# Indirect damping for coupled systems 

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## Outline

## Indirect stabilization of weakly coupled systems

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Abstract set-up

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Systems with standard boundary conditions

Systems with hybrid boundary conditions
Remarks and open problems

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## a conservative system

$\Omega \subset \mathbb{R}^{n}$ bounded the wave equation

$$
\begin{array}{rll}
\partial_{t}^{2} v-\Delta v & =0 & \text { in } \\
v=0 & \text { on } & \partial \Omega \times \mathbb{R},
\end{array}
$$

describes a conservative system


## a conservative system

$\Omega \subset \mathbb{R}^{n}$ bounded the wave equation

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\begin{array}{rll}
\partial_{t}^{2} v-\Delta v & =0 & \text { in } \\
v=0 & \text { on } & \Omega \Omega \times \mathbb{R}, \\
& \partial \Omega \times \mathbb{R}
\end{array}
$$

describes a conservative system: the energy of a solution

$$
E(u(t))=\frac{1}{2} \int_{\Omega}\left(|D u(t, x)|^{2}+\left|\partial_{t} u(t, x)\right|^{2}\right) d x
$$

is constant in $t$

## a dissipative system

the damped wave equation

$$
\begin{array}{rll}
\partial_{t}^{2} u-\Delta u+\partial_{t} u & =0 & \text { in } \\
u & =0 & \text { on }
\end{array} \quad \begin{aligned}
& \quad \Omega \times \mathbb{R} \\
&
\end{aligned}
$$

is exponentially stable as $t \rightarrow \infty$


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$$

is exponentially stable as $t \rightarrow \infty$

$$
E(u(t)) \leq E(u(0)) e^{c(1-t)} \quad(c>0)
$$

## a weakly coupled system

consider the coupling through zero order terms

$$
\left\{\begin{array}{c}
\partial_{t}^{2} u-\Delta u+\partial_{t} u+\alpha v=0 \quad \text { in } \Omega \times \mathbb{R} \\
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$$

any kind of stability for $\alpha \neq 0$ ?

## lack of exponential stability



## lack of exponential stability

recast

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as an evolution equation in $\mathcal{H}=\left[H_{0}^{1}(\Omega) \times L^{2}(\Omega)\right]^{2}$

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$$
\left(\begin{array}{l}
u \\
u^{\prime} \\
v \\
v^{\prime}
\end{array}\right)^{\prime}=\left(\begin{array}{ll}
L_{1} & K \\
K & L_{2}
\end{array}\right)\left(\begin{array}{c}
u \\
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v \\
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\end{array}\right)=: \mathcal{L}\left(\begin{array}{c}
u \\
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v \\
v^{\prime}
\end{array}\right),
$$

- $L_{1}, L_{2}$ generators of $\mathcal{C}_{0}$-semigroups on $H_{0}^{1}(\Omega) \times L^{2}(\Omega)$
- K compact operator in $H_{0}^{1}(\Omega) \times L^{2}(\Omega)$


## Indirect damping for coupled systems

$L_{\text {indirect stabilization }}$

## lack of exponential stability (ctnd)

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- $\omega_{\text {ess }}(\mathcal{L})=$ essential growth bound
(blind to compact perturbations)


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\omega_{0}(\mathcal{L}) \geq \omega_{\text {ess }}(\mathcal{L})=\omega_{\text {ess }}\left(\begin{array}{cc}
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$\Rightarrow$ system cannot be exponentially stable

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## a system of second order evolutions equations



## a system of second order evolutions equations

in a separable Hilbert space $H$

$$
\left\{\begin{array}{l}
u^{\prime \prime}+A_{1} u+B u^{\prime}+\alpha v=0 \\
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$(\mathrm{H} 2) B$ is a bounded linear operator on $H$ such that

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(H1) $A_{i}: D\left(A_{i}\right) \subset H \rightarrow H(i=1,2)$ are densely defined closed linear operators such that

$$
A_{i}=A_{i}^{*}, \quad\left\langle A_{i} u, u\right\rangle \geq \omega_{i}|u|^{2} \quad\left(\omega_{1}, \omega_{2}>0\right)
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$$

(H3) $0<|\alpha|<\sqrt{\omega_{1} \omega_{2}}$

## energies

energies associated to $A_{1}, A_{2}$

$$
E_{i}(u, p)=\frac{1}{2}\left(\left|A_{i}^{1 / 2} u\right|^{2}+|p|^{2}\right)
$$

total energy of the system $U=(u, p, v, q)$

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$$
\mathcal{E}(U):=E_{1}(u, p)+E_{2}(v, q)+\alpha\langle u, v\rangle
$$

assumptions yield

$$
\begin{aligned}
& |u|^{2} \leq \frac{2}{\omega_{i}} E_{i}(u, p) \\
& -\mathcal{E}(U) \geq \nu(\alpha)\left[E_{1}(u, p)+E_{2}(v, q)\right]
\end{aligned}
$$

## reduction to a first order system

## system takes the equivalent form



## reduction to a first order system

$$
\begin{aligned}
\mathcal{H}= & D\left(A_{1}^{1 / 2}\right) \times H \times D\left(A_{2}^{1 / 2}\right) \times H \\
(U \mid \widehat{U})= & \left\langle A_{1}^{1 / 2} u, A_{1}^{1 / 2} \widehat{u}\right\rangle+\langle p, \widehat{p}\rangle \\
& +\left\langle A_{2}^{1 / 2} v, A_{2}^{1 / 2} \widehat{v}\right\rangle+\langle q, \widehat{q}\rangle+\alpha\langle u, \widehat{v}\rangle+\alpha\langle v, \widehat{u}\rangle
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\end{aligned}
$$

system takes the equivalent form

$$
\left\{\begin{array}{l}
U^{\prime}(t)=\mathcal{A} U(t) \\
U(0)=U_{0}:=\left(u^{0}, u^{1}, v^{0}, v^{1}\right) .
\end{array}\right.
$$

with $\mathcal{A}: D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ defined by

$$
\left\{\begin{array}{l}
D(\mathcal{A})=D\left(A_{1}\right) \times D\left(A_{1}^{1 / 2}\right) \times D\left(A_{2}\right) \times D\left(A_{2}^{1 / 2}\right) \\
\mathcal{A} U=\left(p,-A_{1} u-B p-\alpha v, q,-A_{2} v-\alpha u\right)
\end{array}\right.
$$

$L_{\text {standard boundary conditions }}$

## a first stability result

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## a first stability result

Theorem (ACK 2002)
Assume, for some integer $j \geq 2$,

$$
\left|A_{1} u\right| \leq c\left|A_{2}^{j / 2} u\right| \quad \forall u \in D\left(A_{2}^{j / 2}\right)
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## a first stability result

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\left|A_{1} u\right| \leq c\left|A_{2}^{j / 2} u\right| \quad \forall u \in D\left(A_{2}^{j / 2}\right) \tag{ACK}
\end{equation*}
$$

- $U_{0} \in D\left(\mathcal{A}^{n j}\right)($ some $n \geq 1) \Rightarrow \mathcal{E}(U(t)) \leq \frac{c_{n}}{t^{n}} \sum_{k=0}^{n j} \mathcal{E}\left(U^{(k)}(0)\right)$


## observe

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- $U_{0} \in D\left(\mathcal{A}^{n j}\right)($ some $n \geq 1) \Rightarrow \mathcal{E}(U(t)) \leq \frac{c_{n}}{t^{n}} \sum_{k=0}^{n j} \mathcal{E}\left(U^{(k)}(0)\right)$
- $\forall U_{0} \in \mathcal{H}, \quad \mathcal{E}(U(t)) \rightarrow 0 \quad$ as $\quad t \rightarrow \infty$


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- $U_{0} \in D\left(\mathcal{A}^{n j}\right)($ some $n \geq 1) \Rightarrow \mathcal{E}(U(t)) \leq \frac{c_{n}}{t^{n}} \sum_{k=0}^{n j} \mathcal{E}\left(U^{(k)}(0)\right)$
- $\forall U_{0} \in \mathcal{H}, \quad \mathcal{E}(U(t)) \rightarrow 0 \quad$ as $\quad t \rightarrow \infty$ observe
$(A C K) \Longleftrightarrow D\left(A_{2}^{j / 2}\right) \subset D\left(A_{1}\right) \quad \& \quad\left|A_{1} A_{2}^{-j / 2} u\right| \leq c|u|$


## main tools

proof uses

- energy dissipation



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- energy dissipation

$$
\frac{d}{d t} \mathcal{E}(U(t))=-\left|B^{1 / 2} u^{\prime}(t)\right|^{2} \quad\left(U_{0} \in D(\mathcal{A})\right)
$$

- multipliers of the form $A_{2}^{2-j} v$ and $A_{2}^{1-j} A_{1} u$
- an abstract decay lemma


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- an abstract decay lemma
$L_{\text {standard boundary conditions }}$


## abstract decay lemma



## abstract decay lemma

- $A: D(A) \subset H \rightarrow H$ generator of a $\mathcal{C}_{0}$-semigroup
- $L: H \rightarrow[0,+\infty)$ continuous function

$$
\int_{0}^{T} L\left(e^{t A} x\right) d t \leq c \sum_{k=0}^{K} L\left(A^{k} x\right)
$$

## abstract decay lemma

- $A: D(A) \subset H \rightarrow H$ generator of a $\mathcal{C}_{0}$-semigroup
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\begin{gathered}
\int_{0}^{T} L\left(e^{t A} x\right) d t \leq c \sum_{k=0}^{K} L\left(A^{k} x\right) \\
\Rightarrow \forall n \geq 1, \forall x \in D\left(A^{n K}\right), \forall 0 \leq s \leq T \\
\int_{s}^{T} L\left(e^{t A} x\right) \frac{(t-s)^{n-1}}{(n-1)!} d t \leq c^{n}(1+K)^{n-1} \sum_{k=0}^{n K} L\left(e^{s A} A^{k} x\right)
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\int_{s}^{T} L\left(e^{t A} x\right) \frac{(t-s)^{n-1}}{(n-1)!} d t \leq c^{n}(1+K)^{n-1} \sum_{k=0}^{n K} L\left(e^{s A} A^{k} x\right)
\end{gathered}
$$

$$
\text { - } L\left(e^{t A} x\right) \leq L\left(e^{s A} x\right) \Rightarrow L\left(e^{t A} x\right) \leq c^{n}(1+K)^{n-1} \frac{n!}{t^{n}} \sum_{k=0}^{n K} L\left(A^{k} x\right)
$$

$L_{\text {standard boundary conditions }}$

## example 1: Dirichlet boundary conditions



## example 1: Dirichlet boundary conditions

$\Omega \subset \mathbb{R}^{n}$ bounded $\quad \Gamma=\partial \Omega$

$$
\left\{\begin{array}{l}
\partial_{t}^{2} u-\Delta u+\partial_{t} u+\alpha v=0 \\
\partial_{t}^{2} v-\Delta v+\alpha u=0
\end{array} \quad \text { in } \quad \Omega \times(0,+\infty)\right.
$$

with boundary conditions

$$
u(\cdot, t)=0=v(\cdot, t) \quad \text { on } \quad\ulcorner\quad \forall t>0
$$

in this example $A_{1}=A=A_{2}$ with

$$
D(A)=H^{2}(\Omega) \cap H_{0}^{1}(\Omega), \quad A u=-\Delta u
$$

so that $(A C K):\left|A_{1} u\right| \leq c\left|A_{2}^{j / 2} u\right|$ holds with $j=2$
$L_{\text {standard boundary conditions }}$

## example 1: conclusion



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$$
\left.\begin{array}{c}
\left\{\begin{array}{l}
\partial_{t}^{2} u-\Delta u+\partial_{t} u+\alpha v=0 \\
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\end{array} \quad \text { in } \quad \Omega \times(0,+\infty)\right. \\
u(\cdot, t)=0=v(\cdot, t) \quad \text { on } \quad\ulcorner\quad \forall t>0
\end{array}\right\} \begin{aligned}
& \begin{array}{l}
u(x, 0)=u^{0}(x), \quad u^{\prime}(x, 0)=u^{1}(x) \\
v(x, 0)=v^{0}(x), \quad v^{\prime}(x, 0)=v^{1}(x)
\end{array} \quad x \in \Omega
\end{aligned}
$$

## example 1: conclusion

$$
\left.\left.\begin{array}{c} 
\begin{cases}\partial_{t}^{2} u-\Delta u+\partial_{t} u+\alpha v=0 \\
\partial_{t}^{2} v-\Delta v+\alpha u=0\end{cases} \\
\text { in } \quad \Omega \times(0,+\infty)
\end{array}\right\} \begin{array}{ll}
u(\cdot, t)=0=v(\cdot, t) \quad \text { on } \quad \Gamma \quad \forall t>0
\end{array}\right\}
$$

conclusion: for $0<|\alpha|<C_{\Omega}$

$$
\begin{aligned}
& \int_{\Omega}\left(\left|\partial_{t} u\right|^{2}+|\nabla u|^{2}+\left|\partial_{t} v\right|^{2}+|\nabla v|^{2}\right) d x \\
& \leq \frac{c}{t}\left(\left\|u^{0}\right\|_{2, \Omega}^{2}+\left\|u^{1}\right\|_{1, \Omega}^{2}+\left\|v^{0}\right\|_{2, \Omega}^{2}+\left\|v^{1}\right\|_{1, \Omega}^{2}\right)
\end{aligned}
$$

$L_{\text {standard boundary conditions }}$

## example 2: hybrid boundary conditions

Let $\alpha \in \mathbb{R}$ and consider the problem
with boundary conditions


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\end{array} \quad \text { in } \Omega \times(0,+\infty)\right.
$$

with boundary conditions

$$
\begin{aligned}
\left(\frac{\partial u}{\partial \nu}+u\right)(\cdot, t) & =0 \text { on } \Gamma \quad \forall t>0 \\
v(\cdot, t) & =0 \text { on } \Gamma
\end{aligned}
$$

$L_{\text {standard boundary conditions }}$

## ACK does not apply



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$$
\begin{aligned}
& D\left(A_{1}\right)=\left\{u \in H^{2}(\Omega): \frac{\partial u}{\partial \nu}+u=0 \text { on } \Gamma\right\}, A_{1} u=-\Delta u \\
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$$

Lemma (ACG)
$D\left(A_{2}^{k / 2}\right)$ is not included in $D\left(A_{1}\right)$ for any $k \geq 2$

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Lemma (ACG)
$D\left(A_{2}^{k / 2}\right)$ is not included in $D\left(A_{1}\right)$ for any $k \geq 2$
Proof.

$$
\begin{aligned}
& (k=2) \quad v_{0}:\left\{\begin{array}{l}
(-\Delta)^{2} v=1 \\
v_{0}=0=\Delta v_{0}
\end{array} \quad \text { on } \Gamma \quad v_{1}:=-\Delta v_{0}\right.
\end{aligned}
$$

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Lemma (ACG)
$D\left(A_{2}^{k / 2}\right)$ is not included in $D\left(A_{1}\right)$ for any $k \geq 2$
Proof.

$$
\begin{aligned}
D\left(A_{2}\right) & \subset D\left(A_{1}\right) \Rightarrow \frac{\partial v_{0}}{\partial \nu \mid \Gamma}=0 \Rightarrow \int_{\Omega} v_{1} d x=0 \\
& \Rightarrow \int_{\Omega}\left|\nabla v_{1}\right|^{2} d x=\int_{\Omega}\left(-\Delta v_{1}\right) v_{1} d x=0 \text { but }-\Delta v_{1}=1
\end{aligned}
$$

$L_{\text {hybrid boundary conditions }}$

## second stability result

Theorem (ACG 2011) Assume


Then
second stability result
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Assume

$$
D\left(A_{2}\right) \subset D\left(A_{1}^{1 / 2}\right) \quad \& \quad\left|A_{1}^{1 / 2} u\right| \leq c\left|A_{2} u\right| \quad \forall u \in D\left(A_{2}\right) \quad(A C G)
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Then

$$
\text { - } U_{0} \in D\left(\mathcal{A}^{4 n}\right)(\text { some } n \geq 1) \Rightarrow \mathcal{E}(U(t)) \leq \frac{c_{n}}{t^{n}} \sum_{k=0}^{4 n} \mathcal{E}\left(U^{(k)}(0)\right)
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- $\forall U_{0} \in \mathcal{H}, \quad \mathcal{E}(U(t)) \rightarrow 0 \quad$ as $t \rightarrow \infty$


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Then

- $U_{0} \in D\left(\mathcal{A}^{4 n}\right)($ some $n \geq 1) \Rightarrow \mathcal{E}(U(t)) \leq \frac{c_{n}}{t^{n}} \sum_{k=0}^{4 n} \mathcal{E}\left(U^{(k)}(0)\right)$
- $\forall U_{0} \in \mathcal{H}, \quad \mathcal{E}(U(t)) \rightarrow 0 \quad$ as $\quad t \rightarrow \infty$ observe

$$
(A C G) \Longleftrightarrow\left|\left\langle A_{1} u, v\right\rangle\right| \leq c\left|A_{2} v\right|\left\langle A_{1} u, u\right\rangle^{1 / 2}
$$

## Indirect damping for coupled systems

$\left\llcorner_{\text {hybrid boundary conditions }}\right.$

## main tools

proof uses

- energy dissipation



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- energy dissipation

$$
\frac{d}{d t} \mathcal{E}(U(t))=-\left|B^{1 / 2} u^{\prime}(t)\right|^{2} \quad\left(U_{0} \in D(\mathcal{A})\right)
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- multipliers of the form $A_{1}^{-1} v$ and $A_{2}^{-1} u$
- abstract decay lemma


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$L_{\text {hybrid boundary conditions }}$


## use interpolation



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- polynomial decay estimates improved for

$$
U_{0} \in\left(\mathcal{H}, D\left(\mathcal{A}^{4 n}\right)\right)_{\theta, 2} \quad \text { for some } \quad n \geq 1,0<\theta<1
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- since $\mathcal{A}$ generates a $\mathcal{C}_{0}$-semigroup of contractions,


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$$

- since $\mathcal{A}$ generates a $\mathcal{C}_{0}$-semigroup of contractions,

$$
D\left(\mathcal{A}^{m}\right)=\left(\mathcal{H}, D\left(\mathcal{A}^{k}\right)\right)_{\theta, 2}
$$

if $\theta k=m$ for some $0<\theta<1$ and $k, m \in \mathbb{N}$

## Indirect damping for coupled systems

$L_{\text {hybrid boundary conditions }}$

## ACG with data in interpolation spaces

## assume

# ACG with data in interpolation spaces 

assume

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D\left(A_{2}\right) \subset D\left(A_{1}^{1 / 2}\right) \quad \& \quad\left|A_{1}^{1 / 2} u\right| \leq c\left|A_{2} u\right|
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let $\quad n \geq 1, \quad 0<\theta<1$

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\end{equation*}
$$

let $n \geq 1, \quad 0<\theta<1 \quad$ then

- $U_{0} \in\left(\mathcal{H}, D\left(\mathcal{A}^{4 n}\right)\right)_{\theta, 2} \Longrightarrow\|U(t)\|_{\mathcal{H}}^{2} \leq \frac{c_{n, \theta}}{t^{n \theta}}\left\|U_{0}\right\|_{\left(\mathcal{H}, D\left(\mathcal{A}^{4 n}\right)\right)_{\theta, 2}}^{2}$


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- $U_{0} \in\left(\mathcal{H}, D\left(\mathcal{A}^{4 n}\right)\right)_{\theta, 2} \Longrightarrow\|U(t)\|_{\mathcal{H}}^{2} \leq \frac{c_{n, \theta}}{t^{n \theta}}\left\|U_{0}\right\|_{\left(\mathcal{H}, D\left(\mathcal{A}^{4 n}\right)\right)_{\theta, 2}}^{2}$
- $U_{0} \in D\left(\mathcal{A}^{n}\right) \Longrightarrow \mathcal{E}(U(t)) \leq \frac{c_{n}}{t^{n / 4}} \sum_{k=0}^{n} \mathcal{E}\left(U^{(k)}(0)\right)$
$L_{\text {hybrid boundary conditions }}$


## example 2: ACG applies

the energy of the solution to the boundary-value problem

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\partial_{t}^{2} u-\Delta u+\partial_{t} u+\alpha v=0 \\
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\end{array} \quad \text { in } \Omega \times(0,+\infty)\right. \\
& \qquad \begin{aligned}
\left(\frac{\partial u}{\partial \nu}+u\right)(\cdot, t) & =0 \\
v(\cdot, t) & =0
\end{aligned} \quad \text { on } \Gamma \quad \text { on } \Gamma
\end{aligned} \quad \forall t>0 \quad . \quad .
$$

satisfies, for $0<|\alpha|<C_{\Omega}$,

$$
\begin{aligned}
& E_{1}\left(u(t), u^{\prime}(t)\right)+E_{2}\left(v(t), v^{\prime}(t)\right) \\
& \quad \leq \frac{c}{t^{1 / 4}}\left(\left|A_{1} u^{0}\right|^{2}+\left|A_{1}^{1 / 2} u^{1}\right|^{2}+\left|A_{2} v^{0}\right|^{2}+\left|A_{2}^{1 / 2} v^{1}\right|^{2}\right)
\end{aligned}
$$

$L_{\text {hybrid boundary conditions }}$

## proof



## Indirect damping for coupled systems

$L_{\text {hybrid boundary conditions }}$

## proof

- recall

$$
\begin{aligned}
& D\left(A_{1}\right)=\left\{u \in H^{2}(\Omega): \frac{\partial u}{\partial \nu}+u=0 \text { on } \Gamma\right\}, A_{1} u=-\Delta u \\
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\end{aligned}
$$

- to obtain, for all $u \in D\left(A_{1}\right), v \in D\left(A_{2}\right)$,

$$
\left|\left\langle A_{1} u, v\right\rangle\right|=\left|\int_{\Omega} \nabla u \nabla v d x\right| \leq c\left\langle A_{1} u, u\right\rangle^{1 / 2}\left|A_{2} v\right|
$$

since $\left\langle A_{1} u, u\right\rangle=\int_{\Omega}|\nabla u|^{2} d x+\int_{\Gamma}|u|^{2} d S$

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- yields (ACG): $\quad D\left(A_{2}\right) \subset D\left(A_{1}^{1 / 2}\right) \quad \& \quad\left|A_{1}^{1 / 2} u\right| \leq c\left|A_{2} u\right|$


## Indirect damping for coupled systems

$L_{\text {concluding remarks }}$

## operators of higher order

## define



## operators of higher order

define

$$
\begin{aligned}
& D\left(A_{1}\right)=\left\{u \in H^{4}(\Omega): \Delta u=0=\frac{\partial \Delta u}{\partial \nu} \text { on } \Gamma\right\}, \quad A_{1} u=\Delta^{2} u \\
& D\left(A_{2}\right)=H^{2}(\Omega) \cap H_{0}^{1}(\Omega), \quad A_{2} v=-\Delta v
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$$

then

$$
\left|\left\langle A_{1} u, v\right\rangle\right|=\left|\int_{\Omega} \Delta u \Delta v d x\right| \leq c\left\langle A_{1} u, u\right\rangle^{1 / 2}\left|A_{2} v\right|
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$$

deduce (ACG): $\quad D\left(A_{2}\right) \subset D\left(A_{1}^{1 / 2}\right) \quad \& \quad\left|A_{1}^{1 / 2} u\right| \leq c\left|A_{2} u\right|$

## application



## application

consider boundary-value problem

$$
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\partial_{t}^{2} u+\Delta^{2} u+\partial_{t} u+\alpha v=0 \\
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& \Delta u(\cdot, t)=0=\frac{\partial \Delta u}{\partial \nu}(\cdot, t) \text { and } \quad v(\cdot, t)=0 \text { on } \Gamma
\end{aligned}
$$

then, for $0<|\alpha|<C_{\Omega}^{3 / 2}$, energy decays at polynomial rate

$$
\begin{aligned}
& E_{1}\left(u(t), u^{\prime}(t)\right)+E_{2}\left(v(t), v^{\prime}(t)\right) \\
& \quad \leq \frac{c}{t^{1 / 4}}\left(\left\|u^{0}\right\|_{4, \Omega}^{2}+\left\|u^{1}\right\|_{2, \Omega}^{2}+\left\|v^{0}\right\|_{2, \Omega}^{2}+\left\|v^{1}\right\|_{1, \Omega}^{2}\right)
\end{aligned}
$$

## Indirect damping for coupled systems

$L_{\text {concluding remarks }}$

## different coupling parameters



- above results can be generalized replacing $(\mathrm{H} 3)$ with


## different coupling parameters

for general $\alpha_{1}, \alpha_{2} \in \mathbb{R}$ consider

$$
\left\{\begin{array}{l}
u^{\prime \prime}(t)+A_{1} u(t)+B u^{\prime}(t)+\alpha_{1} v(t)=0 \\
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- $\mathcal{E}(\boldsymbol{U}):=\alpha_{2} E_{1}(u, p)+\alpha_{1} E_{2}(v, q)+\alpha_{1} \alpha_{2}\langle u, v\rangle$


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$$

- $\mathcal{E}(U):=\alpha_{2} E_{1}(u, p)+\alpha_{1} E_{2}(v, q)+\alpha_{1} \alpha_{2}\langle\boldsymbol{u}, \boldsymbol{v}\rangle$
- $\frac{d}{d t} \mathcal{E}(U(t))=-\alpha_{2}\left|B^{1 / 2} u^{\prime}(t)\right|^{2}$


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- $\mathcal{E}(U):=\alpha_{2} E_{1}(u, p)+\alpha_{1} E_{2}(v, q)+\alpha_{1} \alpha_{2}\langle\boldsymbol{u}, \boldsymbol{v}\rangle$
- $\frac{d}{d t} \mathcal{E}(U(t))=-\alpha_{2}\left|B^{1 / 2} u^{\prime}(t)\right|^{2}$
- cannot take $\alpha_{1} \alpha_{2}=0$


## why not $\alpha_{1}=0$ ?

with positive eigenvalues $\omega_{k} \rightarrow+\infty$ and eigenspaces $\left(Z_{k}\right)_{k \geq 1}$
the equation $u^{\prime \prime}(t)+A u(t)+2 \beta u^{\prime}(t)=0$ with initial conditions

admits the solution


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- let $A_{1}=A_{2}=: A$ with positive eigenvalues $\omega_{k} \rightarrow+\infty$ and eigenspaces $\left(Z_{k}\right)_{k \geq 1}$
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$$
u(0)=u^{0}=\sum_{k \geq 1} u_{k}^{0}, \quad u^{\prime}(0)=u^{1}=\sum_{k \geq 1} u_{k}^{1}, \quad u_{k}^{i} \in Z_{k}
$$

admits the solution

$$
u(t)=e^{-\beta t} \sum_{k \geq 1}\left[u_{k}^{0} \cos \left(\lambda_{k} t\right)+\frac{u_{k}^{1}+\beta u_{k}^{0}}{\lambda_{k}} \sin \left(\lambda_{k} t\right)\right]
$$

## why not $\alpha_{1}=0$ ?

- let $A_{1}=A_{2}=: A$
with positive eigenvalues $\omega_{k} \rightarrow+\infty$ and eigenspaces $\left(Z_{k}\right)_{k \geq 1}$
- $B=2 \beta I$, with $0<\beta<\sqrt{\omega_{1}}$, and set $\lambda_{k}=\sqrt{\omega_{k}-\beta^{2}}$
- the equation $u^{\prime \prime}(t)+A u(t)+2 \beta u^{\prime}(t)=0$ with initial conditions

$$
u(0)=u^{0}=\sum_{k \geq 1} u_{k}^{0}, \quad u^{\prime}(0)=u^{1}=\sum_{k \geq 1} u_{k}^{1}, \quad u_{k}^{i} \in Z_{k}
$$

admits the solution

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u(t)=e^{-\beta t} \sum_{k \geq 1}\left[u_{k}^{0} \cos \left(\lambda_{k} t\right)+\frac{u_{k}^{1}+\beta u_{k}^{0}}{\lambda_{k}} \sin \left(\lambda_{k} t\right)\right]
$$

- $u(t) \in Z_{1} \quad$ for $\quad u^{0} \in Z_{1} \quad$ and $\quad u^{1} \in Z_{1}$


## on the other hand, the solution to

is given by $v(t)=v_{1}(t)+v_{2}(t) \in Z_{1}+Z_{1}^{\perp}$ where

thus, the energy

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\begin{equation*}
v^{\prime \prime}(t)+\boldsymbol{A v}(t)+\alpha u(t)=0 \tag{1}
\end{equation*}
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$$
\left\{\begin{array}{l}
v_{1}^{\prime \prime}(t)+\omega_{1} v_{1}(t)+\alpha u(t)=0 \\
v_{2}^{\prime \prime}(t)+A v_{2}(t)=0
\end{array}\right.
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E\left(v_{2}(t), v_{2}^{\prime}(t)\right)=\frac{1}{2}\left(\left|v_{2}^{\prime}(t)\right|^{2}+\left\langle A v_{2}(t), v_{2}(t)\right\rangle\right)=\mathrm{const}
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hence, $v^{0} \notin Z_{1}, v^{1} \notin Z_{1}$ ensure that the system is not stabilizable

## open problems

- study localized damping with hybrid boundary conditions
- consider boundary control with hybrid boundary conditions
- obtain similar decay rates for problems in exterior domains



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## Thank you for your attention Danke!

