamma}$ for large $|x|$. 
The general case

**Theorem 2.** Assume \((H)\) and \(\theta > 0\) s.t. \(\theta \Lambda < 1\). Then,

\[
k_t(x, y) \leq Ce^{-\mu_0 t} e^{c t^b} (|x||y|)^{-\beta} e^{-\frac{\sqrt{\theta}}{\gamma} |x|^\gamma} e^{-\frac{\sqrt{\theta}}{\gamma} |y|^\gamma}, \quad t > 0
\]

for large \(|x|, |y|\). Here \(C, c > 0, b > \frac{\alpha+2}{\alpha-2}, \beta = \frac{\alpha}{4} + \frac{n-1}{2}\) and \(\gamma = 1 + \frac{\alpha}{2}\).