PDE motion planning for finite–time multi–agent deployment

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(joint work with M.Krstic)
PDE motion planning for finite–time multi–agent deployment

- Source localization by mobile sensor network ⇒ **PDEs for diffusive field**

- Mobile agent network ⇒ **PDEs for agent position / velocity**

Graph–Laplacian control ⇒ consensus, see e.g. [1]

\[
\partial_t x(n_i, t) = \alpha(x(n_{i+1}, t) - 2x(n_i, t) + x(n_{i-1}, t)), \quad \forall n_i \in S_f
\]

\[
x(n_j, t) = u(n_j, t), \quad \forall n_j \in S_l
\]
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- Source localization by mobile sensor network ⇒ PDEs for diffusive field

- Mobile agent network ⇒ PDEs for agent position / velocity

Graph–Laplacian control ⇒ consensus, see e.g. [1]
\[
\frac{\partial}{\partial t} x(n_i, t) = \alpha (x(n_{i+1}, t) - 2x(n_i, t) + x(n_{i-1}, t)), \quad \forall n_i \in S_f
\]
\[
x(n_i, t) = u(n_j, t), \quad \forall n_j \in S_i
\]

Heat equation ⇒ exp. stable [2]
\[
\frac{\partial}{\partial t} x'(z, t) = \frac{\partial^2}{\partial z^2} x'(z, t), \quad z \in (0, 1)
\]
\[
x'(0, t) = u^a_i(t), \quad x'(1, t) = u^l_i(t)
\]
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- Mobile agent network ⇒ PDEs for agent position / velocity

\[ \text{Graph–Laplacian control} \Rightarrow \text{consensus, see e.g. [1]} \]
\[ \partial_t x(n_i, t) = \alpha (x(n_{i+1}, t) - 2x(n_i, t) + x(n_{i-1}, t)), \quad \forall n_i \in S_f \]
\[ x(n_i, t) = u(n_i, t), \quad \forall n_j \in S_l \]

Heat equation ⇒ exp. stable [2]
\[ \partial_t x^i(z, t) = \partial^2_z x^i(z, t), \quad z \in (0, 1) \]
\[ x^i(0, t) = u^i_a(t), \quad x^i(1, t) = u^i_l(t) \]

- Outline
  - Flatness–based motion planning for leader–enabled deployment
  - Convergence analysis and summability
  - Feedforward formation tracking control
  - Simulation results
Nonlinear time-varying Burgers–type flow model for mobile agent continuum [3, 4]

Finite time deployment into steady state formation profiles for $c^i(t) = c^i = \text{const.}$
Flatness–based motion planning

- Flatness–based trajectory planning
  \[ x^i(z, t) \rightarrow \hat{x}^i(z, t) = \sum_{n=0}^{\infty} \hat{x}_n^i(t) \frac{(z-1/2)^n}{n!} \]

PDE \[ \Rightarrow \hat{x}_n^i(t) = \frac{1}{a_i} \left[ b^i \sum_{i=0}^{n-2} \binom{n-2}{i} \hat{x}_{n-2-i}^i(t) \hat{x}_{i+1}^i(t) - c^i(t) \hat{x}_{n-2}^i(t) + \partial_t \hat{x}_{n-2}^i(t) \right], \quad n \geq 2 \]

Impose \[ \hat{x}_0^i(t) = \hat{x}^i(1/2, t) = y_1^i(t), \quad \hat{x}_1^i(t) = \partial_z \hat{x}^i(1/2, t) = y_2^i(t) \quad \Rightarrow \hat{x}_n^i(t) = \psi_n(y_1^i(t), y_2^i(t)) \]

- Formal state and input parametrization in terms of basic output \((y_1^i(t), y_2^i(t))\)
  \[ x^i(z, t) = \sum_{n=0}^{\infty} \psi_n(y_1^i(t), y_2^i(t)) \frac{(z-1/2)^n}{(n!)}, \quad u_a^i(t) = x^i(0, t), \quad u_b^i(t) = x^i(1, t) \]

- Uniform convergence if \(y_1^i(t), y_2^i(t), c^i(t)\) are Gevrey of order \(\alpha \in (1, 2]\), i.e.
  \[ \sup_{t \in \mathbb{R}^+} |\partial_t^n f(t)| \leq D_f^{n+1}(n!)^\alpha, \quad f(t) \in \{y_1^i(t), y_2^i(t), c^i(t)\} \]

with finite radius of convergence \(\rho = 1/(DA_i(D))\), \(D = \max\{D_y, D_c\}\) in \(|z - 1/2|\), where

\[ A_i(D) = \max \left\{ 1, \sqrt{\frac{2 + b^i}{2a_i}}, \sqrt[3]{\frac{b^i}{6a_i} \left(1 + \frac{3}{2D}\right) + \sqrt{\left(\frac{b^i}{6a_i} \left(1 + \frac{3}{2D}\right)\right)^2 + 2/ Da_i}} \right\} \]
Convergence analysis and summability

- Proof of convergence (sketch), see [4]

(i) \(|\partial_t^l \hat{x}_n(t)| \leq D^{l+n}((1 + n - 1)!)^\alpha F_n^i, \quad n \geq 2\)

with \(F_n^i = \begin{cases} 1, & n = 0, 1 \\ \frac{2 + b^i}{a^i}, & n = 2 \\ \frac{1}{a^i} \left( \frac{2F_{n-2}^i}{D(n-1)^\alpha} + \frac{b^i}{D(n-1)^\alpha} \sum_{i=0}^{n-3} \binom{n-2}{i} \frac{F_{n-2-i}^i F_{i+1}^i}{(\beta^i)^\alpha} + b^i F_{n-1}^i F_0^i \right), & n \geq 3 \end{cases}\)

⇒ Use Leibniz formula, Gevrey assumption for \(y_1(t), y_2(t), c^i(t)\) and induction

(ii) \(F_n^i \leq (A_i(D))^n \frac{n!}{(n-1)!^\alpha}, \quad n \geq 1\)

with \(A_i(D) = \max \left\{ 1, \sqrt{\frac{2 + b^i}{2a^i}}, \frac{b^i}{6a^i} \left( 1 + \frac{3}{2D} \right) + \sqrt{\left( \frac{b^i}{6a^i} \left( 1 + \frac{3}{2D} \right) \right)^2 + \frac{2}{Da^i} } \right\} \)

⇒ Induction

(iii) Apply Cauchy–Hadamard with \(\sup_{t \geq 0} |\hat{x}_n(t)| \leq (DA_i(D))^n n!, \quad n \geq 2\) from (i) and (ii)
Parameter–space ensuring uniform convergence with $\rho > \frac{1}{2}$

Convergence restrictions can be relaxed by **summability methods** [5]

\[
x'^i(z, t) \approx \hat{x}^i(z, t) = \left(S^N_k \hat{x}^i\right)(z, t) = \frac{\sum_{n=0}^{N} s_n^i(z, t) \frac{\xi}{\Gamma(1 + \frac{n}{k})}}{\sum_{n=0}^{N} \frac{\xi}{\Gamma(1 + \frac{n}{k})}}, \quad s_n^i(z, t) = \sum_{j=0}^{n} \hat{x}^i_j(t) \frac{(z - \frac{1}{2})^j}{j!}
\]
**Goal:** Realize finite time deployment into steady state formation profiles

\[ a^i \partial_z^2 x^i_s(z) - b^i x^i_s(z) \partial_z x^i_s(z) + c^i_s x^i_s(z) = 0, \quad z \in (0, 1) \]

\[ x^i_s(0) = u^i_{a,s}, \quad x^i_s(1) = u^i_{l,s} \]

Admits a closed-form solution only for special cases

(1) **Shock–like steady st.** for \( a^i \downarrow \lor b^i \uparrow 

(2) **Eigenf.** for \( b^i = 0, \quad c^i_s = (k\pi)^2, \quad k \in \mathbb{N} \)
**Goal:** Realize finite time deployment into steady state formation profiles

\[
\begin{align*}
\alpha^i \frac{\partial^2 x_s^i(z)}{\partial z^2} - \beta^i x_s^i(z) \frac{\partial x_s^i(z)}{\partial z} + \gamma^i x_s^i(z) &= 0, \quad z \in (0, 1) \\
x_s^i(0) &= u^i_{a,s}, \quad x_s^i(1) = u^i_{l,s}
\end{align*}
\]

Admits a closed–form solution only for special cases

(1)+(2) \implies \text{non–trivial profiles}

⇒ Transitions by means of desired trajectories \((y_1^{*,i}(t), y_2^{*,i}(t))\) for basic output
Trajectory assignment for basic output

- **Goal:** Realize finite time deployment into steady state formation profiles

\[ a^i \partial_z^2 x^i_s(z) - b^i x^i_s(z) \partial_z x^i_s(z) + c^i_s x^i_s(z) = 0, \quad z \in (0, 1) \]

\[ x^i_s(0) = u^i_{a,s}, \quad x^i_s(1) = u^i_{l,s} \]

- Desired trajectories for the basic output

\[ y^{*,i}_1(t) = A_1^0 + (A_1^T - A_1^0) \Phi_{Y,T}(t) \]

\[ y^{*,i}_2(t) = A_2^0 + (A_2^T - A_2^0) \Phi_{Y,T}(t) \]

- \( \Phi_{Y,T}(t) \) non-analytic, i.e. \( \Phi_{Y,T}(t) = 0 \) if \( t \leq 0 \), \( \Phi_{Y,T}(t) = 1 \) if \( t \geq T \), \( \partial_t^n \Phi_{Y,T}(t)|_{t \in \{0, T\}} = 0 \)

- \( A_1^0 = y^{*,i}_1(0) = x^{i,0}_s(1/2), \quad A_2^0 = y^{*,i}_2(0) = \partial_z x^{i,0}_s(1/2), \quad A_1^T = y^{*,i}_1(T) = x^{i,T}_s(1/2), \quad A_2^T = y^{*,i}_2(T) = \partial_z x^{i,T}_s(1/2) \)

  for **consistency** with initial and final steady states \( x^{i,0}_s(z) \) and \( x^{i,T}_s(z) \)

- Temporal path for reaction parameter

\[ c^i(t) = c^i_{s,0} + (c^i_{s,T} - c^i_{s,0}) \Phi_{Y,T}(t) \quad \Rightarrow \text{connect different families of steady states} \]
Feedforward formation tracking control

- **Feedforward controls** for leader and anchor

  \[ u^{*,i}_a(t) = \left( S_k^{N,s} \hat{x} \right)(0, t), \quad u^{*,i}_i(t) = \left( S_k^{N,s} \hat{x} \right)(1, t) \text{ with } x^i(z, t) = \sum_{n=0}^{\infty} \psi_n(y^{*,i}_1(t), y^{*,i}_2(t)) \frac{(z-1/2)^n}{(n)!} \]

  \[ \Rightarrow \text{ independent of communication topology} \]

- **Communication topology** by discretization (continuum to } m \text{ agents} \]

  \[
  \partial_t x_j^i(t) = \frac{2a_i - b_i \Delta z x_j^i(t)}{2\Delta z^2} x_j^i_{j+1}(t) + \left( c_i(t) - \frac{2a_i}{\Delta z^2} \right) x_j^i(t) + \frac{2a_i + b_i \Delta z x_j^i(t)}{2\Delta z^2} x_j^i_{j-1}(t), \quad j = 1, \ldots, m - 1
  \]

  \[ x_0^i(t) = u^{*,i}_a(t), \quad x_m^i(t) = u^{*,i}_l(t) \]
Simulation results (1)

- Finite time deployment into $Z$–shape ($m = 25$)
Simulation results (2)

- Finite time deployment into 8–shape \((m = 25)\)

- Enlarged set of target formations by including unstable steady state profiles

\[ \Rightarrow \text{Realization requires the exponential stabilization of the tracking error} \]

- 2DOF control approach for spatial–temporal systems

\[ \Sigma^* \quad y^* \quad \Sigma_{\infty}^{-1} \quad u^* \quad u \quad \Sigma_{\infty} \quad x \rightarrow x^* \]

Flatness–based motion planning

Backstepping–based state feedback [6, 7]
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Conclusion

- Flatness–based motion planning approach for finite time deployment of mobile agents
- Continuum of agents governed by viscous time–varying Burgers–type PDE
- Applicability of the PDE–based motion planning can be significantly enhanced by summability techniques
- Communication topology for discrete set of agents by finite difference discretization (decentralized)

Ongoing research

- Feedback stabilization with agent localization using backstepping
- Relationship with (approximate) controllability, stabilizability, ...
- Different communication topologies (2D, 3D, ...)
References


