PDE motion planning for finite–time multi–agent deployment

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(joint work with M.Krstic)

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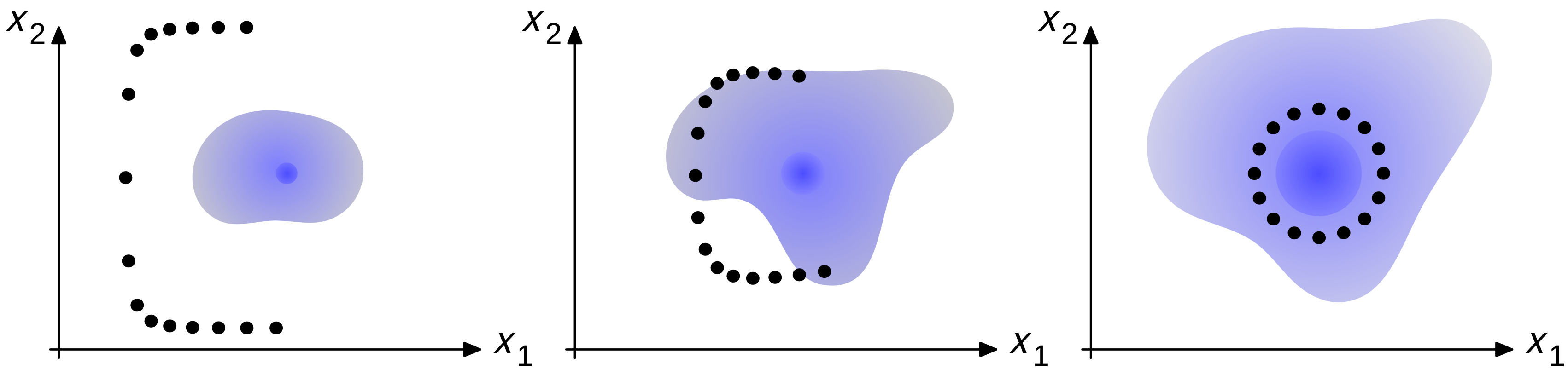
- Source localization by mobile sensor network ⇒ **PDEs for diffusive field**



- Mobile agent network ⇒ **PDEs for agent position / velocity**

Graph–Laplacian control ⇒ consensus, see e.g. [1]

$$\partial_t x(\mathtt{n}_i, t) = \alpha(x(\mathtt{n}_{i+1}, t) - 2x(\mathtt{n}_i, t) + x(\mathtt{n}_{i-1}, t)), \quad \forall \mathtt{n}_i \in \mathtt{S}_f$$

$$x(\mathtt{n}_j, t) = u(\mathtt{n}_j, t), \quad \forall \mathtt{n}_j \in \mathtt{S}_l$$

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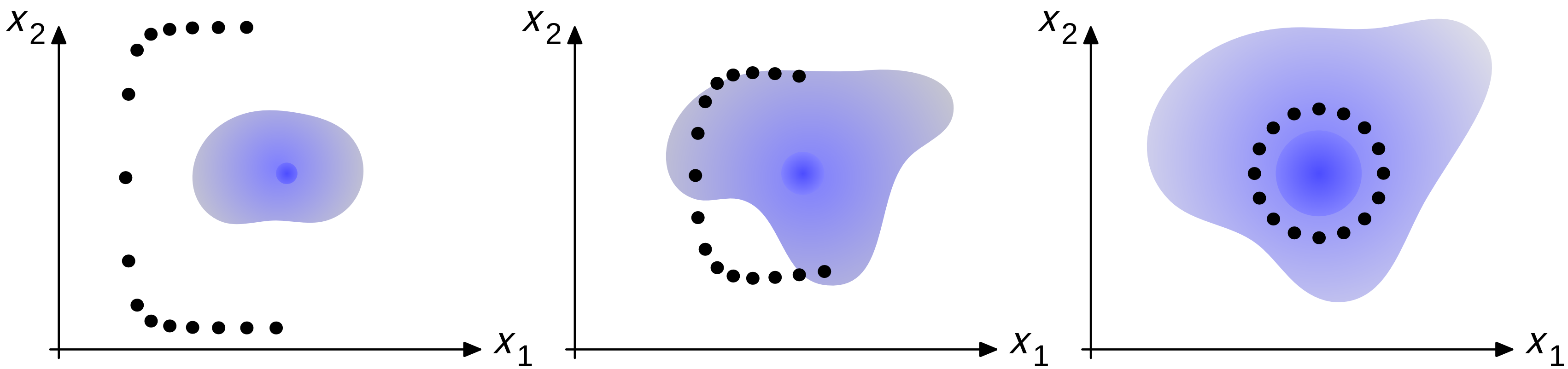

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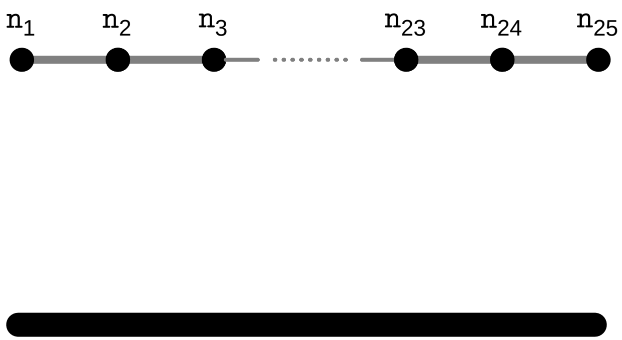

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Heat equation ⇒ exp. stable [2]

$$\partial_t x^i(z,t) = \partial_z^2 x^i(z,t), \qquad z \in (0,1)$$

$$x^i(0,t) = u_a^i(t), \quad x^i(1,t) = u_l^i(t)$$

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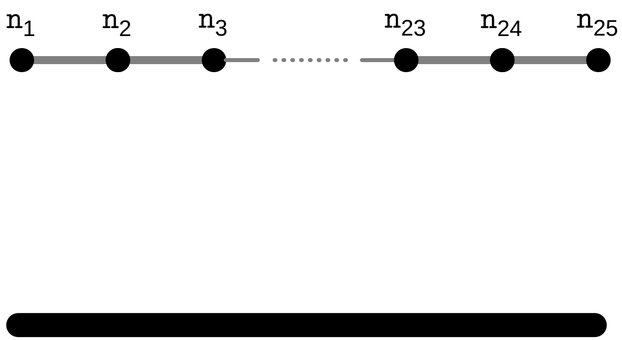

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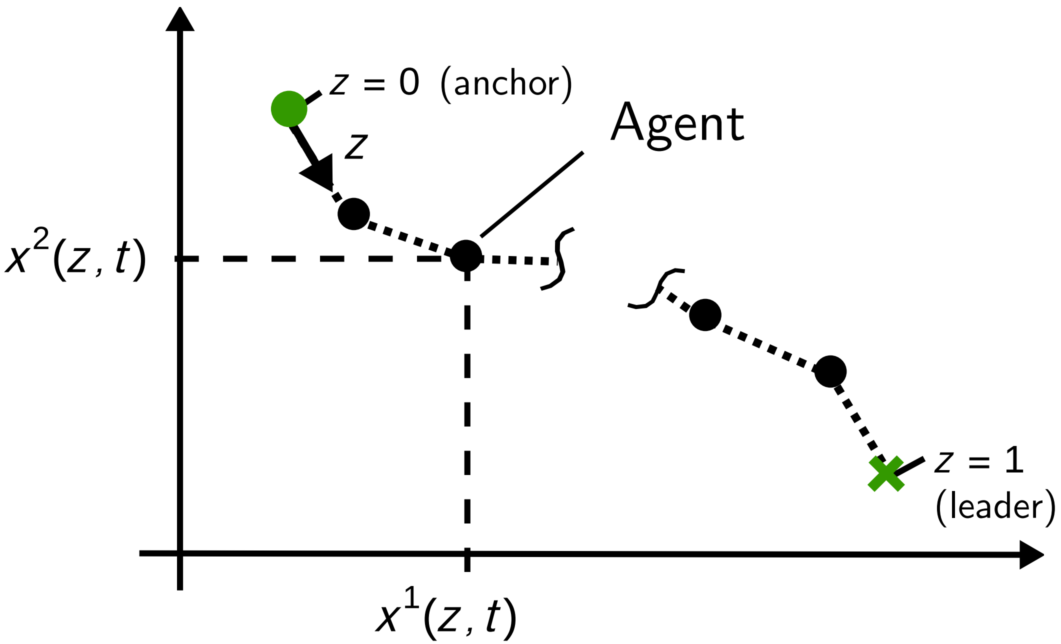
- **Outline**
  - Flatness–based motion planning for leader–enabled deployment
  - Convergence analysis and summability
  - Feedforward formation tracking control
  - Simulation results

Deployment problem

- Nonlinear time–varying Burgers–type flow model for **mobile agent continuum** [3, 4]

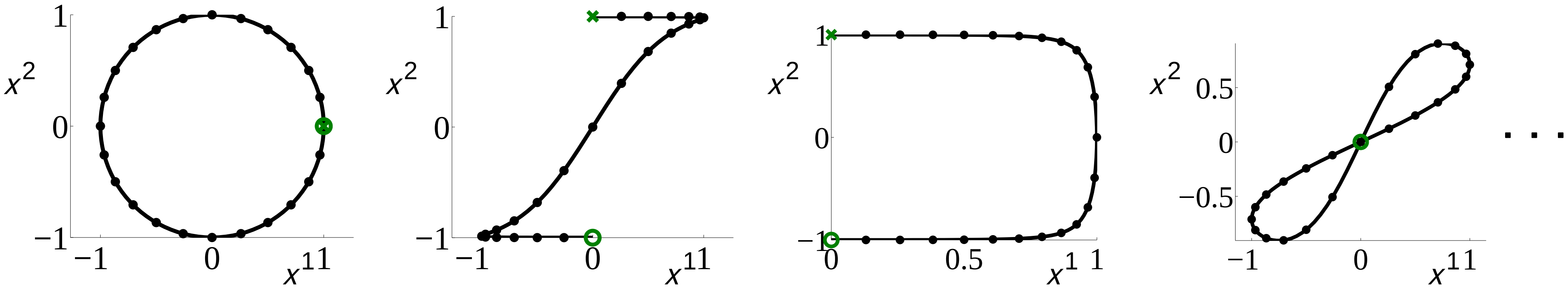


PDE: $\partial_t x^i = a^i \partial_z^2 x^i - b^i x^i \partial_z x^i + c^i(t) x^i$

BCs: $x^i(0,t) = u_a^i(t), \quad x^i(1,t) = u_l^i(t)$

ICs: $x^i(z,0) = x_0^i(z) = 0, \qquad i = 1, 2$

- **Finite time deployment** into steady state formation profiles for $c^i(t) = c^i =$ const.



Flatness–based motion planning

- Flatness–based trajectory planning $x^i(z,t) \to \hat{x}^i(z,t) = \sum_{n=0}^{\infty} \hat{x}^i_n(t)\frac{(z-1/2)^n}{n!}$

PDE $\Rightarrow$ $\hat{x}^i_n(t) = \frac{1}{a^i}\Big[b^i \sum_{i=0}^{n-2}\binom{n-2}{i}\hat{x}^i_{n-2-i}(t)\hat{x}^i_{i+1}(t) - c^i(t)\hat{x}^i_{n-2}(t) + \partial_t \hat{x}^i_{n-2}(t)\Big], \quad n \geq 2$

Impose $\hat{x}^i_0(t) = \hat{x}^i(1/2,t) = y^i_1(t), \quad \hat{x}^i_1(t) = \partial_z \hat{x}^i(1/2,t) = y^i_2(t)$ $\Rightarrow \hat{x}^i_n(t) = \psi_n\big(\boldsymbol{y}^i_{1,n}(t), \boldsymbol{y}^i_{2,n}(t)\big)$

- Formal **state and input parametrization** in terms of **basic output** $(y^i_1(t), y^i_2(t))$

$x^i(z,t) = \sum_{n=0}^{\infty} \psi_n\big(\boldsymbol{y}^i_{1,n}(t), \boldsymbol{y}^i_{2,n}(t)\big)\frac{(z-1/2)^n}{(n)!}, \quad u^i_a(t) = x^i(0,t), \quad u^i_l(t) = x^i(1,t)$

- Uniform convergence if $y^i_1(t)$, $y^i_2(t)$, $c^i(t)$ are Gevrey of order $\alpha \in (1,2]$, i.e.

$$\sup_{t\in\mathbb{R}^+} |\partial^n_t f(t)| \leq D_f^{n+1}(n!)^{\alpha}, \;\; f(t) \in \{y^i_1(t), y^i_2(t), c^i(t)\}$$

with **finite radius of convergence** $\rho = 1/(DA_i(D))$, $D = \max\{D_y, D_c\}$ in $|z - 1/2|$, where

$$A_i(D) = \max\left\{1, \sqrt{\frac{2+b^i}{2a^i}}, \frac{b^i}{6a^i}\left(1+\frac{3}{2D}\right) + \sqrt{\left(\frac{b^i}{6a^i}\left(1+\frac{3}{2D}\right)\right)^2 + \frac{2}{Da^i}}\right\}$$

Convergence analysis and summability

- Proof of convergence (sketch), see [4]

(i) $|\partial_t^l \hat{x}_n^i(t)| \leq D^{l+n}((l+n-1)!)^\alpha F_n^i, \quad n \geq 2$

$$\text{with } F_n^i = \begin{cases} 1, & n = 0,1 \\ \frac{2+b^i}{a^i}, & n = 2 \\ \frac{1}{a^i}\left(\frac{2F_{n-2}^i}{D(n-1)^\alpha} + \frac{b^i}{D(n-1)^\alpha}\sum_{i=0}^{n-3}\binom{n-2}{i}\frac{F_{n-2-i}^i F_{i+1}^i}{(\beta_i^n)^\alpha} + b^i \frac{F_{n-1}^i F_0^i}{(n-1)^\alpha}\right), & n \geq 3 \end{cases}$$

$\Rightarrow$ Use Leibniz formula, Gevrey assumption for $y_1^i(t)$, $y_2^i(t)$, $c^i(t)$ and induction

(ii) $F_n^i \leq \left(A_i(D)\right)^n \frac{n!}{((n-1)!)^\alpha}, \quad n \geq 1$

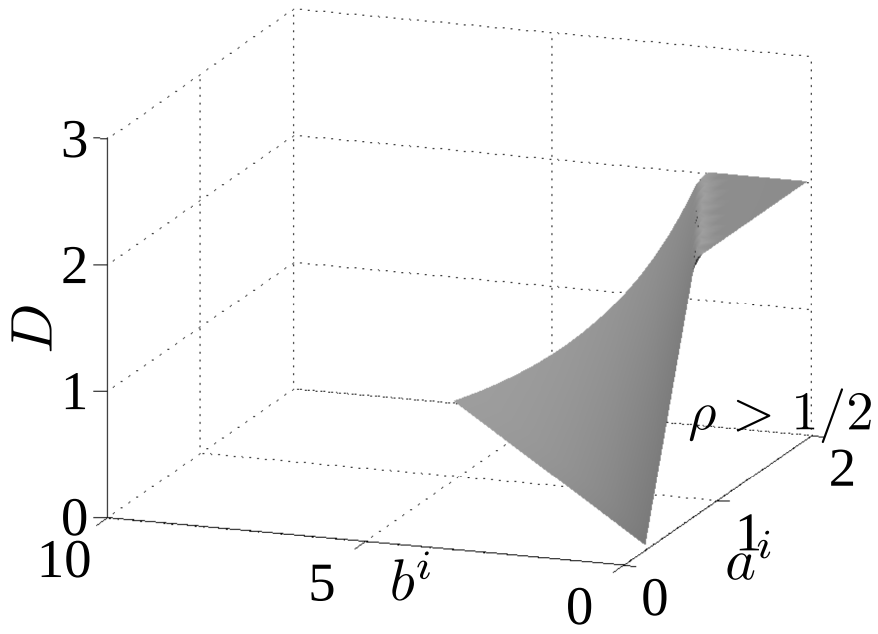
$$\text{with } A_i(D) = \max\left\{1, \sqrt{\frac{2+b^i}{2a^i}}, \frac{b^i}{6a^i}\left(1+\frac{3}{2D}\right) + \sqrt{\left(\frac{b^i}{6a^i}\left(1+\frac{3}{2D}\right)\right)^2 + \frac{2}{Da^i}}\right\}$$

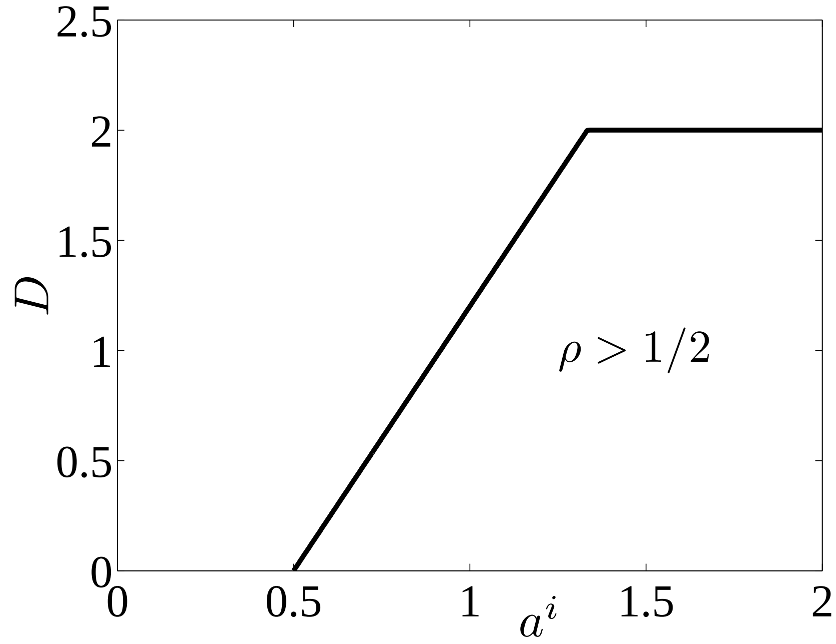
$\Rightarrow$ Induction

(iii) Apply Cauchy–Hadamard with $\sup_{t\geq 0}|\hat{x}_n^i(t)| \leq \left(DA_i(D)\right)^n n!, \quad n \geq 2$ from (i) and (ii)

Convergence analysis and summability

- Parameter–space ensuring uniform convergence with $\rho > 1/2$





(recall $\partial_t x^i = a^i \partial_z^2 x^i - b^i x^i \partial_z x^i + c^i(t) x^i$)

- Convergence restrictions can be relaxed by **summability methods** [5]

$$x^i(z,t) \cong \hat{x}^i(z,t) = \left(\mathcal{S}_k^{N,\xi} \hat{x}^i\right)(z,t) = \frac{\sum_{n=0}^{N} s_n^i(z,t) \frac{\xi}{\Gamma(1+\frac{n}{k})}}{\sum_{n=0}^{N} \frac{\xi}{\Gamma(1+\frac{n}{k})}}, \quad s_n^i(z,t) = \sum_{j=0}^{n} \hat{x}_j^i(t) \frac{(z - 1/2)^j}{j!}$$

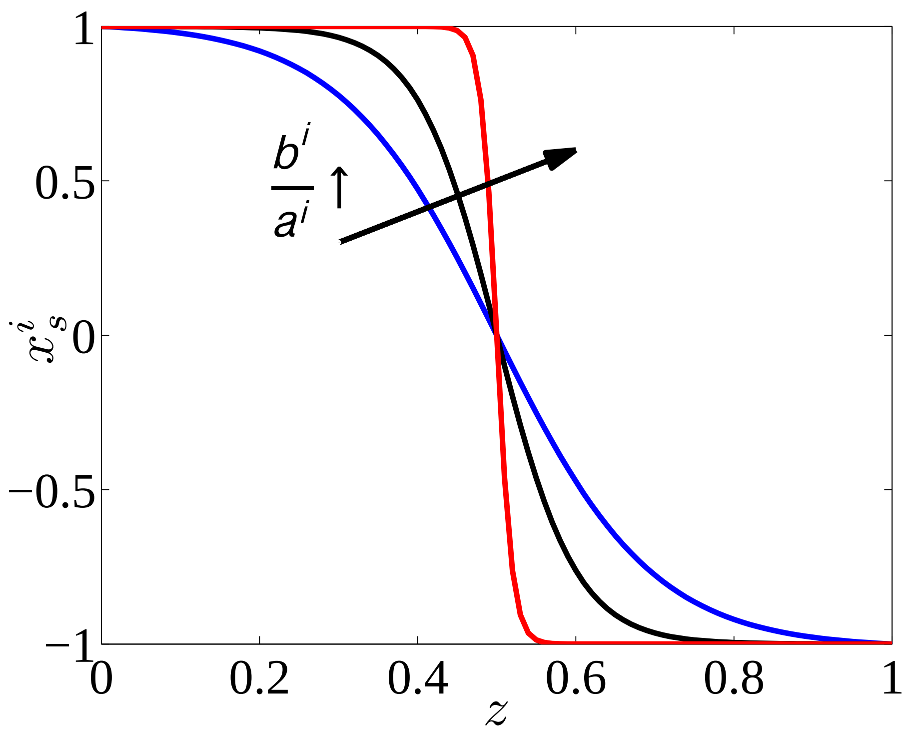
Desired formation profiles

- **Goal:** Realize finite time deployment into steady state formation profiles

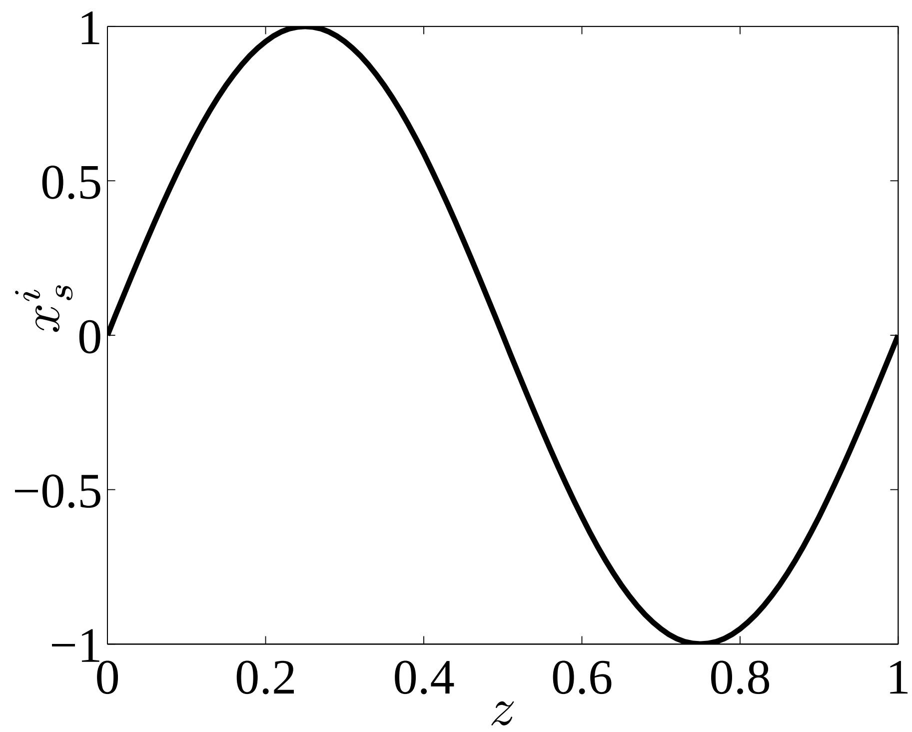
$$a^i\partial_z^2 x_s^i(z) - b^i x_s^i(z)\partial_z x_s^i(z) + c_s^i x_s^i(z) = 0,\ z \in (0,1)$$
$$x_s^i(0) = u_{a,s}^i, \quad x_s^i(1) = u_{l,s}^i$$

Admits a closed–form solution only for special cases

(1) **Shock–like steady st.** for $a^i \downarrow \vee\, b^i \uparrow$



(2) **Eigenf.** for $b^i = 0,\ c_s^i = (k\pi)^2,\ k \in \mathbb{N}$



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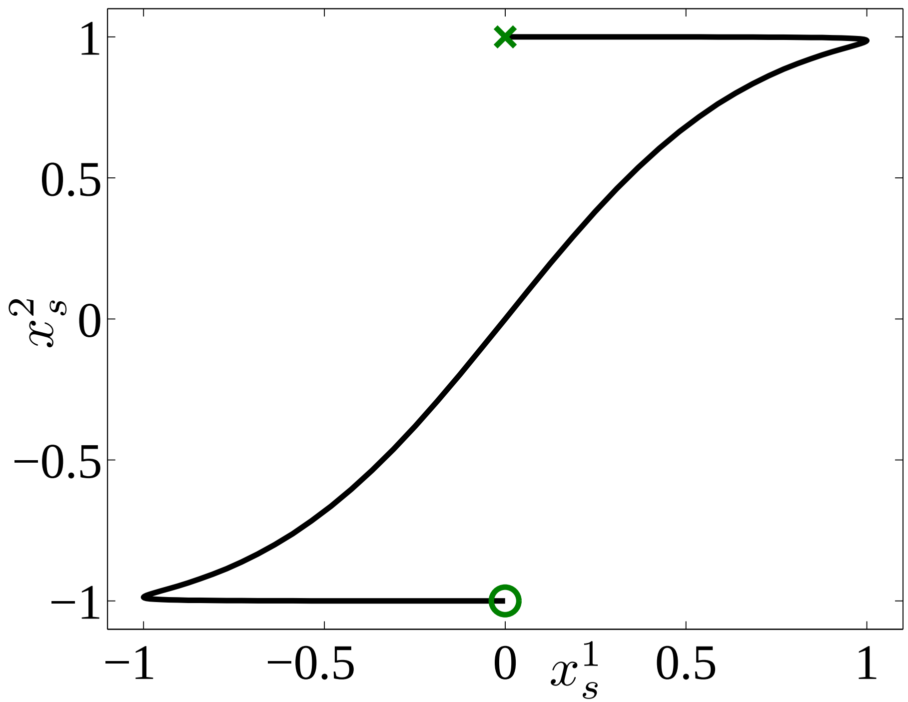
(1)+(2) $\rightsquigarrow$ non–trivial profiles



$\Rightarrow$ Transitions by means of desired trajectories $(y_1^{*,i}(t), y_2^{*,i}(t))$ for basic output

Trajectory assignment for basic output

- **Goal:** Realize finite time deployment into steady state formation profiles

$$a^i \partial_z^2 x_s^i(z) - b^i x_s^i(z)\partial_z x_s^i(z) + c_s^i x_s^i(z) = 0,\; z \in (0,1)$$
$$x_s^i(0) = u_{a,s}^i, \quad x_s^i(1) = u_{l,s}^i$$

Fix $(u_{a,s}^{i,0}, u_{l,s}^{i,0})$ as well as $(u_{a,s}^{i,T}, u_{l,s}^{i,T})$ and solve for $x_s^{i,0}(z)$ and $x_s^{i,T}(z)$

- Desired trajectories for the basic output

$$y_1^{*,i}(t) = A_1^0 + (A_1^T - A_1^0)\Phi_{\gamma,T}(t)$$
$$y_2^{*,i}(t) = A_2^0 + (A_2^T - A_2^0)\Phi_{\gamma,T}(t)$$

  - $\Phi_{\gamma,T}(t)$ non–analytic, i.e. $\Phi_{\gamma,T}(t) = 0$ if $t \leq 0$, $\Phi_{\gamma,T}(t) = 1$ if $t \geq T$, $\partial_t^n \Phi_{\gamma,T}(t)|_{t\in\{0,T\}} = 0$
  - $A_1^0 = y_1^{*,i}(0) = x_s^{i,0}(1/2)$, $A_2^0 = y_2^{*,i}(0) = \partial_z x_s^{i,0}(1/2)$, $A_1^T = y_1^{*,i}(T) = x_s^{i,T}(1/2)$, $A_2^T = y_2^{*,i}(T) = \partial_z x_s^{i,T}(1/2)$ for **consistency** with initial and final steady states $x_s^{i,0}(z)$ and $x_s^{i,T}(z)$

- Temporal path for reaction parameter

$$c^i(t) = c_{s,0}^i + (c_{s,T}^i - c_{s,0}^i)\Phi_{\gamma,T}(t)$$

$\Rightarrow$ connect different families of steady states

Feedforward formation tracking control

- **Feedforward controls** for leader and anchor

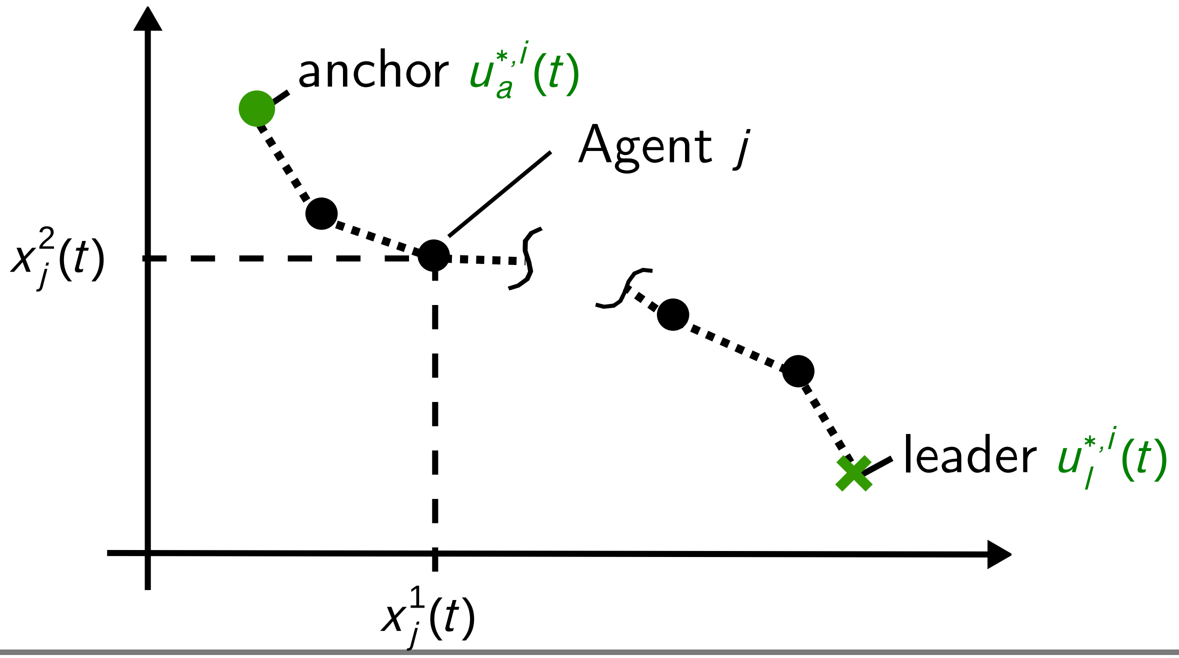
$$u_a^{*,i}(t) = \left(\mathcal{S}_k^{N,\xi}\hat{x}\right)(0,t),\; u_l^{*,i}(t) = \left(\mathcal{S}_k^{N,\xi}\hat{x}\right)(1,t) \text{ with } x^i(z,t) = \sum_{n=0}^{\infty} \psi_n\left(\boldsymbol{y}_{1,n}^{*,i}(t), \boldsymbol{y}_{2,n}^{*,i}(t)\right)\frac{(z-1/2)^n}{(n)!}$$

$\Rightarrow$ independent of communication topology

- **Communication topology** by discretization (continuum to m agents)

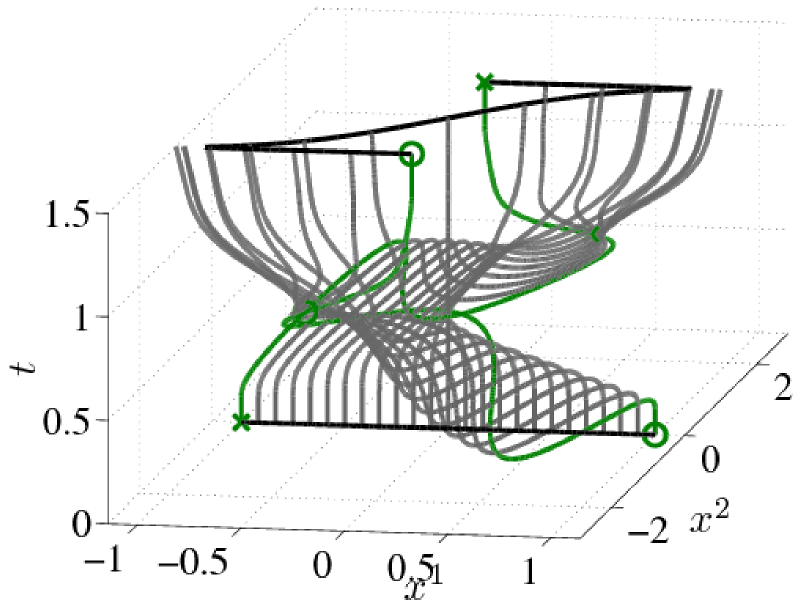
$$\partial_t x_j^i(t) = \frac{2a^i - b^i\Delta z x_j^i(t)}{2\Delta z^2}x_{j+1}^i(t) + \left(c^i(t) - \frac{2a^i}{\Delta z^2}\right)x_j^i(t) + \frac{2a^i + b^i\Delta z x_j^i(t)}{2\Delta z^2}x_{j-1}^i(t), \quad j = 1,\dots,m-1$$

$$x_0^i(t) = u_a^{*,i}(t),\; x_m^i(t) = u_l^{*,i}(t)$$



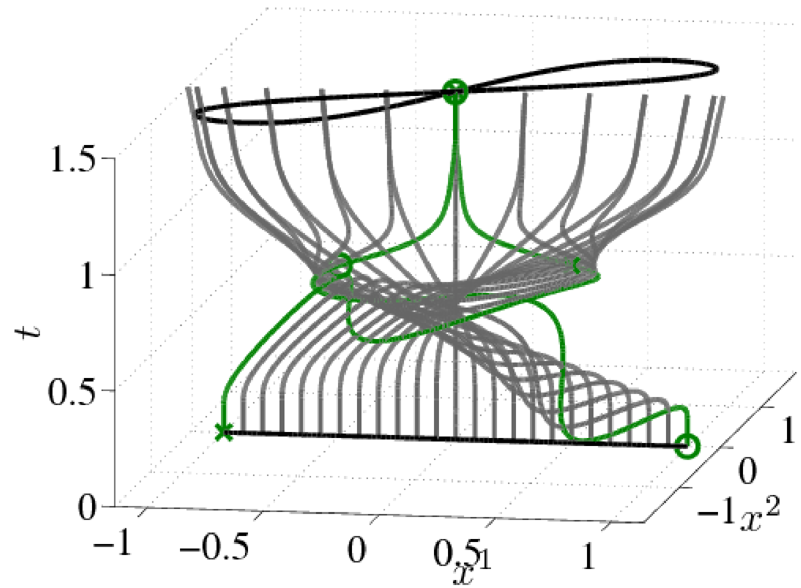
Simulation results (1)

- Finite time deployment into $\mathcal{Z}$–shape ($m = 25$)



Simulation results (2)

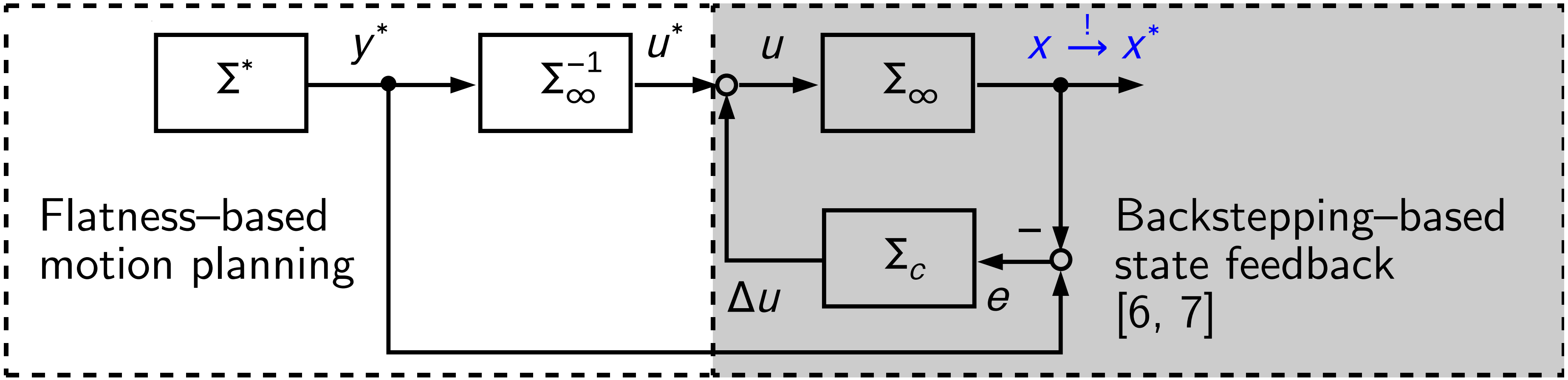
- Finite time deployment into 8–shape ($m = 25$)



Enlarged set of target formations by including **unstable** steady state profiles

⇒ Realization requires the exponential **stabilization of the tracking error**

- 2DOF control approach for spatial–temporal systems



Conclusion

- Flatness–based motion planning approach for finite time deployment of mobile agents
- Continuum of agents governed by viscous time–varying Burgers–type PDE
- Applicability of the PDE–based motion planning can be significantly enhanced by summability techniques
- Communication topology for discrete set of agents by finite difference discretization (decentralized)

Ongoing research

- Feedback stabilization with agent localization using backstepping
- Relationship with (approximate) controllability, stabilizability, ...
- Different communication topologies (2D, 3D, ...)

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