Indirect damping for coupled systems

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Indirect stabilization of weakly coupled systems

Abstract set-up

Systems with standard boundary conditions

Systems with hybrid boundary conditions

Remarks and open problems



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-indirect stabilization

a conservative system

$\Omega \subset \mathbb{R}^n$ bounded the wave equation

$$\begin{array}{ll} \partial_t^2 v - \Delta v = 0 & \text{in} & \Omega \times \mathbb{R} \,, \\ v = 0 & \text{on} & \partial \Omega \times \mathbb{R} \,, \end{array}$$

describes a conservative system: the energy of a solution

$$E(u(t)) = \frac{1}{2} \int_{\Omega} \left(|Du(t,x)|^2 + |\partial_t u(t,x)|^2 \right) dx$$

is constant in t



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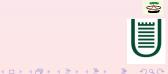
a dissipative system

the damped wave equation

$$\partial_t^2 u - \Delta u + \partial_t u = 0 \qquad \text{in} \qquad \Omega \times \mathbb{R},$$
$$u = 0 \qquad \text{on} \qquad \partial\Omega \times \mathbb{R},$$

is exponentially stable as $t \to \infty$

 $E(u(t)) \le E(u(0))e^{c(1-t)}$ (c > 0)



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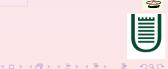
a weakly coupled system

consider the coupling through zero order terms

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0\\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in} \quad \Omega \times \mathbb{R}$$

$$u = 0 = v$$
 on $\partial \Omega \times \mathbb{R}$

any kind of stability for $\alpha \neq 0$?



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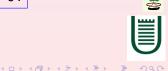
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lack of exponential stability

recast

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as an evolution equation in $\mathcal{H} = [H_0^1(\Omega) \times L^2(\Omega)]^2$

$$\begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix}' = \begin{pmatrix} L_1 & K \\ K & L_2 \end{pmatrix} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix} =: \mathcal{L} \begin{pmatrix} u \\ u' \\ v \\ v' \end{pmatrix}$$

L₁, L₂ generators of C₀-semigroups on H¹₀(Ω) × L²(Ω
 K compact operator in H¹₀(Ω) × L²(Ω)



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lack of exponential stability (ctnd)

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 ω_{ess}(L) = essential growth bound
 - (blind to compact perturbations)

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 \Rightarrow system cannot be exponentially stable



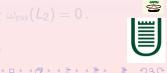
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a system of second order evolutions equations

in a separable Hilbert space *H*

$$\begin{pmatrix} u'' + A_1 u + Bu' + \alpha v = 0 \\ v'' + A_2 v + \alpha u = 0 \end{pmatrix}$$

(H1) $A_i : D(A_i) \subset H \rightarrow H \ (i = 1, 2)$ are densely defined closed linear operators such that

$$A_i = A_i^*$$
, $\langle A_i u, u \rangle \ge \omega_i |u|^2$ $(\omega_1, \omega_2 > 0)$

(H2) B is a bounded linear operator on H such that

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(H3) $0 < |\alpha| < \sqrt{\omega_1 \omega_2}$



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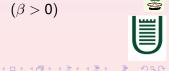
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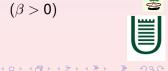
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-abstract set-up

energies

energies associated to A_1, A_2

$$E_i(u,p) = \frac{1}{2} \left(|A_i^{1/2}u|^2 + |p|^2 \right)$$

total energy of the system U = (u, p, v, q)

 $\mathcal{E}(U) := E_1(u, p) + E_2(v, q) + \alpha \langle u, v \rangle$

assumptions yield

$$|u|^{2} \leq \frac{2}{\omega_{i}} E_{i}(u, p)$$

$$\mathcal{E}(U) \geq \nu(\alpha) \Big[E_{1}(u, p) + E_{2}(v, q) \Big]$$

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-abstract set-up

reduction to a first order system

$$\mathcal{H} = D(A_1^{1/2}) \times H \times D(A_2^{1/2}) \times H$$

$$(U|\widehat{U}) = \langle A_1^{1/2}u, A_1^{1/2}\widehat{u} \rangle + \langle p, \widehat{p} \rangle$$

$$+ \langle A_2^{1/2}v, A_2^{1/2}\widehat{v} \rangle + \langle q, \widehat{q} \rangle + \alpha \langle u, \widehat{v} \rangle + \alpha \langle v, \widehat{u} \rangle$$

system takes the equivalent form

$$\begin{cases} U'(t) = \mathcal{A}U(t) \\ U(0) = U_0 := (u^0, u^1, v^0, v^1). \end{cases}$$

with $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ defined by $\begin{cases} D(\mathcal{A}) = D(A_1) \times D(A_1^{1/2}) \times D(A_2) \times D(A_2^{1/2}) \\ \mathcal{A}U = (p, -A_1u - Bp - \alpha v, q, -A_2v - \alpha u) \end{cases}$



-abstract set-up

reduction to a first order system

$$\begin{aligned} \mathcal{H} &= D(\mathcal{A}_1^{1/2}) \times \mathcal{H} \times D(\mathcal{A}_2^{1/2}) \times \mathcal{H} \\ (\mathcal{U}|\widehat{\mathcal{U}}) &= \langle \mathcal{A}_1^{1/2} u, \mathcal{A}_1^{1/2} \widehat{u} \rangle + \langle \mathcal{p}, \widehat{\mathcal{p}} \rangle \\ &+ \langle \mathcal{A}_2^{1/2} v, \mathcal{A}_2^{1/2} \widehat{v} \rangle + \langle \mathcal{q}, \widehat{\mathcal{q}} \rangle + \alpha \langle u, \widehat{v} \rangle + \alpha \langle v, \widehat{u} \rangle \end{aligned}$$

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-standard boundary conditions

a first stability result



-standard boundary conditions

a first stability result

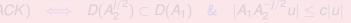
Theorem (ACK 2002) Assume, for some integer j > 2, $|A_1 u| \leq c |A_2^{j/2} u| \qquad \forall u \in D(A_2^{j/2})$ (ACK)



-standard boundary conditions

a first stability result

Theorem (ACK 2002) Assume, for some integer $j \geq 2$, $|A_1 u| \leq c |A_2^{j/2} u| \qquad \forall u \in D(A_2^{j/2})$ (ACK)► $U_0 \in D(\mathcal{A}^{nj}) \text{ (some } n \ge 1) \Rightarrow \mathcal{E}(U(t)) \le \frac{c_n}{t^n} \sum_{i=1}^{nj} \mathcal{E}(U^{(k)}(0))$



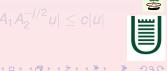


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 $(ACK) \iff D(A_2^{j/2}) \subset D(A_1) \quad \& \quad |A_1A_2^{-j/2}u| \le c|u|$

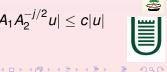


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-standard boundary conditions

main tools

proof uses

energy dissipation

$$\frac{d}{dt}\mathcal{E}(U(t)) = -|B^{1/2}u'(t)|^2 \qquad (U_0 \in D(\mathcal{A}))$$

• multipliers of the form $A_2^{2-j}v$ and $A_2^{1-j}A_1u$

an abstract decay lemma



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-standard boundary conditions

abstract decay lemma

A: D(A) ⊂ H → H generator of a C₀-semigroup
L: H → [0, +∞) continuous function $\int_0^T L(e^{tA}x)dt \le c \sum_{k=0}^K L(A^kx)$ ∀n ≥ 1, ∀x ∈ D(A^{nK}), ∀0 ≤ s ≤ T

$$\int_{s}^{T} L(e^{tA}x) \frac{(t-s)^{n-1}}{(n-1)!} dt \le c^{n}(1+K)^{n-1} \sum_{k=0}^{nK} L(e^{sA}A^{k}x)$$

 $\blacktriangleright L(e^{tA}x) \le L(e^{sA}x) \Rightarrow L(e^{tA}x) \le c^n(1+K)^{n-1} \frac{n!}{t^n} \sum_{k=0}^{m} C_k^{n-1} \sum_{k=0}^{m} C_k^{n-$



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standard boundary conditions

abstract decay lemma

► $A: D(A) \subset H \rightarrow H$ generator of a C_0 -semigroup

• $L: H \rightarrow [0, +\infty)$ continuous function

$$\int_0^T L(e^{t\mathsf{A}}x) dt \leq c \sum_{k=0}^K L(\mathsf{A}^k x)$$

$$\int_{s}^{T} L(e^{tA}x) \ \frac{(t-s)^{n-1}}{(n-1)!} \ dt \le c^{n}(1+K)^{n-1} \sum_{k=0}^{nK} L(e^{sA}A^{k}x)$$

 $L(e^{tA}x) \leq L(e^{sA}x) \Rightarrow L(e^{tA}x) \leq c^n(1+K)^{n-1} \frac{n!}{t^n} \sum_{k=1}^{nK} L(A^kx)$



-standard boundary conditions

abstract decay lemma

► $A: D(A) \subset H \rightarrow H$ generator of a C_0 -semigroup • $L: H \rightarrow [0, +\infty)$ continuous function $\int_0^T L(e^{tA}x)dt \le c \sum_{k=0}^{K} L(A^k x)$ $\Rightarrow \forall n > 1, \forall x \in D(A^{nK}), \forall 0 < s < T$ $\int_{s}^{t} L(e^{tA}x) \frac{(t-s)^{n-1}}{(n-1)!} dt \le c^{n}(1+K)^{n-1} \sum_{k=1}^{nK} L(e^{sA}A^{k}x)$

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-standard boundary conditions

example 1: Dirichlet boundary conditions

$$\Omega \subset \mathbb{R}^n \quad \text{bounded} \quad \Gamma = \partial \Omega$$

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0\\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in} \quad \Omega \times (0, +\infty)$$

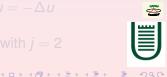
with boundary conditions

$$u(\cdot, t) = 0 = v(\cdot, t)$$
 on $\Gamma \quad \forall t > 0$

in this example $A_1 = A = A_2$ with

 $D(A) = H^2(\Omega) \cap H^1_0(\Omega), \qquad Au = -\Delta u$

so that (ACK) : $|A_1u| \le c |A_2^{j/2}u|$ holds with j = 2



-standard boundary conditions

example 1: Dirichlet boundary conditions

$$\begin{split} \Omega \subset \mathbb{R}^n & \text{bounded} \quad \Gamma = \partial \Omega \\ & \left\{ \begin{array}{l} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{array} \right. & \text{in} \quad \Omega \times (0, +\infty) \end{split}$$

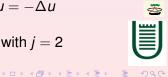
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-standard boundary conditions

example 1: conclusion

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0 \\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in} \quad \Omega \times (0, +\infty) \\ u(\cdot, t) = 0 = v(\cdot, t) \quad \text{on} \quad \Gamma \quad \forall t > 0 \\ \begin{cases} u(x, 0) = u^0(x), & u'(x, 0) = u^1(x) \\ v(x, 0) = v^0(x), & v'(x, 0) = v^1(x) \end{cases} \quad x \in \Omega \\ \text{nclusion: for} \quad 0 < |\alpha| < C_\Omega \end{cases}$$



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-standard boundary conditions

example 1: conclusion

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$$\int_{\Omega} \left(|\partial_t u|^2 + |\nabla u|^2 + |\partial_t v|^2 + |\nabla v|^2 \right) dx$$

$$\leq \frac{c}{t} \left(\|u^0\|_{2,\Omega}^2 + \|u^1\|_{1,\Omega}^2 + \|v^0\|_{2,\Omega}^2 + \|v^1\|_{1,\Omega}^2 \right)$$

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-standard boundary conditions

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-standard boundary conditions

example 2: hybrid boundary conditions

Let $\alpha \in \mathbb{R}$ and consider the problem

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0\\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases}$$

with boundary conditions

$$egin{pmatrix} \displaystyle \left(rac{\partial u}{\partial
u}+u
ight)(\cdot,t)&=0 ext{ on } \Gamma \ v(\cdot,t)&=0 ext{ on } \Gamma \ \end{cases} orall t>0$$



-standard boundary conditions

example 2: hybrid boundary conditions

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with boundary conditions

$$\begin{pmatrix} \frac{\partial u}{\partial \nu} + u \end{pmatrix} (\cdot, t) = 0 \text{ on } \Gamma \\ v(\cdot, t) = 0 \text{ on } \Gamma \end{cases} \quad \forall t > 0$$



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standard boundary conditions

ACK does not apply

$$D(A_1) = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} + u = 0 \text{ on } \Gamma \right\}, \ A_1 u = -\Delta u$$
$$D(A_2) = H^2(\Omega) \cap H^1_0(\Omega), \ A_2 v = -\Delta v$$

$$(k = 2) \quad v_0 : \begin{cases} (-\Delta)^2 v = 1 \\ v_0 = 0 = \Delta v_0 & \text{on } \Gamma \end{cases}$$

$$D(A_2) \subset D(A_1) \Rightarrow \frac{\partial v_0}{\partial \nu}_{|\Gamma} = 0 \Rightarrow \int_{\Omega} v_1 dx = 0$$

$$\Rightarrow \int_{\Omega} |\nabla v_1|^2 dx = \int_{\Omega} (-\Delta v_1) v_1 dx = 0 \text{ but } -\Delta v_1 = 1$$

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-standard boundary conditions

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Lemma (ACG) $D(A_2^{k/2})$ is not included in $D(A_1)$ for any $k \ge 2$

-standard boundary conditions

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Lemma (ACG) $D(A_2^{k/2})$ is not included in $D(A_1)$ for any $k \ge 2$ Proof. roof. $(k = 2) \quad v_0 : \begin{cases} (-\Delta)^2 v = 1 \\ v_0 = 0 = \Delta v_0 & \text{on } \Gamma \\ D(A_2) \subset D(A_1) \Rightarrow \frac{\partial v_0}{\partial \nu} = 0 \Rightarrow \int_{\Omega} v_1 dx = 0 \end{cases}$ $V_1 := -\Delta V_0$

standard boundary conditions

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$$D(A_1) = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} + u = 0 \text{ on } \Gamma \right\}, \ A_1 u = -\Delta u$$
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hybrid boundary conditions

second stability result

Theorem (ACG 2011) Assume

 $D(A_2) \subset D(A_1^{1/2})$ & $|A_1^{1/2}u| \le c|A_2u| \quad \forall u \in D(A_2)$ (ACG)

Then

► $U_0 \in D(\mathcal{A}^{4n}) \text{ (some } n \ge 1) \Rightarrow \mathcal{E}(U(t)) \le \frac{c_n}{t^n} \sum_{k=0}^{4n} \mathcal{E}(U^{(k)}(0))$ ► $\forall U_0 \in \mathcal{H} \quad \mathcal{E}(U(t)) \to 0 \quad \text{as } t \to \infty$

observe

 $(ACG) \iff |\langle A_1 u, v \rangle| \le c |A_2 v| \langle A_1 u, u \rangle^{1/2}$



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hybrid boundary conditions

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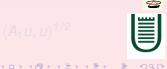
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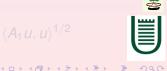
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hybrid boundary conditions

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hybrid boundary conditions

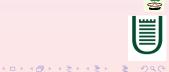
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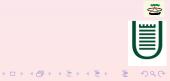
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hybrid boundary conditions

main tools

proof uses

energy dissipation

$$\frac{d}{dt}\mathcal{E}(U(t)) = -|B^{1/2}u'(t)|^2 \qquad (U_0 \in D(\mathcal{A}))$$

- multipliers of the form $A_1^{-1}v$ and $A_2^{-1}u$
- abstract decay lemma



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-hybrid boundary conditions

use interpolation

polynomial decay estimates improved for $U_0 \in (\mathcal{H}, D(\mathcal{A}^{4n}))_{\theta,2} \quad \text{for some} \quad n \geq 1 \ , \ 0 < \theta < 1$

► since A generates a C₀-semigroup of contractions,

 $D(\mathcal{A}^m) = (\mathcal{H}, D(\mathcal{A}^k))_{\theta, 2}$

if heta k = m for some 0 < heta < 1 and $k, m \in \mathbb{N}$

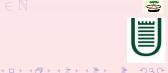


-hybrid boundary conditions

use interpolation

polynomial decay estimates improved for
 U₀ ∈ (H, D(A⁴ⁿ))_{θ,2} for some $n \ge 1$, $0 < \theta < 1$

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-hybrid boundary conditions

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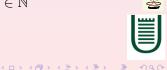
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hybrid boundary conditions

ACG with data in interpolation spaces

assume $D(A_2) \subset D(A_1^{1/2}) \quad \& \quad |A_1^{1/2}u| \leq c|A_2u| \quad (ACG)$ let $n \geq 1, \quad 0 < \theta < 1$ then $\mathbb{V}_0 \in (\mathcal{H}, D(\mathcal{A}^{4n}))_{\theta,2} \Longrightarrow \|U(t)\|_{\mathcal{H}}^2 \leq \frac{c_{n,\theta}}{t^{n\theta}} \|U_0\|_{(\mathcal{H}, D(\mathcal{A}^{4n}))_{\theta,2}}^2$ $\mathbb{V}_0 \in D(\mathcal{A}^n) \Longrightarrow \mathcal{E}(U(t)) \leq \frac{c_n}{t^{n/4}} \sum_{k=0}^n \mathcal{E}(U^{(k)}(0))$



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hybrid boundary conditions

ACG with data in interpolation spaces

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 & $|A_1^{1/2}u| \le c|A_2u|$ (ACG)

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-hybrid boundary conditions

example 2: ACG applies

the energy of the solution to the boundary-value problem

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u + \alpha v = 0\\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in } \Omega \times (0, +\infty)$$

$$\begin{pmatrix} \frac{\partial u}{\partial \nu} + u \end{pmatrix} (\cdot, t) = 0 \quad \text{on } \Gamma \\ v(\cdot, t) = 0 \quad \text{on } \Gamma \end{cases} \quad \forall t > 0$$

satisfies, for $0 < |\alpha| < C_{\Omega}$,

 $E_{1}(u(t), u'(t)) + E_{2}(v(t), v'(t))$ $\leq \frac{c}{t^{1/4}} \left(|A_{1}u^{0}|^{2} + |A_{1}^{1/2}u^{1}|^{2} + |A_{2}v^{0}|^{2} + |A_{2}^{1/2}v^{1}|^{2} \right)$

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hybrid boundary conditions

proof

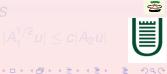
recall

$$D(A_1) = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} + u = 0 \text{ on } \Gamma \right\}, \ A_1 u = -\Delta u$$
$$D(A_2) = H^2(\Omega) \cap H^1_0(\Omega), \ A_2 v = -\Delta v$$

▶ to obtain, for all $u \in D(A_1), v \in D(A_2)$,

$$|\langle A_1 u, v \rangle| = \left| \int_{\Omega} \nabla u \nabla v \, dx \right| \le c \langle A_1 u, u \rangle^{1/2} |A_2 v|$$

since $\langle A_1 u, u \rangle = \int_{\Omega} |\nabla u|^2 dx + \int_{\Gamma} |u|^2 dS$ > yields (ACG): $D(A_2) \subset D(A_1^{1/2})$ & $|A_1^{1/2}u| \le c |A_2 u|$



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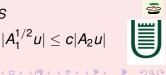
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-concluding remarks

operators of higher order

define

$$D(A_1) = \left\{ u \in H^4(\Omega) : \Delta u = 0 = \frac{\partial \Delta u}{\partial \nu} \text{ on } \Gamma \right\}, \quad A_1 u = \Delta^2 u$$
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$$|\langle A_1 u, v \rangle| = \left| \int_{\Omega} \Delta u \Delta v \, dx \right| \le c \, \langle A_1 u, u \rangle^{1/2} \, |A_2 v|$$

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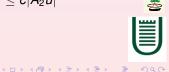
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-concluding remarks

application

consider boundary-value problem

$$\begin{cases} \partial_t^2 u + \Delta^2 u + \partial_t u + \alpha v = 0\\ \partial_t^2 v - \Delta v + \alpha u = 0 \end{cases} \quad \text{in } \Omega \times (0, +\infty)$$

$$\Delta u(\cdot, t) = 0 = \frac{\partial \Delta u}{\partial \nu}(\cdot, t)$$
 and $v(\cdot, t) = 0$ on Γ

then, for 0 $<|\alpha|< {\cal C}_{\Omega}^{3/2},$ energy decays at polynomial rate

$$\begin{split} \mathsf{E}_1(u(t), u'(t)) + \mathsf{E}_2(v(t), v'(t)) \\ & \leq \frac{c}{t^{1/4}} \left(\|u^0\|_{4,\Omega}^2 + \|u^1\|_{2,\Omega}^2 + \|v^0\|_{2,\Omega}^2 + \|v^1\|_{1,\Omega}^2 \right) \end{split}$$



-concluding remarks

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-concluding remarks

different coupling parameters

for general $lpha_{\mathsf{1}}, lpha_{\mathsf{2}} \in \mathbb{R}$ consider

$$\begin{cases} u''(t) + A_1 u(t) + Bu'(t) + \alpha_1 v(t) = 0\\ v''(t) + A_2 v(t) + \alpha_2 u(t) = 0. \end{cases}$$

- ▶ above results can be generalized replacing (H3) with 0 < $\alpha_1 \alpha_2 < \omega_1 \omega_2$
 - $\blacktriangleright \mathcal{E}(\boldsymbol{U}) := \alpha_2 E_1(\boldsymbol{u}, \boldsymbol{p}) + \alpha_1 E_2(\boldsymbol{v}, \boldsymbol{q}) + \alpha_1 \alpha_2 \langle \boldsymbol{u}, \boldsymbol{v} \rangle$
 - $d_{dt} \mathcal{E}(U(t)) = -\alpha_2 |B^{1/2} u'(t)|^2$
- cannot take $\alpha_1 \alpha_2 = 0$



-concluding remarks

different coupling parameters

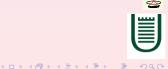
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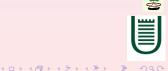
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-concluding remarks

why not $\alpha_1 = 0$?

Iet A₁ = A₂ =: A with positive eigenvalues ω_k → +∞ and eigenspaces (Z_k)_{k≥1}

• $B = 2\beta I$, with $0 < \beta < \sqrt{\omega_1}$, and set $\lambda_k = \sqrt{\omega_k - \beta^2}$

• the equation $u''(t) + Au(t) + 2\beta u'(t) = 0$ with initial conditions

$$u(0) = u^0 = \sum_{k \ge 1} u_k^0, \quad u'(0) = u^1 = \sum_{k \ge 1} u_k^1, \quad u_k^i \in Z_k$$

admits the solution

$$u(t) = e^{-\beta t} \sum_{k \ge 1} \left[u_k^0 \cos(\lambda_k t) + \frac{u_k^1 + \beta u_k^0}{\lambda_k} \sin(\lambda_k t) \right]$$

• $u(t) \in Z_1$ for $u^0 \in Z_1$ and $u^1 \in Z_1$



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-concluding remarks

why not $\alpha_1 = 0$?

▶ let $A_1 = A_2 =: A$

with positive eigenvalues $\omega_k \to +\infty$ and eigenspaces $(Z_k)_{k\geq 1}$

- $B = 2\beta I$, with $0 < \beta < \sqrt{\omega_1}$, and set $\lambda_k = \sqrt{\omega_k \beta^2}$
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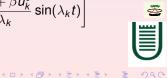
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-concluding remarks

why not $\alpha_1 = 0$?

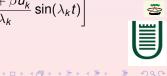
► the equation $u''(t) + Au(t) + 2\beta u'(t) = 0$ with initial conditions

$$u(0) = u^0 = \sum_{k \ge 1} u_k^0, \quad u'(0) = u^1 = \sum_{k \ge 1} u_k^1, \quad u_k^j \in Z_k$$

admits the solution

$$u(t) = e^{-\beta t} \sum_{k \ge 1} \left[u_k^0 \cos(\lambda_k t) + \frac{u_k^1 + \beta u_k^0}{\lambda_k} \sin(\lambda_k t) \right]$$

• $u(t) \in Z_1$ for $u^0 \in Z_1$ and $u^1 \in Z_1$



-concluding remarks

on the other hand, the solution to

$$v''(t) + Av(t) + \alpha u(t) = 0 \tag{1}$$

is given by $v(t) = v_1(t) + v_2(t) \in Z_1 + Z_1^{\perp}$ where

$$\begin{cases} v_1''(t) + \omega_1 v_1(t) + \alpha u(t) = 0\\ v_2''(t) + A v_2(t) = 0 \end{cases}$$

thus, the energy

$$E(v_{2}(t), v_{2}'(t)) = \frac{1}{2} \left(|v_{2}'(t)|^{2} + \langle Av_{2}(t), v_{2}(t) \rangle \right) = \text{const}$$



-concluding remarks

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-concluding remarks

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-concluding remarks

open problems

- study localized damping with hybrid boundary conditions
- consider boundary control with hybrid boundary conditions
- obtain similar decay rates for problems in exterior domains

Thank you for your attention Danke!



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-concluding remarks

open problems

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Thank you for your attention Danke!



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