The circle criterion, balls of stabilizing gains and input-to-state stability

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Further development of joint work (2008) with

B Jayawardhana (Groningen) and E P Ryan (Bath)
Contents

1. Introduction

2. Balls of stabilizing gains: Aizerman version of circle criterion & ISS

3. “Standard” version of circle criterion & ISS

4. ISS with bias (practical ISS)

5. Hysteretic nonlinearities
1 Introduction
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absolute stability, input-to-state stability, and boundedness properties of a feedback interconnection of a well-posed infinite-dimensional MIMO linear system $\Sigma$ and a static nonlinearity $f$

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Well-posed linear systems: Curtain, Salamon, Staffans, Weiss.
Feedback system
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\begin{align*}
\dot{x} &= Ax + Bu, \quad x(0) = x^0 \in X, \\
y &= C_\Lambda \left( x - (s_0 I - A)^{-1} Bu \right) + G(s_0)u, \\
u &= v - f(y)
\end{align*}
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where

$\text{Re } s_0 > \text{exponential growth constant of } A$. 

CDPS 2011 Workshop, Wuppertal, 18-22 July 2011
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Let \(0 < \sigma \leq \infty\). A solution of (FS) on \([0, \sigma)\) is a pair

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(x, y) \in C([0, \sigma), X) \times L_{\text{loc}}^2([0, \sigma), Y)
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The set of all global solutions of (FS) is denoted by \(S(x^0, v)\).
For every solution there exists (by Zorn’s lemma) a maximally defined solution which cannot be extended any further.
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The question of existence requires addressing on a less general basis, taking into account relevant features of the particular system or subclass of systems under consideration.

**Special case:** if $C$ is bounded, $\dim Y < \infty$, feedthrough is equal to 0 and $f$ is continuous, then, for every $(x^0, v) \in X \times L^2_{\text{loc}}(\mathbb{R}^+, Y)$, (FS) has solutions.
The feedback system (FS) is said to be **input-to-state stable (ISS)** if there exist functions $\gamma_1 \in \mathcal{K}\mathcal{L}$ and $\gamma_2 \in \mathcal{K}$ such that, for each $x^0 \in X$, each $v \in L^\infty_{\text{loc}}(\mathbb{R}_+, Y)$ and all solutions in $S(x^0, v)$,

$$\|x(t)\| \leq \gamma_1(t, \|x^0\|) + \gamma_2(\|v\|_{L^\infty_{\text{loc}}(0,t)}), \quad \forall t \in \mathbb{R}_+.$$
The feedback system (FS) is said to be input-to-state stable (ISS) if there exist functions $\gamma_1 \in \mathcal{KL}$ and $\gamma_2 \in \mathcal{K}$ such that, for each $x^0 \in X$, each $v \in L^\infty_{\text{loc}}(\mathbb{R}_+, Y)$ and all solutions in $S(x^0, v)$,

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- $\mathcal{K}\mathcal{L} = \text{all } \gamma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \text{ which are decreasing and converging to 0 in the first variable and of class } \mathcal{K} \text{ in the second variable.}$
2 Balls of stabilizing gains: Aizerman version of circle criterion & ISS
We say that $K \in \mathcal{B}(Y)$ is an admissible feedback operator for $\Sigma$ if there exists $\alpha \in \mathbb{R}$ such that

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**Notation.** For $K \in \mathcal{B}(Y)$ and $r > 0$, define

$$\mathcal{B}(K, r) := \{T \in \mathcal{B}(Y) : \|T - K\| < r\}.$$
Theorem (Aizerman version of circle criterion)
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Let $K \in \mathcal{B}(Y)$ and $r > 0$. Assume that $\Sigma$ is optimizable and estimatable and $\mathcal{B}(K, r) \subset S(G)$. If

$$\sup_{z \neq 0} \frac{\|f(z) - Kz\|}{\|z\|} < r,$$

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Moreover, if $v \in L^\infty_{\text{loc}}(\mathbb{R}_+, Y)$, then, in the above estimate for $x$, the $L^2$-norm of $v$ on $[0, t]$ may be replaced by the $L^\infty$-norm of $v$ on $[0, t]$ (yielding an ISS result).
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Theorem remains true for time-dependent nonlinearities \( f = f(t, z) \) provided the “boundedness” condition on \( f \) holds uniformly in \( t \).
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The assumptions of the above Theorem guarantee that maximal defined solutions are global, provided that (FS) has the blow-up property.
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The assumptions of the above Theorem guarantee that maximal defined solutions are global, provided that (FS) has the blow-up property.

(FS) has the blow-up property if, for every maximally defined solution $(x, y)$ with finite interval of existence $[0, \omega)$,

$$\max \left\{ \limsup_{t \uparrow \omega} \|x(t)\|, \lim_{t \uparrow \omega} \int_0^t \|y(\tau)\|^2 d\tau \right\} = \infty.$$
If $f \circ w \in L^2_{\text{loc}}(\mathbb{R}^+, Y)$ for all $w \in L^2_{\text{loc}}(\mathbb{R}^+, Y)$, then the condition

$$\max \left\{ \limsup_{t \uparrow \omega} \|x(t)\|, \lim_{t \uparrow \omega} \int_0^t \|y(\tau)\|^2 d\tau \right\} = \infty$$

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**Special case:** blow-up property holds if $C$ is bounded, $\dim Y < \infty$, feedthrough is equal to 0 and $f$ satisfies the “ball condition” of the Theorem.
Proof of Aizerman version of circle criterion - main ideas
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Lemma

If $\mathbb{B}(K, r) \subset S(G)$, then $\|G(I + KG)^{-1}\|_{H\infty} \leq 1/r$. 

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- It is important that $Y$ is a complex space - Lemma does not hold in a real setting.
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- Apply small-gain ideas together with exponential weighting technique to output equation and use Lemma to obtain estimate for $y$. 
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Lemma

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- Apply loop shifting with \( K \).

- Apply small-gain ideas together with exponential weighting technique to output equation and use Lemma to obtain estimate for \( y \).

- Use results from theory of well-posed linear systems to obtain estimate for state.
Exponential weighting/small gain: idea is old - goes back to papers by Sandberg and Zames from the 1960s. Was used in input-output setting, but not in state-space context.
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Aizerman conjecture over the complex field: was studied (in a different context) by Hinrichsen & Pritchard (1992).
Let us look at the Lemma again.

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Set $G_K := G(I + KG)^{-1}$. 
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**Lemma**

If $B(K, r) \subset S(G)$, then $\|G(I + KG)^{-1}\|_{H^\infty} \leq 1/r$.

**Proof of Lemma**

Set $G_K := G(I + KG)^{-1}$.

Choose $s_n$ with $\text{Re} s_n > 0$ such that

$$\|G_K\|_{H^\infty} - \|G_K(s_n)\| \leq 1/n.$$  

Can construct operators $Z_n \in B(Y)$ (of rank 1, in general complex, even if the underlying system is real) such that

$$0 \leq \|Z_n\| - 1/\|G_K(s_n)\| \leq 1/n$$

and

$I + Z_n G_K(s_n)$ is not invertible.
Hence $Z_n \notin S(G_K)$ and so $Z_n + K \notin S(G)$. By hypothesis, this implies that $Z_n + K \notin \mathcal{B}(K, r)$ and therefore $\|Z_n\| \geq r$. By the above construction, $\|Z_n\| \to \frac{1}{\|G_K\|_{H^\infty}}$ as $n \to \infty$, showing that

$$\frac{1}{\|G_K\|_{H^\infty}} \geq r,$$

or, equivalently,

$$\|G(I + KG)^{-1}\|_{H^\infty} = \|G_K\|_{H^\infty} \leq \frac{1}{r}.$$
Lemma remains true for real data, provided that

\[ \| G_K \|_{H^\infty} = \sup_{s \in R} \| G_K(s) \|, \quad (\star) \]

where

\[ R := \{ s \in \mathbb{C}_0 : G_K(s) \text{ real} \}. \]
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(In this case, the operators \( Z_n \) can be chosen to be real.)
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In the SISO case, (\(\star\)) means that the maximal distance of the Nyquist diagram of \(G_K\) to the origin is achieved when it “intersects” the real axis.
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In the SISO case, (\( \star \)) means that the maximal distance of the Nyquist diagram of \( G_K \) to the origin is achieved when it “intersects” the real axis.

Under the additional assumption that (\( \star \)) holds, Aizerman version of the circle criterion remains true in a real setting.
3 “Standard” version of circle criterion & ISS
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Theorem ("Standard" version of circle criterion) 

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- \( K_1 \) is an admissible feedback operator, 
- \( K_2 - K_1 \) is invertible, 
- \( (I + K_2 G)(I + K_1 G)^{-1} \) is positive real.
Theorem ("Standard" version of circle criterion) 

Let $K_1, K_2 \in \mathcal{B}(Y)$ and let $\Sigma$ be optimizable and estimatable. Assume that

1. $K_1$ is an admissible feedback operator,
2. $K_2 - K_1$ is invertible,
3. $(I + K_2 G)(I + K_1 G)^{-1}$ is positive real.

Moreover, assume that there exists $\delta > 0$ such that the sector condition

$$\text{Re}\langle f(z) - K_1 z, f(z) - K_2 z \rangle \leq -\delta \|z\|^2 \quad \forall z \in Y$$

holds.
Then there exist positive $\gamma$ and $\Gamma$ such that, for each $(x, y) \in S(x^0, v)$,

$$\|x(t)\| \leq \Gamma \left( \exp(-\gamma t) \|x^0\| + \|v\|_{L^2(0,t)} \right), \quad \forall t \in \mathbb{R}_+$$
Then there exist positive $\gamma$ and $\Gamma$ such that, for each $(x, y) \in S(x^0, v)$,

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$$

Moreover, if $v \in L^\infty_{\text{loc}}(\mathbb{R}_+, Y)$, then, in the above estimate for $x$, the $L^2$-norm of $v$ on $[0, t]$ may be replaced by the $L^\infty$-norm of $v$ on $[0, t]$ (yielding an ISS result).
In the SISO real case, the strict sector condition

$$\text{Re}\langle f(z) - K_1 z, f(z) - K_2 z \rangle \leq -\delta \|z\|^2 \quad \forall z \in Y$$

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In the SISO real case, the strict sector condition

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can be expressed as

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(k_1 + \varepsilon)z^2 \leq f(z)z \leq (k_2 - \varepsilon)z^2 \quad \forall z \in \mathbb{R},
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Proof of “standard” version of circle criterion - main ideas
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Set $L := (K_2 - K_1)/2$
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\[ v \rightarrow + \rightarrow \Sigma \rightarrow y \rightarrow L \rightarrow Ly \]

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Let $\tilde{G}$ denote the transfer function of $\tilde{\Sigma}$
Proof of “standard” version of circle criterion - main ideas

Set \( L := (K_2 - K_1)/2 \)

Consider

\[
\begin{aligned}
\v + \Sigma - & \rightarrow y \\
& \rightarrow L \\
& \rightarrow Ly
\end{aligned}
\]

\( K_1 \)

The system \( \tilde{\Sigma} \)

Let \( \tilde{G} \) denote the transfer function of \( \tilde{\Sigma} \)

Set

\[
\tilde{f}(z) := f(L^{-1}z) - K_1 L^{-1}z \quad \forall z \in Y.
\]
If \((x, y) \in \mathcal{S}(x^0, v)\), then \((x, Ly)\) is a solution of

\[
\begin{array}{c}
v \\
\downarrow \quad + \\
\uparrow \quad - \\
\tilde{\Sigma} \\
\tilde{f} \\
Ly
\end{array}
\]

Lure system given by \(\tilde{\Sigma}\) and \(\tilde{f}\)
If \((x, y) \in \mathcal{S}(x^0, v)\), then \((x, Ly)\) is a solution of

\[
\begin{align*}
v & \quad + \\ \sum & \quad - \\ \tilde{f} & \quad Ly
\end{align*}
\]

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Positive real condition guarantees that \(\mathbb{B}(I, 1) \subset \mathcal{S}(\tilde{G})\).
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& \downarrow - \leftarrow \tilde{f}
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Positive real condition guarantees that \(\mathbb{B}(I, 1) \subset S(\tilde{G})\).

Sector condition guarantees that

\[
\sup_{z \neq 0} \frac{\|\tilde{f}(z) - z\|}{\|z\|} < 1.
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\[
\begin{align*}
v & \quad \rightarrow \quad + \quad \rightarrow \quad \tilde{\Sigma} \\
& \quad \downarrow \quad - \quad \leftarrow \quad \tilde{f} \\
& \quad \rightarrow \quad \downarrow \quad \rightarrow \quad Ly
\end{align*}
\]

Lure system given by \(\tilde{\Sigma}\) and \(\tilde{f}\)

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Aizerman version of circle criterion (with \(K = I\) and \(r = 1\)) applies to above system, proving the claim.

\[\square\]
Theorem extends to the "non-square" case, provided that $K_2 - K_1$ is left invertible.
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  ▷ For finite-dimensional case, see Arcak & Teel (2002) and Jayawardhana, L & Ryan (2009).
4 ISS with bias (practical ISS)
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Theorem (Circle criterion - ISS with bias 1)
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Moreover, assume that there exist $\delta > 0$ and a bounded set $E \subset Y$ such that the generalized sector condition

$$\text{Re}\langle f(z) - K_1 z, f(z) - K_2 z\rangle \leq -\delta \|z\|^2 \quad \forall z \in Y \setminus E$$

holds.
Then there exist $\beta \geq 0$, $\gamma > 0$ and $\Gamma \geq 1$ such that, for each $(x, y) \in S(x^0, v)$,

$$\|x(t)\| \leq \Gamma \left( \exp(-\gamma t)\|x^0\| + \|v\|_{L^\infty(0,t)} + \beta \right), \quad \forall t \in \mathbb{R}_+$$
Then there exist $\beta \geq 0$, $\gamma > 0$ and $\Gamma \geq 1$ such that, for each $(x, y) \in \mathcal{S}(x^0, v)$,

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where the bias $\beta$ depends on $f$, $E$, $K_1$ and $K_2$. 
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where the bias $\beta$ depends on $f$, $E$, $K_1$ and $K_2$.

The bias $\beta$ is a measure of the extent of the violation of the sector condition on the set $E$. A bound for $\beta$ is given by

$$\beta \leq \sup_{z \in E} \|f(z) - \frac{1}{2}(K_1 + K_2)z\|.$$
SISO nonlinearity $f$ satisfying a generalized sector condition with $E = [-1, 1]$. 
Proof of Theorem - main ideas
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- Construct nonlinearity $\tilde{f}$ such that $\tilde{f}$ satisfies a “proper” sector condition with sector data $K_1$ and $K_2$,

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- Replace $f$ by $\tilde{f}$ and absorb error into input, that is, replace $v$ by

\[ \tilde{v}(t) := v(t) + \tilde{f}(y(t)) - f(y(t)). \]
Proof of Theorem - main ideas

- Construct nonlinearity \( \tilde{f} \) such that \( \tilde{f} \) satisfies a “proper” sector condition with sector data \( K_1 \) and \( K_2 \),

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\tilde{v}(t) := v(t) + \tilde{f}(y(t)) - f(y(t)).
\]

- Apply ISS version of circle criterion to Lure system with nonlinearity \( \tilde{f} \).
5 Hysteretic nonlinearities

Replace static nonlinearity \( f : Y \to Y \) by a \textit{causal} nonlinear operator

\[
F : \text{dom}(F) \subset L^2_{\text{loc}}(\mathbb{R}^+, Y) \to L^2_{\text{loc}}(\mathbb{R}^+, Y).
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Theorem on ISS with bias extends to this case.

In what sense?
Theorem (Circle criterion - ISS with bias 2)

Let $K_1, K_2 \in \mathcal{B}(Y)$, let $v \in L_\infty^{\infty}(\mathbb{R}^+, Y)$ and let $\Sigma$ be optimizable and estimatable. Assume that
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- $K_1$ is an admissible feedback operator,
- $K_2 - K_1$ is invertible,
- $(I + K_2 G)(I + K_1 G)^{-1}$ is positive real.

Moreover, assume that there exist $\delta > 0$, a bounded set $E \subset Y$ and $b > 0$ such that

$$\text{Re}\langle (F(w))(t) - K_1 w(t), (F(w))(t) - K_2 w(t) \rangle \leq -\delta \|w(t)\|^2$$

$$\forall (t, w) \in \mathbb{R}_+ \times \text{dom}(F) \text{ s.t. } w(t) \in Y \setminus E$$

and

$$\|F(w))(t)\| \leq b \quad \forall (t, w) \in \mathbb{R}_+ \times \text{dom}(F) \text{ s.t. } w(t) \in E.$$
Then there exist $\beta \geq 0$, $\gamma > 0$ and $\Gamma \geq 1$ such that, for each $(x, y) \in S(x^0, v),

\|x(t)\| \leq \Gamma \left( \exp(-\gamma t) \|x^0\| + \|v\|_{L^\infty(0,t)} + \beta \right), \quad \forall t \in \mathbb{R}_+$
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Are there any non-static nonlinearities which satisfy the relevant conditions?
Then there exist $\beta \geq 0$, $\gamma > 0$ and $\Gamma \geq 1$ such that, for each $(x, y) \in S(x^0, v)$,

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Are there any non-static nonlinearities which satisfy the relevant conditions?

Yes: hysteretic nonlinearities!
In the following: $Y = \mathbb{R}$. 

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- \( F : C'((\mathbb{R}^+)) \rightarrow C'((\mathbb{R}^+)) \) is a hysteresis operator if \( F \) is causal and rate-independent.
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- $F : C(\mathbb{R}_+) \rightarrow C(\mathbb{R}_+)$ is a **hysteresis operator** if $F$ is **causal** and rate-independent.

- **Rate independence** means that

\[ F(w \circ \zeta) = F(w) \circ \zeta \]

for every $w \in C(\mathbb{R}_+)$ and every time transformation $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ (continuous, non-decreasing and surjective).
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- A basic hysteresis operator is the **backlash or play** operator:

  ![Diagram](attachment:image.png)
Denote backlash operator by $B_{\sigma, \eta}$, where $\eta \in \mathbb{R}$ plays the role of an initial condition.
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It is clear from diagram that $\mathcal{B}_{\sigma, \eta}$ satisfies a generalized sector condition.
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$B_{\sigma, \eta}$ is the basic building block for other hysteresis operators, such as the Prandtl and Preisach operators which are “weighted sums” of backlash operators and exhibit nested hysteresis loops.
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Prandtl operator:

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(\mathcal{P}_{\xi}(w))(t) = \int_0^\infty (\mathcal{B}_{\sigma, \xi(\sigma)}(w))(t)\mu(d\sigma),
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Denote backlash operator by $B_{\sigma,\eta}$, where $\eta \in \mathbb{R}$ plays the role of an initial condition.

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$$ (P_\xi(w))(t) = \int_0^\infty (B_{\sigma,\xi(\sigma)(w)}(w))(t) \mu(d\sigma), $$

where

$\xi : \mathbb{R}_+ \rightarrow \mathbb{R}$ is globally Lipschitz and compactly supported.
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where

- $\xi : \mathbb{R}_+ \rightarrow \mathbb{R}$ is globally Lipschitz and compactly supported.
- $\mu$ is a finite Borel measure on $\mathbb{R}_+$ such that $\int_0^\infty \sigma\mu(d\sigma) < \infty$. 

CDPS 2011 Workshop, Wuppertal, 18-22 July 2011
Illustration

- $\xi(\sigma) \equiv 0$
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- $\mu(S) = \lambda(S \cap [0, 5])$ for every Borel set $S \subset \mathbb{R}_+$
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for every $\alpha > 0$,

$$(k - \alpha)w^2(t) \leq (P_\xi(w))(t)w(t) \leq (k + \alpha)w^2(t)$$

$$\forall (t, w) \in \mathbb{R}_+ \times C(\mathbb{R}_+) \text{ s.t. } |w(t)| \geq l/\alpha$$
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$$ k := \mu(\mathbb{R}_+), \quad l := \int_0^{\infty} \sigma \mu(d\sigma). $$

Conclusion: circle criterion (ISS with bias 2) applies.