Geometric invariant theory

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(0.1) Action on polynomial algebras. Let K be a field, let $n \in \mathbb{N}_0$, and let $K[\mathcal{X}] = K[X_1, \ldots, X_n]$ be the polynomial K-algebra in the indeterminates $\mathcal{X} := \{X_1, \ldots, X_n\}$. We have $K[\mathcal{X}] = \bigoplus_{d\in\mathbb{N}_0} K[\mathcal{X}]_d$ as graded K-algebras, where $K[\mathcal{X}]_d \leq K[\mathcal{X}]$ is the subspace of homogeneous polynomials of degree d.

The general linear group $\mathrm{GL}_n(K)$ acts naturally (from the right) K-linearly on the K-vector space K^n. Hence $\mathrm{GL}_n(K)$ acts K-linearly by pre-composition on the space of **linear forms** $K[\mathcal{X}]_1 \leq K[\mathcal{X}]$, and thus by the universal property of polynomial K-algebras we obtain graded K-algebra isomorphisms on $K[\mathcal{X}]$:

For $A = [a_{ij}]_{ij} \in \mathrm{GL}_n(K)$ we have $({}^A X_j)(x_1, \ldots, x_n) = X_j([x_1, \ldots, x_n] \cdot A) = \sum_{i=1}^n x_i a_{ij} = (\sum_{i=1}^n a_{ij} X_i)(x_1, \ldots, x_n)$, for $x_1, \ldots, x_n \in K$, saying that ${}^A X_j = \sum_{i=1}^n X_i a_{ij}$, in other words we have $\mathcal{X} \mapsto \mathcal{X} \cdot A$. Thus in terms of the K-basis $\mathcal{X}$ of $K[\mathcal{X}]_1$ the K-linear map induced by A is given by $A^{\mathrm{tr}} \in K^{n\times n}$. Hence in order to get an action of $\mathrm{GL}_n(K)$ we let $(fA)(\mathcal{X}) := f(\mathcal{X} \cdot A^{-1})$, for $f \in K[\mathcal{X}]$; in particular the K-linear map on $K[\mathcal{X}]_1$ induced by A is given by $A^{-\mathrm{tr}} \in K^{n\times n}$.

(0.2) Quadratic forms. Let K be a field such that $\mathrm{char}(K) \neq 2$, and let $n \in \mathbb{N}$. A polynomial $q = q(\mathcal{X}) = q(X_1, \ldots, X_n) := \sum_{1\leq i\leq j\leq n} q_{ij} X_i X_j \in K[\mathcal{X}]_2$ is called an **(n-ary) quadratic form**; we have $\dim_K(K[\mathcal{X}]_2) = \frac{n(n+1)}{2}$. We may identify q with the associated map $K^n \to K\colon x = [x_1, \ldots, x_n] \mapsto q(x) = q(x_1, \ldots, x_n)$, which with a slight abuse is also called a quadratic form; thus the map $K^n \times K^n \to K\colon [x, y] \to \frac{1}{2}(q(x+y) - q(x) - q(y))$ is a symmetric K-bilinear form, and we have the name-giving property $q(\lambda x) = \lambda^2 \cdot q(x)$, for $\lambda \in K$.

Let $K^{n\times n}_{\mathrm{sym}} := \{A \in K^{n\times n}; A^{\mathrm{tr}} = A\} \leq K^{n\times n}$ the K-subspace of symmetric matrices; we have $\dim_K(K^{n\times n}_{\mathrm{sym}}) = \frac{n(n+1)}{2}$. The quadratic form q is associated with the **Gram matrix** $Q = Q_q := [q'_{ij}]_{ij} \in K^{n\times n}_{\mathrm{sym}}$, where $q'_{ii} = q_{ii}$, and $q'_{ij} = q'_{ji} = \frac{1}{2} \cdot q_{ij}$ for $i < j$. This gives rise to an isomorphism of K-vector spaces $K[\mathcal{X}]_2 \to K^{n\times n}_{\mathrm{sym}}\colon q \mapsto Q_q$, such that conversely $q(\mathcal{X}) = \mathcal{X} \cdot Q_q \cdot \mathcal{X}^{\mathrm{tr}}$.

For $A \in \mathrm{GL}_n(K)$ we get $(qA)(\mathcal{X}) = (\mathcal{X} \cdot A^{-1}) \cdot Q \cdot (A^{-\mathrm{tr}} \cdot \mathcal{X}^{\mathrm{tr}})$, thus we have $Q_{qA} = A^{-1} \cdot Q \cdot A^{-\mathrm{tr}}$; recall that applying A amounts to applying base change of K^n. Quadratic forms q and q' with associated Gram matrices Q and Q', respectively, are called **equivalent** if there is $A \in \mathrm{GL}_n(K)$ such that $qA = q'$, or equivalently $Q' = A^{-1} \cdot Q \cdot A^{-\mathrm{tr}}$.

Then $\Delta(q) := \det(Q_q) \in K$ is called the **discriminant** of q **[Sylvester, 1852]**, and $\mathrm{rk}(q) := \mathrm{rk}(Q_q) \in \{0, \ldots, n\}$ is called the **rank** of q. Thus applying $A \in \mathrm{GL}_n(K)$ yields $\mathrm{rk}(qA) = \mathrm{rk}(q)$ and $\Delta(qA) = \det(A^{-1} \cdot Q \cdot A^{-\mathrm{tr}}) = \det(A)^{-2} \cdot \det(Q) = \det(A)^{-2} \cdot \Delta(q)$. In particular, the rank is a $\mathrm{GL}_n(K)$-invariant of quadratic forms, while the the discriminant of quadratic forms is invariant with respect to the special linear group $\mathrm{SL}_n(K)$.

(0.3) Classification of quadratic forms. Let $\mathbb{K}$ be an algebraically closed field such that $\mathrm{char}(\mathbb{K}) \neq 2$, and let $n \in \mathbb{N}$. Given a quadratic form $q \in \mathbb{K}[\mathcal{X}]_2$, let $[q] \subseteq \mathbb{K}[\mathcal{X}]_2$ be its equivalence class with respect to the action of $\mathbf{SL}_n = \mathbf{SL}_n(\mathbb{K})$, rather than $\mathbf{GL}_n = \mathbf{GL}_n(\mathbb{K})$. These equivalence classes are as follows:

Theorem. Any n-ary quadratic form is $\mathbf{SL}_n$-equivalent to precisely one of:

i) $q_{n,\delta} := \delta X_n^2 + \sum_{i=1}^{n-1} X_i^2$, where $\delta \neq 0$; we have $\mathrm{rk}(q_{n,\delta}) = n$ and $\Delta(q_{n,\delta}) = \delta$.
ii) $q_r := \sum_{i=1}^{r} X_i^2$, where $r \in \{0, \ldots, n-1\}$; we have $\mathrm{rk}(q_r) = r$ and $\Delta(q_r) = 0$.

Moreover, all the forms $q_{n,\delta}$ for $\delta \neq 0$ are $\mathbf{GL}_n$-equivalent.

Proof. We show that the Gram matrix Q of any quadratic form q of rank $r := \mathrm{rk}(q)$ is $\mathbf{SL}_n$-diagonalizable: By induction we may assume that $n \geq 2$ and $q \neq 0$. Since $\mathbf{SL}_n$ acts transitively on $\mathbb{K}^n \setminus \{0\}$, we may choose a $\mathbb{K}$-basis of $\mathbb{K}^n$ whose first element, v say, is non-isotropic. Since any unitriangular matrix belongs to $\mathbf{SL}_n$, by the standard orthogonalization procedure we may complement this by a $\mathbb{K}$-basis of the orthogonal complement $\langle v \rangle_{\mathbb{K}}^{\perp} \leq \mathbb{K}^n$. (So far the argument works for any field K such that $\mathrm{char}(K) \neq 2$.)

Hence we may assume that $q = \sum_{i=1}^{r} \delta_i X_i^2$, where $\delta_i \neq 0$. If $r < n$, letting $A := \mathrm{diag}[\epsilon_1, \ldots, \epsilon_r, 1, \ldots, 1, (\prod_{i=1}^{r} \epsilon_i)^{-1}] \in \mathbf{SL}_n$, where $\epsilon_i^2 = \delta_i$ for $i \in \{1, \ldots, r\}$, we get $qA = \sum_{i=1}^{r} \delta_i \epsilon_i^{-2} X_i^2 = q_r$. If $r = n$, letting $A := \mathrm{diag}[\epsilon_1, \ldots, \epsilon_{n-1}, \epsilon^{-1}] \in \mathbf{SL}_n$, where $\epsilon_i^2 = \delta_i$ for $i \in \{1, \ldots, n-1\}$, and $\epsilon := \prod_{i=1}^{n-1} \epsilon_i$, we get $qA = \delta_n \epsilon^2 X_n^2 + \sum_{i=1}^{n-1} \delta_i \epsilon_i^{-2} X_i^2 = q_{n,\delta_n \epsilon^2}$. Finally, letting $A := \mathrm{diag}[1, \ldots, 1, \epsilon] \in \mathbf{GL}_n$, where $\epsilon^2 = \delta$, we get $q_{n,\delta} A = \delta \epsilon^{-2} X_n^2 + \sum_{i=1}^{n-1} X_i^2 = q_{n,1}$. ♯

We may view the discriminant Δ as a regular map on the affine variety $\mathbb{K}[\mathcal{X}]_2$. Its fibre associated with $\delta \in \mathbb{K}$ is the hypersurface $\Delta^{-1}(\delta) \subseteq \mathbb{K}[\mathcal{X}]_2$. Since Δ is $\mathbf{SL}_n$-invariant, we conclude that $\Delta^{-1}(\delta)$ consists of a union of equivalence classes. More precisely, for $\delta \neq 0$ the fibre $\Delta^{-1}(\delta) = [q_{n,\delta}]$ is a single equivalence class, while the fibre $\Delta^{-1}(0) = \coprod_{r=0}^{n-1} [q_r]$ is a union of equivalence classes, for $n \geq 2$; note that $[q_0] = \{q_0\}$ is a singleton set.

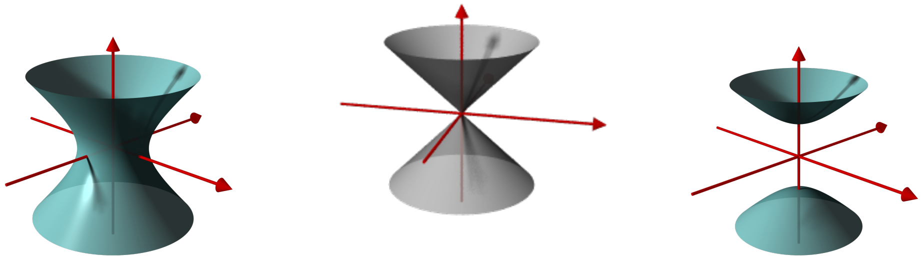
Since Δ is continuous with respect to the Zariski topology, we conclude that the fibre $\Delta^{-1}(\delta) \subseteq \mathbb{K}[\mathcal{X}]_2$ is closed. This implies that the equivalence class $[q_{n,\delta}]$ is closed for $\delta \neq 0$. But for $\delta = 0$ this is different, where for $r \in \{0, \ldots, n-1\}$ the closure of $[q_r]$ equals $\overline{[q_r]} = \coprod_{s=0}^{r} [q_s] \subseteq \mathbb{K}[\mathcal{X}]_2$:

For the time being, we are only able to present an argument which is valid for the case $\mathbb{K} = \mathbb{C}$ and $\mathbb{C}[\mathcal{X}]_2$ carrying the complex metric topology instead of the Zariski topology, but we will show in (3.2) that it carries over to the Zariski topology over any algebraically closed field. Now, since $\mathbf{SL}_n(\mathbb{C})$ acts by homeomorphisms, $\overline{[q_r]}$ is $\mathbf{SL}_n(\mathbb{C})$-invariant as well, hence is a union of equivalence classes. Since $\{M \in \mathbb{C}^{n \times n}; \mathrm{rk}(M) \leq r\} \subseteq \mathbb{C}^{n \times n}$ coincides with the set of all matrices whose $\big((r+1) \times (r+1)\big)$-minors all vanish, we conclude that the latter set is closed. Hence $\{M \in \mathbb{C}^{n \times n}_{\mathrm{sym}}; \mathrm{rk}(M) \leq r\} \subseteq \mathbb{C}^{n \times n}_{\mathrm{sym}}$ is closed as well, in other words $\coprod_{s=0}^{r} [q_s]$ is closed, whence $\overline{[q_r]} \subseteq \coprod_{s=0}^{r} [q_s]$. Conversely, for $r = 0$ we have $\overline{[q_0]} = [q_0]$. For $r \in \{1, \ldots, n-1\}$ and $\epsilon \in \mathbb{C}$ let $q_{r,\epsilon} := \epsilon X_r^2 + \sum_{i=1}^{r-1} X_i^2$; in particular $q_{r,1} = q_r$. Then we have $q_{r,\epsilon} \in [q_r]$ for $\epsilon \neq 0$, and $\lim_{\epsilon \to 0} q_{r,\epsilon} = q_{r,0} = q_{r-1}$, which entails $[q_{r-1}] \subseteq \overline{[q_r]}$. This implies $\coprod_{s=0}^{r} [q_s] \subseteq \overline{[q_r]}$, thus equality. ♯

Letting $\mathbb{K}$ be an arbitrary algebraically closed field such that $\mathrm{char}(\mathbb{K}) \neq 2$ again, in particular we have $\Delta^{-1}(0) = \overline{[q_{n-1}]}$, implying that any $\mathbf{SL}_n$-invariant regular map on $\Delta^{-1}(0)$ is constant, hence the equivalence classes contained in $\Delta^{-1}(0)$ cannot be separated by these maps.

This also entails that any $\mathbf{SL}_n$-invariant regular map F on $\mathbb{K}[\mathcal{X}]_2$ is constant on the fibres of Δ, that is we have $F(q) = f(\Delta(q))$ for some map $f \colon \mathbb{K} \to \mathbb{K}$.

Table 1: Hyperboloids for $\delta < 0$ and $\delta = 0$ and $\delta > 0$.



Moreover, Δ admits the section $s\colon \mathbb{K} \to \mathbb{K}[\mathcal{X}]_2\colon \delta \mapsto q_{n,\delta}$, where $q_{n,0} := q_{n-1}$, that is $s \cdot \Delta = \mathrm{id}_{\mathbb{K}}$. This yields $F(s(\delta)) = f(\Delta(s(\delta))) = f(\delta)$, thus $f = s \cdot F$. Since s is a morphism, f likewise is, implying that F is a polynomial in Δ. $\sharp$

Similarly, in the case $\mathbb{K} = \mathbb{C}$ and $\mathbb{C}[\mathcal{X}]_2$ carrying the complex metric topology, the above argument shows that any $\mathbf{SL}_n(\mathbb{C})$-invariant continuous complex-valued function on $\Delta^{-1}(0)$ is constant; and that any $\mathbf{SL}_n(\mathbb{C})$-invariant continuous complex-valued function F on $\mathbb{C}[\mathcal{X}]_2$ is constant on the fibres of Δ, which since s is continuous entails that F is a continuous function of Δ.

(0.4) Binary quadratic forms. In particular, we consider **binary forms**, that is the case $n = 2$, and let $\mathcal{X} := \{X, Y\}$ and $\mathcal{V} := \mathbb{K}[X, Y]_2$. We consider the $\mathbb{K}$-bases $\{X^2, 2XY, Y^2\} \subseteq \mathcal{V}$ and $\{X^2+Y^2, 2XY, X^2-Y^2\} \subseteq \mathcal{V}$. Letting A, B, C and U, W, V be the the associated coordinate functions, using the base change matrix $M := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ we get $[A, B, C] = [U, W, V] \cdot M = [U+V, W, U-V]$ and $[U, W, V] = [A, B, C] \cdot M^{-1} = [\frac{A+C}{2}, B, \frac{A-C}{2}]$. This yields identifications of $\mathcal{V}$ with $\mathbb{K}^3$, with coordinate algebra $\mathbb{K}[\mathcal{V}] = \mathbb{K}[A, B, C] = \mathbb{K}[U, W, V]$.

Let $q := aX^2 + 2bXY + cY^2 \in \mathcal{V}$, having Gram matrix $Q = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in \mathbb{K}^{2\times 2}_{\mathrm{sym}}$, and thus $\Delta(q) = \det(Q) = ac - b^2 \in \mathbb{K}$ **[Lagrange; Gauß, 1801]**. Thus as a regular map on $\mathcal{V}$ we get $\Delta = AC - B^2 = U^2 - V^2 - W^2 \in \mathbb{K}[\mathcal{V}]$.

For $\delta \in \mathbb{K}$ the fibre $\Delta^{-1}(\delta) \subseteq \mathcal{V}$ is, with respect to the above identifications, given as $\{[a, b, c] \in \mathbb{K}^3; ac - b^2 = \delta\}$ and $\{[u, w, v] \in \mathbb{K}^3; v^2 + w^2 = u^2 - \delta\}$, respectively. The Jacobian $[\frac{\partial\Delta}{\partial U}, \frac{\partial\Delta}{\partial W}, \frac{\partial\Delta}{\partial V}] = 2 \cdot [U, -W, -V]$ shows that $\Delta^{-1}(\delta)$ is smooth for $\delta \neq 0$, while for $\delta = 0$ we get the unique singular point $q_0 \in \Delta^{-1}(0)$.

Geometrically, letting $\mathbb{K} = \mathbb{C}$, for $\delta \in \mathbb{R}$ considering $\Delta^{-1}(\delta) \cap \mathbb{R}^3$ in the second picture, we get a **single-shell hyperboloid** for $\delta < 0$, a **double-shell hyperboloid** for $\delta > 0$, and a **cone** for $\delta = 0$; see Table 1, where the u-axis is the vertical one. In the 'degenerate' case $\delta = 0$, the cone consists of two $\mathbf{SL}_2$-equivalence classes, namely $[q_0] = \{q_0\}$ and $[q_1]$, where $q_0 = 0$ and $q_1 = X^2$.

1 Algebraic groups

(1.1) Algebraic groups. a) Let $\mathbb{K}$ be an algebraically closed field. A $\mathbb{K}$-variety $\mathbf{G}$ endowed with a group structure, such that the multiplication map $\mu = \mu_{\mathbf{G}} \colon \mathbf{G} \times \mathbf{G} \to \mathbf{G} \colon [x, y] \mapsto xy$ and the inversion map $\iota = \iota_{\mathbf{G}} \colon \mathbf{G} \to \mathbf{G} \colon x \mapsto x^{-1}$ are morphisms of varieties, is called an **algebraic group** over $\mathbb{K}$. If $\mathbf{G}$ is an affine variety, then $\mathbf{G}$ is called an **affine** algebraic group.

Note that, since the Zariski topology on $\mathbf{G} \times \mathbf{G}$ is finer than the product topology of the Zariski topologies, multiplication is not necessarily continuous with respect to the product topology, so that $\mathbf{G}$ is not necessarily a topological group.

If $\mathbf{H}$ is an algebraic group, then a morphism $\varphi \colon \mathbf{G} \to \mathbf{H}$ of varieties which also is a group homomorphism is called a **homomorphism** of algebraic groups. If φ additionally is an isomorphism of varieties, then it is called an **isomorphism** of algebraic groups; note that here bijectivity of φ is necessary but not sufficient.

For example, we have the homomorphisms of algebraic groups $\epsilon = \epsilon_{\mathbf{G}} \colon \{1_{\mathbf{G}}\} \to \mathbf{G} \colon 1_{\mathbf{G}} \mapsto 1_{\mathbf{G}}$ and $\nu = \nu_{\mathbf{G}} \colon \mathbf{G} \to \{1_{\mathbf{G}}\} \colon g \mapsto 1_{\mathbf{G}}$.

b) The group laws of associativity, for the identity and for the properties of inverses can be translated into commutative diagrams of morphisms of varieties. Hence for affine algebraic groups these laws can be equivalently reformulated in terms of coordinate algebras and comorphisms as follows:

i) Associativity: For $x, y, z \in \mathbf{G}$ we have $(xy)z = x(yz)$.

$$\begin{tikzcd} \mathbf{G} \times \mathbf{G} \times \mathbf{G} \arrow[r, "\mu \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \mu"'] & \mathbf{G} \times \mathbf{G} \arrow[d, "\mu"] \\ \mathbf{G} \times \mathbf{G} \arrow[r, "\mu"'] & \mathbf{G} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{K}[\mathbf{G}] \otimes_{\mathbb{K}} \mathbb{K}[\mathbf{G}] \otimes_{\mathbb{K}} \mathbb{K}[\mathbf{G}] & \mathbb{K}[\mathbf{G}] \otimes_{\mathbb{K}} \mathbb{K}[\mathbf{G}] \arrow[l, "\mu^* \otimes \mathrm{id}^*"'] \\ \mathbb{K}[\mathbf{G}] \otimes_{\mathbb{K}} \mathbb{K}[\mathbf{G}] \arrow[u, "\mathrm{id}^* \otimes \mu^*"] & \mathbb{K}[\mathbf{G}] \arrow[l, "\mu^*"] \arrow[u, "\mu^*"'] \end{tikzcd}$$

ii) Identity: For $x \in \mathbf{G}$ we have $x \cdot 1_{\mathbf{G}} = x = 1_{\mathbf{G}} \cdot x$.

$$\begin{tikzcd} \mathbf{G} \arrow[r, "\epsilon\nu \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \epsilon\nu"'] \arrow[dr, "\mathrm{id}"] & \mathbf{G} \times \mathbf{G} \arrow[d, "\mu"] \\ \mathbf{G} \times \mathbf{G} \arrow[r, "\mu"'] & \mathbf{G} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{K}[\mathbf{G}] & \mathbb{K}[\mathbf{G}] \otimes_{\mathbb{K}} \mathbb{K}[\mathbf{G}] \arrow[l, "(\epsilon\nu)^* \otimes \mathrm{id}^*"'] \\ \mathbb{K}[\mathbf{G}] \otimes_{\mathbb{K}} \mathbb{K}[\mathbf{G}] \arrow[u, "\mathrm{id}^* \otimes (\epsilon\nu)^*"] & \mathbb{K}[\mathbf{G}] \arrow[l, "\mu^*"] \arrow[u, "\mu^*"'] \arrow[ul, "\mathrm{id}^*"'] \end{tikzcd}$$

iii) Inversion: For $x \in \mathbf{G}$ we have $x \cdot x^{-1} = 1_{\mathbf{G}} = x^{-1} \cdot x$.

$$\begin{tikzcd} \mathbf{G} \arrow[r, "\iota \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \iota"'] \arrow[dr, "\epsilon\nu"] & \mathbf{G} \times \mathbf{G} \arrow[d, "\mu"] \\ \mathbf{G} \times \mathbf{G} \arrow[r, "\mu"'] & \mathbf{G} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{K}[\mathbf{G}] & \mathbb{K}[\mathbf{G}] \otimes_{\mathbb{K}} \mathbb{K}[\mathbf{G}] \arrow[l, "\iota^* \otimes \mathrm{id}^*"'] \\ \mathbb{K}[\mathbf{G}] \otimes_{\mathbb{K}} \mathbb{K}[\mathbf{G}] \arrow[u, "\mathrm{id}^* \otimes \iota^*"] & \mathbb{K}[\mathbf{G}] \arrow[l, "\mu^*"] \arrow[u, "\mu^*"'] \arrow[ul, "(\epsilon\nu)^*"'] \end{tikzcd}$$

(1.2) Example: Additive and multiplicative groups. a) Let $n \in \mathbb{N}_0$. Then $\mathbb{K}^n$ is an affine algebraic group, having multiplication $\mu \colon \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^n \colon [x, y] \mapsto x + y$, inversion $\iota \colon \mathbb{K}^n \to \mathbb{K}^n \colon x \mapsto -x$, and identity element

$\epsilon\colon \{0\} \to \mathbb{K}^n$; hence $\mathbb{K}^n$ is irreducible of dimension n. For $n = 1$, the **additive group** $\mathbf{G}_a := \mathbb{K}$ is an irreducible affine algebraic group of dimension 1.

Going over to the coordinate algebra $\mathbb{K}[\mathcal{X}]$, where $\mathcal{X} = \{X_1, \ldots, X_n\}$, yields $\mu^*\colon \mathbb{K}[\mathcal{X}] \to \mathbb{K}[\mathcal{X}] \otimes_{\mathbb{K}} \mathbb{K}[\mathcal{X}]\colon X_i \mapsto (X_i \otimes 1) + (1 \otimes X_i)$, and $\iota^*\colon \mathbb{K}[\mathcal{X}] \to \mathbb{K}[\mathcal{X}]\colon X_i \mapsto -X_i$, as well as $\epsilon^*\colon \mathbb{K}[\mathcal{X}] \to \mathbb{K}\colon X_i \mapsto 0$, where $\mathbb{K}$ is the coordinate algebra associated with $\{0\}$.

b) The **multiplicative group** $\mathbf{G}_m := \mathbb{K} \setminus \{0\}$ coincides with the principal open subset $\mathbb{K}_X := \{x \in \mathbb{K}; X(x) \neq 0\} \subseteq \mathbb{K}$, and thus is an irreducible affine variety of dimension 1. The associated coordinate algebra is given as $\mathbb{K}[\mathbf{G}_m] = \mathbb{K}[X]_X = \mathbb{K}[X, X^{-1}] := \mathbb{K}[X, T]/\langle XT - 1\rangle$; it can be seen as the localisation of the coordinate algebra $\mathbb{K}[X]$ of $\mathbb{K}$ at the multiplicatively closed set generated by X, where $\langle X\rangle \trianglelefteq \mathbb{K}[X]$ is the maximal ideal belonging to the point $0 \in \mathbb{K}$.

Then $\mathbf{G}_m$ becomes an affine algebraic group with respect to multiplication $\mu\colon \mathbf{G}_m \times \mathbf{G}_m \to \mathbf{G}_m\colon [x, y] \mapsto xy$, inversion $\iota\colon \mathbf{G}_m \to \mathbf{G}_m\colon x \mapsto x^{-1}$, and identity element $\epsilon\colon \{1\} \to \mathbf{G}_m$. Going over to the coordinate algebra $\mathbb{K}[X]_X$ yields $\mu^*\colon \mathbb{K}[X]_X \to \mathbb{K}[X]_X \otimes_{\mathbb{K}} \mathbb{K}[X]_X\colon X \mapsto X \otimes X$ and $\iota^*\colon \mathbb{K}[X]_X \to \mathbb{K}[X]_X\colon X \mapsto X^{-1}$, as well as $\epsilon^*\colon \mathbb{K}[X]_X \to \mathbb{K}\colon X \mapsto 1$. In particular this shows that inversion indeed is a morphism.

For $n \in \mathbb{Z}$ the map $\varphi_n\colon \mathbf{G}_m \to \mathbf{G}_m\colon x \mapsto x^n$, thus $\varphi_n^*\colon \mathbb{K}[X]_X \to \mathbb{K}[X]_X\colon X \mapsto X^n$, is a homomorphism of algebraic groups. If $\mathrm{char}(\mathbb{K}) = p > 0$ and $q = p^f$, for some $f \in \mathbb{N}$, then the **Frobenius morphism** φ_q is a group isomorphism, but since φ_q^* is not surjective, φ_q is not an isomorphism of algebraic groups.

(It can be shown that $\mathbf{G}_a$ and $\mathbf{G}_m$ are not isomorphic as algebraic groups, and that they are the only irreducible affine algebraic groups of dimension 1.)

(1.3) General and special linear groups. a) Let $n \in \mathbb{N}_0$. We consider the affine variety $\mathbb{K}^{n\times n}$, with coordinate algebra $\mathbb{K}[\mathcal{X}]$, where $\mathcal{X} := \{X_{11}, \ldots, X_{nn}\}$. Let $\det = \det_n := \sum_{\sigma\in\mathcal{S}_n} \left(\mathrm{sgn}(\sigma) \cdot \prod_{i=1}^n X_{i,i^\sigma}\right) \in \mathbb{K}[\mathcal{X}]$ be the n-th **determinant polynomial**; in particular we have $\det_1 = X$ and $\det_0 = 1$.

The principal open subset $\mathbf{GL}_n = \mathbf{GL}_n(\mathbb{K}) := (\mathbb{K}^{n\times n})_{\det} = \{A = [a_{ij}]_{ij} \in \mathbb{K}^{n\times n}; \det(A) = \det_n(a_{11}, a_{12}, \ldots, a_{nn}) \neq 0\} \subseteq \mathbb{K}^{n\times n}$ is called the **general linear group**; in particular we have $\mathbf{GL}_1 = \mathbf{G}_m$. Its coordinate algebra is $\mathbb{K}[\mathbf{GL}_n] = \mathbb{K}[\mathcal{X}]_{\det} = \mathbb{K}[\mathcal{X}, \det_n^{-1}]$, and together with its natural abstract group structure $\mathbf{GL}_n$ is an affine algebraic group:

Multiplication $\mu\colon \mathbf{GL}_n \times \mathbf{GL}_n \to \mathbf{GL}_n\colon [[a_{ij}]_{ij}, [b_{jk}]_{jk}] \mapsto [\sum_{j=1}^n a_{ij}b_{jk}]_{ik}$ yields $\mu^*\colon \mathbb{K}[\mathcal{X}]_{\det} \to \mathbb{K}[\mathcal{X}]_{\det} \otimes_{\mathbb{K}} \mathbb{K}[\mathcal{X}]_{\det}\colon X_{ik} \mapsto \sum_{j=1}^n X_{ij} \otimes X_{jk}$. Moreover, using the adjoint matrix, inversion can be written as $\iota\colon \mathbf{GL}_n \to \mathbf{GL}_n\colon A \mapsto A^{-1} = \det(A)^{-1} \cdot \mathrm{adj}(A)$, where $\mathrm{adj}(A) := [(-1)^{i+j} \cdot \det([a_{kl}]_{k\neq j, l\neq i})]_{ij} \in \mathbb{K}^{n\times n}$ and we let $\mathrm{adj}([a_{11}]) = [1]$. This yields $\iota^*\colon \mathbb{K}[\mathcal{X}]_{\det} \to \mathbb{K}[\mathcal{X}]_{\det}\colon X_{ij} \mapsto (-1)^{i+j} \cdot \det_n^{-1}(\mathcal{X}) \cdot \det_{n-1}(\{X_{kl}; k \neq j, l \neq i\})$, in particular showing that inversion is a morphism. Finally, the identitiy element $\epsilon\colon \{E_n\} \to \mathbf{GL}_n$ yields $\epsilon^*\colon \mathbb{K}[\mathcal{X}]_{\det} \to \mathbb{K}\colon X_{ij} \mapsto \delta_{ij}$, the Kronecker δ-function. $\sharp$

Since $\mathbb{K}^{n\times n}$ is irreducible such that $\dim(\mathbb{K}^{n\times n}) = n^2$, these statements also hold for $\mathbf{GL}_n$. The map $\varphi_{\det}\colon \mathbf{GL}_n \to \mathbf{G}_m\colon A \mapsto \det(A)$ is a homomorphism of algebraic groups, such that $\varphi_{\det}^*\colon \mathbb{K}[X]_X \to \mathbb{K}[\mathcal{X}]_{\det}\colon X \mapsto \det$.

b) Similarly, $\mathbf{SL}_n = \mathbf{SL}_n(\mathbb{K}) := \mathcal{V}(\det_n - 1) = \{A = [a_{ij}]_{ij} \in \mathbb{K}^{n\times n}; \det(A) = \det_n(a_{11}, a_{12}, \ldots, a_{nn}) = 1\} \subseteq \mathbb{K}^{n\times n}$ is called the **special linear group**.

Proposition. $\det_n - a \in \mathbb{K}[\mathcal{X}]$ is irreducible, for any $n \in \mathbb{N}$ and $a \in \mathbb{K}$.

Proof. We first consider the case $a = 0$, and show by induction that $\det_n \in \mathbb{K}[\mathcal{X}]$ is irreducible, which holds for $n = 1$. For $n \geq 2$ assume to the contrary that $\det_n$ is reducible. Expansion with respect to the n-th row yields $\det_n = \det_{n-1} \cdot X_{nn} + \delta_n$, where $\delta_n := \sum_{i=1}^{n-1} (-1)^{n-i} \cdot \det_{n-1}(\{X_{kl}; k \neq n, l \neq i\}) \cdot X_{ni}$. Since $\deg_{X_{nn}}(\det_n) = 1$, and by induction $\det_{n-1} \in \mathbb{K}[\{X_{kl}; k \neq n, l \neq n\}] \subseteq \mathbb{K}[\mathcal{X}]$ is irreducible, this implies that $\det_{n-1}$ divides δ_n. By specifying $X_{nj} \mapsto 0$, for $j \in \{1, \ldots, n-1\} \setminus \{i\}$, this yields that $\det_{n-1}$ divides $\det_{n-1}(\{X_{kl}; k \neq n, l \neq i\})$, for $i \in \{1, \ldots, n-1\}$, which is a contradiction.

Now let $a \neq 0$, and assume to the contrary that $\det_n - a$ is reducible. Then we conclude similarly that $\det_{n-1}$ divides $\delta_n - a$, which by specifying $X_{ni} \mapsto 0$, for $i \in \{1, \ldots, n-1\}$, is a contradiction. ♯

This implies that $\langle \det_n - 1 \rangle \trianglelefteq \mathbb{K}[\mathcal{X}]$ is prime, for $n \in \mathbb{N}_0$, and thus the coordinate algebra of $\mathbf{SL}_n$ is $\mathbb{K}[\mathbf{SL}_n] \cong \mathbb{K}[\mathcal{X}]/\langle \det_n - 1 \rangle$; in particular $\mathbb{K}[\mathbf{SL}_n]$ is a domain, or equivalently $\mathbf{SL}_n$ is irreducible. Moreover, since the prime ideal $\langle \det_n - 1 \rangle$ has height 1, we conclude that $\dim(\mathbf{SL}_n) = \dim(\mathbb{K}^{n\times n}) - 1 = n^2 - 1$.

Since $\mathbf{SL}_n \subseteq \mathbf{GL}_n$, we conclude that $\mathbf{SL}_n \leq \mathbf{GL}_n$ is a closed subgroup; alternatively, this also follows from $\mathbf{SL}_n = \ker(\varphi_{\det}) \leq \mathbf{GL}_n$. The associated inclusion morphism has comorphism $\mathbb{K}[\mathcal{X}]_{\det} \to \mathbb{K}[\mathcal{X}]/\langle \det_n - 1 \rangle \colon X_{ij} \mapsto X_{ij}$, $\det_n^{-1} \mapsto 1$.

(1.4) Linear algebraic groups. Let $n \in \mathbb{N}_0$. Any closed subgroup of $\mathbf{GL}_n$ is an affine variety, such that the structure morphisms carry over from $\mathbf{GL}_n$, thus is an affine algebraic group in its own right, being called a **linear** algebraic group. (We will show in (2.7) that any affine algebraic group is isomorphic as algebraic groups to a linear algebraic group, so that these notions coincide.)

Example. We have the following linear algebraic groups, where $n \in \mathbb{N}_0$:

i) The **scalar group** $\mathbf{Z}_n := \{a \cdot E_n \in \mathbf{GL}_n; a \neq 0\} \cong \mathbf{G}_m$, where $\mathbf{Z}_n = Z(\mathbf{GL}_n)$; and the group of diagonal matrices or **torus** $\mathbf{T}_n := \{[a_{ij}]_{ij} \in \mathbf{GL}_n; a_{ij} = 0 \text{ for } i \neq j\} \cong (\mathbf{G}_m)^n$, the n-fold direct product of $\mathbf{G}_m$ with itself.

ii) The group of upper unitriangular matrices or **unipotent group** $\mathbf{U}_n := \{[a_{ij}]_{ij} \in \mathbf{GL}_n; a_{ij} = 0 \text{ for } i > j, a_{ii} = 1\}$; and the group of upper triangular matrices or **Borel group** $\mathbf{B}_n := \{[a_{ij}]_{ij} \in \mathbf{GL}_n; a_{ij} = 0 \text{ for } i > j\}$, where $\mathbf{B}_n = N_{\mathbf{GL}_n}(\mathbf{U}_n)$ and $\mathbf{B}_n \cong \mathbf{T}_n \ltimes \mathbf{U}_n$ as abstract groups (at least).

In particular, we have $\mathbf{U}_2 := \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in \mathbf{GL}_2; x \in \mathbb{K} \right\}$, with coordinate algebra $\mathbb{K}[X_{11}, X_{12}, X_{21}, X_{22}]/\langle X_{11} - 1, X_{22} - 1, X_{21} \rangle \cong \mathbb{K}[\overline{X}_{12}]$. Thus the map $\varphi \colon \mathbf{G}_a \to \mathbf{U}_2 \colon x \mapsto \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$, which has comorphism $\varphi^* \colon \mathbb{K}[\overline{X}_{12}] \to \mathbb{K}[X] \colon \overline{X}_{12} \mapsto X$, is an isomorphism of algebraic groups.

iii) The group of permutation matrices or **Weyl group** $W_n \leq \mathbf{GL}_n$, where as abstract groups $W_n \cong \mathcal{S}_n$ is isomorphic to the symmetric group on n letters;

and the group $\mathbf{N}_n \leq \mathbf{GL}_n$ of **monomial** matrices, where $\mathbf{N}_n = N_{\mathbf{GL}_n}(\mathbf{T}_n)$ and $\mathbf{N}_n \cong W_n \ltimes \mathbf{T}_n$ as abstract groups (at least).

In particular, since by Cayley's Theorem any finite group is isomorphic to a subgroup of a finite symmetric group, and thus to a group of permutation matrices, any finite group can be considered as a linear algebraic group.

2 Basic properties

(2.1) Theorem. a) Let $\mathbf{G}$ be an affine algebraic group. Then there is a unique irreducible component $\mathbf{G}^\circ$ of $\mathbf{G}$ containing $1_\mathbf{G}$. The **identity component** $\mathbf{G}^\circ \trianglelefteq \mathbf{G}$ is a closed normal subgroup of finite index, being contained in any closed subgroup of finite index, and containing any irreducible closed subgroup.

b) The finite set $\mathbf{G}^\circ\backslash\mathbf{G} := \{\mathbf{G}^\circ g; g \in \mathbf{G}\}$ of (right) cosets of $\mathbf{G}^\circ$ in $\mathbf{G}$ coincides with the irreducible components of $\mathbf{G}$, which in turn coincide with the connected components of $\mathbf{G}$. In particular, $\mathbf{G}$ is equidimensional such that $\dim(\mathbf{G}) = \dim(\mathbf{G}^\circ)$, and $\mathbf{G}$ is irreducible if and only if it is connected; in this case the affine algebraic group $\mathbf{G}$ is called **connected**.

Proof. a) Let first $V, W \subseteq \mathbf{G}$ be irreducible components such that $1_\mathbf{G} \in V \cap W$. Multiplication $\mu\colon \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ yields that $VW = \mu(V \times W) \subseteq \mathbf{G}$ is irreducible, hence $\overline{VW} \subseteq \mathbf{G}$ is irreducible as well. Since both $V \subseteq \overline{VW}$ and $W \subseteq \overline{VW}$, we conclude that $V = \overline{VW} = W$. This shows that $\mathbf{G}^\circ$ is well-defined.

In particular, we have $\mathbf{G}^\circ\mathbf{G}^\circ = \mathbf{G}^\circ$. Since inversion $\iota\colon \mathbf{G} \to \mathbf{G}$ is an automorphism of varieties, $(\mathbf{G}^\circ)^{-1} = \iota(\mathbf{G}^\circ) \subseteq \mathbf{G}$ is an irreducible component containing $1_\mathbf{G}$, implying that $(\mathbf{G}^\circ)^{-1} = \mathbf{G}^\circ$. Thus $\mathbf{G}^\circ \leq \mathbf{G}$ is a subgroup. For any $g \in \mathbf{G}$ conjugation $\kappa_g\colon \mathbf{G} \to \mathbf{G}\colon x \mapsto x^g := g^{-1}xg$ is an automorphism of varieties, hence $(\mathbf{G}^\circ)^g = \kappa_g(\mathbf{G}^\circ) \subseteq \mathbf{G}$ is an irreducible component containing $1_\mathbf{G}$, thus $(\mathbf{G}^\circ)^g = \mathbf{G}^\circ$, implying that $\mathbf{G}^\circ \trianglelefteq \mathbf{G}$ is normal. This proves the first half of a).

b) For any $g \in \mathbf{G}$ right translation $\rho_g\colon \mathbf{G} \to \mathbf{G}\colon x \mapsto xg$ is an automorphism of varieties, hence $\mathbf{G}^\circ g = \rho_g(\mathbf{G}^\circ) \subseteq \mathbf{G}$ is an irreducible component, in particular is connected. Since $\mathbf{G}$ is Noetherian, $\mathbf{G}^\circ\backslash\mathbf{G}$ is a finite set; in particular $\mathbf{G}^\circ$ has finite index in $\mathbf{G}$. From $\mathbf{G} = \coprod_{g \in \mathbf{G}^\circ\backslash\mathbf{G}} \mathbf{G}^\circ g$ we conclude that all the sets $\mathbf{G}^\circ g \subseteq \mathbf{G}$ are open and closed, hence are the connected components of $\mathbf{G}$.

If $V \subseteq \mathbf{G}$ is an irreducible component, then from $V = \coprod_{g \in \mathbf{G}^\circ\backslash\mathbf{G}} (V \cap \mathbf{G}^\circ g)$ we conclude that $V = V \cap \mathbf{G}^\circ g$, hence $V = \mathbf{G}^\circ g$, for some $g \in \mathbf{G}$. This proves b).

a) (cont.) Let $\mathbf{H} \leq \mathbf{G}$ be closed of finite index. Hence $\mathbf{G} = \coprod_{g \in \mathbf{H}\backslash\mathbf{G}} \mathbf{H}g$ is a finite union of open and closed subsets. Thus we have $\mathbf{G}^\circ = \coprod_{g \in \mathbf{H}\backslash\mathbf{G}} (\mathbf{G}^\circ \cap \mathbf{H}g)$, and since $1_\mathbf{G} \in \mathbf{G}^\circ \cap \mathbf{H}$ this implies $\mathbf{G}^\circ = \mathbf{G}^\circ \cap \mathbf{H}$, hence $\mathbf{G}^\circ \leq \mathbf{H}$.

Let $\mathbf{H} \leq \mathbf{G}$ be closed and irreducible. Then $\mathbf{H} \cap \mathbf{G}^\circ \trianglelefteq \mathbf{H}$ has finite index, hence $\mathbf{H} = \mathbf{H}^\circ \leq \mathbf{H} \cap \mathbf{G}^\circ$, implying $\mathbf{H} \leq \mathbf{G}^\circ$. This proves the second half of a). ♯

Example. Let $K := \mathbb{C}$ and $J := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $\mathbf{O}_2 := \{A \in \mathbf{GL}_2; AJA^{\mathrm{tr}} = J\}$ be the **2-dimensional orthogonal group**; hence $\mathbf{O}_2 \leq \mathbf{GL}_2$ is closed. For $A \in \mathbf{O}_2$ we have $\det(A)^2 = 1$, hence $\det\colon \mathbf{O}_2 \to \{\pm 1\}$ is a surjective homomorphism of algebraic groups; note that $J \in \mathbf{O}_2$ such that $\det(J) = -1$. Thus its kernel

$\mathbf{SO}_2 := \mathbf{O}_2 \cap \mathbf{SL}_2 \trianglelefteq \mathbf{O}_2$ is a closed normal subgroup of index 2, being called the 2-**dimensional special orthogonal group**; in particular $\mathbf{O}_2$ is not conntected.

Letting $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{GL}_2$, from $JA^{\mathrm{tr}}J = A^{-1} = \frac{1}{\det(A)} \cdot \mathrm{adj}(A)$ we get $\begin{bmatrix} d & b \\ c & a \end{bmatrix} = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Hence we have $\mathbf{SO}_2 = \{\mathrm{diag}[a, a^{-1}] \in \mathbf{GL}_2; a \neq 0\} \cong \mathbf{G}_m$ as algebraic groups; in particular $\mathbf{SO}_2$ is conntected. Thus we have $\mathbf{O}_2^\circ = \mathbf{SO}_2$ and $\mathbf{O}_2 = \mathbf{SO}_2 \,\dot\cup\, \mathbf{SO}_2 \cdot J$. ♯

(2.2) Lemma. Let $\mathbf{G}$ be an affine algebraic group.
a) Let $\emptyset \neq V \subseteq \mathbf{G}$ be open and $W \subseteq \mathbf{G}$ be dense. Then $VW = \mathbf{G} = WV$.
b) Let $H \leq \mathbf{G}$ be a subgroup. Then $\overline{H} \leq \mathbf{G}$ is a subgroup as well. If moreover H contains a non-empty open subset of $\overline{H}$, then we have $H = \overline{H}$.

Proof. a) Recall that a dense subset of a topological space intersects non-trivially with any non-empty open subset. Now let $g \in \mathbf{G}$. Then $V^{-1}g \subseteq \mathbf{G}$ and $gV^{-1} \subseteq \mathbf{G}$ are open as well. Hence we have $V^{-1}g \cap W \neq \emptyset$, implying that there is $v^{-1}g = w \in V^{-1}g \cap W$, for some $v \in V$ and $w \in W$, thus $g = vw$. Similarly, we have $gV^{-1} \cap W \neq \emptyset$, implying that there is $gv^{-1} = w \in gV^{-1} \cap W$, for some $v \in V$ and $w \in W$, thus $g = wv$.

b) We have $\overline{H}^{-1} = \overline{H^{-1}} = \overline{H}$. Moreover, for any $h \in H$ we have $\overline{H}h = \overline{Hh} = \overline{H}$, implying $\overline{H}H \subseteq \overline{H}$. Thus for any $g \in \overline{H}$ we have $gH \subseteq \overline{H}$, implying $g\overline{H} = \overline{gH} \subseteq \overline{H}$, thus $\overline{H}\,\overline{H} \subseteq \overline{H}$. This shows that $\overline{H} \leq \mathbf{G}$ is a closed subgroup.

Moreover, if $\emptyset \neq U \subseteq \overline{H}$ is open such that $U \subseteq H$, then $H = \bigcup\{Uh; h \in H\} \subseteq \overline{H}$ is open and dense, thus $H = HH = \overline{H}$. ♯

Recall that any constructible subset of an affine variety contains a dense open subset of its closure, and that the image of any morphism is constructible.

(2.3) Theorem. Let $\varphi\colon \mathbf{G} \to \mathbf{H}$ be a homomorphism of affine algebraic groups.
a) Then the kernel $\ker(\varphi) \trianglelefteq \mathbf{G}$ and the image $\varphi(\mathbf{G}) \leq \mathbf{H}$ are closed subgroups, and we have the dimension formula $\dim(\mathbf{G}) = \dim(\ker(\varphi)) + \dim(\varphi(\mathbf{G}))$.
b) We have $\varphi(\mathbf{G}^\circ) = \varphi(\mathbf{G})^\circ$.

Proof. a) Since $\{1_{\mathbf{H}}\} \subseteq \mathbf{H}$ is closed, $\ker(\varphi) = \varphi^{-1}(\{1_{\mathbf{H}}\}) \subseteq \mathbf{G}$ is closed as well. Moreover, since φ is a morphism of varieties, $\varphi(\mathbf{G})$ fulfills the assumptions of (2.2)b). Hence $\varphi(\mathbf{G}) \leq \mathbf{H}$ is closed. This proves the first half of a).

b) Since $\varphi(\mathbf{G}) \leq \mathbf{H}$ is closed, it is an affine algebraic group. Since $\varphi(\mathbf{G}^\circ) \leq \varphi(\mathbf{G})$ is closed and irreducible, we have $\varphi(\mathbf{G}^\circ) \leq \varphi(\mathbf{G})^\circ$. Conversely, since $\mathbf{G}^\circ \leq \mathbf{G}$ is a subgroup of finite index, $\varphi(\mathbf{G}^\circ) \leq \varphi(\mathbf{G})$ is a subgroup of finite index as well, implying $\varphi(\mathbf{G})^\circ \leq \varphi(\mathbf{G}^\circ)$. This proves b).

a) (cont.) In order to proceed towards the dimension formula, since $\varphi(\mathbf{G}) \leq \mathbf{H}$ is closed, we may assume that φ is surjective. Hence we have the restriction $\varphi_0 := \varphi|_{\mathbf{G}^\circ}\colon \mathbf{G}^\circ \to \mathbf{H}^\circ$, which is a surjective morphism between irreducible varieties. The fibres of φ_0 are the cosets of $\ker(\varphi_0)$ in $\mathbf{G}^\circ$, thus are all isomorphic to $\ker(\varphi_0)$ as varieties. Hence the dimension formula for morphisms yields $\dim(\mathbf{G}^\circ) = \dim(\ker(\varphi_0)) + \dim(\mathbf{H}^\circ)$. Moreover, $\ker(\varphi_0) = \ker(\varphi) \cap$

$\mathbf{G}^\circ \trianglelefteq \ker(\varphi)$ has finite index, hence we have $\ker(\varphi)^\circ \leq \ker(\varphi_0)$, implying that $\dim(\ker(\varphi)^\circ) = \dim(\ker(\varphi_0)) = \dim(\ker(\varphi))$. Since $\dim(\mathbf{G}^\circ) = \dim(\mathbf{G})$ and $\dim(\mathbf{H}^\circ) = \dim(\mathbf{H})$, this proves the second half of a). $\sharp$

(2.4) Action on varieties. a) Let $\mathbf{G}$ be an affine algebraic group, and let $V \neq \emptyset$ be a variety. A (right) group action $\alpha\colon V \times \mathbf{G} \to V\colon [x,g] \mapsto xg$, such that α is a morphism, is called a **morphical** or **regular** action, and V is called a **G-variety**. If $\mathbf{G}$ acts morphically on W as well, then a morphism $\varphi\colon V \to W$ is called **G-equivariant** if $\varphi(xg) = \varphi(x)g$, for $x \in V$ and $g \in \mathbf{G}$.

For any $g \in \mathbf{G}$ we have the automorphism of varieties $\alpha_g\colon V \to V\colon x \mapsto xg$. If V is affine, then we get the associated automorphism of $\mathbb{K}$-algebras $\alpha_g^*\colon \mathbb{K}[V] \to \mathbb{K}[V]\colon f \mapsto (x \mapsto f(xg))$, also called the induced **translation of functions**. This gives rise to the associated representation of $\mathbf{G}$ on $\mathbb{K}[V]$ defined as $\alpha^\vee\colon \mathbf{G} \to \mathrm{Aut}_{\mathbb{K}}(\mathbb{K}[V])\colon g \mapsto \alpha_g^\vee := \alpha_{g^{-1}}^* = (\alpha_g^*)^{-1}$.

For any $x \in V$ we have the **orbit morphism** $\alpha_x\colon \mathbf{G} \to V\colon g \mapsto xg$, whose image $x\mathbf{G} = \alpha_x(\mathbf{G}) \subseteq V$ is called the associated **G-orbit**. If $\mathbf{G}$ acts transitively, that is $x\mathbf{G} = V$ for some $x \in V$, then V is called **homogeneous**.

Example. The affine algebraic group $\mathbf{G}$ acts morphically on itself by **right translation** $\rho = \mu\colon \mathbf{G} \times \mathbf{G} \to \mathbf{G}\colon [x,g] \mapsto xg$, as well as by **left translation** $\lambda\colon \mathbf{G} \times \mathbf{G} \to \mathbf{G}\colon [x,g] \mapsto g^{-1}x$, where $\mathbf{G}$ is homogeneous for either action.

Moreover, $\mathbf{G}$ acts morphically on itself by **conjugation** or **inner automorphisms** $\kappa\colon \mathbf{G} \times \mathbf{G} \to \mathbf{G}\colon [x,g] \mapsto x^g := g^{-1}xg$; note that $\kappa_g = \rho_g\lambda_g = \lambda_g\rho_g$ for $g \in \mathbf{G}$. For $x \in \mathbf{G}$ the orbit $x^{\mathbf{G}} \subseteq \mathbf{G}$ is called the associated **conjugacy class**.

(2.5) Stabilisers and fixed points. a) Let $\mathbf{G}$ be an affine algebraic group, let V be a $\mathbf{G}$-variety, let $U \subseteq V$ be a subset, and let $W \subseteq V$ be closed.

Then the **transporter** $\mathrm{Tran}_{\mathbf{G}}(U,W) := \{g \in \mathbf{G}; Ug \subseteq W\} = \bigcap_{x \in U} \alpha_x^{-1}(W) \subseteq \mathbf{G}$ is a closed subset. Moreover, the **normaliser** $N_{\mathbf{G}}(W) := \{g \in \mathbf{G}; Wg = W\} = \mathrm{Tran}_{\mathbf{G}}(W,W) \cap \mathrm{Tran}_{\mathbf{G}}(W,W)^{-1} \leq \mathbf{G}$ is a closed subgroup.

In particular, for any $x \in V$ the **isotropy group** or **centraliser** or **stabiliser** $\mathbf{G}_x = C_{\mathbf{G}}(x) = \mathrm{Stab}_{\mathbf{G}}(x) := \{g \in \mathbf{G}; xg = x\} = \mathrm{Tran}_{\mathbf{G}}(\{x\},\{x\}) \leq \mathbf{G}$ is a closed subgroup, hence $C_{\mathbf{G}}(U) := \bigcap_{x \in U} \mathbf{G}_x \leq \mathbf{G}$ is a closed subgroup as well.

b) For any $g \in \mathbf{G}$ the set of **fixed points** $V^g = \mathrm{Fix}_V(g) := \{x \in V; xg = x\} \subseteq V$ is closed, implying that $V^{\mathbf{G}} = \mathrm{Fix}_V(\mathbf{G}) := \bigcap_{g \in \mathbf{G}} V^g \subseteq V$ is closed as well:

Since V is a variety, the **diagonal** $\Delta(V) := \{[x,x] \in V \times V; x \in V\} \subseteq V \times V$ is closed; note that this is clear anyway if V is affine. Hence using the **graph** of α_g, that is the morphism $\gamma_g\colon V \to V \times V\colon x \mapsto [x, xg]$, we infer that $\gamma_g^{-1}(\Delta(V)) = V^g \subseteq V$ is closed as well. $\sharp$

c) Each irreducible component of $W \subseteq V$ is $\mathbf{G}^\circ$-invariant; in particular, if V is finite then $\mathbf{G}^\circ$ acts trivially:

The group $\mathbf{G}$, acting by automorphisms of varieties, permutes the finitely many irreducible components of V, hence $N_{\mathbf{G}}(W) \leq \mathbf{G}$ is a closed subgroup of finite index, thus contains $\mathbf{G}^\circ$. $\sharp$

Example. For the right and left translation actions of $\mathbf{G}$ on itself, for $x \in \mathbf{G}$ the associated isotropy groups are trivial, and for $g \in \mathbf{G} \setminus \{1_{\mathbf{G}}\}$ the associated fixed point sets are empty.

For the conjugation action, the isotropy group of $x \in \mathbf{G}$ is given as $C_{\mathbf{G}}(x) = \{g \in \mathbf{G}; g^{-1}xg = x\} = \{g \in \mathbf{G}; xg = gx\}$, which hence is a closed subgroup of $\mathbf{G}$. Similarly, for $g \in \mathbf{G}$ we have $\mathrm{Fix}_{\mathbf{G}}(g) = \{x \in \mathbf{G}; g^{-1}xg = x\} = \{x \in \mathbf{G}; xg = gx\} = C_{\mathbf{G}}(g)$. Thus the center $Z(\mathbf{G}) := C_{\mathbf{G}}(\mathbf{G}) = \bigcap_{x\in\mathbf{G}} C_{\mathbf{G}}(x) = \{g \in \mathbf{G}; xg = gx \text{ for all } x \in \mathbf{G}\} = \{x \in \mathbf{G}; xg = gx \text{ for all } g \in \mathbf{G}\} = \bigcap_{g\in\mathbf{G}} \mathrm{Fix}_{\mathbf{G}}(g) = \mathrm{Fix}_{\mathbf{G}}(\mathbf{G})$ is a closed subgroup of $\mathbf{G}$ as well.

If $\mathbf{H} \leq \mathbf{G}$ is closed, then both $C_{\mathbf{G}}(\mathbf{H}) \leq \mathbf{G}$ are $N_{\mathbf{G}}(\mathbf{H}) \leq \mathbf{G}$ are closed.

(2.6) Proposition. Let $\mathbf{G}$ be an affine algebraic group acting morphically via α on an affine variety V, and let $U \leq \mathbb{K}[V]$ be a finitely generated $\mathbb{K}$-subspace.
a) Then there is a finitely generated $\mathbf{G}$-invariant $\mathbb{K}$-subspace of $\mathbb{K}[V]$ encompassing U; that is the $\mathbf{G}$-action is **locally finite**.
b) The $\mathbb{K}$-subspace U is $\mathbf{G}$-invariant if and only if $\alpha^*(U) \leq U \otimes_{\mathbb{K}} \mathbb{K}[\mathbf{G}]$.

Proof. a) Since $\mathbf{G}$ acts by $\mathbb{K}$-linear maps on $\mathbb{K}[V]$, we may assume that $U = \langle f \rangle_{\mathbb{K}}$, for some $0 \neq f \in \mathbb{K}[V]$. Hence we have $\alpha^*(f) = \sum_{i=1}^r f_i \otimes g_i \in \mathbb{K}[V] \otimes_{\mathbb{K}} \mathbb{K}[\mathbf{G}]$, where $r \in \mathbb{N}$ and $f_i \in \mathbb{K}[V]$ and $g_i \in \mathbb{K}[\mathbf{G}]$. For $g \in \mathbf{G}$ and $x \in V$ we have $(\alpha_g^*(f))(x) = f(\alpha_g(x)) = f(xg) = f(\alpha([x, g])) = (\alpha^*(f))([x, g]) = \sum_{i=1}^r f_i(x)g_i(g)$, which implies that $\alpha_g^*(f) = \sum_{i=1}^r f_i \cdot g_i(g) \in \mathbb{K}[V]$.

Hence we conclude that $\langle \alpha_g^*(f); g \in \mathbf{G}\rangle_{\mathbb{K}} \leq \langle f_1, \ldots, f_r\rangle_{\mathbb{K}} \leq \mathbb{K}[V]$ is a finitely generated $\mathbf{G}$-invariant $\mathbb{K}$-subspace which contains $f = \alpha_1^*(f)$. Note that the latter is the smallest $\mathbb{K}$-subspace of $\mathbb{K}[V]$ having these properties, thus it is called the $\mathbf{G}$-invariant subspace **generated** by f.

b) If $\alpha^*(U) \leq U \otimes_{\mathbb{K}} \mathbb{K}[\mathbf{G}]$ holds, then the above computation shows that $\alpha_g^*(U) \leq U$, for $g \in \mathbf{G}$, that is U is $\mathbf{G}$-invariant.

Conversely, let $U \leq \mathbb{K}[V]$ be $\mathbf{G}$-invariant. Then let $\{f_1, \ldots, f_s, f_{s+1}, \ldots\} \subseteq \mathbb{K}[V]$ be a $\mathbb{K}$-basis, where $\{f_1, \ldots, f_s\} \subseteq U$ is a $\mathbb{K}$-basis and $s := \dim_{\mathbb{K}}(U) \in \mathbb{N}_0$. For $f \in U$ we have $\alpha^*(f) = \sum_{i=1}^r f_i \otimes g_i$, where $s \leq r \in \mathbb{N}_0$ and $g_i \in \mathbb{K}[\mathbf{G}]$. For $g \in \mathbf{G}$ this yields $\alpha_g^*(f) = \sum_{i=1}^r f_i \cdot g_i(g)$. Since $\alpha_g^*(f) \in U$, from $\mathbb{K}$-linear independence we infer that $g_i(g) = 0$ for $i \geq s+1$. This being true for all $g \in \mathbf{G}$, we deduce that $g_i = 0 \in \mathbb{K}[\mathbf{G}]$ for $i \geq s+1$. Thus we have $\alpha^*(f) = \sum_{i=1}^s f_i \otimes g_i \in U \otimes_{\mathbb{K}} \mathbb{K}[\mathbf{G}]$. ♯

(2.7) Theorem: Linearisation of actions. Let $\mathbf{G}$ be an affine algebraic group, and let V be an affine $\mathbf{G}$-variety. Then there is a closed embedding $\varphi\colon V \to \mathbb{K}^n$, for some $n \in \mathbb{N}_0$, and a homomorphism of algebraic groups $\delta\colon \mathbf{G} \to \mathbf{GL}_n$, such that we have $\mathbf{G}$-equivariance $\varphi(xg) = \varphi(x)\delta(g)$, for $x \in V$ and $g \in \mathbf{G}$.

Proof. Let $\{f_1, \ldots, f_n\} \subseteq \mathbb{K}[V]$, where $n \in \mathbb{N}_0$, be a $\mathbb{K}$-linear independent $\mathbb{K}$-algebra generating set such that additionally, by (2.6), the $\mathbb{K}$-subspace $U := \langle f_1, \ldots, f_n\rangle_{\mathbb{K}} \leq \mathbb{K}[V]$ is $\mathbf{G}$-invariant. Letting α be the action morphism, we have $\alpha^*(f_i) = \sum_{j=1}^n f_j \otimes g_{ji} \in \mathbb{K}[V] \otimes_{\mathbb{K}} \mathbb{K}[\mathbf{G}]$, where the $g_{ij} \in \mathbb{K}[\mathbf{G}]$ are uniquely defined. Thus we get $\alpha_g^*(f_i) = \sum_{j=1}^n f_j \cdot g_{ji}(g) \in \mathbb{K}[\mathbf{G}]$, for $g \in \mathbf{G}$. Recall that this yields a left $\mathbf{G}$-action on U, that is we have $\alpha_{hg}^*(f) = \alpha_h^*(\alpha_g^*(f))$ for

$g, h \in \mathbf{G}$ and $f \in U$. Thus in terms of matrices we have $[g_{kj}(h)]_{kj} \cdot [g_{ji}(g)]_{ji} = [g_{ki}(hg)]_{ki}$. Hence we get a morphism of varieties $\delta\colon \mathbf{G} \to \mathbf{GL}_n\colon g \mapsto [g_{ji}(g)]_{ji}$ with comorphism $\mathbb{K}[\mathbf{GL}_n] \to \mathbb{K}[\mathbf{G}]\colon X_{ji} \mapsto g_{ji}$, such that $\delta(g)\delta(h) = \delta(gh)$. Thus δ is a homomorphism of algebraic groups.

Next, Let $U^\vee := \mathrm{Hom}_{\mathbb{K}}(U, \mathbb{K})$ be the dual $\mathbb{K}$-space of U, and let $\{\lambda_1, \ldots, \lambda_n\} \subseteq U^\vee$ be the $\mathbb{K}$-basis dual to the $\mathbb{K}$-basis $\{f_1, \ldots, f_n\} \subseteq U$, that is $\lambda_j(f_i) = \delta_{ij}$ for $i, j \in \{1, \ldots, n\}$. In other words, we may view the evaluation map $f_i^\bullet$ at f_i as coordinate functions on $U^\vee$, for $i \in \{1, \ldots, n\}$, hence we have $\mathbb{K}[U^\vee] = \mathbb{K}[f_1^\bullet, \ldots, f_n^\bullet]$. Moreover, $U^\vee$ carries a (right) $\mathbf{G}$-action such that $g \in \mathbf{G}$ maps $\lambda \in U^\vee$ to $\lambda^g\colon f \mapsto \lambda(\alpha_g^*(f))$. Hence, with respect to the above $\mathbb{K}$-basis of $U^\vee$, this action is given by δ.

Now let $\varphi\colon V \to U^\vee\colon x \mapsto \iota_x^*$, where $\iota_x^*\colon f \mapsto f(x)$ is the comorphism associated with $\iota_x\colon \{x\} \to V$. Then φ is a closed embedding: For $i \in \{1, \ldots, n\}$ and $x \in V$ we have $(\varphi^*(f_i^\bullet))(x) = f_i^\bullet(\varphi(x)) = f_i^\bullet(\iota_x^*) = \iota_x^*(f_i) = f_i(x)$, hence $\varphi^*(f_i^\bullet) = f_i \in \mathbb{K}[V]$, showing that φ is a morphism; moreover, since $\{f_1, \ldots, f_n\} \subseteq \mathbb{K}[V]$ is a $\mathbb{K}$-algebra generating set, from $\varphi^*\colon \mathbb{K}[U^\vee] \to \mathbb{K}[V]\colon f_i^\bullet \to f_i$ we conclude that φ is a surjective homomorphism of $\mathbb{K}$-algebras. Finally, for $g \in \mathbf{G}$ and $f \in U$ we have $\varphi(xg)(f) = \iota_{xg}^*(f) = f(xg) = (\alpha_g^*(f))(x) = \iota_x^*(\alpha_g^*(f)) = \varphi(x)(\alpha_g^*(f)) = \varphi(x)^g(f)$, hence $\varphi(xg) = \varphi(x)^g \in U^\vee$, that is φ is $\mathbf{G}$-equivariant. ♯

Corollary: Linearisation of groups. Any affine algebraic group $\mathbf{G}$ is isomorphic as an algebraic group to a closed subgroup of $\mathbf{GL}_n$, for some $n \in \mathbb{N}_0$.

Proof. We consider the right translation action $\rho = \mu\colon \mathbf{G} \times \mathbf{G} \to \mathbf{G}$, and let $\{f_1, \ldots, f_n\} \subseteq \mathbb{K}[\mathbf{G}]$ with associated map $\delta\colon \mathbf{G} \to \mathbf{GL}_n\colon g \mapsto [g_{ji}(g)]_{ji}$ be as above. It remains to be shown that δ is a closed embedding:

Since $f_i(g) = f_i(1_{\mathbf{G}} \cdot g) = (\rho_g^*(f_i))(1_{\mathbf{G}}) = \sum_{j=1}^n f_j(1_{\mathbf{G}}) \cdot g_{ji}(g)$, for $g \in \mathbf{G}$, we get $f_i = \sum_{j=1}^n f_j(1_{\mathbf{G}}) \cdot g_{ji} \in \mathbb{K}[\mathbf{G}]$, implying that $\{g_{ji}; i, j \in \{1, \ldots, n\}\} \subseteq \mathbb{K}[\mathbf{G}]$ is a $\mathbb{K}$-algebra generating set, thus $\delta^*\colon \mathbb{K}[\mathbf{GL}_n] \to \mathbb{K}[\mathbf{G}]$ is surjective. ♯

3 Orbits

(3.1) Theorem: Closed orbit lemma. a) Let $\mathbf{G}$ be an affine algebraic group, let V be a $\mathbf{G}$-variety, and let $O \subseteq V$ be a $\mathbf{G}$-orbit. Then $\overline{O} \subseteq V$ is $\mathbf{G}$-invariant, $O \subseteq \overline{O}$ is open, and if $O \neq \overline{O}$ then $\dim(\overline{O} \setminus O) < \dim(\overline{O})$.
b) For $\mathbf{G}$-orbits $O, O' \subseteq V$ such that $O' \subseteq \overline{O}$ we write $O' \preceq O$. Then the **orbit closure relation** $\preceq$ is a partial order on the set of $\mathbf{G}$-orbits in V. Moreover, there are $\preceq$-minimal $\mathbf{G}$-orbits, all of which are closed. In particular, any $\mathbf{G}$-orbit contains a closed $\mathbf{G}$-orbit in its closure.

Proof. a) Letting $\mathbf{G}$ act via α, since α_g is an isomorphism for any $g \in \mathbf{G}$, from $\alpha_g(O) = O$ we get $O = \alpha_g^{-1}(O) \subseteq \alpha_g^{-1}(\overline{O})$, where the latter is closed, hence $\overline{O} \subseteq \alpha_g^{-1}(\overline{O})$, thus $\alpha_g(\overline{O}) \subseteq \overline{O}$, hence $\overline{O}$ is $\mathbf{G}$-invariant.

Let $O = x\mathbf{G}$, for some $x \in V$, let $\emptyset \neq U \subseteq \overline{O}$ be open such that $U \subseteq O$, and let $h \in \mathbf{G}$ such that $xh \in U$. Thus $x \in Uh^{-1}$, implying $O = x\mathbf{G} \subseteq \bigcup_{g \in \mathbf{G}} Ug \subseteq O$, and hence $O = \bigcup_{g \in \mathbf{G}} Ug$, where $Ug \subseteq \overline{O}$ is open for all $g \in \mathbf{G}$.

Let $\overline{O} = \bigcup_{i=1}^r W_i$, where $r \in \mathbb{N}$ and the $W_i \subseteq \overline{O}$ are the irreducible components. Assume that $W_i \cap O = \emptyset$ for some $i \in \{1, \ldots, r\}$, then $O \subseteq \bigcup_{j \neq i} W_i$, hence $W_i \subseteq \overline{O} \subseteq \bigcup_{j \neq i} W_i$, a contradiction. Hence we have $\overline{O} \setminus O = \bigcup_{i=1}^r (W_i \setminus O)$, where $W_i \setminus O \neq W_i$ for $i \in \{1, \ldots, r\}$. Thus if $W_i \not\subseteq O$ then $\dim(W_i \setminus O) < \dim(W_i) \leq \dim(\overline{O})$, while if $W_i \subseteq O$ then $W_i \setminus O = \emptyset$ anyway. Hence we get $\dim(\overline{O} \setminus O) = \max\{\dim(W_i \setminus O) \in \mathbb{N}_0; i \in \{1, \ldots, r\}, W_i \not\subseteq O\} < \dim(\overline{O})$.

b) We have to show that $\preceq$ is reflexive, transitive and anti-symmetric: We have $O \subseteq \overline{O}$. Moreover, $O'' \subseteq \overline{O'}$ and $O' \subseteq \overline{O}$ imply $O'' \subseteq \overline{O'} \subseteq \overline{O}$. Finally, let $O' \subseteq \overline{O}$ and $O \subseteq \overline{O'}$, hence we have $\overline{O'} \subseteq \overline{O} \subseteq \overline{O'}$, and both $O, O' \subseteq \overline{O} = \overline{O'}$ being open and dense implies that $O \cap O' \neq \emptyset$, thus $O = O'$.

The existence of $\preceq$-minimal orbits follows from a) by induction on dimension. $\sharp$

Example. The natural action α of $\mathbf{GL}_n$ on $\mathbb{K}^n$, for $n \in \mathbb{N}_0$, is morphical: We have $\alpha^* \colon \mathbb{K}[X_1, \ldots, X_n] \to \mathbb{K}[X_1, \ldots, X_n] \otimes_{\mathbb{K}} \mathbb{K}[X_{11}, \ldots, X_{nn}]_{\det} \colon X_j \mapsto \sum_{i=1}^n X_i \otimes X_{ij}$; for $n = 1$ we recover the right translation action of $\mathbf{G}_m$ on $\mathbf{G}_a$.

Since $\mathbf{GL}_n$ acts transitively on the non-zero vectors in $\mathbb{K}^n$, for $n \in \mathbb{N}$, we get the orbits $O_0 := \{0\}$ and $O_1 := \mathbb{K}^n \setminus \{0\}$, where O_0 is closed of dimension 0, and O_1 is open of dimension n. Since $O_0 \subseteq \overline{O_1} = \mathbb{K}^n$ we conclude that $O_0 \preceq O_1$.

(3.2) Proposition. Let $\mathbf{G}$ be an affine algebraic group, let V be a $\mathbf{G}$-variety, and let $O, O' \subseteq V$ be $\mathbf{G}$-orbits. Moreover, let $\varphi \colon \mathbb{K} \to V$ be a morphism such that there is $\emptyset \neq U \subseteq \mathbb{K}$ open fulfilling $\varphi(U) \subseteq O$, and $\varphi(a) \in O'$ for some $a \in \mathbb{K}$. Then we have $O' \subseteq \overline{O}$, that is $O' \preceq O$.

Proof. Recall that for a continuous map $f \colon X \to Y$ between topological spaces, and a subset $S \subseteq X$, we have $f(\overline{S}) \subseteq \overline{f(S)}$: Indeed, we have $S \subseteq f^{-1}(f(S)) \subseteq f^{-1}(\overline{f(S)})$, where the latter is closed in X, implying $\overline{S} \subseteq f^{-1}(\overline{f(S)})$.

Letting $\mathbf{G}$ act via α, we get the morphism $\beta := (\varphi \times \mathrm{id})\alpha \colon \mathbb{K} \times \mathbf{G} \to V \times \mathbf{G} \to V$. Writing $\mathbf{G} = \coprod_{g \in \mathbf{G}^\circ \backslash \mathbf{G}} \mathbf{G}^\circ g$, we get $U \times \mathbf{G} = \coprod_{g \in \mathbf{G}^\circ \backslash \mathbf{G}} (U \times \mathbf{G}^\circ g) \subseteq \coprod_{g \in \mathbf{G}^\circ \backslash \mathbf{G}} (\mathbb{K} \times \mathbf{G}^\circ g) = \mathbb{K} \times \mathbf{G}$. The irreducibility of $\mathbb{K} \times \mathbf{G}^\circ$ implies that $U \times \mathbf{G}^\circ g \subseteq \mathbb{K} \times \mathbf{G}^\circ g$ is dense, hence $U \times \mathbf{G} \subseteq \mathbb{K} \times \mathbf{G}$ is dense. Hence we have $O' = \varphi(a)\mathbf{G} = \beta(\{a\} \times \mathbf{G}) \subseteq \beta(\mathbb{K} \times \mathbf{G}) = \beta(\overline{U \times \mathbf{G}}) \subseteq \overline{\beta(U \times \mathbf{G})} = \overline{\varphi(U)\mathbf{G}} = \overline{O}$. $\sharp$

Most often, this is applied for $U := \mathbb{K} \setminus \{a\}$, in which case, by abuse of notation, we also write $\lim_{t \to a} \varphi(t) := \varphi(a) \in V$.

(3.3) Proposition. Let $\mathbf{G}$ be an affine algebraic group, let V be a $\mathbf{G}$-variety, and let $x \in V$. Then $x\mathbf{G}$ consists of finitely many $\mathbf{G}^\circ$-orbits, all of which are irreducible, and open and closed in $x\mathbf{G}$. Moreover, the closures of the latter constitute are precisely the irreducible components of $\overline{x\mathbf{G}}$, in particular entailing that $\overline{x\mathbf{G}}$ is equidimensional such that $\dim(\overline{x\mathbf{G}}) = \dim(\overline{x\mathbf{G}^\circ})$, Finally, we have $\dim(C_{\mathbf{G}}(x)) = \dim(C_{\mathbf{G}^\circ}(x))$, and we have the orbit-stabiliser dimension formula $\dim(\mathbf{G}) = \dim(C_{\mathbf{G}}(x)) + \dim(\overline{x\mathbf{G}})$.

Proof. Since $\mathbf{G}^\circ \trianglelefteq \mathbf{G}$ has finite index, we have $x\mathbf{G} = \coprod_{i=1}^r xg_i\mathbf{G}^\circ$, for some $g_1, \ldots, g_r \in \mathbf{G}$ and $r \in \mathbb{N}$, where the $xg_i\mathbf{G}^\circ = x\mathbf{G}^\circ \cdot g_i$ are $\mathbf{G}^\circ$-invariant, pairwise isomorphic irreducible subsets. Hence we get $\overline{x\mathbf{G}} = \bigcup_{i=1}^r \overline{xg_i\mathbf{G}^\circ}$, where the $\overline{xg_i\mathbf{G}^\circ} = \overline{x\mathbf{G}^\circ} \cdot g_i$ are $\mathbf{G}^\circ$-invariant, pairwise isomorphic irreducible closed subsets. Considering dimensions shows $xg_j\mathbf{G}^\circ \not\subseteq \overline{xg_i\mathbf{G}^\circ}$ whenever $j \neq i$, hence we get $\overline{xg_i\mathbf{G}^\circ} \cap xg_j\mathbf{G}^\circ = \emptyset$. This yields $\overline{xg_i\mathbf{G}^\circ} \cap x\mathbf{G} = xg_i\mathbf{G}^\circ$, showing that $xg_i\mathbf{G}^\circ \subseteq x\mathbf{G}$ is closed; since $\coprod_{j\neq i} xg_j\mathbf{G}^\circ \subseteq x\mathbf{G}$ is closed, we conclude that $xg_i\mathbf{G}^\circ \subseteq x\mathbf{G}$ is open as well. Moreover, this also shows that $\overline{x\mathbf{G}} = \bigcup_{i=1}^r \overline{xg_i\mathbf{G}^\circ}$ is irredundant, hence the $\overline{xg_i\mathbf{G}^\circ}$ are precisely the irreducible components of $\overline{x\mathbf{G}}$.

Moreover, since $C_{\mathbf{G}^\circ}(x) \leq C_{\mathbf{G}}(x)$ is a closed subgroup of finite index, we have $\dim(C_{\mathbf{G}}(x)) = \dim(C_{\mathbf{G}^\circ}(x))$. Hence to show the last assertion, we may assume that $\mathbf{G}$ is connected. Letting $\mathbf{G}$ act via α, then the orbit map $\alpha_x \colon \mathbf{G} \to \overline{x\mathbf{G}}$ is a dominant morphism between irreducible varieties. Hence there is $\emptyset \neq U \subseteq \overline{x\mathbf{G}}$ such that $U \subseteq x\mathbf{G}$, and such that $\dim(\alpha_x^{-1}(y)) = \dim(\mathbf{G}) - \dim(\overline{x\mathbf{G}})$ for $y \in U$. For any $y \in U$ we have $\alpha_x^{-1}(y) = \{h \in \mathbf{G}; xh = y\} = C_{\mathbf{G}}(x)g \subseteq \mathbf{G}$, where $g \in \mathbf{G}$ is fixed such that $y = xg$, implying $\dim(\alpha_x^{-1}(y)) = \dim(C_{\mathbf{G}}(x))$. ♯

(3.4) Example: Matrix equivalence. For $n \in \mathbb{N}_0$ let $\mathcal{M} := \mathbb{K}^{n\times n}$ and $\mathbf{G} := \mathbf{GL}_n$. Then $\mathbf{G}$ acts by conjugation on $\mathcal{M}$, that is via $\mathcal{M} \times \mathbf{G} \to \mathcal{M} \colon [A, T] \mapsto T^{-1}AT$. Matrices $A, B \in \mathcal{M}$ are called **equivalent** if they belong to the same $\mathbf{G}$-orbit; recall that this holds if and only if their Jordan normal forms coincide. Note that since $\mathbf{G} = \mathbf{Z}_n\mathbf{SL}_n$, where $\mathbf{Z}_n$ acts trivially on $\mathcal{M}$, the $\mathbf{G}$-orbits and the $\mathbf{SL}_n$-orbits on $\mathcal{M}$ coincide.

For $A \in \mathcal{M}$ let $\chi(A) := \det(XE_n - A) = X^n + \sum_{i=1}^n (-1)^i \epsilon_i(A) X^{n-i} \in \mathbb{K}[X]$ be the associated characteristic polynomial. Here, $\epsilon_i(A)$ coincides with the i-th elementary symmetric polynomial in the eigenvalues of A. Thus, in terms of the coordinate algebra $\mathbb{K}[X_{11}, \ldots, X_{nn}]$ of $\mathcal{M}$, we have $\epsilon_i \in \mathbb{K}[X_{11}, \ldots, X_{nn}]_i$, for $i \in \{1, \ldots, n\}$; in particular, we have $\epsilon_1 = \sum_{i=1}^n X_{ii}$ and $\epsilon_n = \det_n$.

This gives rise to the morphism $\epsilon \colon \mathcal{M} \to \mathbb{K}^n \colon A \mapsto [\epsilon_1(A), \ldots, \epsilon_n(A)]$. Moreover, we have the morphism $\gamma \colon \mathbb{K}^n \to \mathcal{M}$ mapping $x = [x_1, \ldots, x_n] \in \mathbb{K}^n$ to

$$\gamma(x) := \begin{bmatrix} . & 1 & & & & \\ & . & 1 & & & \\ & & . & 1 & & \\ & & & \ddots & \ddots & \\ & & & & . & 1 \\ (-1)^{n-1}x_n & (-1)^{n-2}x_{n-1} & & \ldots & -x_2 & x_1 \end{bmatrix} \in \mathcal{M};$$

note that in particular $J_n = J_n(0) := \gamma(0)$ is a Jordan block of size n with respect to the eigenvalue 0. From $\gamma(x)$ being a companion matrix, we infer $\chi(\gamma(x)) = X^n + \sum_{i=1}^n (-1)^i x_i X^{n-i}$, entailing that $\epsilon(\gamma(x)) = [x_1, \ldots, x_n]$. Hence we have $\gamma\epsilon = \mathrm{id}_{\mathbb{K}^n}$, in particular ϵ is surjective.

Since the characteristic polynomial of a matrix is invariant under base change, we conclude that ϵ is $\mathbf{G}$-invariant, that is constant on $\mathbf{G}$-orbits. Hence for $x \in \mathbb{K}^n$ the fibre $\epsilon^{-1}(x) \subseteq \mathcal{M}$ is a closed union of $\mathbf{G}$-orbits.

Since the eigenvalues of a matrix, together with their multiplicities, are uniquely determined by its characteristic polynomial, the fibre $\epsilon^{-1}(x)$ consists of only finitely many $\mathbf{G}$-orbits, being parametrised by the possible Jordan normal forms. In particular, any fibre $\epsilon^{-1}(x)$ contains a unique **semisimple** $\mathbf{G}$-orbit, that is a $\mathbf{G}$-orbit consisting of diagonalisable matrices.

Proposition. For any $\mathbf{G}$-orbit $\mathcal{O} \subseteq \mathcal{M}$ we have:
a) There is a unique semisimple $\mathbf{G}$-orbit contained in $\overline{\mathcal{O}}$.
b) The $\mathbf{G}$-orbit $\mathcal{O}$ is closed if and only if it is semisimple.

Proof. a) We first consider $A := aE_n + tJ_n \in \mathcal{M}$, for $a, t \in \mathbb{K}$. Hence we have $\chi(A) = (X - a)^n$, where A is semisimple if $t = 0$. Moreover, whenever $t \neq 0$ we have $\mathrm{rk}((A - aE_n)^i) = \mathrm{rk}(t^i J_n^i) = n - i$ for $i \in \{0, \ldots, n\}$. Thus in this case A is equivalent to a Jordan block of size n with respect to the eigenvalue a.

Now, considering Jordan normal forms shows that there is matrix $D + N \in \mathcal{O}$, where $D \in \mathcal{M}$ is a diagonal matrix, and $N = \bigoplus_{i=1}^{l} J_{\lambda_i} := \mathrm{diag}[J_{\lambda_1}, \ldots, J_{\lambda_l}] \in \mathcal{M}$, where $l \in \mathbb{N}_0$ and $\lambda_i \in \mathbb{N}$ such that $\sum_{i=1}^{l} \lambda_i = n$. This gives rise to the morphism $\varphi \colon \mathbb{K} \to \mathcal{M} \colon t \mapsto D + tN$. The above observation shows that $\varphi(t) \in \mathcal{O}$ whenenever $t \neq 0$, while the $\mathbf{G}$-orbit containing $\lim_{t \to 0} \varphi(t) = D$ is semisimple. Hence by (3.2) we conclude that the $\mathbf{G}$-orbit of D is contained in $\overline{\mathcal{O}}$.

To show uniqueness, let $\mathcal{V} := \epsilon^{-1}(\epsilon(\mathcal{O})) \subseteq \mathcal{M}$ be the fibre of ϵ containing $\mathcal{O}$. Since $\mathcal{V} \subseteq \mathcal{M}$ is closed we have $\overline{\mathcal{O}} \subseteq \mathcal{V}$, where we have already seen that $\mathcal{V}$ contains a unique semisimple $\mathbf{G}$-orbit.

b) Let $\mathcal{O}$ be closed, that is we have $\mathcal{O} = \overline{\mathcal{O}}$. By a) there is a semisimple $\mathbf{G}$-orbit contained in $\overline{\mathcal{O}}$, hence $\mathcal{O}$ is semisimple.

Conversely, let $\mathcal{O}$ be semisimple. Then by the closed orbit lemma there is a closed $\mathbf{G}$-orbit $\mathcal{O}_0$ contained in $\overline{\mathcal{O}}$, where we have just seen that $\mathcal{O}_0$ is semisimple. Hence we have $\mathcal{O} \cup \mathcal{O}_0 \subseteq \overline{\mathcal{O}}$, where by a) the latter contains a unique semisimple $\mathbf{G}$-orbit. Thus $\mathcal{O} = \mathcal{O}_0$ is closed. ♯

This facilitates a description of the $\mathbf{G}$-invariant regular maps $\varphi \colon \mathcal{M} \to \mathbb{K}$; recall that $\mathbf{G}$-invariance is equivalent to being constant on $\mathbf{G}$-orbits:

Firstly, any such φ is constant on the fibres of ϵ; hence the $\mathbf{G}$-orbits contained in one and the same fibre of ϵ cannot be separated by $\mathbf{G}$-invariant regular maps:

For $x \in \mathbb{K}^n$ let $\mathcal{V} := \epsilon^{-1}(x) \subseteq \mathcal{M}$ be the associated fibre, and let $\mathcal{O}_0 \subseteq \mathcal{V}$ be the unique semisimple $\mathbf{G}$-orbit. Then for any $\mathbf{G}$-orbit $\mathcal{O} \subseteq \mathcal{V}$ we have $\mathcal{O}_0 \subseteq \overline{\mathcal{O}}$. Recalling that $\varphi(\mathcal{O})$ is a singleton set, from $\mathcal{O} \subseteq \varphi^{-1}(\varphi(\mathcal{O}))$, where the latter is closed, we get $\mathcal{O}_0 \subseteq \overline{\mathcal{O}} \subseteq \varphi^{-1}(\varphi(\mathcal{O}))$. Hence we have $\varphi(\mathcal{O}) = \varphi(\mathcal{O}_0)$. ♯

Now, it follows that φ is a polynomial in $\{\epsilon_1, \ldots, \epsilon_n\}$: Since φ is constant on the fibres of ϵ, there is a map $\widetilde{\varphi} \colon \mathbb{K}^n \to \mathbb{K}$ such that $\varphi = \epsilon\widetilde{\varphi}$, where from $\widetilde{\varphi} = \mathrm{id}_{\mathbb{K}^n} \cdot \widetilde{\varphi} = \gamma\epsilon\widetilde{\varphi} = \gamma\varphi$ we infer that $\widetilde{\varphi}$ is regular.

(3.5) Dominance order on partitions. a) We will need some combinatorics of partitions, which we collect next: We consider the set $P(n)$ of partitions of $n \in \mathbb{N}_0$. For a partition $\lambda := [\lambda_1, \ldots, \lambda_l] \vdash n$ with $l \in \{0, \ldots, n\}$ parts we let $\lambda_i := 0$ for $i > l$, thus we may write $\lambda = [\lambda_1, \ldots, \lambda_n]$. Then $P(n)$ is partially ordered by **dominance** $\trianglelefteq$, where $\lambda = [\lambda_1, \ldots, \lambda_n] \vdash n$ is said to **dominate** $\mu = [\mu_1, \ldots, \mu_n] \vdash n$ if $\sum_{i=1}^{k} \mu_i \leq \sum_{i=1}^{k} \lambda_i$ for all $k \in \{0, \ldots, n\}$.

It is immediate that reflexivity and transitivity hold. Moreover, from $\mu \trianglelefteq \lambda$ and $\lambda \trianglelefteq \mu$ we get $\sum_{i=1}^{k} \mu_i = \sum_{i=1}^{k} \lambda_i$ for $k \in \{1, \ldots, n\}$, which successively entails

$\mu_i = \lambda_i$ for $k \in \{1, \ldots, n\}$. Hence we have antisymmetry as well, showing that dominance $\trianglelefteq$ indeed is a partial order.

b) We describe the associated covering relation: Given $\mu \vdash n$, we have $\mu \lessdot \lambda$, that is $\mu \lhd \nu \trianglelefteq \lambda$ already implies $\nu = \lambda$, if and only if

$$\lambda = [\mu_1, \ldots, \mu_{r-1}, \mu_r + 1, \mu_{r+1}, \ldots, \mu_{s-1}, \mu_s - 1, \mu_{s+1}, \ldots, \mu_n],$$

where $1 \leq r < s \leq n$ such that $\mu_s > \mu_{s+1}$, and $\mu_{r-1} > \mu_r$ if $r > 1$, and such that either $s = r + 1$, or $s > r + 1$ and $\mu_r = \mu_s$:

If $\mu \lessdot \lambda$, then let $r := \min\{i \in \{1, \ldots, n\}; \mu_i \neq \lambda_i\}$ and $s := \min\{k \in \{r + 1, \ldots, n\}; \sum_{i=1}^{k} \mu_i = \sum_{i=1}^{k} \lambda_i\}$, thus $1 \leq r < s \leq n$. Hence we have $\mu_r < \lambda_r$, and $\lambda_r \leq \lambda_{r-1} = \mu_{r-1}$ if $r > 1$, as well as $\mu_s > \lambda_s \geq \lambda_{s+1} \geq \mu_{s+1}$. This yields

$$\mu \lhd \nu := [\mu_1, \ldots, \mu_{r-1}, \mu_r + 1, \mu_{r+1}, \ldots, \mu_{s-1}, \mu_s - 1, \mu_{s+1}, \ldots, \mu_n] \trianglelefteq \lambda,$$

hence $\nu = \lambda$. It remains to show $\mu_r = \mu_s$ whenever $s > r + 1$: Assume to the contrary that $\mu_r > \mu_s$, and let $r < t := \min\{i \in \{r + 1, \ldots, s\}; \mu_{i-1} > \mu_i\} \leq s$. If $t = s$ then

$$\mu \lhd [\mu_1, \ldots, \mu_{r-1}, \mu_r + 1, \mu_{r+1}, \ldots, \mu_{s-2}, \mu_{s-1} - 1, \mu_s, \ldots, \mu_n] \lhd \nu = \lambda,$$

while if $t < s$ then

$$\mu \lhd [\mu_1, \ldots, \mu_r, \ldots, \mu_{t-1}, \mu_t + 1, \mu_{t+1}, \ldots, \mu_{s-1}, \mu_s - 1, \mu_{s+1}, \ldots, \mu_n] \lhd \nu = \lambda,$$

a contradiction.

Let conversely λ be as asserted, and let $\nu = [\nu_1, \ldots, \nu_n] \vdash n$ such that $\mu \lhd \nu \trianglelefteq \lambda$. Hence for $i \notin \{r, \ldots, s\}$ we have $\nu_i = \mu_i$. Thus if $s = r+1$ we conclude $\nu_r = \mu_r + 1$ and $\nu_{r+1} = \mu_{r+1} - 1$, thus $\nu = \lambda$. If $s > r + 1$ and hence $\mu_r = \mu_s$, then there are $r \leq r' < s' \leq s$ such that $\nu_i = \mu_i$ for $i \notin \{r', s'\}$ as well as $\nu_{r'} = \mu_{r'} + 1$ and $\nu_{s'} = \mu_{s'} - 1$. Since $\mu_{r'} = \nu_{r'} - 1 \leq \nu_{r'-1} - 1 = \mu_{r'-1} - 1 < \mu_{r'-1}$, whenever $r' > 1$, and $\mu_{s'} = \nu_{s'} + 1 \geq \nu_{s'+1} + 1 = \mu_{s'+1} + 1 > \mu_{s'+1}$, this implies $r' = r$ and $s' = s$, hence $\nu = \lambda$ in this case as well. ♯

For example, we have $\lambda \trianglelefteq [n]$ and $[1^n] \trianglelefteq \lambda$ for all $\lambda \vdash n$ and $n \in \mathbb{N}_0$, and $[n-1, 1] \lessdot [n]$ for $n \geq 2$, and $[1^3] \lessdot [2, 1] \lessdot [3]$ and $[1^4] \lessdot [2, 1^2] \lessdot [2^2] \lessdot [3, 1] \lessdot [4]$ and $[1^4] \lessdot [2, 1^3] \lessdot [2^2, 1] \lessdot [3, 2] \lessdot [4, 1] \lessdot [5]$, and

$$[1^6] \lessdot [2, 1^4] \lessdot [2^2, 1^2] \lessdot \{[3, 1^3], [2^3]\} \lessdot [3, 2, 1] \lessdot \{[4, 1^2], [3^2]\} \lessdot [4, 2] \lessdot [5, 1] \lessdot [6],$$

where $\{[3, 1^3], [2^3]\}$ and $\{[4, 1^2], [3^2]\}$ are non-comparable.

c) Identifying a partition $\lambda \vdash n$ having l parts with the set $\{[i, j] \in \mathbb{N}^2; i \in \{1, \ldots, l\}, j \in \{1, \ldots, \lambda_i\}\}$, it can be depicted by a **Young diagram**, that is rectangular array of boxes consisting of l rows, where row i contains λ_i boxes. Using this, the **conjugate** partition $\lambda' \vdash n$ is defined as the partition belonging to the diagram obtained by reflecting the original one along its main diagonal.

Formally, if $\lambda = [\lambda_1, \ldots, \lambda_n] \vdash n$ then letting $\lambda'_i := |\{j \in \mathbb{N}; \lambda_j \geq i\}| \in \mathbb{N}_0$ for $i \in \mathbb{N}$, we have $\lambda'_1 \geq \cdots \geq \lambda'_n \geq 0$ and

$$\sum_{i=1}^{n} \lambda'_i = \sum_{j=1}^{n} |\{i \in \{1, \ldots, n\}; i \leq \lambda_j\}| = \sum_{j=1}^{n} \lambda_j = n,$$

hence indeed $\lambda' := [\lambda'_1, \ldots, \lambda'_n] \vdash n$ is a partition of n.

Moreover, conjugating twice yields $\lambda'' \vdash n$, where $\lambda''_i = |\{j \in \mathbb{N}; \lambda'_j \geq i\}| = \big|\{j \in \mathbb{N}; |\{k \in \mathbb{N}; \lambda_k \geq j\}| \geq i\}\big| = \big|\{j \in \mathbb{N}; \{1, \ldots, i\} \subseteq \{k \in \mathbb{N}; \lambda_k \geq j\}\}\big| = |\{j \in \mathbb{N}; \lambda_i \geq j\}| = |\{1, \ldots, \lambda_i\}| = \lambda_i$, that is we indeed have $\lambda'' = \lambda$.

Recalling that in terms of multiplicities we also write $\lambda = [n^{a_n}, \ldots, 1^{a_1}]$, where $a_i = a_i(\lambda) := |\{j \in \{1, \ldots, l\}; \lambda_j = i\}| \in \mathbb{N}_0$, the fastest way to compute conjugate partitions is given as follows: Writing $\lambda' = [n^{a'_n}, \ldots, 1^{a'_1}] \vdash n$, for $i \in \{1, \ldots, n\}$ we have

$$\begin{aligned} a'_i &= |\{j \in \mathbb{N}; \lambda'_j = i\}| \\ &= \big|\{j \in \mathbb{N}; |\{k \in \mathbb{N}; \lambda_k \geq j\}| = i\}\big| \\ &= \big|\{j \in \mathbb{N}; \{k \in \mathbb{N}; \lambda_k \geq j\} = \{1, \ldots, i\}\}\big| \\ &= |\{j \in \mathbb{N}; \lambda_i \geq j, \lambda_{i+1} < j\}| \\ &= |\{\lambda_{i+1} + 1, \ldots, \lambda_i\}| \\ &= \lambda_i - \lambda_{i+1}. \end{aligned}$$

For example, we have $[n]' = [1^n]$ for $n \in \mathbb{N}_0$, and $[n-1, 1]' = [2, 1^{n-2}]$ for $n \geq 2$, as well as $[2^2]' = [2^2]$ and $[3, 2]' = [2^2, 1]$ and $[3, 1^2]' = [3, 1^2]$.

d) Then we have $\mu \trianglelefteq \lambda$ if and only if $\lambda' \trianglelefteq \mu'$, in other words conjugating partitions inverts the dominance partial order:

To show this, it suffices to assume to the contrary that $\mu \trianglelefteq \lambda$ but $\lambda' \not\trianglelefteq \mu'$. Then for some $k \in \mathbb{N}$ we have $\sum_{i=1}^{j} \lambda'_i \leq \sum_{i=1}^{j} \mu'_i$ for $j \in \{1, \ldots, k-1\}$, and $\sum_{i=1}^{k} \lambda'_i > \sum_{i=1}^{k} \mu'_i$. Hence we have $\lambda'_k > \mu'_k$ and $\sum_{i=k+1}^{n} \lambda'_i < \sum_{i=k+1}^{n} \mu'_i$. Now we have

$$\sum_{i=k+1}^{n} \lambda'_i = \sum_{i=k+1}^{n} |\{j \in \mathbb{N}; i \leq \lambda_j\}| = \sum_{j=1}^{\lambda'_k} (\lambda_j - k),$$

and similarly we get $\sum_{i=k+1}^{n} \mu'_i = \sum_{j=1}^{\mu'_k} (\mu_j - k)$; note that $\lambda_j \geq k$ for $j \in \{1, \ldots, \lambda'_k\}$. This implies that $\sum_{j=1}^{\mu'_k} (\mu_j - k) > \sum_{j=1}^{\lambda'_k} (\lambda_j - k) \geq \sum_{j=1}^{\mu'_k} (\lambda_j - k)$, thus we have $\mu \not\trianglelefteq \lambda$, a contradiction. ♯

(3.6) Example: Nilpotent matrices. a) We keep the notation of (3.4) and (3.5), and let $\mathcal{N} := \epsilon^{-1}(0) = \{A \in \mathcal{M}; \chi(A) = X^n\} = \{A \in \mathcal{M}; A^n = 0\} \subseteq \mathcal{M}$ be the **nilpotent variety**. Hence the **G**-orbits in $\mathcal{N}$ are parametrised by the Jordan normal forms with respect to the eigenvalue 0, that is block diagonal matrices $\bigoplus_{i=1}^{l} J_{\lambda_i}$, where $l \in \mathbb{N}_0$ and $\lambda_i \in \mathbb{N}$ such that $\sum_{i=1}^{l} \lambda_i = n$.

Assuming that $\lambda_1 \geq \cdots \geq \lambda_l$, we infer that the Jordan normal forms in turn are parametrised by the partitions $\lambda := [\lambda_1, \ldots, \lambda_l] \vdash n$. Thus $\mathcal{N}$ can be written as a disjoint union of **G**-orbits as $\mathcal{N} = \coprod_{\lambda \vdash n} \mathcal{N}_\lambda$, where $\mathcal{N}_{[\lambda_1, \ldots, \lambda_l]}$ contains $\bigoplus_{i=1}^{l} J_{\lambda_i}$. The orbit closure relation on $\mathcal{N}$ induces a partial order on $P(n)$. We are going to show that the latter coincides with the dominance partial order $\trianglelefteq$ on $P(n)$, and thus has a purely combinatorial description:

i) We show that $\overline{\mathcal{N}}_\lambda \subseteq \mathcal{N}_{\trianglelefteq \lambda} := \coprod_{\mu \trianglelefteq \lambda} \mathcal{N}_\mu \subseteq \mathcal{N}$: For a Jordan block $J_i \in \mathbb{C}^{i \times i}$, for some $i \in \mathbb{N}_0$, we have $\mathrm{rk}(J_i^k) = i - k$ for $k \in \{0, \ldots, i\}$. Thus for $A \in \mathcal{N}_\lambda$, where

$\lambda = [n^{a_n}, \ldots, 1^{a_1}] \vdash n$, we have $\mathrm{rk}(A^k) = \sum_{i=k+1}^{n} (i-k)a_i = \sum_{i=k+1}^{n} \sum_{j=i}^{n} a_j = \sum_{i=k+1}^{n} \sum_{j=i}^{n} (\lambda'_i - \lambda'_{i+1}) = \sum_{i=k+1}^{n} \lambda'_i = n - \sum_{i=1}^{k} \lambda'_i$, for $k \in \{0, \ldots, n\}$. Hence λ and thus $\mathcal{N}_\lambda$ are uniquely determined by the rank sequence $[\mathrm{rk}(A^k) \in \mathbb{N}_0; k \in \{0, \ldots, n\}]$; note that we have $\mathrm{rk}(A^0) = n$ and $\mathrm{rk}(A^n) = 0$ anyway.

Now let $\mu \vdash n$ and $B \in \mathcal{N}_\mu$. Then we have $\mu \trianglelefteq \lambda$ if and only if $\lambda' \trianglelefteq \mu'$, hence in terms of matrices this holds if and only if $\mathrm{rk}(A^k) \geq \mathrm{rk}(B^k)$ for all $k \in \{0, \ldots, n\}$. Thus we have $B \in \mathcal{N}_{\trianglelefteq\lambda}$ if and only if $\mathrm{rk}(B^k) \leq n - \sum_{i=1}^{k} \lambda'_i$ for all $k \in \{0, \ldots, n\}$. In other words, letting $\mathcal{N}_{\leq k} := \{C \in \mathcal{N}; \mathrm{rk}(C) \leq k\} \subseteq \mathcal{N}$, we have $B \in \mathcal{N}_{\trianglelefteq\lambda}$ if and only if $B^k \in \mathcal{N}_{\leq (n - \sum_{i=1}^{k} \lambda'_i)}$ for all $k \in \{0, \ldots, n\}$.

Recall that $\mathrm{rk}(C) \in \mathbb{N}_0$ equals the smallest $k \in \mathbb{N}_0$ such that all $\big((k+1) \times (k+1)\big)$-minors of $C \in \mathcal{M}$ vanish; in this case all larger minors of C vanish as well. Hence we conclude that $\mathcal{N}_{\leq k} \subseteq \mathcal{N}$ is closed, for $k \in \mathbb{N}_0$. Thus, since taking matrix powers is a morphism, we conclude that $\mathcal{N}_{\trianglelefteq\lambda} \subseteq \mathcal{N}$ is closed as well. From this, since $\mathcal{N}_{\trianglelefteq\lambda}$ contains $\mathcal{N}_\lambda$, we infer that $\overline{\mathcal{N}}_\lambda \subseteq \mathcal{N}_{\trianglelefteq\lambda}$.

ii) For the converse $\mathcal{N}_{\trianglelefteq\lambda} \subseteq \overline{\mathcal{N}}_\lambda$, we have to show that $\mu \trianglelefteq \lambda$ implies $\mathcal{N}_\mu \subseteq \overline{\mathcal{N}}_\lambda$. In order to do so, by the transitivity of the closure relation we may assume that

$$\mu := [\lambda_1, \ldots, \lambda_{r-1}, \lambda_r - 1, \lambda_{r+1}, \ldots, \lambda_{s-1}, \lambda_s + 1, \lambda_{s+1}, \ldots, \lambda_n] \lessdot \lambda,$$

for some $1 \leq r < s \leq n$. Letting $a := \lambda_r$ and $b := \lambda_s$, hence $a-1 \geq b+1 \geq 1$, we have $J_a \oplus J_b \oplus N \in \mathcal{N}_\lambda$ and $J_{a-1} \oplus J_{b+1} \oplus N \in \mathcal{N}_\mu$, where $N \in \mathbb{K}^{(n-a-b)\times(n-a-b)}$. Hence we may assume that $\lambda = [a, b] \vdash n$ and $\mu = [a-1, b+1] \vdash n$, and let

$$N_\epsilon = \left[\begin{array}{ccccc|ccccc} . & \epsilon & & & & 1 & & & & \\ & . & 1 & & & & & & & \\ & & \ddots & \ddots & & & & & & \\ & & & . & 1 & & & & & \\ & & & & . & & & & & \\ \hline & & & & & . & 1 & & & \\ & & & & & & \ddots & \ddots & & \\ & & & & & & & . & 1 & \\ & & & & & & & & . & \end{array}\right] \in \mathcal{N},$$

for $\epsilon \in \mathbb{K}$, where the upper left and lower right hand corners have size $a \times a$ and $b \times b$, respectively. We show that $N_\epsilon \in \mathcal{N}_\lambda$ if $\epsilon \neq 0$, while $\lim_{\epsilon \to 0} N_\epsilon = N_0 \in \mathcal{N}_\mu$:

If $b = 0$, then for $\epsilon \neq 0$ the unit vector $e_1 \in \mathbb{K}^n$ has minimum polynomial $X^a \in \mathbb{K}[X]$ with respect to N_ϵ, hence N_ϵ has Jordan normal form J_a; moreover, N_0 has Jordan normal form $J_{a-1} \oplus J_1$. Hence we may assume that $b > 0$.

If $\epsilon \neq 0$ then, since $a > b$, the unit vector $e_1 \in \mathbb{K}^n$ has minimum polynomial $X^a \in \mathbb{K}[X]$ with respect to N_ϵ. Moreover, the unit vector $e_{a+1} \in \mathbb{K}^n$ has minimum polynomial $X^b \in \mathbb{K}[X]$. From $\langle e_{a+1} \rangle_{N_\epsilon} = \langle e_{a+1}, \ldots, e_n \rangle_{\mathbb{K}}$ and $\langle e_1 \rangle_{N_\epsilon} \cap \langle e_{a+1}, \ldots, e_n \rangle_{\mathbb{K}} = \{0\}$ we conclude $\mathbb{K}^n = \langle e_1 \rangle_{N_\epsilon} \oplus \langle e_{a+1} \rangle_{N_\epsilon}$, hence N_ϵ has Jordan normal form $J_a \oplus J_b$.

If $\epsilon = 0$ then $e_2 \in \mathbb{K}^n$ and $e_1 \in \mathbb{K}^n$ have minimum polynomials $X^{a-1} \in \mathbb{K}[X]$ and $X^{b+1} \in \mathbb{K}[X]$, respectively, with respect to N_0. From $\langle e_2 \rangle_{N_0} = \langle e_2, \ldots, e_a \rangle_{\mathbb{K}}$ and $\langle e_1 \rangle_{N_0} = \langle e_1, e_{a+1}, \ldots, e_n \rangle_{\mathbb{K}}$ we conclude $\mathbb{K}^n = \langle e_2 \rangle_{N_0} \oplus \langle e_1 \rangle_{N_0}$, hence N_0 has Jordan normal form $J_{a-1} \oplus J_{b+1}$. ♯

b) We show that $\mathcal{N}$ is irreducible such that $\dim(\mathcal{N}) = n(n-1)$:

We may assume that $n \geq 1$. We have $\lambda \trianglelefteq [n]$ for all $\lambda \vdash n$. Hence we have $\overline{\mathcal{N}_{[n]}} = \mathcal{N}$, thus $J_n \cdot \mathbf{G} = \mathcal{N}_{[n]} \subseteq \mathcal{N}$ is open and dense; the elements of $\mathcal{N}_{[n]}$ are called **regular nilpotent**. In particular, since $\mathbf{G}$ is connected, $\mathcal{N}_{[n]}$ is irreducible, and thus $\mathcal{N}$ is as well.

Moreover, we have $\dim(\mathcal{N}) = \dim(\mathbf{G}) - \dim(C_{\mathbf{G}}(J_n)) = n^2 - \dim(C_{\mathbf{G}}(J_n))$. Let $\mathcal{C} := C_{\mathcal{M}}(J_n) := \{A \in \mathcal{M}; AJ_n = J_nA\}$, which is both a closed subset and a $\mathbb{K}$-subalgebra of $\mathcal{M}$, thus $\dim(\mathcal{C}) = \dim_{\mathbb{K}}(\mathcal{C})$. Since $C_{\mathbf{G}}(J_n) = \mathcal{C} \cap \mathbf{G} \subseteq \mathcal{C}$ is open, we conclude that $\dim(C_{\mathbf{G}}(J_n)) = \dim(\mathcal{C})$. Thus we have $\dim(\mathcal{N}) = n^2 - \dim_{\mathbb{K}}(\mathcal{C})$, and it remains to be shown that $\dim_{\mathbb{K}}(\mathcal{C}) = n$:

Let $\mathcal{A} := \mathbb{K}[J_n] \cong \mathbb{K}[X]/\langle X^n \rangle$ be the $\mathbb{K}$-subalgebra of $\mathcal{M}$ generated by J_n; recall that J_n has minimum polynomial $X^n \in \mathbb{K}[X]$. Since $\langle J_n \rangle \trianglelefteq \mathcal{A}$ is nilpotent, we conclude that $\mathcal{A}/\mathrm{rad}(\mathcal{A}) = \mathcal{A}/\langle J_n \rangle \cong \mathbb{K}$, so that the unique simple $\mathcal{A}$-module is given by $J_n \mapsto 0 \in \mathbb{K}^{1\times 1}$. Now the $\mathcal{A}$-module $\mathbb{K}^n$ is generated as an $\mathcal{A}$-module by the unit vector e_1, and since $\dim_{\mathbb{K}}(\mathcal{A}) = n$ we conclude that $\mathbb{K}^n$ is isomorphic to the regular $\mathcal{A}$-module, the latter being the projective cover of the unique simple $\mathcal{A}$-module. Thus we have $\mathcal{C} = \mathrm{End}_{\mathcal{A}}(\mathbb{K}^n) \cong \mathcal{A}^{\circ}$, the opposite $\mathbb{K}$-algebra of $\mathcal{A}$, and since $\mathcal{A}$ is commutative we get $\mathcal{C} = \mathcal{A}$. (Indeed, the latter being a local $\mathbb{K}$-algebra, we have $C_{\mathbf{G}}(J_n) = \mathcal{C} \setminus \mathrm{rad}(\mathcal{C})$, thus choosing the $\mathbb{K}$-basis $\{E_n, J_n, \ldots, J_n^{n-1}\} \subseteq \mathcal{C}$ we get $C_{\mathbf{G}}(J_n) = \{\sum_{i=0}^{n-1} a_i J_n^i \in \mathcal{M}; a_i \in \mathbb{K}, a_0 \neq 0\}$.) $\sharp$

4 Representations

(4.1) Representations. Let $\mathbf{G}$ be an affine algebraic group, and let $\delta\colon \mathbf{G} \to \mathbf{GL}_n\colon g \mapsto [g_{ij}(g)]_{ij}$, where $n \in \mathbb{N}_0$, be a **(matrix) representation**, that is a group homomorphism. Hence we get an action of $\mathbf{G}$ on the affine variety $V := \mathbb{K}^n$ by $\mathbb{K}$-linear maps by letting $\alpha\colon V \times \mathbf{G} \to V\colon [x, g] \mapsto x\delta(g)$.

The map δ is a morphism of varieties if and only if $\delta^*\colon \mathbb{K}[X_{11}, \ldots, X_{nn}]_{\det} \to \mathbb{K}[\mathbf{G}]\colon X_{ij} \mapsto g_{ij}$ defines a homomorphism of $\mathbb{K}$-algebras, which in turn holds if and only if the coordinate functions g_{ij} are regular maps, for $i, j \in \{1, \ldots, n\}$; note that $\det_n(g_{11}, \ldots, g_{nn}) \neq 0$ anyway. Recall that in this case δ is called an **algebraic** or **rational representation** of $\mathbf{G}$ of **degree** n. For example, for $n = 1$, letting $\mathbf{G} \to \mathbf{G}_m\colon g \mapsto 1$ defines the **trivial** representation.

Moreover, let $\mathbb{K}[V] = \mathbb{K}[\mathcal{X}] = \bigoplus_{d \in \mathbb{N}_0} \mathbb{K}[\mathcal{X}]_d = \bigoplus_{d \in \mathbb{N}_0} \mathbb{K}[V]_d$ be the coordinate algebra of V, where $\mathcal{X} := \{X_1, \ldots, X_n\}$. In order to determine the comorphism associated with α, for $i \in \{1, \ldots, n\}$ we observe $X_i([x_1, \ldots, x_n] \cdot \delta(g)) = X_i([\sum_{j=1}^n x_j g_{jk}(g)]_k) = \sum_{j=1}^n x_j g_{ji}(g) = (\sum_{j=1}^n X_j \otimes g_{ji})(x_1, \ldots, x_n; g)$, for all $x_1, \ldots, x_n \in \mathbb{K}$ and $g \in \mathbf{G}$. This entails $\alpha^*\colon \mathbb{K}[V] \to \mathbb{K}[V] \otimes_{\mathbb{K}} \mathbb{K}[\mathbf{G}]\colon X_i \mapsto \sum_{j=1}^n X_j \otimes g_{ji}$. Hence we conclude that δ is algebraic if and only if α is a morphical action. In this case V becomes a $\mathbf{G}$-variety, on which $\mathbf{G}$ acts by $\mathbb{K}$-linear maps, and as such is called a **G-module**.

(4.2) Homomorphisms. a) Let $\mathbf{G}$ be an affine algebraic group, and let V and W be $\mathbf{G}$-modules, with associated algebraic representations $\delta\colon \mathbf{G} \to \mathbf{GL}_n$ and $\delta'\colon \mathbf{G} \to \mathbf{GL}_m$, respectively. Then a $\mathbb{K}$-linear map $\varphi\colon V \to W$ which is $\mathbf{G}$-equivariant in the sense of $\varphi(x\delta(g)) = \varphi(x)\delta'(g)$, for $x \in V$ and $g \in \mathbf{G}$, is called a **homomorphism** of $\mathbf{G}$-modules; in particular φ is a morphism of varieties.

The $\mathbf{G}$-modules V and W are called **isomorphic** if there is a bijective homomorphism $\varphi\colon V \to W$ of $\mathbf{G}$-modules; in this case, $\varphi^{-1}\colon W \to V$ is a homomorphism of $\mathbf{G}$-modules as well, and we write $V \cong W$. In terms of representations, this is equivalent to saying that there is $\alpha \in \mathbf{GL}_n$ such that $\delta'(g) = \alpha^{-1}\delta(g)\alpha$, for $g \in \mathbf{G}$; in this case δ and δ' are called **equivalent**.

Let $\mathrm{Hom}_{\mathbf{G}}(V,W)$ be the $\mathbb{K}$-vector space of all $\mathbf{G}$-equivariant $\mathbb{K}$-linear maps from V to W. Similarly, we get the $\mathbb{K}$-vector space $\mathrm{End}_{\mathbf{G}}(V) := \mathrm{Hom}_{\mathbf{G}}(V,V)$ of all $\mathbf{G}$-equivariant $\mathbb{K}$-endomorphisms of V, and the (affine algebraic) group $\mathrm{Aut}_{\mathbf{G}}(V) := \mathrm{End}_{\mathbf{G}}(V) \cap \mathbf{GL}(V))$ of all $\mathbf{G}$-equivariant $\mathbb{K}$-automorphisms of V.

b) Let $U \leq V$ be a $\mathbf{G}$-invariant $\mathbb{K}$-subspace; for example, $U = \{0\}$ and $U = V$. Then, by choosing a $\mathbb{K}$-basis of V containing a $\mathbb{K}$-basis of U, and going over to an equivalent representation, we get algebraic representations $\mathbf{G} \to \mathbf{GL}(U)$ and $\mathbf{G} \to \mathbf{GL}(V/U)$ on the $\mathbb{K}$-subspace U of V, and the associated quotient $\mathbb{K}$-vector space V/U, respectively, such that the natural maps $\iota_U\colon U \to V$ and $\nu_U\colon V \to V/U$ are $\mathbf{G}$-equivariant. Note that the **G-submodule** U and the **quotient G-module** V/U are only determined up to $\mathbf{G}$-isomorphism.

In particular, given $\varphi \in \mathrm{Hom}_{\mathbf{G}}(V,W)$, the kernel $\ker(\varphi) \leq V$ and the image $\varphi(V) \leq W$ are $\mathbf{G}$-submodules, such that $V/\ker(\varphi) \cong \varphi(V)$ as $\mathbf{G}$-modules.

c) If $\{U_i \leq V; i \in \mathcal{I}\}$ are $\mathbf{G}$-submodules, where $\mathcal{I} \neq \emptyset$ is an index set, then so are their intersection $\bigcap_{i\in\mathcal{I}} U_i \leq V$ and their sum $\sum_{i\in\mathcal{I}} U_i \leq V$.

If $S \subseteq V$ is a subset, then $\langle S\rangle_{\mathbf{G}} := \bigcap\{U \leq V\ \mathbf{G}\text{-submodule}; S \subseteq U\} \leq V$ is called the $\mathbf{G}$-submodule **generated** by S; note that since $S \subseteq V$ the intersection is taken over a non-empty set. In particular, if $\{U_i \leq V; i \in \mathcal{I}\}$ are $\mathbf{G}$-submodules then we have $\langle U_i; i \in \mathcal{I}\rangle_{\mathbf{G}} = \sum_{i\in\mathcal{I}} U_i$.

(4.3) Constructions. a) Let $\mathbf{G}$ be an affine algebraic group, and let V and W be $\mathbf{G}$-modules, with associated algebraic representations $\delta\colon \mathbf{G} \to \mathbf{GL}_n$ and $\delta'\colon \mathbf{G} \to \mathbf{GL}_m$, respectively. Then $V \oplus W$ becomes a $\mathbf{G}$-module with respect to the algebraic representation $\mathbf{G} \to \mathbf{GL}_{n+m}\colon g \mapsto \delta(g) \oplus \delta'(g) := \mathrm{diag}[\delta(g), \delta'(g)]$.

b) Let $\mathbf{H}$ be an affine algebraic group, and let U be an $\mathbf{H}$-module, with associated algebraic representation $\epsilon\colon \mathbf{H} \to \mathbf{GL}_r$. Then $V \otimes_{\mathbb{K}} U$ becomes a $(\mathbf{G} \times \mathbf{H})$-module, with respect to the algebraic representation $\mathbf{G} \times \mathbf{H} \to \mathbf{GL}_{nr}\colon [g,h] \mapsto \delta(g) \otimes \epsilon(h)$, the latter denoting the Kronecker product, having coordinate functions $[\delta(g) \otimes \epsilon(h)]_{ij,kl} = \delta(g)_{ik} \cdot \epsilon(h)_{jl}$, for $i,k \in \{1,\ldots,n\}$ and $j,l \in \{1,\ldots,r\}$.

In particular, $V \otimes_{\mathbb{K}} W$ becomes a $(\mathbf{G} \times \mathbf{G})$-module. Thus, restricting along the diagonal embedding $\mathbf{G} \to \mathbf{G} \times \mathbf{G}$ of algebraic groups, it becomes a $\mathbf{G}$-module with respect to the algebraic representation $\mathbf{G} \to \mathbf{GL}_{nm}\colon g \mapsto \delta(g) \otimes \delta'(g)$.

c) The dual $\mathbb{K}$-space $V^\vee := \mathrm{Hom}_{\mathbb{K}}(V,\mathbb{K})$ of V becomes a $\mathbf{G}$-module with respect to the $\mathbf{G}$-action such that $g \in \mathbf{G}$ maps $\lambda \in V^\vee$ to $\lambda^g\colon x \mapsto \lambda(xg^{-1})$. With respect to the $\mathbb{K}$-basis $B^\vee \subseteq V^\vee$ dual to the standard $\mathbb{K}$-basis $B \subseteq V$, the $\mathbf{G}$-action is given by the algebraic representation $\delta^\vee\colon \mathbf{G} \to \mathbf{GL}_n\colon g \mapsto \delta(g)^{-\mathrm{tr}}$; recall that inversion is an automorphism of $\mathbf{G}$ as a variety. The representation $\delta^\vee$ is called the representation **contragredient** to δ.

More generally, $\mathrm{Hom}_{\mathbb{K}}(V,U)$ becomes a $(\mathbf{G}\times\mathbf{H})$-module by letting $[g,h] \in \mathbf{G}\times\mathbf{H}$ act by mapping $\varphi \in \mathrm{Hom}_{\mathbb{K}}(V,U)$ to $\varphi^{[g,h]}\colon V \to U\colon x \mapsto x \cdot g^{-1}\varphi h$. In other words, in terms of the standard $\mathbb{K}$-bases $B \subseteq V$ and $C \subseteq U$, respectively, we get $_B(\varphi^{[g,h]})_C = \delta(g)^{-1} \cdot {_B\varphi_C} \cdot \epsilon(h)$, in particular showing that this indeed is an algebraic representation.

Employing the natural isomorphism $\mathrm{Hom}_{\mathbb{K}}(V,U) \cong V^\vee \otimes_{\mathbb{K}} U$ of $\mathbb{K}$-vector spaces, in terms of the standard $\mathbb{K}$-basis $B^\vee \otimes C \subseteq V^\vee \otimes_{\mathbb{K}} U$ the action of $[g,h] \in \mathbf{G} \times \mathbf{H}$ translates into the Kronecker product $\delta(g)^{-\mathrm{tr}} \otimes \epsilon(h)$. Thus in conclusion we have $\mathrm{Hom}_{\mathbb{K}}(V,U) \cong V^\vee \otimes_{\mathbb{K}} U$ as $(\mathbf{G} \times \mathbf{H})$-modules.

In particular, $\mathrm{Hom}_{\mathbb{K}}(V,W)$ becomes a $(\mathbf{G} \times \mathbf{G})$-module. Thus, restricting along the diagonal embedding $\mathbf{G} \to \mathbf{G} \times \mathbf{G}$ of algebraic groups, $\mathrm{Hom}_{\mathbb{K}}(V,W)$ becomes a $\mathbf{G}$-module, for which we have $\mathrm{Hom}_{\mathbb{K}}(V,W)^{\mathbf{G}} = \{\varphi \in \mathrm{Hom}_{\mathbb{K}}(V,W); \varphi^g = \varphi \text{ for all } g \in \mathbf{G}\} = \{\varphi \in \mathrm{Hom}_{\mathbb{K}}(V,W); g\varphi = \varphi g \text{ for all } g \in \mathbf{G}\} = \mathrm{Hom}_{\mathbf{G}}(V,W)$.

(4.4) Simple modules. a) Let $\mathbf{G}$ be an affine algebraic group, and let $V \neq \{0\}$ be a $\mathbf{G}$-module, with algebraic representation δ. Then V is called **simple**, and δ is called **irreducible**, if $\{0\}$ and V are the only $\mathbf{G}$-submodules of V.

The property of being simple is an invariant of the $\mathbf{G}$-isomorphism class of V. Hence let $\Sigma_{\mathbf{G}}$ be the set of $\mathbf{G}$-isomorphism classes of simple $\mathbf{G}$-modules; note that $\Sigma_{\mathbf{G}}$ is not necessarily finite.

Proposition: Schur's Lemma. Let V and W be simple $\mathbf{G}$-modules.
i) If $V \not\cong W$ then we have $\mathrm{Hom}_{\mathbf{G}}(V,W) = \{0\}$. **ii)** We have $\mathrm{End}_{\mathbf{G}}(V) = \mathbb{K} \cdot \mathrm{id}_V$.

Proof. i) Assume there is $0 \neq \varphi \in \mathrm{Hom}_K(V,W)$. Then $\{0\} \neq \varphi(V) \leq W$ is a $\mathbf{G}$-submodule, hence W being simple we have $\varphi(V) = W$. Moreover, $\ker(\varphi) < V$ is a proper $\mathbf{G}$-submodule, hence V being simple we have $\ker(\varphi) = \{0\}$. Hence φ is bijective, and thus a $\mathbf{G}$-isomorphism, a contradiction.

ii) Let φ be a $\mathbf{G}$-endomorphism of V. Since $\mathbb{K}$ is algebraically closed, φ has an eigenvalue $a \in \mathbb{K}$. Hence $\psi := \varphi - a \cdot \mathrm{id}_V$ is a $\mathbf{G}$-endomorphism, such that $\ker(\psi) \neq \{0\}$. Hence V being simple we have $\ker(\psi) = V$, that is $\varphi = a \cdot \mathrm{id}_V$. ♯

b) In order to give a characterisation of simple $\mathbf{G}$-modules, let $\mathcal{A}_{\mathbf{G},V} \subseteq \mathrm{End}_{\mathbb{K}}(V)$ be the (finite dimensional) $\mathbb{K}$-algebra generated by $\delta(\mathbf{G}) \leq \mathbf{GL}(V)$. Then V is simple if and only if $\mathcal{A}_{\mathbf{G},V} = \mathrm{End}_{\mathbb{K}}(V)$:

If V is simple, then since $\mathrm{End}_{\mathbf{G}}(V) = \mathbb{K} \cdot \mathrm{id}_V$ it follows from **Wedderburn's Theorem** (which we are not able to prove here) that $\mathcal{A}_{\mathbf{G},V} = \mathrm{End}_{\mathbb{K}}(V)$. Conversely, the equality $\mathcal{A}_{\mathbf{G},V} = \mathrm{End}_{\mathbb{K}}(V)$ implies that V cannot possibly have a proper non-zero $\mathbf{G}$-invariant $\mathbb{K}$-subspace, hence V is simple. ♯

c) We show that V is simple if and only if $V^\vee$ is so; note that $\mathcal{A}_{\mathbf{G},V} = \mathrm{End}_{\mathbb{K}}(V)$ if and only if $\mathcal{A}_{\mathbf{G},V^\vee} = \mathrm{End}_{\mathbb{K}}(V^\vee)$, but here is a direct proof:

Since $V^{\vee\vee} \cong V$ as $\mathbf{G}$-modules, it suffices to show that if V is not simple then $V^\vee$ neither is. Hence letting $U \leq V$ be a $\mathbf{G}$-submodule, the natural $\mathbf{G}$-monomorphism $\iota \colon U \to V$ induces the $\mathbf{G}$-epimorphism $\iota^* \colon V^\vee \to U^\vee \colon \lambda \mapsto \iota\lambda$, where we only have to show that ι^* indeed is $\mathbf{G}$-equivariant: Since ι is $\mathbf{G}$-equivariant, for $g \in \mathbf{G}$ and $y \in U$ we have $(y\iota \cdot g^{-1})\lambda = yg^{-1} \cdot \iota\lambda$, thus $\iota^*(\lambda^g) = \iota \cdot \lambda^g = (\iota\lambda)^g = \iota^*(\lambda)^g$. Finally, by a consideration of $\mathbb{K}$-dimensions we have $\{0\} \neq U < V$ if and only if $\{0\} \neq \ker(\iota^*) < V^\vee$. ♯

(4.5) Semisimple modules. a) Let $\mathbf{G}$ be an affine algebraic group, and let V be a $\mathbf{G}$-module, with associated algebraic representation δ. Then V is called **semisimple**, and δ is called **completely reducible**, if $V = \bigoplus_{i=1}^{r} V_i$ is the direct sum of simple $\mathbf{G}$-submodules $V_i \leq V$, for some $r \in \mathbb{N}_0$.

Proposition. The following statements are equivalent:
i) V is semisimple. **ii)** V is a (possibly empty) sum of simple $\mathbf{G}$-submodules.
iii) For any $\mathbf{G}$-submodule $U \leq V$ there is a $\mathbf{G}$-invariant complement $W \leq V$, that is we have $V = U \oplus W$ as $\mathbf{G}$-modules.

Proof. The implication **i)**$\Rightarrow$**ii)** is trivial. For **ii)**$\Rightarrow$**iii)** we proceed by induction on $\dim_{\mathbb{K}}(V) - \dim_{\mathbb{K}}(U) \in \mathbb{N}_0$, where the case $U = V$ is trivial. Hence let $U < V$. Since V is generated by simple $\mathbf{G}$-submodules, there is a simple $\mathbf{G}$-submodule $S \leq V$ such that $S \not\leq U$. Since S is simple, we have $U \cap S = \{0\}$, and hence $U \oplus S \leq V$. By induction the latter has a $\mathbf{G}$-invariant complement $W \leq V$, thus we get $V = U \oplus (S \oplus W)$ as $\mathbf{G}$-modules.

To show **iii)**$\Rightarrow$**ii)** let $U \leq V$ be a maximal semisimple $\mathbf{G}$-submodule, and assume that $U < V$. Then U has a $\mathbf{G}$-invariant complement $\{0\} \neq W \leq V$. Let $S \leq W$ be a simple $\mathbf{G}$-submodule. Then we have $U \oplus S \leq V$, where the latter is a semisimple $\mathbf{G}$-submodule, a contradiction. Hence we conclude that $U = V$, that is V is semisimple. $\sharp$

In particular, if V is semisimple and $U \leq V$ is a $\mathbf{G}$-submodule, then both U and V/U are semisimple again: If $U' \leq U$ is a $\mathbf{G}$-submodule, then letting $W \leq V$ be a $\mathbf{G}$-invariant complement for U' we get $U = U' \oplus (W \cap U)$; and since V is a sum of simple $\mathbf{G}$-submodule this also holds for the quotient $\mathbf{G}$-module V/U.

b) For a $\mathbf{G}$-isomorphism class $\sigma \in \Sigma_{\mathbf{G}}$ let $V_\sigma := \sum\{S \leq V \text{ } \mathbf{G}\text{-submodule}; S \in \sigma\}$. Thus $V_\sigma \leq V$ is a semisimple $\mathbf{G}$-submodule, being called the **σ-isotypic socle** of V. In particular, the isotypic socle associated with the trivial $\mathbf{G}$-module is $\{v \in V; vg = v \text{ for all } g \in \mathbf{G}\} = V^{\mathbf{G}}$, that is the set of $\mathbf{G}$-fixed points in V.

Proposition. Let $S \in \sigma$. Then $V_\sigma \cong S^r$ as $\mathbf{G}$-modules for some $r \in \mathbb{N}_0$, and the map $\epsilon_S \colon S \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathbf{G}}(S, V) \to V_\sigma \colon v \otimes \varphi \mapsto \varphi(v)$ is a $\mathbf{G}$-isomorphism, where $\mathbf{G}$ acts trivially on $\mathrm{Hom}_{\mathbf{G}}(S, V) = \mathrm{Hom}_{\mathbb{K}}(S, V)^{\mathbf{G}}$. Moreover, any simple $\mathbf{G}$-submodule of V_σ is $\mathbf{G}$-isomorphic to S.

Proof. Let $U \leq V_\sigma$ be a maximal $\mathbf{G}$-submodule with respect to being a direct sum $U = \bigoplus_{i=1}^{r} S_i$, such that $S_i \cong S$ for all $i \in \{1, \ldots, r\}$, and assume that $U < V_\sigma$. By the definition of V_σ there is $S \cong S_{r+1} \leq V_\sigma$ such that $S \not\leq U$, thus $U \oplus S \leq V_\sigma$, a contradiction. Hence $U = V_\sigma$, showing the first assertion.

For any $0 \neq \varphi \in \mathrm{Hom}_{\mathbf{G}}(S, V)$ we have $S \cong \varphi(S) \leq V_\sigma$. Hence ϵ_S is well-defined and $\mathbb{K}$-linear. Moreover, for $g \in \mathbf{G}$ we have $\epsilon_S((v \otimes \varphi)g) = \epsilon_S(vg \otimes \varphi) = \varphi(vg) = \varphi(v)g = \epsilon_S(v \otimes \varphi)g$, showing that ϵ_S is $\mathbf{G}$-equivariant. Now we have $\mathrm{Hom}_{\mathbf{G}}(S, V) \cong \mathrm{Hom}_{\mathbf{G}}(S, V_\sigma) \cong \bigoplus_{i=1}^{r} \mathrm{End}_{\mathbf{G}}(S)$ as $\mathbb{K}$-vector spaces, thus $\dim_{\mathbb{K}}(\mathrm{Hom}_{\mathbf{G}}(S, V)) = r$, hence $\dim_{\mathbb{K}}(S \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathbf{G}}(S, V)) = \dim_{\mathbb{K}}(S) \cdot r = \dim_{\mathbb{K}}(V_\sigma)$. Thus it suffices to show that ϵ_S is surjective: Letting $V_\sigma = \bigoplus_{i=1}^{r} S_i$ be a fixed direct sum decomposition, for any $i \in \{1, \ldots, r\}$ there is an associated $\mathbf{G}$-embedding $\iota_i \colon S \to S_i$, entailing $\epsilon_S(S \otimes \iota_i) = S_i$.

Finally, in order to show the last assertion, let $T \leq V_\sigma$ be a simple **G**-submodule. Then we have $\{0\} \neq \mathrm{Hom}_{\mathbf{G}}(T, V_\sigma) \cong \bigoplus_{i=1}^r \mathrm{Hom}_{\mathbf{G}}(T, S)$, implying $T \cong S$. ♯

Let $\mathrm{soc}(V) := \sum_{\sigma \in \Sigma_{\mathbf{G}}} V_\sigma = \sum\{S \leq V\ \mathbf{G}\text{-submodule}\} \leq V$ be the largest semisimple **G**-submodule of V, being called the **socle** of V.

If W is a **G**-module and $\varphi \in \mathrm{Hom}_{\mathbf{G}}(V, W)$, then we have $\varphi(\mathrm{soc}(V)) \leq \mathrm{soc}(W)$. Actually, since for simple **G**-modules S and T we have $\mathrm{Hom}_{\mathbf{G}}(S, T) \neq \{0\}$ if and only if $S \cong T$, we conclude that $\varphi(V_\sigma) \leq W_\sigma$, for $\sigma \in \Sigma_{\mathbf{G}}$.

Proposition. We have the direct sum decomposition $\mathrm{soc}(V) = \bigoplus_{\sigma \in \Sigma_{\mathbf{G}}} V_\sigma$, with only finitely many non-zero summands.

Proof. Letting $\{\sigma_1, \ldots, \sigma_k\} \subseteq \Sigma_{\mathbf{G}}$, we show by induction on $k \in \mathbb{N}_0$ that the sum $W := \sum_{i=1}^k V_{\sigma_i} \leq \mathrm{soc}(V)$ is direct; since $\mathrm{soc}(V)$ is the sum of its isotypic components and a finitely generated $\mathbb{K}$-vector space, this implies the assertion:

The case $k = 0$ being trivial, let $k \geq 1$. For any $j \in \{1, \ldots, k\}$ by induction we have $W_j := \bigoplus_{j \neq i \in \{1,\ldots,k\}} V_{\sigma_i} \leq W$. Assume that $W_j \cap V_{\sigma_j} \neq \{0\}$, then there is a simple **G**-submodule $S \leq W_j$ such that $S \in \sigma_j$. Hence we have $\{0\} \neq \mathrm{Hom}_{\mathbf{G}}(S, W_j) \cong \bigoplus_{j \neq i \in \{1,\ldots,k\}} \mathrm{Hom}_{\mathbf{G}}(S, V_{\sigma_i})$, thus there is V_{σ_i} having a **G**-submodule isomorphic to S, where $j \neq i \in \{1, \ldots, k\}$, a contradiction. Hence we infer $W_j \cap V_{\sigma_j} = \{0\}$, so that the sum defining W is direct. ♯

In particular, V is semisimple if and only if $\mathrm{soc}(V) = V$. In this case, $V = \bigoplus_{\sigma \in \Sigma_{\mathbf{G}}} V_\sigma$, where V_σ is called the ***σ*-isotypic component** of V, and $[V : S] := \frac{\dim_{\mathbb{K}}(V_\sigma)}{\dim_{\mathbb{K}}(S)} \in \mathbb{N}_0$ is called the **multiplicity** of $S \in \sigma$ in V. Moreover, if W is a semisimple **G**-module, then we have $\mathrm{Hom}_{\mathbf{G}}(V, W) = \bigoplus_{\sigma \in \Sigma_{\mathbf{G}}} \mathrm{Hom}_{\mathbf{G}}(V_\sigma, W_\sigma)$.

(4.6) Action on coordinate algebras. a) Let **G** be an affine algebraic group, let V be an affine **G**-variety, and let $\mathbb{K}[V]$ be the coordinate algebra of V. Hence **G** acts from the left on $\mathbb{K}[V]$, by $\mathbb{K}$-algebra automorphisms. Thus pre-composing with inversion yields a (right) **G**-action on $\mathbb{K}[V]$; recall that the latter is not finitely generated as a $\mathbb{K}$-vector space. Given a finitely generated **G**-invariant $\mathbb{K}$-subspace $U \leq \mathbb{K}[V]$, the proof of (2.7) shows that **G**-acts morphically and $\mathbb{K}$-linearly on U. Hence, U is a **G**-module, and by abuse of terminology it is called a **G**-submodule of $\mathbb{K}[V]$. Moreover, by (2.6), **G** acts locally finitely on $\mathbb{K}[V]$, so that $\mathbb{K}[V]$ is the union of its **G**-submodules.

Now let V be a **G**-module, with associated algebraic representation $\mathbf{G} \to \mathbf{GL}_n : g \mapsto [g_{ij}(g)]_{ij}$, and let $\mathbb{K}[V] = \mathbb{K}[X_1, \ldots, X_n]$. Since we have $X_i \mapsto \sum_{j=1}^n X_j \otimes g_{ji}$, for $i \in \{1, \ldots, n\}$, we conclude that all the homogeneous components $\mathbb{K}[V]_d \leq \mathbb{K}[V]$, for $d \in \mathbb{N}_0$, are **G**-submodules. In particular, for $d = 0$ we have $\mathbb{K}[V]_0 = \mathbb{K}$, on which **G** acts by the trivial representation. For $d = 1$, with the respect to the $\mathbb{K}$-basis $\mathcal{X} \subseteq \mathbb{K}[V]_1$ the matrix of the action of $g \in \mathbf{G}$ is given by $\delta(g)^{\mathrm{tr}}$, hence $\mathbb{K}[V]_1$ is a **G**-module via $g \mapsto \delta(g)^{-\mathrm{tr}}$, thus carries the contragredient representation associated with δ.

b) In particular, **G** acts on itself and thus on $\mathbb{K}[\mathbf{G}]$ by right translation; in this sense, again by abuse of terminology, $\mathbb{K}[\mathbf{G}]$ is called the **(right) regular G**-module. This is of importance in view of the following:

Theorem. Let $V^\vee = \langle \lambda_1, \ldots, \lambda_r \rangle_{\mathbf{G}}$, where $r \in \mathbb{N}_0$. Then V is isomorphic to a $\mathbf{G}$-submodule of the r-fold direct sum $\mathbb{K}[\mathbf{G}]^r$ of the regular $\mathbf{G}$-module.

Proof. Let $\lambda \in V^\vee$. Then for $x \in V$ let $\varphi_{\lambda,x} \colon \mathbf{G} \to \mathbb{K} \colon g \mapsto \lambda^g(x) = \lambda(xg^{-1})$. Since the orbit map associated with x is a morphism and λ is a regular map, we conclude that $\varphi_{\lambda,x} \in \mathbb{K}[\mathbf{G}]$. Then the map $\varphi_\lambda \colon V \to \mathbb{K}[\mathbf{G}] \colon x \mapsto \varphi_{\lambda,x}$ is $\mathbb{K}$-linear and $\mathbf{G}$-equivariant: For $g \in \mathbf{G}$ and $x, y \in V$ and $a \in K$ we have $\varphi_{\lambda,ax+y}(g) = \lambda^g(ax+y) = a\lambda^g(x) + \lambda^g(y) = a\varphi_{\lambda,x}(g) + \varphi_{\lambda,y}(g)$, thus $\varphi_{\lambda,ax+y} = a\varphi_{\lambda,x} + \varphi_{\lambda,y}$; and for $h \in \mathbf{G}$ we have $\varphi_{\lambda,xh}(g) = \lambda(xh \cdot g^{-1}) = \lambda(x \cdot hg^{-1}) = \varphi_{\lambda,x}(gh^{-1}) = (\varphi_{\lambda,x})^h(g)$, thus $\varphi_{\lambda,xh} = (\varphi_{\lambda,x})^h$.

Now, for $i \in \{1, \ldots, r\}$ let φ_i be the map associated with λ_i as above, and let $\varphi := \bigoplus_{i=1}^r \varphi_i \colon V \to \mathbb{K}[\mathbf{G}]^r \colon x \mapsto [\varphi_{1,x}, \ldots, \varphi_{r,x}]$. Hence φ is $\mathbb{K}$-linear and $\mathbf{G}$-equivariant. Finally, φ is injective: If $\varphi(x) = 0$, then we have $\lambda_i^g(x) = 0$ for all $i \in \{1, \ldots, r\}$ and $g \in \mathbf{G}$; hence from $\langle \lambda_i^g; i \in \{1, \ldots, r\}, g \in \mathbf{G} \rangle_{\mathbb{K}} = \langle \lambda_1, \ldots, \lambda_r \rangle_{\mathbf{G}} = V^\vee$ we conclude that $V^\vee(x) = \{0\}$, implying $x = 0$. ♯

In particular, if V is simple then $V^\vee = \langle \lambda \rangle_{\mathbf{G}}$ for any $0 \neq \lambda \in V^\vee$, so that in this case V is isomorphic to a $\mathbf{G}$-submodule of the regular $\mathbf{G}$-module $\mathbb{K}[\mathbf{G}]$.

5 Linear reductivity

(5.1) Linear reductivity. An affine algebraic group $\mathbf{G}$ is called **linearly reductive** if any $\mathbf{G}$-module is semisimple; this is equivalent to saying that any algebraic representation of $\mathbf{G}$ is completely reducible.

We look for examples: Recall that if G is a finite group, then the associated coordinate algebra $\mathbb{K}[G]$ is the set of all maps from G to $\mathbb{K}$, that is $\mathbb{K}[G]$ coincides with the conventional group algebra of G. Hence any conventional representation of G is algebraic. Thus from representation theory of finite groups the following is well-known:

Theorem: Maschke. Let G be a finite group. Then G is linearly reductive if and only if $\mathrm{char}(\mathbb{K}) \nmid |G|$. ♯

As a generalisation of this, the following theorem says that in positive characteristic there are not too many linearly reductive groups either. Actually, in (5.7) we show that the question whether $\mathbf{G}$ is linearly reductive can be reduced to the identity component $\mathbf{G}^\circ$ and the finite quotient $\mathbf{G}/\mathbf{G}^\circ$, and in (5.2) we show that tori are linearly reductive in any characteristic; recall that a torus is an algebraic group isomorphic to $(\mathbf{G}_m)^n$, for some $n \in \mathbb{N}_0$:

Theorem: Nagata [1961]. Let $\mathrm{char}(\mathbb{K}) = p > 0$. Then $\mathbf{G}$ is linearly reductive if and only if $\mathbf{G}^\circ$ is a torus and $p \nmid [\mathbf{G} \colon \mathbf{G}^\circ]$. ♯

The situation is completely different if $\mathrm{char}(\mathbb{K}) = 0$, where there are many more linearly reductive groups: For example, $\mathbf{GL}_n$ and $\mathbf{SL}_n$, where $n \in \mathbb{N}$, are linearly reductive; unfortunately, we are not able to prove this here for $n \geq 2$, while $\mathbf{SL}_1 = \{1\}$ is trivial, and $\mathbf{GL}_1 = \mathbf{G}_m$ is covered in (5.2). Actually, in any characteristic linearly reductive groups are necessarily group theoretically reductive, see (6.6), where for $\mathrm{char}(\mathbb{K}) = 0$ the latter property is also sufficient, and thus provides a rich source of linearly reductive groups. In (6.6) we indicate that in any characteristic the groups $\mathbf{GL}_n$ and $\mathbf{SL}_n$ are reductive indeed, while the following are typical examples of non-reductive groups:

Example. Let $n \geq 2$, and let $\mathbf{G} := \mathbf{B}_n$ or $\mathbf{G} := \mathbf{U}_n$; recall that in particular $\mathbf{U}_2 \cong \mathbf{G}_a$. Let $V := \mathbb{K}^n$ be the natural $\mathbf{G}$-module. Then $U := \langle e_n \rangle \leq V$ is a $\mathbf{G}$-submodule. We show that U does not have a $\mathbf{G}$-invariant complement in V, so that V is not semisimple and thus $\mathbf{G}$ is not linearly reductive:

Assume to the contrary that $V = U \oplus W$ as $\mathbf{G}$-modules. Let $J := J_n(1) \in \mathbf{G}$ be a Jordan block of size n with respect to the eigenvalue 1. Then J has characteristic polynomial $\chi(J) = (X-1)^n \in \mathbb{K}[X]$, and thus has $1 \in \mathbb{K}$ as its only eigenvalue on both U and W. But we have $\ker_V(J - E_n) = \langle e_n \rangle = U$, so that $\ker_W(J - E_n) = \{0\}$, a contradiction. ♯

(5.2) Example: Tori. We consider the torus $\mathbf{T}_n \cong (\mathbf{G}_m)^n$, where $n \in \mathbb{N}$. We show that it is linearly reductive, and determine its simple modules:

a) In order to show that $\mathbf{T}_n$ is linearly reductive, let V be a $\mathbf{T}_n$-module with associated algebraic representation $\delta\colon \mathbf{T}_n \to \mathbf{GL}_m$, where $m := \dim_{\mathbb{K}}(V) \in \mathbb{N}_0$. We show that δ is not only completely reducible, but even diagonalisable, that is equivalent to an algebraic representation δ' such that $\delta'(\mathbf{T}_n) \subseteq \mathbf{T}_m$:

Since $\mathbf{T}_n \cong (\mathbf{G}_m)^n$ is commutative, it is sufficient to show that $\mathbf{T}_1 = \mathbf{G}_m$ is diagonalisable, that is to consider the case $n = 1$. To this end, let $H := \{t \in \mathbf{G}_m; t^k = 1 \text{ for some } k \in \mathbb{Z} \text{ such that } \gcd(p,k) = 1\}$, where $p := 1$ if $\mathrm{char}(\mathbb{K}) = 0$, and $p := \mathrm{char}(\mathbb{K})$ otherwise; in other words H consists of all elements of $\mathbf{G}_m$ of finite order coprime to p. Since $\mathbf{G}_m$ is commutative we infer that $H \leq \mathbf{G}_m$ is a subgroup. Since $\mathbb{K}$ is algebraically closed, we conclude that H is infinite, thus $\overline{H} \subseteq \mathbf{G}_m$ is a closed subset (actually a subgroup) of non-zero dimension. Hence, since $\mathbf{G}_m$ is connected of dimension 1, we have $\overline{H} = \mathbf{G}_m$. (Note that for $\mathbb{K} = \mathbb{C}$ this does not work with respect to the metric topology.)

Let $t \in H$ of order $k \in \mathbb{N}$. Then the minimum polynomial of $\delta(t)$ divides $T^k - 1 \in \mathbb{K}[T]$, and thus is multiplicity-free. Hence $\delta(t) \in \mathbf{GL}_m$ is diagonalisable. Since H is commutative it follows that $\delta(H)$ is diagonalisable, thus we may assume that $\overline{\delta(H)} \subseteq \mathbf{T}_m \subseteq \mathbf{GL}_m$, where the latter is closed. From $H \subseteq \delta^{-1}(\overline{\delta(H)}) \subseteq \mathbf{G}_m$, where since δ is continuous the latter is closed, we infer that $\overline{H} \subseteq \delta^{-1}(\overline{\delta(H)})$, that is $\delta(\overline{H}) \subseteq \overline{\delta(H)}$. This yields $\delta(\mathbf{G}) = \delta(\overline{H}) \subseteq \overline{\delta(H)} \subseteq \mathbf{T}_m$. ♯

b) We proceed to determine the simple $\mathbf{T}_n$-modules, where by the above we already know that these are precisely those of $\mathbb{K}$-dimension 1. We first stick to the case $n = 1$, and continue to consider $\mathbf{T}_1 = \mathbf{G}_m$:

Recalling that any simple module is a submodule of the regular module, we consider the coordinate algebra $\mathbb{K}[\mathbf{G}_m] \cong K[X]_X \cong \mathbb{K}[X, X^{-1}]$. The latter is $\mathbb{Z}$-graded, where letting $S_d := \langle X^{-d} \rangle_{\mathbb{K}}$, for $d \in \mathbb{Z}$, we have $\mathbb{K}[\mathbf{G}_m] = \bigoplus_{d \in \mathbb{Z}} S_d$ as $\mathbb{K}$-vector spaces. The right translation action $\mathbf{G}_m \times \mathbf{G}_m \to \mathbf{G}_m\colon [x,t] \mapsto xt$ has comorphism $\mathbb{K}[\mathbf{G}_m] \to \mathbb{K}[\mathbf{G}_m] \otimes_{\mathbb{K}} \mathbb{K}[\mathbf{G}_m]\colon X \mapsto X \otimes T$. Hence we have $X^d \mapsto X^d \otimes T^d$, for $d \in \mathbb{Z}$, from which we conclude that the $\mathbb{K}$-subspaces $S_d \leq \mathbb{K}[\mathbf{G}_m]$ are $\mathbf{G}_m$-submodules, with respect to $\mathbf{G}_m \to \mathbf{GL}_1 = \mathbf{G}_m\colon t \mapsto t^d$. Hence the above direct sum decomposition of $\mathbb{K}[\mathbf{G}_m]$ holds as $\mathbf{G}_m$-modules, where the S_d are pairwise non-isomorphic simple $\mathbf{G}_m$-submodules.

We show that $\Sigma_{\mathbf{G}_m} = \{\sigma_d; d \in \mathbb{Z}\}$, where $S_d \in \sigma_d$; in other words, $\Sigma_{\mathbf{G}_m}$ can be naturally identified with $\mathbb{Z}$: To this end let $S \leq \mathbb{K}[\mathbf{G}_m]$ be a simple $\mathbf{G}_m$-submodule. Since S is finitely generated, there is a finite subset $\mathcal{I} \subseteq \mathbb{Z}$ such that $S \leq \bigoplus_{d\in\mathcal{I}} S_d =: U$. Since U is semisimple with isotypic components $U_{\sigma_d} = S_d$, for $d \in \mathcal{I}$, we conclude that $S = S_d$ for some $d \in \mathcal{I}$. $\sharp$

Let now $n \in \mathbb{N}$ be arbitrary. Then any simple $\mathbf{T}_n$-module is uniquely determined by its **weight** $[d_1, \ldots, d_n] \in \mathbb{Z}^n$, where the action is given as $\mathbf{T}_n \cong (\mathbf{G}_m)^n \to \mathbf{GL}_1 = \mathbf{G}_m\colon [t_1, \ldots, t_n] \mapsto \prod_{i=1}^n t_i^{d_i}$; in other words, for the associated simple $\mathbf{T}_n$-module we have $S_{[d_1,\ldots,d_n]} \cong S_{d_1} \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} S_{d_n}$. Conversely, any choice of a weight in $\mathbb{Z}^n$ gives rise to a simple $\mathbf{T}_n$-module by way of the above formulae. Moreover, simple $\mathbf{T}_n$-modules are isomorphic if and only if their weights coincide, so that in conclusion $\Sigma_{\mathbf{T}_n}$ can be naturally identified with $\mathbb{Z}^n$.

(5.3) Action on coordinate algebras again. a) Let $\mathbf{G}$ be an affine algebraic group, and let V be an affine $\mathbf{G}$-variety. We consider the associated $\mathbf{G}$-action on the coordinate algebra $\mathbb{K}[V]$ of V. Recall from (4.6) that $\mathbb{K}[V] = \bigcup\{U \leq \mathbb{K}[V]\ \mathbf{G}\text{-submodule}\}$ is the union of its $\mathbf{G}$-submodules.

Note first that for $\mathbf{G}$-modules $U \leq U'$ and any subset $\Sigma \subseteq \Sigma_{\mathbf{G}}$ we have $(\bigoplus_{\sigma\in\Sigma} U'_\sigma) \cap U = \bigoplus_{\sigma\in\Sigma} U_\sigma$; in particular for $\sigma \in \Sigma$ we have $U'_\sigma \cap U = U_\sigma$: We only have to show that $\leq$ holds; but the left hand side is semisimple, where any of its simple submodules is contained in U and has an isomorphism type in Σ.

Hence for $\sigma \in \Sigma_{\mathbf{G}}$ we have $\mathbb{K}[V]_\sigma := \bigcup\{U_\sigma; U \leq \mathbb{K}[V]\ \mathbf{G}\text{-submodule}\} = \sum\{U_\sigma; U \leq \mathbb{K}[V]\ \mathbf{G}\text{-submodule}\} = \sum\{S \leq \mathbb{K}[V]\ \mathbf{G}\text{-submodule}; S \in \sigma\}$. Thus $\mathbb{K}[V]_\sigma \leq \mathbb{K}[V]$ is a $\mathbf{G}$-invariant $\mathbb{K}$-subspace, being called the **σ-isotypic socle** of $\mathbb{K}[V]$; if it is finitely generated, then letting $S \in \sigma$ we have $\mathbb{K}[V]_\sigma \cong S^r$, for some $r \in \mathbb{N}_0$. Let $\mathrm{soc}(\mathbb{K}[V]) := \sum_{\sigma\in\Sigma_{\mathbf{G}}} \mathbb{K}[V]_\sigma = \sum\{S \leq \mathbb{K}[V]\ \mathbf{G}\text{-submodule}\} = \sum\{\mathrm{soc}(U); U \leq \mathbb{K}[V]\ \mathbf{G}\text{-submodule}\} = \bigcup\{\mathrm{soc}(U); U \leq \mathbb{K}[V]\ \mathbf{G}\text{-submodule}\} \leq \mathbb{K}[V]$ be the **socle** of V.

Note that for any $\mathbf{G}$-submodule $U \leq \mathbb{K}[V]$ and any subset $\Sigma \subseteq \Sigma_{\mathbf{G}}$ we have $(\sum_{\sigma\in\Sigma} \mathbb{K}[V]_\sigma) \cap U = \bigoplus_{\sigma\in\Sigma} U_\sigma$; hence in particular $\mathrm{soc}(\mathbb{K}[V]) \cap U = \mathrm{soc}(U)$: We only have to show that $\leq$ holds; but since $\sum_{\sigma\in\Sigma} \mathbb{K}[V]_\sigma = \bigcup\{\sum_{\sigma\in\Sigma} U'_\sigma; U' \leq \mathbb{K}[V]\ \mathbf{G}\text{-submodule}\}$ the left hand side is semisimple, where any of its simple submodules is contained in U and has an isomorphism type in Σ.

Moreover, we have the direct sum decomposition $\mathrm{soc}(\mathbb{K}[V]) = \bigoplus_{\sigma\in\Sigma_{\mathbf{G}}} \mathbb{K}[V]_\sigma$, with possibly infinitely many non-zero summands: Assume there is $\sigma \in \Sigma_{\mathbf{G}}$ such that $\mathbb{K}[V]_\sigma \cap \sum_{\sigma\neq\tau\in\Sigma_{\mathbf{G}}} \mathbb{K}[V]_\tau \neq \{0\}$, then there is a $\mathbf{G}$-submodule $U \leq \mathbb{K}[V]$ such that $U_\sigma \cap \sum_{\sigma\neq\tau\in\Sigma_{\mathbf{G}}} U_\tau \neq \{0\}$, a contradiction.

In particular, if V is a $\mathbf{G}$-module, then the decomposition $\mathbb{K}[V] = \bigoplus_{d\in\mathbb{N}_0} \mathbb{K}[V]_d$ into homogeneous components is a direct sum of $\mathbf{G}$-submodules. Thus we have $\mathbb{K}[V]_\sigma = \bigoplus_{d\in\mathbb{N}_0} \mathbb{K}[V]_{d,\sigma}$, for $\sigma \in \Sigma_{\mathbf{G}}$, and $\mathrm{soc}(\mathbb{K}[V]_d) = \bigoplus_{\sigma\in\Sigma_{\mathbf{G}}} \mathbb{K}[V]_{d,\sigma}$, for $d \in \mathbb{N}_0$, hence we get $\mathrm{soc}(\mathbb{K}[V]) = \bigoplus_{d\in\mathbb{N}_0} \mathrm{soc}(\mathbb{K}[V]_d) = \bigoplus_{\sigma\in\Sigma_{\mathbf{G}}} \mathbb{K}[V]_\sigma = \bigoplus_{d\in\mathbb{N}_0} \bigoplus_{\sigma\in\Sigma_{\mathbf{G}}} \mathbb{K}[V]_{d,\sigma} = \bigoplus_{\sigma\in\Sigma_{\mathbf{G}}} \bigoplus_{d\in\mathbb{N}_0} \mathbb{K}[V]_{d,\sigma}$.

b) If $U \leq \mathbb{K}[V]$ is a $\mathbf{G}$-invariant (not necessarily finitely generated) $\mathbb{K}$-subspace, then both U and $\mathbb{K}[V]/U$ carry locally finite $\mathbf{G}$-actions. Hence the natural $\mathbb{K}$-linear map $\nu_U\colon \mathbb{K}[V] \to \mathbb{K}[V]/U$ is $\mathbf{G}$-equivariant, and for $\sigma \in \Sigma_{\mathbf{G}}$ we have

$\nu_U(\mathbb{K}[V]_\sigma) \leq \nu_U(\mathbb{K}[V])_\sigma = (\mathbb{K}[V]/U)_\sigma$ and $U_\sigma = U \cap \mathbb{K}[V]_\sigma$.

In particular, if W is an affine **G**-variety and $\varphi\colon W \to V$ is a **G**-equivariant morphism, then $\varphi^*\colon \mathbb{K}[V] \to \mathbb{K}[W]$ is a **G**-equivariant homomorphism of $\mathbb{K}$-algebras: For $f \in \mathbb{K}[V]$ and $g \in \mathbf{G}$ we have $\varphi^*(f^g)(w) = f^g(\varphi(w)) = f(\varphi(w) \cdot g^{-1}) = f(\varphi(wg^{-1})) = \varphi^*(f)(wg^{-1}) = \varphi^*(f)^g(w)$, for $w \in W$, thus $\varphi^*(f^g) = \varphi^*(f)^g$. Hence the above applies with $U := \ker(\varphi^*)$ and $\mathbb{K}[V]/U \cong \varphi^*(\mathbb{K}[V])$ as $\mathbb{K}$-algebras, entailing that $\varphi^*(\mathbb{K}[V]_\sigma) \leq \mathbb{K}[W]_\sigma$.

c) Now assume that $\mathbb{K}[V] = \mathrm{soc}(\mathbb{K}[V]) = \bigoplus_{\sigma \in \Sigma_{\mathbf{G}}} \mathbb{K}[V]_\sigma$, or equivalently that $\mathrm{soc}(U) = U$ for all **G**-submodules $U \leq \mathbb{K}[V]$; hence in particular this holds whenever **G** is linearly reductive. Then $\mathbb{K}[V]_\sigma$ is called the **σ-isotypic component** of $\mathbb{K}[V]$. If $\mathbb{K}[V]_\sigma$ is finitely generated, then $\mathbb{K}[V]_\sigma \cong S^r$, where $S \in \sigma$, and $[\mathbb{K}[V]\colon S] := r = \frac{\dim_{\mathbb{K}}(\mathbb{K}[V]_\sigma)}{\dim_{\mathbb{K}}(S)} \in \mathbb{N}_0$, is called the **multiplicity** of S in V; otherwise we let $[\mathbb{K}[V]\colon S] := \infty$.

If $U \leq \mathbb{K}[V]$ is a **G**-invariant $\mathbb{K}$-subspace, then local finiteness implies that $U = \mathrm{soc}(U) = \bigoplus_{\sigma \in \Sigma_{\mathbf{G}}} U_\sigma = \bigoplus_{\sigma \in \Sigma_{\mathbf{G}}} (U \cap \mathbb{K}[V]_\sigma)$. Moreover, if $S \leq \mathbb{K}[V]/U$ is a simple **G**-submodule, then picking a $\mathbb{K}$-vector space complement of U in $\nu_U^{-1}(S)$, by local finiteness there is a simple **G**-submodule $T \leq \mathbb{K}[V]$ such that $S = \nu_U(T)$. This entails $\nu_U(\mathbb{K}[V]_\sigma) = (\mathbb{K}[V]/U)_\sigma$, hence we get $(\mathbb{K}[V]/U)_\sigma \cong \mathbb{K}[V]_\sigma/(U \cap \mathbb{K}[V]_\sigma) = \mathbb{K}[V]_\sigma/U_\sigma$, for $\sigma \in \Sigma_{\mathbf{G}}$, and thus $\mathbb{K}[V]/U = \mathrm{soc}(\mathbb{K}[V]/U)$.

In particular, if W is an affine **G**-variety and $\varphi\colon W \to V$ is a **G**-equivariant closed embedding, then for the associated epimorphism $\varphi^*\colon \mathbb{K}[V] \to \mathbb{K}[W]$ of coordinate algebras we have $\varphi^*(\mathbb{K}[V]_\sigma) = \mathbb{K}[W]_\sigma$.

(5.4) Invariant algebras. Let **G** be an affine algebraic group, and let V be an affine **G**-variety. The isotypic socle of $\mathbb{K}[V]$ associated with the trivial module is given as $\{f \in \mathbb{K}[V]; f^g = f \text{ for all } g \in \mathbf{G}\} = \mathbb{K}[V]^{\mathbf{G}}$, that is the set of **G-invariant** regular maps on V; recall that the latter are the regular maps being constant on **G**-orbits.

Since the constant maps are contained in $\mathbb{K}[V]^{\mathbf{G}}$, and for $f, f' \in \mathbb{K}[V]^{\mathbf{G}}$ we have $ff' \in \mathbb{K}[V]^{\mathbf{G}}$ as well, we conclude that $\mathbb{K}[V]^{\mathbf{G}}$ is a $\mathbb{K}$-subalgebra of $\mathbb{K}[V]$, being called the associated **invariant algebra**.

Then any isotypic socle $\mathbb{K}[V]_\sigma$ of $\mathbb{K}[V]$, for $\sigma \in \Sigma_{\mathbf{G}}$, is a $\mathbb{K}[V]^{\mathbf{G}}$-module: For the $\mathbb{K}$-linear multiplication map $\rho_f\colon \mathbb{K}[V] \to \mathbb{K}[V]\colon h \mapsto hf$ associated with $f \in \mathbb{K}[V]^{\mathbf{G}}$ we have $(hf)^g = h^g f^g = h^g f$, for $g \in \mathbf{G}$, showing that ρ_f is **G**-equivariant, entailing that indeed $\mathbb{K}[V]_\sigma \cdot f \leq \mathbb{K}[V]_\sigma$.

Moreover, if **G** is linearly reductive, then letting $\mathbb{K}^n$, where $n \in \mathbb{N}_0$, be a **G**-module such that there is a **G**-equivariant closed embedding $\varphi\colon V \to \mathbb{K}^n$, then for the associated epimorphism $\varphi^*\colon \mathbb{K}[\mathcal{X}] \to \mathbb{K}[V]$ of coordinate algebras we have $\varphi^*(\mathbb{K}[\mathcal{X}]^{\mathbf{G}}) = \mathbb{K}[V]^{\mathbf{G}}$. Thus the determination of invariant algebras arising from arbitrary affine **G**-varieties can be reduced to the case of **G**-modules.

(5.5) Coordinate algebras of affine algebraic groups. Let **G** be an affine algebraic group. Then **G** acts on itself both by right and by left translation. Hence **G** acts on $\mathbb{K}[\mathbf{G}]$ both by right and by left translation, where $\mathbb{K}[\mathbf{G}]$ becomes the regular **G**-module with respect to the former action, while with respect to the latter $\mathbb{K}[\mathbf{G}]$ is called the **left regular G**-module.

We consider both actions at the same time: From $h^{-1}(xg) = (h^{-1}x)g$, for $x, g, h \in \mathbf{G}$, we conclude that $\mathbf{G} \times \mathbf{G}$ acts morphically on $\mathbf{G}$ by $[h, g] \colon \mathbf{G} \to \mathbf{G} \colon x \mapsto h^{-1}xg$. Hence we get an induced action of $\mathbf{G} \times \mathbf{G}$ on the coordinate algebra $\mathbb{K}[\mathbf{G}]$ by $[h, g] \colon \mathbb{K}[\mathbf{G}] \to \mathbb{K}[\mathbf{G}] \colon f \mapsto \big(x \mapsto f(hxg^{-1})\big)$; in view of this $\mathbb{K}[\mathbf{G}]$ is also called the **bi-regular** $\mathbf{G}$-module.

We proceed to describe the σ-isotypic socle $\mathbb{K}[\mathbf{G}]_\sigma \leq \mathbb{K}[\mathbf{G}]$ explicitly; recall that any simple $\mathbf{G}$-module occurs as a $\mathbf{G}$-submodule of $\mathbb{K}[\mathbf{G}]$, so that $\mathbb{K}[\mathbf{G}]_\sigma \neq \{0\}$:

Theorem. Let $\sigma \in \Sigma_{\mathbf{G}}$ and $S \in \sigma$. Then $\mathbb{K}[\mathbf{G}]_\sigma \leq \mathbb{K}[\mathbf{G}]$ is a (finitely generated) $(\mathbf{G} \times \mathbf{G})$-submodule, and we have $\mathbb{K}[\mathbf{G}]_\sigma \cong S^\vee \otimes_{\mathbb{K}} S \cong \mathrm{End}_{\mathbb{K}}(S)$ as $(\mathbf{G} \times \mathbf{G})$-modules; in particular we have $\mathbb{K}[\mathbf{G}]_\sigma \cong S^{\dim_{\mathbb{K}}(S)}$ as $\mathbf{G}$-modules.

Proof. We provide an explicit $(\mathbf{G} \times \mathbf{G})$-equivariant embedding of $S^\vee \otimes_{\mathbb{K}} S$ into $\mathbb{K}[\mathbf{G}]$: For $x \in S$ and $\lambda \in S^\vee$ let $\varphi_{\lambda,x} \colon \mathbf{G} \to \mathbb{K} \colon g \mapsto \lambda^g(x) = \lambda(xg^{-1})$; see also (4.6). Since the orbit map associated with x is a morphism and λ is a regular map, we conclude that $\varphi_{\lambda,x} \in \mathbb{K}[\mathbf{G}]$. Then the map $\varphi \colon S^\vee \otimes_{\mathbb{K}} S \to \mathbb{K}[\mathbf{G}] \colon \lambda \otimes x \mapsto \varphi_{\lambda,x}$ is $\mathbb{K}$-linear and $(\mathbf{G} \times \mathbf{G})$-equivariant:

Firstly, for $g \in \mathbf{G}$ and $x, y \in S$ and $\lambda, \mu \in S^\vee$ and $a \in K$ we have $\varphi_{\lambda,ax+y}(g) = \lambda^g(ax + y) = a\lambda^g(x) + \lambda^g(y) = a\varphi_{\lambda,x}(g) + \varphi_{\lambda,y}(g)$, thus $\varphi_{\lambda,ax+y} = a\varphi_{\lambda,x} + \varphi_{\lambda,y}$, and $\varphi_{a\lambda+\mu,x}(g) = (a\lambda + \mu)^g(x) = (a\lambda^g + \mu^g)(x) = a\varphi_{\lambda,x}(g) + \varphi_{\mu,x}(g)$, thus $\varphi_{a\lambda+\mu,x} = a\varphi_{\lambda,x} + \varphi_{\mu,x}$, showing $\mathbb{K}$-bilinearity of f on $S^\vee \times S$. Secondly, for $h, t \in \mathbf{G}$ we have $\varphi_{\lambda^t,xh}(g) = \lambda^t(xh \cdot g^{-1}) = \lambda(xhg^{-1}t^{-1}) = \lambda(x \cdot (tgh^{-1})^{-1}) = \lambda^{tgh^{-1}}(x) = \varphi_{\lambda,x}(tgh^{-1}) = (\varphi_{\lambda,x})^{[t,h]}(g)$, thus we have $\varphi_{\lambda^t,xh} = (\varphi_{\lambda,x})^{[t,h]}$.

Hence $\varphi(S^\vee \otimes_{\mathbb{K}} S) \leq \mathbb{K}[\mathbf{G}]$ is a $(\mathbf{G} \times \mathbf{G})$-submodule. We show that the image $\varphi(S^\vee \otimes_{\mathbb{K}} S)$ only depends on the isomorphism class σ of S: To this end, let $\alpha \colon S \to T$ be a $\mathbf{G}$-isomorphism. Then the map $\alpha^* \colon T^\vee \to S^\vee \colon \tau \mapsto \alpha\tau$ is a $\mathbf{G}$-isomorphism as well. Thus for $y \in T$ we have $\tau^g(y) = \tau^g(\alpha(\alpha^{-1}(y))) = (\tau^g)^{\alpha^*}(\alpha^{-1}(y)) = (\tau^{\alpha^*})^g(\alpha^{-1}(y))$, showing that $\varphi_{\tau,y} = \varphi_{\alpha^*(\tau),\alpha^{-1}(y)} \in \mathbb{K}[\mathbf{G}]$. Hence we have $\varphi(T^\vee \otimes_{\mathbb{K}} T) \leq \varphi(S^\vee \otimes_{\mathbb{K}} S)$, by symmetry entailing equality.

Next, $S^\vee \otimes_{\mathbb{K}} S$ is a simple $(\mathbf{G} \times \mathbf{G})$-module: Given a $\mathbf{G}$-module V, let $\mathcal{A}_{\mathbf{G},V}$ be the (finite dimensional) $\mathbb{K}$-algebra generated by the $\mathbb{K}$-endomorphisms of V afforded by $\mathbf{G}$. Then, since $\mathbb{K}$ is algebraically closed, it is well-known from representation theory that V is a simple $\mathbf{G}$-module if and only if $\mathcal{A}_{\mathbf{G},V} = \mathrm{End}_{\mathbb{K}}(V)$. Now, since S and $S^\vee$ are simple $\mathbf{G}$-modules, we have $\mathcal{A}_{\mathbf{G},S} = \mathrm{End}_{\mathbb{K}}(S)$ and $\mathcal{A}_{\mathbf{G},S^\vee} = \mathrm{End}_{\mathbb{K}}(S^\vee)$, entailing that $\mathcal{A}_{\mathbf{G}\times\mathbf{G},S^\vee\otimes_{\mathbb{K}}S} \cong \mathcal{A}_{\mathbf{G},S^\vee} \otimes_{\mathbb{K}} \mathcal{A}_{\mathbf{G},S} = \mathrm{End}_{\mathbb{K}}(S^\vee) \otimes_{\mathbb{K}} \mathrm{End}_{\mathbb{K}}(S) \cong \mathrm{End}_{\mathbb{K}}(S^\vee \otimes_{\mathbb{K}} S)$, thus $S^\vee \otimes_{\mathbb{K}} S$ is a simple $(\mathbf{G} \times \mathbf{G})$-module.

Now, choosing $x \in S$ and $\lambda \in S^\vee$ such that $\lambda(x) \neq 0$ shows that $\varphi \neq 0$. Hence, by simplicity, φ is injective. Moreover, restricting along the embedding of algebraic groups $\mathbf{G} \to \mathbf{G} \times \mathbf{G} \colon g \mapsto [1_{\mathbf{G}}, g]$ yields $S^\vee \otimes_{\mathbb{K}} S \cong S^{\dim_{\mathbb{K}}(S)}$ as $\mathbf{G}$-modules, thus $\varphi(S^\vee \otimes_{\mathbb{K}} S) \leq \mathbb{K}[\mathbf{G}]_\sigma$.

Finally, if $T \leq \mathbb{K}[\mathbf{G}]$ is a $\mathbf{G}$-submodule such that $T \cong S$, then let $\tau \in T^\vee$ be defined by $\tau \colon T \to \mathbb{K} \colon y \mapsto y(1_{\mathbf{G}})$. Then for $y \in T$ and we have $\varphi_{\tau,y}(g) = \tau^g(y) = \tau(y^{g^{-1}}) = y^{g^{-1}}(1_{\mathbf{G}}) = y(1_{\mathbf{G}} \cdot g) = y(g)$, for $g \in \mathbf{G}$, thus $y = \varphi_{\tau,y} \in \varphi(T^\vee \otimes_{\mathbb{K}} T)$, and hence $T \leq \varphi(T^\vee \otimes_{\mathbb{K}} T)$. Thus, since $\varphi(S^\vee \otimes_{\mathbb{K}} S)$ only depends on the isomorphism class σ, we indeed have $\varphi(S^\vee \otimes_{\mathbb{K}} S) = \mathbb{K}[\mathbf{G}]_\sigma$. ♯

An alternative description of the $(\mathbf{G}\times\mathbf{G})$-equivariant map φ defined above, using the $(\mathbf{G}\times\mathbf{G})$-isomorphism $S^\vee \otimes_{\mathbb{K}} S \cong \mathrm{End}_{\mathbb{K}}(S)$, is given as follows: Let $B := \{x_1,\ldots,x_n\} \subseteq S$ be a $\mathbb{K}$-basis, where $n := \dim_{\mathbb{K}}(S)$, let $B^\vee := \{\lambda_1,\ldots,\lambda_n\} \subseteq S^\vee$ be the associated dual $\mathbb{K}$-basis, that is $\lambda_i(x_j) = \delta_{ij}$ for $i,j \in \{1,\ldots,n\}$, and let $\delta\colon \mathbf{G} \to \mathbf{GL}_n\colon g \mapsto [g_{ij}(g)]_{ij}$ be the algebraic representation associated with S, with respect to the $\mathbb{K}$-basis B. Then for $\varphi_{ij} := \varphi_{\lambda_i,x_j} \in \mathbb{K}[\mathbf{G}]$ we have $\varphi_{ij}(g) = \lambda_i(x_j g^{-1}) = g_{ji}(g^{-1})$, for $g \in \mathbf{G}$.

Let $\alpha \in \mathrm{End}_{\mathbb{K}}(S)$, having matrix ${}_B\alpha_B = [a_{ij}]_{ij} \in \mathbb{K}^{n\times n}$. Then α can be identified with $\sum_{i=1}^n\sum_{j=1}^n a_{ij}\lambda_i\otimes x_j \in S^\vee\otimes_{\mathbb{K}} S$, thus we have $\varphi_\alpha = \sum_{i=1}^n\sum_{j=1}^n a_{ij}\varphi_{ij}$, entailing $\varphi_\alpha(g) = \sum_{i=1}^n\sum_{j=1}^n a_{ij}g_{ji}(g^{-1}) = \sum_{i=1}^n (\alpha\delta(g^{-1}))_{ii} = \mathrm{Tr}(\alpha\delta(g^{-1}))$, for $g \in \mathbf{G}$. Note that this again shows that the image of φ only depends on the isomorphism class σ of S. $\sharp$

Theorem. The group $\mathbf{G}$ is linearly reductive if and only if $\mathbb{K}[\mathbf{G}] = \mathrm{soc}(\mathbb{K}[\mathbf{G}])$.

Proof. If $\mathbf{G}$ is linearly reductive, then any $\mathbf{G}$-submodule $U \leq \mathbb{K}[\mathbf{G}]$ is semisimple, and thus we have $\mathbb{K}[\mathbf{G}] = \sum\{U \leq \mathbb{K}[\mathbf{G}]\ \mathbf{G}\text{-submodule}\} = \sum\{\mathrm{soc}(U) \leq \mathbb{K}[\mathbf{G}]\ \mathbf{G}\text{-submodule}\} = \mathrm{soc}(\mathbb{K}[\mathbf{G}])$.

Let conversely $\mathbb{K}[\mathbf{G}] = \mathrm{soc}(\mathbb{K}[\mathbf{G}])$. Then for any $\mathbf{G}$-submodule $U \leq \mathbb{K}[\mathbf{G}]$ we have $\mathrm{soc}(U) = U \cap \mathrm{soc}(\mathbb{K}[\mathbf{G}]) = U$, that is U is semisimple. Next, for any $n \in \mathbb{N}$ and any $\mathbf{G}$-submodule $U \leq \mathbb{K}[\mathbf{G}]^n$ there are $\mathbf{G}$-submodules $U_1,\ldots,U_n \leq \mathbb{K}[\mathbf{G}]$ such that $U \leq \bigoplus_{i=1}^n U_i \leq \mathbb{K}[\mathbf{G}]^n$, where $\bigoplus_{i=1}^n U_i$ being semisimple entails that U is semisimple as well. Finally, recall that any $\mathbf{G}$-module is isomorphic to a $\mathbf{G}$-submodule of $\mathbb{K}[\mathbf{G}]^n$ for some $n \in \mathbb{N}$. $\sharp$

In this case we have the decomposition $\mathbb{K}[\mathbf{G}] = \mathrm{soc}(\mathbb{K}[\mathbf{G}]) = \bigoplus_{\sigma\in\Sigma_{\mathbf{G}}} \mathbb{K}[\mathbf{G}]_\sigma$ into isotypic components, with multiplicities $[\mathbb{K}[\mathbf{G}]\colon S] = \dim_{\mathbb{K}}(S)$, for $S \in \sigma$.

(5.6) Theorem. Let $\mathbf{G}$ be an affine algebraic group. Then $\mathbf{G}$ is linearly reductive, if and only if for any $\mathbf{G}$-epimorphism $\varphi\colon V \to W$, where V and W are $\mathbf{G}$-modules, the induced map $\varphi^{\mathbf{G}}\colon V^{\mathbf{G}} \to W^{\mathbf{G}}$ is surjective as well.

Proof. Note first that for any $\varphi \in \mathrm{Hom}_{\mathbb{K}}(V,W)$ we have $\varphi(V^{\mathbf{G}}) \leq W^{\mathbf{G}}$, so that $\varphi^{\mathbf{G}}$ is well-defined in any case. Now let $\mathbf{G}$ be linearly reductive. Then we have $\mathrm{Hom}_{\mathbf{G}}(V,W) = \bigoplus_{\sigma\in\Sigma_{\mathbf{G}}} \mathrm{Hom}_{\mathbf{G}}(V_\sigma,W_\sigma)$, implying that $\varphi \in \mathrm{Hom}_{\mathbf{G}}(V,W)$ is surjective if and only if all its components $\varphi_\sigma\colon V_\sigma \to W_\sigma$ are so, where $\varphi^{\mathbf{G}}$ is the component of φ associated with the trivial $\mathbf{G}$-module.

Let conversely $\mathbf{G}$ have the asserted property, let V be a $\mathbf{G}$-module, and let $U \leq V$ be a $\mathbf{G}$-submodule. We show that U has a $\mathbf{G}$-invariant complement in V: To this end, we consider the $\mathbf{G}$-modules $\mathrm{Hom}_{\mathbb{K}}(V,U)$ and $\mathrm{End}_{\mathbb{K}}(U)$, and the $\mathbf{G}$-equivariant map $\Phi\colon \mathrm{Hom}_{\mathbb{K}}(V,U) \to \mathrm{End}_{\mathbb{K}}(U)\colon \alpha \mapsto \alpha|_U$. Choosing a $\mathbb{K}$-vector space complement of U in V shows that Φ is surjective.

Hence $\Phi^{\mathbf{G}}\colon \mathrm{Hom}_{\mathbf{G}}(V,U) = \mathrm{Hom}_{\mathbb{K}}(V,U)^{\mathbf{G}} \to \mathrm{End}_{\mathbb{K}}(U)^{\mathbf{G}} = \mathrm{End}_{\mathbf{G}}(U)$ is surjective as well. Thus in particular there is $\varphi \in \mathrm{Hom}_{\mathbf{G}}(V,U)$ such that $\varphi|_U = \mathrm{id}_U \in \mathrm{End}_{\mathbf{G}}(U)$. Hence we have $U \cap \ker(\varphi) = \{0\}$. Moreover, for $v \in V$ we have $\varphi(\varphi(v)) = \mathrm{id}_U(\varphi(v)) = \varphi(v)$, thus $\varphi(v - \varphi(v)) = 0$, hence $v - \varphi(v) \in \ker(\varphi)$, thus $v \in U + \ker(\varphi)$. This shows that $V = U \oplus \ker(\varphi)$ as $\mathbf{G}$-modules. $\sharp$

Corollary. The group $\mathbf{G}$ is linearly reductive, if and only if any $\mathbf{G}$-module V has a (unique) $\mathbf{G}$-submodule $V' \leq V$ such that $V = V^{\mathbf{G}} \oplus V'$ and $(V'^{\vee})^{\mathbf{G}} = \{0\}$.

Proof. Let $\mathbf{G}$ be linearly reductive, and let V be a $\mathbf{G}$-module. Then V being semisimple we have the isotypic decomposition $V = V^{\mathbf{G}} \oplus V'$, where $V' := \bigoplus_{\sigma \in \Sigma'_{\mathbf{G}}} V_\sigma$, where in turn $\Sigma'_{\mathbf{G}} \subseteq \Sigma_{\mathbf{G}}$ is the union of the isomorphism classes of simple $\mathbf{G}$-modules being non-isomorphic to the trivial $\mathbf{G}$-module $\mathbb{K}$.

Assume that $(V'^{\vee})^{\mathbf{G}} \neq \{0\}$, then there a $\mathbf{G}$-monomorphism $\varphi \colon \mathbb{K} \to V'^{\vee}$. Hence the induced map $\varphi^* \colon V' = V'^{\vee\vee} \to \mathbb{K}^{\vee} \cong \mathbb{K}$ is a $\mathbf{G}$-epimorphism, which since $\mathrm{Hom}_{\mathbf{G}}(V', \mathbb{K}) \cong \bigoplus_{\sigma \in \Sigma'_{\mathbf{G}}} \mathrm{Hom}_{\mathbf{G}}(V_\sigma, \mathbb{K}) = \{0\}$ is a contradiction.

Note that $V' \leq V$ is unique: If $V = V^{\mathbf{G}} \oplus U$ as $\mathbf{G}$-modules, where we may assume that $U \neq \{0\}$, then letting $S \leq U$ be a simple $\mathbf{G}$-submodule we conclude that $S \not\cong \mathbb{K}$, entailing $S \leq V'$. Hence we infer that $U \leq V'$, which entails equality.

Let conversely $\mathbf{G}$ have the asserted property, and let $V = V^{\mathbf{G}} \oplus V'$ and $W = W^{\mathbf{G}} \oplus W'$ be $\mathbf{G}$-modules, where $(V'^{\vee})^{\mathbf{G}} = \{0\}$. Assume that $\mathrm{Hom}_{\mathbf{G}}(V', W^{\mathbf{G}}) \neq \{0\}$. Then, since $W^{\mathbf{G}} \cong \mathbb{K}^r$ for some $r \in \mathbb{N}$, we get $(V'^{\vee})^{\mathbf{G}} \cong \mathrm{Hom}_{\mathbb{K}}(V', \mathbb{K})^{\mathbf{G}} = \mathrm{Hom}_{\mathbf{G}}(V', \mathbb{K}) \neq \{0\}$, a constradiction. Hence we have $\mathrm{Hom}_{\mathbf{G}}(V', W^{\mathbf{G}}) = \{0\}$.

Now let $\varphi \colon V \to W$ be a $\mathbf{G}$–epimorphism, let $y \in W^{\mathbf{G}}$, and let $x \in V$ such that $\varphi(x) = y$. Writing $x = \widetilde{x} + x'$, where $\widetilde{x} \in V^{\mathbf{G}}$ and $x' \in V'$, we get $\varphi(\widetilde{x}) \in W^{\mathbf{G}}$ and $\varphi(x') \in W'$. Thus from $\varphi(\widetilde{x}) + \varphi(x') = y \in W^{\mathbf{G}}$ we conclude that $\varphi(x') = 0$ and $y = \varphi(\widetilde{x}) \in \varphi(V^{\mathbf{G}})$. Hence $\varphi^{\mathbf{G}} \colon V^{\mathbf{G}} \to W^{\mathbf{G}}$ is surjective. $\sharp$

(5.7) Theorem. Let $\mathbf{G}$ be an affine algebraic group.
a) Let $\mathbf{G}$ be linearly reductive. Then any closed normal subgroup of $\mathbf{G}$ and any homomorphic image of $\mathbf{G}$ are linearly reductive as well.
b) Let $\pi \colon \mathbf{G} \to \mathbf{H}$ a homomorphism of affine algebraic groups. If both $\ker(\pi) \trianglelefteq \mathbf{G}$ and $\pi(\mathbf{G}) \leq \mathbf{H}$ are linearly reductive, then $\mathbf{G}$ is linearly reductive as well.

Proof. a) Let $\varphi \colon \mathbf{G} \to \mathbf{H}$ be an epimorphism of affine algebraic groups, and let V be an $\mathbf{H}$-module with action α. Then V becomes a $\mathbf{G}$-module via $(\mathrm{id}_V \times \varphi)\alpha$. Since the $\mathbf{G}$-submodules of V coincide with its $\mathbf{H}$-submodules, the semisimplicity of V as an $\mathbf{H}$-module follows from its semisimplicity as a $\mathbf{G}$-module.

Let $\mathbf{M} \trianglelefteq \mathbf{G}$ be a closed normal subgroup. We show that $\mathrm{soc}(\mathbb{K}[\mathbf{M}]) = \mathbb{K}[\mathbf{M}]$, to which end we let $U \leq \mathbb{K}[\mathbf{M}]$ be an $\mathbf{M}$-submodule, and proceed to show that U is semisimple: Now we observe that $\rho \colon \mathbb{K}[\mathbf{G}] \to \mathbb{K}[\mathbf{M}] \colon f \mapsto f|_{\mathbf{M}}$ is an $\mathbf{M}$-equivariant epimorphism of $\mathbb{K}$-algebras, with respect to the right translation action of $\mathbf{M}$ on $\mathbb{K}[\mathbf{G}]$, where $\ker(\rho) = \mathcal{I}(\mathbf{M}) \trianglelefteq \mathbb{K}[\mathbf{G}]$ is the vanishing ideal of $\mathbf{M}$.

Picking a $\mathbb{K}$-vector space complement of $\ker(\rho)$ in $\rho^{-1}(U)$, and using local finiteness of the $\mathbf{G}$-action on $\mathbb{K}[\mathbf{G}]$, we conclude that there is a $\mathbf{G}$-submodule $W \leq \mathbb{K}[\mathbf{G}]$ such that $U \leq \rho(W)$. Hence it suffices to show that W is semisimple as an $\mathbf{M}$-module. By assumption, W is semisimple as a $\mathbf{G}$-module, thus is a sum of simple $\mathbf{G}$-modules. Hence it suffices to show that the restriction of any simple $\mathbf{G}$-module S to $\mathbf{M}$ is semisimple:

Let $T \leq S$ be a simple $\mathbf{M}$-submodule. From $\mathbf{M} \trianglelefteq \mathbf{G}$ we conclude that $Tg \leq S$ is $\mathbf{M}$-invariant as well, thus is a simple $\mathbf{M}$-submodule, for $g \in \mathbf{G}$. Moreover,

$\{0\} \neq \sum_{g\in\mathbf{G}} Tg \leq S$ is a **G**-submodule, which implies equality, and thus shows that S as an **M**-module is a sum of simple **M**-submodules.

b) We may assume that $\pi(\mathbf{G}) = \mathbf{H}$, and let $\mathbf{M} := \ker(\pi) \trianglelefteq \mathbf{G}$. Letting $\varphi\colon V \to W$ be a **G**-epimorphism, by assumption the induced map $\varphi^{\mathbf{M}}\colon V^{\mathbf{M}} \to W^{\mathbf{M}}$ is surjective. Moreover, $V^{\mathbf{M}} = \{x \in V; xh = x \text{ for all } h \in \mathbf{M}\} \leq V$ is **G**-invariant and carries the trivial **M**-action, hence the **G**-action factors through the group homomorphism π, so that $V^{\mathbf{M}}$ carries a $\mathbb{K}$-linear **H**-action. We show that thus $V^{\mathbf{M}}$ becomes an **H**-module, that is **H** acts morphically:

Identifying **H** with the set $\mathbf{M}\backslash\mathbf{G}$ of right **M**-cosets in **G**, it follows from the linear reductivity of **M** and (7.7) (which actually has been proven for this case), that the epimorphism $\pi\colon \mathbf{G} \to \mathbf{H}$ coincides with the quotient morphism associated with the left translation action of **M** on **G**. Hence the associated injective comorphism $\pi^*\colon \mathbb{K}[\mathbf{H}] \to \mathbb{K}[\mathbf{G}]$ fulfills $\pi^*(\mathbb{K}[\mathbf{H}]) = \mathbb{K}[\mathbf{G}]^{\mathbf{M}} = \{f \in \mathbb{K}[\mathbf{G}]; f(hg) = f(g) \text{ for all } g \in \mathbf{G}, h \in \mathbf{M}\}$; note that since π is **M**-invariant it is immediate that $\pi^*(\mathbb{K}[\mathbf{H}]) \subseteq \mathbb{K}[\mathbf{G}]^{\mathbf{M}}$, but equality is not.

Let $\{x_1, \ldots, x_n\} \subseteq V^{\mathbf{M}}$ be a $\mathbb{K}$-basis, where $n := \dim_{\mathbb{K}}(V^{\mathbf{M}}) \in \mathbb{N}_0$, and let $\mathbf{G} \to \mathbf{GL}_n\colon g \mapsto [g_{ij}(g)]_{ij}$ be the associated algebraic representation, with matrix coordinate functions $g_{ij} \in \mathbb{K}[\mathbf{G}]$. Then letting $\mathcal{X} := \{X_1, \ldots, X_n\}$ be the associated coordinate functions, the **G**-action on $V^{\mathbf{M}}$ has comorphism $\mathbb{K}[\mathcal{X}] \to \mathbb{K}[\mathcal{X}] \otimes_{\mathbb{K}} \mathbb{K}[\mathbf{G}]\colon X_i \mapsto \sum_{j=1}^{n} X_j \otimes g_{ji}$. Since **M** acts trivially on $V^{\mathbf{M}}$, we have $g_{ij}(hg) = g_{ij}(g)$, for $g \in \mathbf{G}$ and $h \in \mathbf{M}$, that is $g_{ij} \in \mathbb{K}[\mathbf{G}]^{\mathbf{M}} = \pi^*(\mathbb{K}[\mathbf{H}])$, for $i, j \in \{1, \ldots, n\}$. Thus we have a comorphism $\mathbb{K}[\mathcal{X}] \to \mathbb{K}[\mathcal{X}] \otimes_{\mathbb{K}} \pi^*(\mathbb{K}[\mathbf{H}])$, in other words the matrix coordinate functions give rise to regular maps on **H**.

This proves that $V^{\mathbf{M}}$ is an **H**-module. Similarly $W^{\mathbf{M}} \leq W$ is, and $\varphi^{\mathbf{M}}$ is **H**-equivariant. Hence by assumption the induced map $\varphi^{\mathbf{G}} = (\varphi^{\mathbf{M}})^{\mathbf{H}}\colon V^{\mathbf{G}} = (V^{\mathbf{M}})^{\mathbf{H}} \to (W^{\mathbf{M}})^{\mathbf{H}} = W^{\mathbf{G}}$ is surjective. Thus **G** is linearly reductive. ♯

We will show in (7.7) that for any linearly reductive closed normal subgroup $\mathbf{M} \trianglelefteq \mathbf{G}$ the quotient group $\mathbf{G}/\mathbf{M}$ carries the structure of an affine algebraic group, such that the natural map $\mathbf{G} \to \mathbf{G}/\mathbf{M}$ is a homomorphism of algebraic groups. Thus b) actually is the converse of a), saying that if $\mathbf{M} \trianglelefteq \mathbf{G}$ is any closed normal subgroup such that both **M** and $\mathbf{G}/\mathbf{M}$ are linearly reductive, then **G** is linearly reductive as well.

Corollary. The group **G** is linearly reductive if and only if $\mathbf{G}^\circ$ is linearly reductive and $\mathrm{char}(\mathbb{K}) \nmid [\mathbf{G}\colon \mathbf{G}^\circ]$.

Proof. Since $\mathbf{G}^\circ \trianglelefteq \mathbf{G}$ has finite index, $H := \mathbf{G}/\mathbf{G}^\circ$ is an affine algebraic group, and the natural quotient map $\varphi\colon \mathbf{G} \to H$ is a morphism. Moreover, since φ is constant on $\mathbf{G}^\circ$-orbits we have $\pi^*(\mathbb{K}[H]) \subseteq \mathbb{K}[\mathbf{G}]^{\mathbf{G}^\circ}$. Since H is finite, $\mathbb{K}[H]$ is the set of all maps from H to $\mathbb{K}$. Now any element of $\mathbb{K}[\mathbf{G}]^{\mathbf{G}^\circ}$ naturally induces such a map, hence it follows without further ado that $\pi^*(\mathbb{K}[H]) = \mathbb{K}[\mathbf{G}]^{\mathbf{G}^\circ}$. Then the assertion follows from Maschke's Theorem. ♯

6 Reductivity

(6.1) Reynolds operators. We proceed to a further rephrasement of linear reductivity. To this end, we need a couple of new notions. In order to introduce the first one let $\mathbf{G}$ be an affine algebraic group.

Let V be an affine $\mathbf{G}$-variety. Then a **Reynolds operator** on $\mathbb{K}[V]$ is a $\mathbf{G}$-equivariant $\mathbb{K}$-linear **projection** $\mathcal{R} = \mathcal{R}_{\mathbf{G},V} \colon \mathbb{K}[V] \to \mathbb{K}[V]^{\mathbf{G}}$ onto $\mathbb{K}[V]^{\mathbf{G}}$, that is $\mathcal{R}|_{\mathbb{K}[V]^{\mathbf{G}}} = \mathrm{id}_{\mathbb{K}[V]^{\mathbf{G}}}$. Recall that hence $\mathbb{K}[V]^{\mathbf{G}} \cap \ker(\mathcal{R}) = \{0\}$, and for $f \in \mathbb{K}[V]$ we have $\mathcal{R}(f - \mathcal{R}(f)) = 0$, thus $f \in \mathbb{K}[V]^{\mathbf{G}} + \ker(\mathcal{R})$, showing that $\mathbb{K}[V] = \mathbb{K}[V]^{\mathbf{G}} \oplus \ker(\mathcal{R})$ as $\mathbf{G}$-invariant $\mathbb{K}$-subspaces.

Example. If $\mathbf{G}$ is finite such that $\mathrm{char}(\mathbb{K}) \nmid |\mathbf{G}|$, then a Reynolds operator is given by averaging $\mathcal{R} \colon f \mapsto \frac{1}{|\mathbf{G}|} \cdot \sum_{g \in \mathbf{G}} f^g$: For $g \in \mathbf{G}$ and $f \in \mathbb{K}[V]$ we have $\mathcal{R}(f)^g = \frac{1}{|\mathbf{G}|} \cdot \sum_{h \in \mathbf{G}} f^{hg} = \frac{1}{|\mathbf{G}|} \cdot \sum_{h \in \mathbf{G}} f^h = \mathcal{R}(f)$, showing that $\mathcal{R}(f) \in \mathbb{K}[V]^{\mathbf{G}}$; moreover, we have $\mathcal{R}(f^g) = \frac{1}{|\mathbf{G}|} \cdot \sum_{h \in \mathbf{G}} f^{gh} = \frac{1}{|\mathbf{G}|} \cdot \sum_{h \in \mathbf{G}} f^h = \mathcal{R}(f) = \mathcal{R}(f)^g$, showing that $\mathcal{R}$ is $\mathbf{G}$-equivariant; finally, for $f \in \mathbb{K}[V]^{\mathbf{G}}$ we have $\mathcal{R}(f) = \frac{1}{|\mathbf{G}|} \cdot \sum_{h \in \mathbf{G}} f^h = \frac{1}{|\mathbf{G}|} \cdot \sum_{h \in \mathbf{G}} f = \frac{|\mathbf{G}|}{|\mathbf{G}|} \cdot f = f$, showing that $\mathcal{R}|_{\mathbb{K}[V]^{\mathbf{G}}} = \mathrm{id}_{\mathbb{K}[V]^{\mathbf{G}}}$.

Example. We consider the multiplicative group $\mathbf{G}_m$, having coordinate algebra $\mathbb{K}[T, T^{-1}]$. For an affine $\mathbf{G}_m$-variety V with action $\alpha \colon V \times \mathbf{G}_m \to V$ we get the comorphism $\alpha^* \colon \mathbb{K}[V] \to \mathbb{K}[V] \otimes_{\mathbb{K}} \mathbb{K}[T, T^{-1}] \colon f \mapsto \sum_{i \in \mathbb{Z}} f_i \otimes T^i$, with uniquely determined coefficients $f_i \in \mathbb{K}[V]$, that is $f^{t^{-1}}(x) = f(xt) = f(\alpha(x,t)) = \alpha^*(f)(x,t) = (\sum_{i \in \mathbb{Z}} f_i \otimes T^i)(x,t) = \sum_{i \in \mathbb{Z}} f_i(x) t^i$, for $x \in V$ and $t \in \mathbf{G}_m$.

Then a Reynolds operator is given as $\mathcal{R}(f) := f_0 \in \mathbb{K}[V]$, for $f \in \mathbb{K}[V]$:

For $x \in V$ and $s \in \mathbf{G}_m$ we have $\sum_{i \in \mathbb{Z}} f_i(xt) s^i = f(xt \cdot s) = f(x \cdot ts) = \sum_{i \in \mathbb{Z}} f_i(x) t^i s^i$, showing that $(f_i)^t = f_i t^{-i}$ for $i \in \mathbb{Z}$ and $t \in \mathbf{G}_m$; in particular we have $\mathcal{R}(f) = f_0 \in \mathbb{K}[V]^{\mathbf{G}_m}$. Moreover, we have $\sum_{i \in \mathbb{Z}} (f^t)_i(x) s^i = f^t(xs) = f(xs \cdot t^{-1}) = f(x \cdot st^{-1}) = \sum_{i \in \mathbb{Z}} f_i(x) t^{-i} s^i$, showing that $(f^t)_i = f_i t^{-i} = (f_i)^t$; in particular we have $\mathcal{R}(f^t) = (f^t)_0 = (f_0)^t = \mathcal{R}(f)^t$, thus $\mathcal{R}$ is $\mathbf{G}_m$-equivariant. Finally, for $f \in \mathbb{K}[V]^{\mathbf{G}_m}$ we have $\sum_{i \in \mathbb{Z}} f_i(x) t^{-i} = f(xt) = f(x) = \sum_{i \in \mathbb{Z}} f_i(x)$, showing that $f_i = 0$ for $i \neq 0$, hence we have $\mathcal{R}(f) = f_0 = f$. ♯

(6.2) Characterisation of linear reductivity. Let $\mathbf{G}$ be an affine algebraic group. Before proceeding we introduce a second new notion:

The group $\mathbf{G}$ is called **linear-geometrically reductive**, if for any $\mathbf{G}$-module V and any $0 \neq v \in V^{\mathbf{G}}$ there is $f \in (V^\vee)^{\mathbf{G}} = \mathbb{K}[V]_1^{\mathbf{G}}$ such that $f(v) \neq 0$.

Theorem. The following are equivalent:
i) The group $\mathbf{G}$ is linearly reductive, that is any $\mathbf{G}$-module is semisimple.
ii) For any affine $\mathbf{G}$-variety V there is a (unique) Reynolds operator on $\mathbb{K}[V]$.
iii) The group $\mathbf{G}$ is linear-geometrically reductive.

Proof. i)⇒ii): Since any $\mathbf{G}$-module is semisimple, any $\mathbf{G}$-submodule $U \leq \mathbb{K}[V]$ has a decomposition $U = U^{\mathbf{G}} \oplus U'$, where $U' = \bigoplus_{\sigma \in \Sigma'_{\mathbf{G}}} U_\sigma$. Hence let $\mathcal{R}_U \colon U \to U^{\mathbf{G}}$ be the associated $\mathbf{G}$-equivariant projection onto $U^{\mathbf{G}}$, that is $\mathcal{R}_U|_{U^{\mathbf{G}}} = \mathrm{id}_{U^{\mathbf{G}}}$

and $\ker(\mathcal{R}_U) = U'$. Then, whenever $W \leq \mathbb{K}[V]$ is a $\mathbf{G}$-submodule such that $U \leq W$, we have $\mathcal{R}_W|_U = \mathcal{R}_U$: We have $U^{\mathbf{G}} \leq W^{\mathbf{G}}$ and $U' \leq W'$, hence $\mathcal{R}_W|_{U^{\mathbf{G}}} = \mathrm{id}_{U^{\mathbf{G}}} = \mathcal{R}_U|_{U^{\mathbf{G}}}$ and $\mathcal{R}_W|_{U'} = 0 = \mathcal{R}_U|_{U'}$.

Thus we may define a $\mathbb{K}$-linear map $\mathcal{R}\colon \mathbb{K}[V] \to \mathbb{K}[V]$ as follows: For any $f \in \mathbb{K}[V]$ there is a $\mathbf{G}$-submodule $U \leq \mathbb{K}[V]$ containing f, and we let $\mathcal{R}(f) := \mathcal{R}_U(f)$. Indeed, by the compatibility property shown above this is independent of the choice of U. Then $\mathcal{R}$ is a Reynolds operator on $\mathbb{K}[V]$:

Since $\mathcal{R}_U$ is $\mathbf{G}$-equivariant, for all $\mathbf{G}$-submodules $U \leq \mathbb{K}[V]$, we conclude that $\mathcal{R}$ is $\mathbf{G}$-equivariant as well. From $\mathcal{R}_U(U) = U^{\mathbf{G}}$ we conclude that $\mathcal{R}(\mathbb{K}[V]) \leq \mathbb{K}[V]^{\mathbf{G}}$, from which, since $\mathbb{K}[V]^{\mathbf{G}}$ is the union of such $U^{\mathbf{G}}$, we infer equality. Finally, since $\mathcal{R}_U|_{U^{\mathbf{G}}} = \mathrm{id}_{U^{\mathbf{G}}}$ we get $\mathcal{R}|_{\mathbb{K}[V]^{\mathbf{G}}} = \mathrm{id}_{\mathbb{K}[V]^{\mathbf{G}}}$.

Note that $\mathcal{R}$ is the only possible choice how a Reynolds operator might look like: Let $\mathcal{R}'$ be a Reynolds operator on $\mathbb{K}[V]$, and let $U \leq \mathbb{K}[V]$ be a $\mathbf{G}$-submodule. Then $\mathcal{R}'|_{\mathbb{K}[V]^{\mathbf{G}}} = \mathrm{id}_{\mathbb{K}[V]^{\mathbf{G}}}$ implies that $\mathcal{R}'|_{U^{\mathbf{G}}} = \mathrm{id}_{U^{\mathbf{G}}}$. Moreover, we have $\mathcal{R}'(U') \leq \mathbb{K}[V]^{\mathbf{G}}$, where $\mathcal{R}'(U')$ has zero isotypic component with respect to any non-trivial simple $\mathbf{G}$-module, thus we have $\mathcal{R}'(U') = \{0\}$. This shows that $\mathcal{R}'|_U = \mathcal{R}_U$, and since $\mathbb{K}[V]$ is the union of such U we infer $\mathcal{R}' = \mathcal{R}$.

ii)$\Rightarrow$iii): Let V be a $\mathbf{G}$-module. Then we have $V \cong V^{\vee\vee} = \mathbb{K}[V^\vee]_1$ as $\mathbf{G}$-modules, where $v \in V$ is identified with the evaluation map $v^\bullet\colon V^\vee \to \mathbb{K}\colon \lambda \mapsto \lambda(v)$. For $0 \neq v \in V^{\mathbf{G}}$, that is $v^\bullet \in \mathbb{K}[V^\vee]_1^{\mathbf{G}}$, we may choose a $\mathbb{K}$-linear form $\lambda\colon \mathbb{K}[V^\vee]_1^{\mathbf{G}} \to \mathbb{K}$ such that $\lambda(v^\bullet) \neq 0$. Then the Reynolds operator $\mathcal{R}$ yields the $\mathbf{G}$-equivariant map $f := {}^\bullet \cdot \mathcal{R} \cdot \lambda\colon V \to \mathbb{K}[V^\vee]_1 \to \mathbb{K}[V^\vee]_1^{\mathbf{G}} \to \mathbb{K}$. Hence we have $f \in (V^\vee)^{\mathbf{G}}$ such that $f(v) = \lambda(\mathcal{R}(v^\bullet)) = \lambda(v^\bullet) \neq 0$.

iii)$\Rightarrow$i): Recall that for $v \in V$ and $\lambda \in V^\vee$ and $g \in \mathbf{G}$ we have $\lambda^g(v) = \lambda(vg^{-1})$. Hence for any $\mathbf{G}$-submodule $U \leq V$ we get the $\mathbf{G}$-submodule ${}^\perp U := \{\lambda \in V^\vee; U \leq \ker(\lambda)\} \leq V^\vee$, and similarly for any $\mathbf{G}$-submodule $W \leq V^\vee$ we get the $\mathbf{G}$-submodule $W^\perp := \{v \in V; W \leq \ker(v^\bullet)\} \leq V$. Recall that as $\mathbf{G}$-modules we have $(V/U)^\vee \cong {}^\perp U$ and $U^\vee \cong V^\vee/{}^\perp U$, as well as $(V^\vee/W)^\vee \cong (W^\perp)^\bullet \cong W^\perp$ and $W^\vee \cong (V/W^\perp)^\bullet \cong V/W^\perp$.

We consider the $\mathbf{G}$-equivariant $\mathbb{K}$-linear map $\rho\colon (V^\vee)^{\mathbf{G}} \to (V^{\mathbf{G}})^\vee\colon \lambda \mapsto \lambda|_{V^{\mathbf{G}}}$. Then, by applying the assumption to $V^\vee$, we get $\ker(\rho) = (V^\vee)^{\mathbf{G}} \cap {}^\perp(V^{\mathbf{G}}) = \{0\}$, that is ρ is injective. For the associated map $\rho^*\colon V^{\mathbf{G}} \to ((V^\vee)^{\mathbf{G}})^\vee\colon v \mapsto (v^\bullet\colon \lambda \mapsto \lambda(v))$, by applying the assumption to V, we get $\ker(\rho^*) = V^{\mathbf{G}} \cap ((V^\vee)^{\mathbf{G}})^\perp = \{0\}$, that is ρ^* is injective, or equivalently ρ is surjective.

Now we have $V^\vee = (V^\vee)^{\mathbf{G}} \oplus {}^\perp(V^{\mathbf{G}})$ as $\mathbf{G}$-modules: We have already seen that $(V^\vee)^{\mathbf{G}} \cap {}^\perp(V^{\mathbf{G}}) = \{0\}$, so that the sum is direct; moreover we have $V^\vee/{}^\perp(V^{\mathbf{G}}) \cong (V^{\mathbf{G}})^\vee \cong (V^\vee)^{\mathbf{G}}$ as $\mathbf{G}$-modules, so that the sum equals $V^\vee$.

In order to show the required fixed point property of ${}^\perp(V^{\mathbf{G}})$, we first observe that similarly $V = V^{\mathbf{G}} \oplus ((V^\vee)^{\mathbf{G}})^\perp$ as $\mathbf{G}$-modules: We have $V^{\mathbf{G}} \cap ((V^\vee)^{\mathbf{G}})^\perp = \{0\}$, so that the sum is direct; moreover we have $V/((V^\vee)^{\mathbf{G}})^\perp \cong ((V^\vee)^{\mathbf{G}})^\vee \cong V^{\mathbf{G}}$ as $\mathbf{G}$-modules, so that the sum equals V.

Thus we have $({}^\perp(V^{\mathbf{G}}))^\vee \cong (V/V^{\mathbf{G}})^{\vee\vee} \cong V/V^{\mathbf{G}} \cong ((V^\vee)^{\mathbf{G}})^\perp$, where the latter is a complement of $V^{\mathbf{G}}$ in V, so that $(({}^\perp(V^{\mathbf{G}}))^\vee)^{\mathbf{G}} \cong (((V^\vee)^{\mathbf{G}})^\perp)^{\mathbf{G}} = \{0\}$. $\sharp$

Corollary. If $\mathbb{K}[V] = \mathrm{soc}(\mathbb{K}[V])$ then the Reynolds operator $\mathcal{R}$ exists, and viewing $\mathbb{K}[V]$ as a $\mathbb{K}[V]^{\mathbf{G}}$-module then $\mathcal{R}$ is a $\mathbb{K}[V]^{\mathbf{G}}$-module homomorphism.

Proof. The first assertion is what is actually proved in the implication **i)**$\Rightarrow$**ii)**.

Then note that we have shown that any $\mathbf{G}$-submodule $U \leq \mathbb{K}[V]$ is $\mathcal{R}$-invariant, and $\mathcal{R}|_U = \mathcal{R}_U \colon U \to U^{\mathbf{G}}$ is the $\mathbf{G}$-equivariant projection onto $U^{\mathbf{G}}$, along the unique complement $U' = \bigoplus_{\sigma \in \Sigma'_{\mathbf{G}}} U_\sigma$ of $U^{\mathbf{G}}$ in U.

Next, recall that any $f \in \mathbb{K}[V]^{\mathbf{G}}$ entails the $\mathbf{G}$-equivariant $\mathbb{K}$-linear multiplication map $\rho_f \colon \mathbb{K}[V] \to \mathbb{K}[V] \colon h \mapsto hf$. Hence $Uf \leq \mathbb{K}[V]$ is a $\mathbf{G}$-submodule such that $(Uf)^{\mathbf{G}} = U^{\mathbf{G}} f$ and $(Uf)' = U'f$, thus $Uf = U^{\mathbf{G}} f \oplus U'f$ as $\mathbf{G}$-modules.

Now we show that $\mathcal{R}(hf) = \mathcal{R}(h)f$, for any $f \in \mathbb{K}[V]^{\mathbf{G}}$ and $h \in \mathbb{K}[V]$: Let $U \leq \mathbb{K}[V]$ be a $\mathbf{G}$-submodule containing h. If $h \in U^{\mathbf{G}}$ then we have $hf \in (Uf)^{\mathbf{G}}$, and thus $\mathcal{R}(hf) = \mathcal{R}_{Uf}(hf) = hf = \mathcal{R}_U(h)f = \mathcal{R}(h)f$. If $h \in U'$ then we have $hf \in (Uf)'$, and thus $\mathcal{R}(hf) = \mathcal{R}_{Uf}(hf) = 0 = \mathcal{R}_U(h')f = \mathcal{R}(h)f$. ♯

(6.3) Theorem: Hilbert's Finiteness Theorem [1890]. Let $\mathbf{G}$ be linearly reductive. Then $\mathbb{K}[V]^{\mathbf{G}}$ is a finitely generated $\mathbb{K}$-algebra, for any $\mathbf{G}$-module V.

Proof. Let $I := \langle \bigoplus_{d \in \mathbb{N}} \mathbb{K}[V]^{\mathbf{G}}_d \rangle \trianglelefteq \mathbb{K}[V]$ be the **Hilbert ideal**, that is the (homogeneous) ideal of $\mathbb{K}[V]$ generated by the maximal homogeneous ideal of $\mathbb{K}[V]^{\mathbf{G}}$. Since $\mathbb{K}[V]$ is Noetherian, there are elements $f_i \in \mathbb{K}[V]^{\mathbf{G}}_{d_i}$, where $d_i \in \mathbb{N}$, for $i \in \{1, \ldots, r\}$ and $r \in \mathbb{N}_0$, such that $I = \langle f_1, \ldots, f_r \rangle \trianglelefteq \mathbb{K}[V]$. We show that any element of $\mathbb{K}[V]^{\mathbf{G}}_d$, where $d \in \mathbb{N}_0$, is a polynomial in $\{f_1, \ldots, f_r\}$:

We proceed by induction on $d \in \mathbb{N}_0$, the case $d = 0$ being trivial we let $d \geq 1$. Letting $h \in \mathbb{K}[V]^{\mathbf{G}}_d$, we may write $h = \sum_{i=1}^r g_i f_i \in I$, with homogeneous elements $g_i \in \mathbb{K}[V]_{d-d_i}$. Applying the Reynolds operator yields $h = \mathcal{R}(h) = \mathcal{R}(\sum_{i=1}^r g_i f_i) = \sum_{i=1}^r \mathcal{R}(g_i f_i) = \sum_{i=1}^r \mathcal{R}(g_i) f_i$. Since $\mathbb{K}[V]_{d-d_i} \leq \mathbb{K}[V]$ is a $\mathbf{G}$-submodule, we conclude that $\mathcal{R}(g_i) \in \mathcal{R}(\mathbb{K}[V]_{d-d_i}) = \mathbb{K}[V]^{\mathbf{G}}_{d-d_i}$ is homogeneous of degree $< d$, and thus by induction is a polynomial in $\{f_1, \ldots, f_r\}$. ♯

Corollary. Let V be an affine $\mathbf{G}$-variety V with coordinate algebra $\mathbb{K}[V]$.
a) Then $\mathbb{K}[V]^{\mathbf{G}}$ is a finitely generated $\mathbb{K}$-algebra.
b) For any $\sigma \in \Sigma_{\mathbf{G}}$, the component $\mathbb{K}[V]_\sigma$ is a finitely generated $\mathbb{K}[V]^{\mathbf{G}}$-module.

Proof. a) Let $\mathbb{K}^n$, where $n \in \mathbb{N}_0$, be a $\mathbf{G}$-module such that there is a $\mathbf{G}$-equivariant closed embedding $\varphi \colon V \to \mathbb{K}^n$. Then for the $\mathbf{G}$-equivariant surjective comorphism $\varphi^* \colon \mathbb{K}[\mathcal{X}] \to \mathbb{K}[V]$ we have $\varphi^*(\mathbb{K}[\mathcal{X}]^{\mathbf{G}}) = \mathbb{K}[V]^{\mathbf{G}}$. Hence, since $\mathbb{K}[\mathcal{X}]^{\mathbf{G}}$ is finitely generated, $\mathbb{K}[V]^{\mathbf{G}}$ is so as well.

b) Recall that $\mathbb{K}[V]$ has an isotypic decomposition, with $\mathbb{K}[V]^{\mathbf{G}}$-invariant components. Let $S \in \sigma$ be a simple $\mathbf{G}$-module. Since $\mathbf{G}$ acts locally finitely on $\mathbb{K}[V]$, the map $\epsilon_S \colon S \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathbf{G}}(S, \mathbb{K}[V]) \to \mathbb{K}[V]_\sigma \colon v \otimes \varphi \mapsto \varphi(v)$ is a $\mathbf{G}$-equivariant bijection, with trivial $\mathbf{G}$-action on $\mathrm{Hom}_{\mathbf{G}}(S, \mathbb{K}[V])$. Moreover, $\mathrm{Hom}_{\mathbf{G}}(S, \mathbb{K}[V])$ becomes a $\mathbb{K}[V]^{\mathbf{G}}$-module by $\varphi \mapsto \varphi\rho_f$, for $\varphi \in \mathrm{Hom}_{\mathbf{G}}(S, \mathbb{K}[V])$ and $f \in \mathbb{K}[V]^{\mathbf{G}}$; recall that $\rho_f \colon \mathbb{K}[V] \to \mathbb{K}[V] \colon h \mapsto hf$ is $\mathbf{G}$-equivariant. Hence ϵ_S is an isomorphism of $\mathbb{K}[V]^{\mathbf{G}}$-modules, $\mathbb{K}[V]^{\mathbf{G}} \cong \mathbb{K} \otimes_{\mathbb{K}} \mathbb{K}[V]^{\mathbf{G}}$ acting 'trivially' on S.

Now we have $\mathbb{K}[S] = \bigoplus_{d \in \mathbb{N}_0} \mathbb{K}[S]_d$, with $\mathbf{G}$-invariant homogeneous components. Then $S \times V$ is an affine $\mathbf{G}$-variety with respect to diagonal action, so that $\mathbb{K}[S \times V]^{\mathbf{G}} \cong (\mathbb{K}[S] \otimes_{\mathbb{K}} \mathbb{K}[V])^{\mathbf{G}}$ is a finitely generated $\mathbb{K}$-algebra. We have $(\mathbb{K}[S] \otimes_{\mathbb{K}} \mathbb{K}[V])^{\mathbf{G}} = \bigoplus_{d \in \mathbb{N}_0} (\mathbb{K}[S]_d \otimes_{\mathbb{K}} \mathbb{K}[V])^{\mathbf{G}}$ as graded $\mathbb{K}$-algebras, so that in particular $(\mathbb{K}[S]_1 \otimes_{\mathbb{K}} \mathbb{K}[V])^{\mathbf{G}}$ is a finitely generated $(\mathbb{K}[S]_0 \otimes_{\mathbb{K}} \mathbb{K}[V])^{\mathbf{G}}$-module, with module structure inherited from multiplication in $\mathbb{K}[S] \otimes_{\mathbb{K}} \mathbb{K}[V]$. Finally, we have $(\mathbb{K}[S]_0 \otimes_{\mathbb{K}} \mathbb{K}[V])^{\mathbf{G}} \cong \mathbb{K}[V]^{\mathbf{G}}$, and thus $(\mathbb{K}[S]_1 \otimes_{\mathbb{K}} \mathbb{K}[V])^{\mathbf{G}} \cong (S^\vee \otimes_{\mathbb{K}} \mathbb{K}[V])^{\mathbf{G}} \cong \mathrm{Hom}_{\mathbb{K}}(S, \mathbb{K}[V])^{\mathbf{G}} = \mathrm{Hom}_{\mathbf{G}}(S, \mathbb{K}[V])$ as $\mathbb{K}[V]^{\mathbf{G}}$-modules. ♯

Despite appearance, and actually being part of **Hilbert's 14th problem**, the invariant algebra associated with an affine $\mathbf{G}$-variety is not necessarily finitely generated, not even for $\mathbf{G}$-modules for $\mathrm{char}(\mathbb{K}) = 0$; see Exercise **??** for the famous counterexample by [**Nagata, 1959**].

(6.4) Geometrical reductivity. An affine algebraic group $\mathbf{G}$ is called **geometrically reductive**, if for any $\mathbf{G}$-module V and any $0 \neq v \in V^{\mathbf{G}}$ there is $f \in \mathbb{K}[V]_d^{\mathbf{G}}$, for some $d \in \mathbb{N}$, such that $f(v) \neq 0$.

Hence if $\mathbf{G}$ is linearly reductive then it is geometrically reductive. But if $\mathrm{char}(\mathbb{K}) > 0$ the converse does not hold, so that in this case geometrical reductivity is a genuine generalisation of linear reductivity. To see this we observe below that any finite group G is geometrically reductive, but recall that G is linearly reductive if and only if $\mathrm{char}(\mathbb{K}) \nmid |G|$.

Theorem: Nagata–Miyata [1963]. If $\mathrm{char}(\mathbb{K}) = 0$, then $\mathbf{G}$ is geometrically reductive if and only if it is linearly reductive. ♯

The relevance of the notion of geometrical reductivity is elucidated by the following theorem, whose 'only if' direction generalises Hilbert's Finiteness Theorem, while its 'if' direction provides the converse of the generalised version; for the proof of the 'only if' direction by [**Nagata**] see [2, Sect.3.2]:

Theorem: Nagata [1963]; Popov [1979]. The group $\mathbf{G}$ is geometrically reductive if and only if $\mathbb{K}[V]^{\mathbf{G}}$ is finitely generated for any affine $\mathbf{G}$-variety V. ♯

At least we are able to prove the following:

Theorem. Let G be a finite group. Then G is geometrically reductive, and $\mathbb{K}[V]^G$ is a finitely generated $\mathbb{K}$-algebra for any affine G-variety V.

Proof. i) Let V be a G-module and $0 \neq v \in V^G$. Letting $\mathbb{K}[V]$ be the associated coordinate algebra, we apply **Dade's trick** to find a homogeneous invariant f of positive degree such that $f(v) \neq 0$:

Let $\lambda \in \mathbb{K}[V]_1$ such that $\lambda(v) \neq 0$. Then we have $\lambda^g(v) = \lambda(vg^{-1}) = \lambda(v)$, for $g \in G$. Moreover, for $f_\lambda := \prod_{g \in G} \lambda^g \in \mathbb{K}[V]_{|G|}$ we get $(f_\lambda)^h = (\prod_{g \in G} \lambda^g)^h = \prod_{g \in G} \lambda^{gh} = \prod_{g \in G} \lambda^g = f_\lambda$, for $h \in G$. Hence we conclude that $f_\lambda \in \mathbb{K}[V]_{|G|}^G$ such that $f_\lambda(v) = \lambda(v)^{|G|} \neq 0$.

ii) Let V be an affine $\mathbf{G}$-variety, and let $\{f_1, \ldots, f_n\} \subseteq \mathbb{K}[V]$ be a $\mathbb{K}$-algebra generating set, for some $n \in \mathbb{N}_0$. Then letting G act trivially on $\mathbb{K}[T]$, let $p_i := \prod_{g \in G}(T - (f_i)^g) = T^j + \sum_{j=0}^{|G|-1} a_{ij} T^j \in \mathbb{K}[V][T] \cong \mathbb{K}[V] \otimes_{\mathbb{K}} \mathbb{K}[T]$, for $i \in \{1, \ldots, n\}$. Hence we get $(p_i)^g = \prod_{h \in G}(T - (f_i)^h)^g = \prod_{h \in G}(T - (f_i)^{hg}) = \prod_{h \in G}(T - (f_i)^h) = p_i$, for $g \in G$, implying that $p_i \in \mathbb{K}[V,T]^G = (\mathbb{K}[V]^G)[T]$.

Hence we have $a_{ij} \in \mathbb{K}[V]^G$, and let $A := \mathbb{K}[a_{ij}; i \in \{1, \ldots, n\}, j \in \{0, \ldots, |G|-1\}] \subseteq \mathbb{K}[V]^G \subseteq \mathbb{K}[V]$. Since p_i is monic, we infer that f_i is integral over A. Thus $\mathbb{K}[V]$ is a finitely generated A-module. Since A is a finitely generated $\mathbb{K}$-algebra, it is Noetherian. Hence $\mathbb{K}[V]$ is a Noetherian A-module, and thus $\mathbb{K}[V]^G \subseteq \mathbb{K}[V]$ is a finitely generated A-module. From this we conclude that $\mathbb{K}[V]^G$ is a finitely generated $\mathbb{K}$-algebra. $\sharp$

Note that this also shows that $\mathbb{K}[V]$ is a finitely generated $\mathbb{K}[V]^G$-module, thus $\mathbb{K}[V]$ is a finite algebra extension of $\mathbb{K}[V]^G$.

(6.5) Example. We consider a couple of examples, which are not linearly reductive: Firstly, let $\mathrm{char}(\mathbb{K}) = p > 0$ and let $G = \langle g \rangle \cong C_p$ be the finite cyclic group of order p, which hence is geometrically reductive. Then $V := \mathbb{K}^2$ becomes a G-module by letting $g \mapsto J := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$; recall that $J^p = E_2$. Hence g acts on V by $[x, y] \mapsto [x, y + x]$, for $x, y \in \mathbb{K}$. Moreover, we get $V^G = \ker(J - E_2) = \langle e_2 \rangle_{\mathbb{K}}$. We show that there is a homogeneous invariant of degree p not annihilating e_2, but there is no such invariant of smaller positive degree:

Let $\mathbb{K}[V] = \mathbb{K}[X, Y]$ be the associated coordinate algebra, with coordinate functions $\{X, Y\}$, on which g acts (from the right) by $X \mapsto X$ and $Y \mapsto Y - X$. Then Dade's trick yields $f_Y \in \mathbb{K}[X, Y]_p^G$ as $f_Y = \prod_{i=0}^{p-1} Y^{g^i} = \prod_{i=0}^{p-1}(Y - iX) = X^p \cdot \prod_{i=0}^{p-1}(\frac{Y}{X} - i) = X^p \cdot \big((\frac{Y}{X})^p - \frac{Y}{X}\big) = Y^p - YX^{p-1} \in \mathbb{K}[X, Y]_X$.

Thus we have $f_Y(e_2) = 1$. To see that we cannot do better we show that $\mathbb{K}[X, Y]^G = \mathbb{K}[X, f_Y]$; then, since $\mathbb{K}[X, Y]^G \subseteq \mathbb{K}[X, Y]$ is a finite algebra extension, we have $\dim(\mathbb{K}[X, Y]^G) = \dim(\mathbb{K}[X, Y]) = 2$, hence $\{X, f_Y\}$ are algebraically independent, so that $\mathbb{K}[X, f_Y]$ actually is a polynomial algebra:

To show that $\mathbb{K}[X, Y]^G \subseteq \mathbb{K}[X, f_Y]$, for $d \in \mathbb{N}$ let $f := \sum_{i=0}^d a_i X^i Y^{d-i} \in \mathbb{K}[X, Y]_d^G$, where $a_i \in \mathbb{K}$. If $a_0 = 0$, then $f = X \cdot f'$, where $f' \in \mathbb{K}[X, Y]_{d-1}^G$. Hence we may assume that $a_0 = 1$, implying $f = X^d \cdot \big((\frac{Y}{X})^d + \sum_{i=1}^d a_i (\frac{Y}{X})^{d-i}\big) = X^d \cdot \prod_{i=1}^d (\frac{Y}{X} - \alpha_i) = \prod_{i=1}^d (Y - \alpha_i X) \in \mathbb{K}[X, Y]_X$, for suitable $\alpha_i \in \mathbb{K}$. Now we have $f = f^g = \prod_{i=1}^d \big(Y - (\alpha_i + 1)X\big)$. Hence there are $\beta_j \in \{\alpha_1, \ldots, \alpha_d\}$ such that $f = \prod_{j=1}^k f_{Y - \beta_j X}$, for some $k \in \mathbb{N}$, where $f_{Y - \beta X} = \prod_{i=0}^{p-1}(Y - \beta X)^{g^i} = \prod_{i=0}^{p-1} \big(Y - (\beta + i)X\big) = \prod_{i=0}^{p-1} \big((Y - \beta X) - iX\big) = f_Y(X, Y - \beta X) = (Y - \beta X)^p - (Y - \beta X)X^{p-1} = (Y^p - YX^{p-1}) - (\beta^p - \beta)X^p = f_Y - f_Y(X, \beta X)$, for $\beta \in \mathbb{K}$. This shows that $f = \prod_{j=1}^k \big(f_Y - f_Y(X, \beta_j X)\big) \in \mathbb{K}[X, f_Y]$. $\sharp$

Example. Secondly, we show that the additive group $\mathbf{G}_a$ is not even geometrically reductive; note that this is contrary to the behaviour of the finite cyclic group C_p of order p, which if $\mathrm{char}(\mathbb{K}) = p > 0$ is a subgroup of $\mathbf{G}_a$:

We use the isomorphism $\mathbf{G}_a \to \mathbf{U}_2 \colon t \mapsto \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ of algebraic groups. Hence let $V := \mathbb{K}^2$ be the natural $\mathbf{G}_a$-module, on which $t \in \mathbb{K}$ acts by $[x, y] \mapsto [x, y + tx]$, for $x, y \in \mathbb{K}$. In particular, we have $V^{\mathbf{G}_a} = \langle e_2 \rangle_{\mathbb{K}}$.

Let $\mathbb{K}[V] = \mathbb{K}[X, Y]$ be the associated coordinate algebra, with coordinate functions $\{X, Y\}$, on which $t \in \mathbb{K}$ acts (on the right) by the $\mathbb{K}$-algebra automorphism defined by $X \mapsto X$ and $Y \mapsto Y - tX$. We proceed to show that $\mathbb{K}[X,Y]^{\mathbf{G}_a} = \mathbb{K}[X]$; then we have $f|_{V^{\mathbf{G}_a}} = 0$ for all $f \in \mathbb{K}[X,Y]_d^{\mathbf{G}_a}$ and $d \in \mathbb{N}$:

For $d \in \mathbb{N}$ let $f := \sum_{i=0}^d a_i X^i Y^{d-i} \in \mathbb{K}[X,Y]_d$, for $a_i \in \mathbb{K}$. Then we have $f^{(-t)} = \sum_{i=0}^d a_i X^i (Y + tX)^{d-i} = \sum_{i=0}^d \sum_{j=0}^{d-i} a_i t^j \binom{d-i}{j} X^{i+j} Y^{d-i-j}$. Hence letting $k = i + j$ we get $f^{(-t)} = \sum_{k=0}^d \left(\sum_{i=0}^k a_i t^{k-i} \binom{d-i}{k-i} \right) X^k Y^{d-k} \in \mathbb{K}[X,Y]_d$. Thus we have $f \in \mathbb{K}[X,Y]_d^{\mathbf{G}_a}$ if and only if $\sum_{i=0}^k a_i \binom{d-i}{k-i} T^{k-i} = a_k \in \mathbb{K}[T]$. This is equivalent to $a_i \binom{d-i}{k-i} = 0$ for $0 \leq i < k \leq d$; note that the summand for $i = k$ indeed equals a_k. Choosing $k = d$ this entails $a_i = 0$ for $i \in \{0, \ldots, d-1\}$, in other words $f = a_d X^d \in \mathbb{K}[X]_d \subseteq \mathbb{K}[X,Y]_d^{\mathbf{G}_a}$. ♯

Hence there are affine $\mathbf{G}_a$-varieties whose associated invariant algebra is not finitely generated; see Exercise **??** for an example of a non-linear $\mathbf{G}_a$-action on $\mathbb{K}^5$ where $\mathrm{char}(\mathbb{K}) = 0$ [**Daigle–Freudenburg, 1999**]. Still, in many cases the invariant algebra associated with an affine $\mathbf{G}_a$-variety is finitely generated, for example, this is the case for any $\mathbf{G}_a$-action on $\mathbb{K}^n$ where $n \leq 3$ [**Zariski, 1954**], and any $\mathbf{G}_a$-module where $\mathrm{char}(\mathbb{K}) = 0$ [**Weitzenböck, 1932**].

(6.6) Reductivity. We proceed to rephrase geometrical reductivity in terms of group theory. To this end, we recall a few notions and facts (without proofs) from general theory of affine algebraic groups:

Let $\mathbf{G}$ be an affine algebraic group. For $g, h \in \mathbf{G}$ let $[g,h] := g^{-1}h^{-1}gh \in \mathbf{G}$ be their **commutator**. For subgroups $U, H \leq \mathbf{G}$ let $[U,H] := \langle [u,h]; u \in U, h \in H \rangle \leq \mathbf{G}$ be the associated **commutator subgroup**.

The weakly descending **derived series** of normal subgroups of $\mathbf{G}$ is defined as $\mathbf{G}^{(0)} := \mathbf{G}$, and $\mathbf{G}^{(i)} := [\mathbf{G}^{(i-1)}, \mathbf{G}^{(i-1)}] \trianglelefteq \mathbf{G}$ for $i \in \mathbb{N}$. Then $\mathbf{G}^{(i)}$ is closed indeed, and if $\mathbf{G}$ is connected then so are all the subgroups $\mathbf{G}^{(i)}$. Moreover, $\mathbf{G}$ is called **solvable** if there is $l \in \mathbb{N}_0$ such that $\mathbf{G}^{(l)} = \{1\}$.

Let the **(solvable) radical** $R(\mathbf{G})$ of $\mathbf{G}$ be the subgroup generated by all closed connected solvable normal subgroups of $\mathbf{G}$. Then $R(\mathbf{G}) \trianglelefteq \mathbf{G}$ is the unique maximal closed connected solvable normal subgroup of $\mathbf{G}$.

Then $\mathbf{G}$ is called **(group theoretically) reductive** if $R(\mathbf{G})$ is a torus, that is isomorphic to $(\mathbf{G}_m)^n$ for some $n \in \mathbb{N}_0$. In particular, $\mathbf{G}$ is called **semisimple** if $R(\mathbf{G}) = \{1\}$. Hence, since $R(\mathbf{G}) \trianglelefteq \mathbf{G}^\circ$ and thus $R(\mathbf{G}) = R(\mathbf{G}^\circ)$, we conclude that $\mathbf{G}$ is reductive (semisimple) if and only $\mathbf{G}^\circ$ is reductive (semisimple); note that we do not assume $\mathbf{G}$ to be connected here.

For example, any torus is reductive, and any finite group is semisimple, while $\mathbf{G}_a \cong \mathbf{U}_2$ is not reductive; recall that $\mathbf{G}_a$ is not isomorphic to $\mathbf{G}_m$, see Exercise **??**. We have already noted that $\mathbf{GL}_n$ and $\mathbf{SL}_n$, for $n \in \mathbb{N}$, are linearly reductive if $\mathrm{char}(\mathbb{K}) = 0$, hence are geometrically reductive, or equivalently reductive by the theorem below; reductivity of $\mathbf{GL}_n$ and semisimplicity of $\mathbf{SL}_n$ hold in arbitrary characteristc, and can be verified straightforwardly, see Exercise **??**.

Theorem: Mumford's Conjecture; Nagata–Miyata [1963]; Haboush [1975]. The group $\mathbf{G}$ is geometrically reductive if and only if it is reductive. ♯

7 Quotients

(7.1) Algebraic quotients. a) Let $\mathbf{G}$ be an affine algebraic group and let V be an affine $\mathbf{G}$-variety. We first observe the following:

If W is an affine variety and $\varphi\colon V \to W$ is a morphism, then φ is constant on $\mathbf{G}$-orbits if and only if $\varphi^*(\mathbb{K}[W]) \subseteq \mathbb{K}[V]^{\mathbf{G}}$:

If φ is constant on $\mathbf{G}$-orbits, then for $f \in \mathbb{K}[W]$ and $g \in \mathbf{G}$ we get $\varphi^*(f)^g(v) = \varphi^*(f)(vg^{-1}) = f(\varphi(vg^{-1})) = f(\varphi(v)) = \varphi^*(f)(v)$ for $v \in V$, showing that $\varphi^*(f)^g = \varphi^*(f)$. Conversely, if $\varphi^*(\mathbb{K}[W]) \subseteq \mathbb{K}[V]^{\mathbf{G}}$ then we get $f(\varphi(vg^{-1})) = \varphi^*(f)(vg^{-1}) = \varphi^*(f)^g(v) = \varphi^*(f)(v) = f(\varphi(v))$ for $f \in \mathbb{K}[W]$, which says that $\varphi(vg^{-1}) = \varphi(v)$, for $v \in V$ and $g \in \mathbf{G}$. ♯

b) Now we assume additionally that the invariant algebra $\mathbb{K}[V]^{\mathbf{G}}$ is a finitely generated $\mathbb{K}$-algebra. Recall that this is always the case if $\mathbf{G}$ is (geometrically) reductive or even linearly reductive (where we have only proved this here for the latter case), or if $\mathbf{G}$ is finite.

Then there is an affine variety Z, such that there is a morphism $\pi\colon V \to Z$ whose associated comorphism $\pi^*\colon \mathbb{K}[Z] \to \mathbb{K}[V]$ is an embedding with image $\pi^*(\mathbb{K}[Z]) = \mathbb{K}[V]^{\mathbf{G}}$. Then π is dominant and constant on $\mathbf{G}$-orbits; in particular, if V is irreducible then so is Z. Moreover, Z is uniquely determined up to isomorphism, being called the (**algebraic** or **categorical**) **quotient** variety of V with respect to $\mathbf{G}$. Then Z has the following universal property:

Proposition. If $\varphi\colon V \to W$ is constant on $\mathbf{G}$-orbits, then there is a unique morphism $\overline{\varphi}\colon Z \to W$ such that $\varphi = \pi\overline{\varphi}$:

Proof. Since φ is constant on $\mathbf{G}$-orbits we have $\varphi^*(\mathbb{K}[W]) \subseteq \mathbb{K}[V]^{\mathbf{G}} \subseteq \mathbb{K}[V]$. Letting $(\pi^*)'\colon \mathbb{K}[Z] \to \mathbb{K}[V]^{\mathbf{G}}$ be the isomorphism induced by π^*, we thus get the $\mathbb{K}$-algebra homomorphism $\varphi^*(\pi^*)'^{-1}\colon \mathbb{K}[W] \to \mathbb{K}[Z]$. Thus let $\overline{\varphi}\colon Z \to W$ be the morphism with associated comorphism $\overline{\varphi}^* = \varphi^*(\pi^*)'^{-1}$. Then we have $(\pi\overline{\varphi})^* = \overline{\varphi}^*\pi^* = \varphi^*(\pi^*)'^{-1}\pi^* = \varphi^*$, that is $\pi\overline{\varphi} = \varphi$.

We show uniqueness: Let $\widetilde{\varphi}\colon Z \to W$ be a morphism such that $\pi\widetilde{\varphi} = \varphi$, then $\widetilde{\varphi}^*\pi^* = \varphi^* = \overline{\varphi}^*\pi^*$, since π^* being injective, implies $\widetilde{\varphi}^* = \overline{\varphi}^*$, that is $\widetilde{\varphi} = \overline{\varphi}$. ♯

In particular, if $\pi'\colon V \to Z$ is any morphism as above, there are unique morphisms $\psi, \psi'\colon Z \to Z$ such that $\pi' = \pi\psi$ and $\pi = \pi'\psi'$, where $\pi' = \pi'\psi'\psi = \pi' \cdot \mathrm{id}_Z$ and $\pi = \pi\psi\psi' = \pi \cdot \mathrm{id}_Z$ imply that $\psi\psi' = \mathrm{id}_Z = \psi\psi'$, thus ψ is an isomorphism such that $\psi' = \psi^{-1}$. Hence, denoting the quotient variety by $V /\!\!/ \mathbf{G}$, the **quotient morphism** $\pi\colon V \to V /\!\!/ \mathbf{G}$ is unique up to a unique isomorphism, and we may identify π^* with the natural embedding $\mathbb{K}[V]^{\mathbf{G}} \subseteq \mathbb{K}[V]$.

Example. We consider a non-reductive example: The additive group $\mathbf{G}_a$ acts on $V := \mathbb{K}^2$ via $\mathbf{G}_a \to \mathbf{GL}_2\colon t \mapsto \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$, that is $t \in \mathbb{K}$ acts by $[x, y] \mapsto [x, y+tx]$, for $x, y \in \mathbb{K}$; see (6.5). Thus the $\mathbf{G}_a$-orbits in V are uniquely given as $[a, 0]\cdot\mathbf{G}_a = \{[a, y] \in V; y \in \mathbb{K}\}$ for $a \in \mathbb{K} \setminus \{0\}$, where $\dim([a, 0] \cdot \mathbf{G}_a) = \dim(\mathbf{G}_a) = 1$; and $[0, b]\cdot\mathbf{G}_a = \{[0, b]\}$ for $b \in \mathbb{K}$, where $\dim([0, b]\cdot\mathbf{G}_a) = 0$ and indeed $V^{\mathbf{G}_a} = \langle e_2 \rangle_{\mathbb{K}}$.

Let $\mathbb{K}[V] = \mathbb{K}[X, Y]$ be the associated coordinate algebra, on which $t \in \mathbb{K}$ acts by $X \mapsto X$ and $Y \mapsto Y - tX$. We have already shown that $\mathbb{K}[X,Y]^{\mathbf{G}_a} = \mathbb{K}[X]$. Since the latter is a finitely generated $\mathbb{K}$-algebra (although $\mathbf{G}_a$ is not reductive) the quotient exists: Indeed, $\mathbb{K}[X,Y]^{\mathbf{G}_a}$ being a univariate polynomial algebra the quotient variety is $V/\!\!/ \mathbf{G}_a \cong \mathbb{K}$, and its embedding into $\mathbb{K}[X, Y]$ shows that the (surjective) quotient morphism is given as $\pi = X : V \to \mathbb{K} : [x, y] \mapsto x$. Note that $\dim(V) - \dim(V/\!\!/ \mathbf{G}_a) = 1$, which coincides with the maximum dimension of a $\mathbf{G}_a$-orbit occurring.

Hence for $a \in \mathbb{K} \setminus \{0\}$ the fibre $\pi^{-1}(a) = \{[a, y] \in V; y \in \mathbb{K}\} = [a, 0] \cdot \mathbf{G}_a$ consists of a single $\mathbf{G}_a$-orbit, which hence is closed. But the fibre $\pi^{-1}(0) = \{[0, y] \in V; y \in \mathbb{K}\} = V^{\mathbf{G}_a} = \coprod_{b\in\mathbb{K}}\{[0, b]\}$ consists of infinitely many $\mathbf{G}_a$-orbits, which are all closed. In particular, π does not separate the $\mathbf{G}_a$-orbits. $\sharp$

(7.2) Properties of quotients. a) We first collect some general observations: Let $\mathbf{G}$ be an affine algebraic group, and let V be a $\mathbf{G}$-variety. Recall that any $\mathbf{G}$-orbit $O \subseteq V$ is open in its closure $\overline{O} \subseteq V$, so that we have $\dim(O) = \dim(\overline{O})$.

Lemma. If all $\mathbf{G}$-orbits have the same dimension, then all $\mathbf{G}$-orbits are closed.

Proof. Assume there are orbits $O \neq O'$ in V such that $O' \preceq O$, then we have $O' \subseteq \overline{O} \setminus O$, hence $\dim(\overline{O}') < \dim(\overline{O})$, a contradiction. Hence we conclude that all $\mathbf{G}$-orbits are $\preceq$-minimal, thus are all closed. $\sharp$

Proposition. The ($\mathbf{G}$-invariant) subset $V^{(\geq n)} := \{v \in V; \dim(\mathbf{G}_v) \geq n\} \subseteq V$ is closed, for any $n \in \mathbb{N}_0$.

Proof. Recall first **Chevalley's Theorem** on **upper semicontinuity** of dimension: Given a morphism of varieties $\varphi : W \to U$, and for $x \in W$ letting $\dim_x(\varphi^{-1}(\varphi(x)))$ be the maximal dimension of an irreducible component of $\varphi^{-1}(\varphi(x))$ containing x, then for $n \in \mathbb{N}_0$ the set $\{x \in W; \dim_x(\varphi^{-1}(\varphi(x))) \geq n\} \subseteq W$ is closed. (Note that this is typically stated for W irreducible and φ dominant, but the general case follows straightforwardly from this.)

Now, considering the graph morphism $\gamma : V \times \mathbf{G} \to V \times V : [v, g] \mapsto [v, vg]$, we get the closed subset $\widehat{V} := \gamma^{-1}(\Delta(V)) = \{[v, g] \in V \times \mathbf{G}; vg = v\} \subseteq V \times \mathbf{G}$. Letting $\nu : \widehat{V} \to V$ be the projection onto the left hand factor, we get $\nu^{-1}(\nu([v, 1_{\mathbf{G}}])) \cong \{v\} \times \mathbf{G}_v$. Since the irreducible component of the latter containing $[v, 1_{\mathbf{G}}]$ equals $\{v\} \times (\mathbf{G}_v)^\circ$, from $\dim(\mathbf{G}) = \dim(\mathbf{G}^\circ)$ we conclude that $\{[v, g] \in \widehat{V}; \dim(\mathbf{G}_v) \geq n\} \subseteq \widehat{V}$ is closed. Finally, using the section morphism $\sigma : V \to \widehat{V} : v \mapsto [v, 1_{\mathbf{G}}]$ of ν, we infer that $V^{(\geq n)} = \sigma^{-1}(\{[v, g] \in \widehat{V}; \dim(\mathbf{G}_v) \geq n\}) \subseteq V$ is closed. $\sharp$

In particular, if $m \in \{0, \ldots, \dim(\mathbf{G})\}$ is minimal such that $V^{(\geq m)} \neq \emptyset$, then the ($\mathbf{G}$-invariant) **stratum** $V^{(m)} := \{v \in V; \dim(\mathbf{G}_v) = m\} = V \setminus V^{(\geq m+1)} \subseteq V$, consisting of the $\mathbf{G}$-orbits of maximal dimension, is open.

b) Using this, we get a dimension formula for quotients: Let V be an affine $\mathbf{G}$-variety such that $\mathbb{K}[V]^{\mathbf{G}}$ is a finitely generated $\mathbb{K}$-algebra, and let $\pi : V \to V/\!\!/ \mathbf{G}$ be the associated quotient.

Proposition. Let V be irreducible. Then we have

$$\dim(V) - \dim(V/\!\!/\, \mathbf{G}) \geq \max\{\dim(O) \in \mathbb{N}_0; O \subseteq V \ \mathbf{G}\text{-orbit}\}.$$

Moreover, if π is geometric in the sense of (7.3) then we have equality.

Proof. By the dimension formula for morphisms, there is an open subset $\emptyset \neq U \subseteq \pi(V) \subseteq V/\!\!/\, \mathbf{G}$, such that $\dim(V) - \dim(V/\!\!/\, \mathbf{G}) = \dim(\pi^{-1}(\pi(v)))$ for $v \in \pi^{-1}(U) \subseteq V$. Letting $V^{(m)} \subseteq V$ be the open stratum as above, we have $\dim(\pi^{-1}(\pi(v))) \geq \dim(\overline{v\mathbf{G}}) = \dim(v\mathbf{G}) = \dim(\mathbf{G}) - m$ for $v \in V^{(m)}$. Since V is irreducible, we have $\pi^{-1}(U) \cap V^{(m)} \neq \emptyset$, hence for $v \in \pi^{-1}(U) \cap V^{(m)}$ we obtain $\dim(V) - \dim(V/\!\!/\, \mathbf{G}) \geq \dim(\mathbf{G}) - m = \max\{\dim(O) \in \mathbb{N}_0; O \subseteq V \ \mathbf{G}\text{-orbit}\}$.

Moreover, if $v \in \pi^{-1}(U) \cap V^{(m)}$ can be chosen such that $\pi^{-1}(\pi(v)) \subseteq V$ consists of a single $\mathbf{G}$-orbit, then we have equality $\dim(V) - \dim(V/\!\!/\, \mathbf{G}) = \dim(\mathbf{G}) - m$. In particular, this condition is fulfilled if π is geometric. ♯

c) Finally, we observe that quotients behave well with respect to affine open subsets: Let $\pi\colon V \to V/\!\!/\, \mathbf{G}$ be a quotient as above. Given $0 \neq f \in \mathbb{K}[V]^{\mathbf{G}}$, then $V_f \subseteq V$ and $(V/\!\!/\, \mathbf{G})_f \subseteq V/\!\!/\, \mathbf{G}$ are affine varieties with coordinate algebras $\mathbb{K}[V]_f$ and $(\mathbb{K}[V]^{\mathbf{G}})_f$, respectively; moreover, $V_f \subseteq V$ is $\mathbf{G}$-invariant.

Proposition. Let V be irreducible, and $0 \neq f \in \mathbb{K}[V]^{\mathbf{G}}$. Then we have $\pi^{-1}((V/\!\!/\, \mathbf{G})_f) = V_f$ and $\pi|_{V_f}\colon V_f \to (V/\!\!/\, \mathbf{G})_f$ is a quotient.

Proof. For $v \in V$ with associated maximal ideal $I_v \lhd \mathbb{K}[V]$, and $z \in V/\!\!/\, \mathbf{G}$ with associated maximal ideal $J_z \lhd \mathbb{K}[V]^{\mathbf{G}}$, we have $v \in \pi^{-1}(z)$ if and only if $J_z = I_v \cap \mathbb{K}[V]^{\mathbf{G}}$. Moreover, the maximal ideals of $\mathbb{K}[V]_f$ and $(\mathbb{K}[V]^{\mathbf{G}})_f$, respectively, can be identified by localisation with the maximal ideals of $\mathbb{K}[V]$ and $\mathbb{K}[V]^{\mathbf{G}}$, respectively, not containing f. Hence we have $\pi^{-1}((V/\!\!/\, \mathbf{G})_f) = V_f$, so that π restricts to a $\mathbf{G}$-equivariant morphism $V_f \to (V/\!\!/\, \mathbf{G})_f$ of affine varieties.

Since $\mathbb{K}[V]$ is an integral domain, the associated comorphism is the natural embedding $(\mathbb{K}[V]^{\mathbf{G}})_f \subseteq \mathbb{K}[V]_f \subseteq \mathbb{K}(V)$, where actually $(\mathbb{K}[V]^{\mathbf{G}})_f \subseteq (\mathbb{K}[V]_f)^{\mathbf{G}}$. If $h \in \mathbb{K}[V]$ and $i \in \mathbb{N}_0$ such that $\frac{h}{f^i} \in (\mathbb{K}[V]_f)^{\mathbf{G}}$, then we have $\frac{h}{f^i} = \left(\frac{h}{f^i}\right)^g = \frac{h^g}{f^i}$, that is $h = h^g$, for $g \in \mathbf{G}$. Hence we infer $h \in \mathbb{K}[V]^{\mathbf{G}}$, so that $\frac{h}{f^i} \in (\mathbb{K}[V]^{\mathbf{G}})_f$. ♯

(7.3) Geometric quotients. a) Let $\mathbf{G}$ be an affine algebraic group, let V be an affine $\mathbf{G}$-variety such that $\mathbb{K}[V]^{\mathbf{G}}$ is a finitely generated $\mathbb{K}$-algebra. and let $\pi\colon V \to V/\!\!/\, \mathbf{G}$ be the associated (algebraic) quotient.

Then π is called **geometric** if it induces a bijection between the $\mathbf{G}$-orbits in V and the points of $V/\!\!/\, \mathbf{G}$, that is π is surjective and any fibre of π consists of a single $\mathbf{G}$-orbit. In particular, if π is geometric then all $\mathbf{G}$-orbits are closed.

Proposition. Let V be irreducible and let π be geometric. Then all (closed) $\mathbf{G}$-orbits in V have one and the same dimension.

Proof. Since the $\mathbf{G}$-orbits coincide with the fibres of π, the dimension formula for $\mathbf{G}$-orbits yields $\dim(\pi^{-1}(\pi(v))) = \dim(v\mathbf{G}) = \dim(\mathbf{G}) - \dim(\mathbf{G}_v)$, for $v \in V$.

We apply Chevalley's Theorem on upper semicontinuity of dimension: Since the irreducible component of $\pi^{-1}(\pi(v)) = v\mathbf{G}$ containing v equals $v\mathbf{G}^\circ$, from $\dim(v\mathbf{G}) = \dim(v\mathbf{G}^\circ)$ we conclude that $V^{(\leq n)} := \{v \in V; \dim(\mathbf{G}_v) \leq n\} \subseteq V$ is closed, for $n \in \mathbb{N}_0$. Since the subset $V^{(\geq n)} \subseteq V$ is closed anyway, we conclude that the stratum $V^{(n)} = V^{(\leq n)} \cap V^{(\geq n)} = \{v \in V; \dim(\mathbf{G}_v) = n\} \subseteq V$ is closed.

This yields the finite decomposition $V = \coprod_{n=0}^{\dim(\mathbf{G})} V^{(n)}$ of V into open and closed subsets. Since V is irreducible, and hence connected, we conclude that there is a unique $m \in \{0, \ldots, \dim(\mathbf{G})\}$ such that $V^{(m)} \neq \emptyset$, thus $V = V^{(m)}$ says that for all $v \in V$ we have $\dim(\mathbf{G}_v) = m$, or equivalently $\dim(v\mathbf{G}) = \dim(\mathbf{G}) - m$. $\sharp$

Example. Let $\mathbf{G}$ be connected, and let V be irreducible possessing a $\mathbf{G}$-fixed point $v \in V$; note that the latter conditions are fulfilled if V is a $\mathbf{G}$-module for $v := 0$. Then V has a geometric quotient if and only if $\mathbf{G}$ acts trivially on V:

If $\mathbf{G}$ acts trivially on V then we have $\mathbb{K}[V]^{\mathbf{G}} = \mathbb{K}[V]$, implying that id_V is a (geometric) quotient. Conversely, if V has a geometric quotient, then from $\mathbf{G}_v = \mathbf{G}$ we infer that $V = V^{(\dim(\mathbf{G}))}$, thus all $\mathbf{G}$-orbits are finite, and hence by connectedness of $\mathbf{G}$ consist entirely of $\mathbf{G}$-fixed points. $\sharp$

b) Hence the notion of geometric quotients might be too restrictive, so that the situation might improve if we went down to suitable open subsets:

An element $v \in V$ is called **$\mathbf{G}$-regular** if there is $0 \neq f \in \mathbb{K}[V]^{\mathbf{G}}$ such that $v \in V_f \subseteq V^{(n)}$, for some $n \in \mathbb{N}_0$; in other words orbit dimension is constant on an open neighbourhood of $v \in V$. Letting $V^{\mathbf{G}\text{-reg}} \subseteq V$ be the set of $\mathbf{G}$-regular elements, we have $V^{\mathbf{G}\text{-reg}} = \coprod_{n=0}^{\dim(\mathbf{G})} (V^{\mathbf{G}\text{-reg}} \cap V^{(n)})$, where $V^{\mathbf{G}\text{-reg}} \cap V^{(n)} \subseteq V$ is open; in particular $V^{\mathbf{G}\text{-reg}} \subseteq V$ is open. Since the stratum $V^{(m)} \subseteq V$, consisting of the $\mathbf{G}$-orbits of maximal dimension, is open, we have $\emptyset \neq V^{(m)} \subseteq V^{\mathbf{G}\text{-reg}}$.

Letting $v \in V^{\mathbf{G}\text{-reg}}$ such that there is $w \in \overline{v\mathbf{G}} \setminus v\mathbf{G}$, then from $\dim(w\mathbf{G}) < \dim(v\mathbf{G})$ and any open neighbourhood of w intersecting $v\mathbf{G}$ we infer that $w \notin V^{\mathbf{G}\text{-reg}}$. Hence the $\mathbf{G}$-orbits in $V^{\mathbf{G}\text{-reg}}$ are closed in $V^{\mathbf{G}\text{-reg}}$. Thus affine open subsets of $V^{\mathbf{G}\text{-reg}}$ should be good candidates to possess a geometric quotient.

Now let V be irreducible. Then $V^{\mathbf{G}\text{-reg}}$ is dense and hence irreducible as well, implying that $V^{\mathbf{G}\text{-reg}} = V^{(m)}$. Moreover, if $0 \neq f \in \mathbb{K}[V]^{\mathbf{G}}$ such that the quotient $V_f \to (V/\!\!/\mathbf{G})_f$ is geometric, then we have $V_f \subseteq V^{(n)}$, where $n \in \mathbb{N}_0$ is the common dimension of the $\mathbf{G}$-orbits in V_f, and hence in this case we necessarily $V_f \subseteq V^{\mathbf{G}\text{-reg}} = V^{(m)}$.

(7.4) Quotients for linearly reductive groups. In order to collect the 'good' properties of quotients in this case, we first consider the relationship between the ideals of a coordinate algebra and the ideals of an associated invariant algebra. To this end, let $\mathbf{G}$ be an affine algebraic group and let V be an affine $\mathbf{G}$-variety such that $\mathbb{K}[V] = \mathrm{soc}(\mathbb{K}[V])$, which in particular entails that the Reynolds operator $\mathcal{R}\colon \mathbb{K}[V] \to \mathbb{K}[V]^{\mathbf{G}}$ exists. (For a moment we do not assume that $\mathbf{G}$ is linearly reductive, and neither that $\mathbb{K}[V]^{\mathbf{G}}$ is finitely generated.)

Lemma. a) If $I \trianglelefteq \mathbb{K}[V]$ is $\mathbf{G}$-invariant, then $\mathbb{K}[V]^{\mathbf{G}}/(I \cap \mathbb{K}[V]^{\mathbf{G}}) \cong (\mathbb{K}[V]/I)^{\mathbf{G}}$.
b) If $J \trianglelefteq \mathbb{K}[V]^{\mathbf{G}}$ is an ideal, then we have $(J \cdot \mathbb{K}[V]) \cap \mathbb{K}[V]^{\mathbf{G}} = J$.
c) If $\{I_j \trianglelefteq \mathbb{K}[V]; j \in \mathcal{J}\}$, where $\mathcal{J}$ is an index set, are $\mathbf{G}$-invariant ideals, then we have $(\sum_{j \in \mathcal{J}} I_j) \cap \mathbb{K}[V]^{\mathbf{G}} = \sum_{j \in \mathcal{J}} (I_j \cap \mathbb{K}[V]^{\mathbf{G}}) \trianglelefteq \mathbb{K}[V]^{\mathbf{G}}$.

Note that, since $\mathbb{K}[V]$ is Noetherian, it follows from b) that $\mathbb{K}[V]^{\mathbf{G}}$ is Noetherian.

Proof. a) The natural $\mathbf{G}$-equivariant $\mathbb{K}$-epimorphism $\mathbb{K}[V] \to \mathbb{K}[V]/I$ implies $\mathbb{K}[V]^{\mathbf{G}}/(I \cap \mathbb{K}[V]^{\mathbf{G}}) = \mathbb{K}[V]^{\mathbf{G}}/I^{\mathbf{G}} = \mathcal{R}(\mathbb{K}[V])/\mathcal{R}(I) \cong \mathcal{R}(\mathbb{K}[V]/I) = (\mathbb{K}[V]/I)^{\mathbf{G}}$.

b) Since $J \trianglelefteq \mathbb{K}[V]^{\mathbf{G}}$ is $\mathbf{G}$-invariant, so is $J \cdot \mathbb{K}[V] \trianglelefteq \mathbb{K}[V]$. Hence we have $(J \cdot \mathbb{K}[V]) \cap \mathbb{K}[V]^{\mathbf{G}} = (J \cdot \mathbb{K}[V])^{\mathbf{G}} = \mathcal{R}(J \cdot \mathbb{K}[V]) = J \cdot \mathcal{R}(\mathbb{K}[V]) = J \cdot \mathbb{K}[V]^{\mathbf{G}} = J$.

c) The ideals I_j being $\mathbf{G}$-invariant, we get $(\sum_{j \in \mathcal{J}} I_j) \cap \mathbb{K}[V]^{\mathbf{G}} = (\sum_{j \in \mathcal{J}} I_j)^{\mathbf{G}} = \mathcal{R}(\sum_{j \in \mathcal{J}} I_j) = \sum_{j \in \mathcal{J}} \mathcal{R}(I_j) = \sum_{j \in \mathcal{J}} I_j^{\mathbf{G}} = \sum_{j \in \mathcal{J}} (I_j \cap \mathbb{K}[V]^{\mathbf{G}})$. ♯

Theorem: 'Good' properties. Assume additionally that $\mathbb{K}[V]^{\mathbf{G}}$ is finitely generated, so that the quotient $\pi \colon V \to V /\!\!/ \mathbf{G}$ exists.

a) Closedness. If $W \subseteq V$ is a $\mathbf{G}$-invariant closed subset, then the subset $\pi(W) \subseteq V /\!\!/ \mathbf{G}$ is closed as well, and $\pi|_W \colon W \to \pi(W)$ is a quotient.

In particular, π is surjective and $V /\!\!/ \mathbf{G}$ carries the associated quotient topology.

b) Separation. If $\{W_j \subseteq V; j \in \mathcal{J}\}$ are $\mathbf{G}$-invariant closed subsets, where $\mathcal{J}$ is an index set, then we have $\pi(\bigcap_{j \in \mathcal{J}} W_j) = \bigcap_{j \in \mathcal{J}} \pi(W_j)$.

In particular, for $\mathbf{G}$-invariant closed subsets $W, W' \subseteq V$ such that $W \cap W' = \emptyset$ we have $\pi(W) \cap \pi(W') = \emptyset$; and any fibre of π contains a unique closed $\mathbf{G}$-orbit. Hence π induces a bijection between the closed $\mathbf{G}$-orbits and the points of $V /\!\!/ \mathbf{G}$.

Proof. Recall that $\pi^* \colon \mathbb{K}[V /\!\!/ \mathbf{G}] \cong \mathbb{K}[V]^{\mathbf{G}} \to \mathbb{K}[V]$ is the natural embedding.

a) Let $\mathcal{I}(W) \trianglelefteq \mathbb{K}[V]$ be the vanishing ideal of $W \subseteq V$; hence we have $\mathbb{K}[W] \cong \mathbb{K}[V]/\mathcal{I}(W)$. Then the vanishing ideal of $\overline{\pi(W)} \subseteq V /\!\!/ \mathbf{G}$ is given as $\mathcal{I}(\overline{\pi(W)}) = (\pi^*)^{-1}(\mathcal{I}(W)) = \mathcal{I}(W) \cap \mathbb{K}[V]^{\mathbf{G}} \trianglelefteq \mathbb{K}[V]^{\mathbf{G}}$, from which we get $\mathbb{K}[\overline{\pi(W)}] \cong \mathbb{K}[V]^{\mathbf{G}}/(\mathcal{I}(W) \cap \mathbb{K}[V]^{\mathbf{G}})$. Thus we conclude that the comorphism associated with $\pi|_W \colon W \to \overline{\pi(W)}$ is given as the natural embedding $(\pi|_W)^* \colon \mathbb{K}[\overline{\pi(W)}] \cong \mathbb{K}[V]^{\mathbf{G}}/(\mathcal{I}(W) \cap \mathbb{K}[V]^{\mathbf{G}}) \to \mathbb{K}[V]/\mathcal{I}(W) \cong \mathbb{K}[W]$.

Since W is $\mathbf{G}$-invariant, the ideal $\mathcal{I}(W)$ is $\mathbf{G}$-invariant as well: We have $f^g(w) = f(wg^{-1}) = 0$, for $w \in W$, hence $f^g \in \mathcal{I}(W)$, for $f \in \mathcal{I}(W)$ and $g \in \mathbf{G}$. Thus we get $\mathbb{K}[V]^{\mathbf{G}}/(\mathcal{I}(W) \cap \mathbb{K}[V]^{\mathbf{G}}) \cong (\mathbb{K}[V]/\mathcal{I}(W))^{\mathbf{G}}$. This shows that $(\pi|_W)^* \colon (\mathbb{K}[V]/\mathcal{I}(W))^{\mathbf{G}} \to \mathbb{K}[V]/\mathcal{I}(W)$, thus $\pi|_W \colon W \to \overline{\pi(W)}$ is a quotient.

Thus to show closedness, it suffices to consider the case $W = V$ and to show that π is surjective: Let $z \in V /\!\!/ \mathbf{G}$, with associated maximal ideal $J_z \triangleleft \mathbb{K}[V]^{\mathbf{G}}$. Then from $(J_z \cdot \mathbb{K}[V]) \cap \mathbb{K}[V]^{\mathbf{G}} = J_z$ we infer that $J_z \cdot \mathbb{K}[V] \triangleleft \mathbb{K}[V]$ is a proper ideal. Hence there is $v \in V$ such that for the associated maximal ideal we have $J_z \cdot \mathbb{K}[V] \subseteq I_v \triangleleft \mathbb{K}[V]$, thus $(\pi^*)^{-1}(I_v) = I_v \cap \mathbb{K}[V]^{\mathbf{G}} = J_z$, that is $\pi(v) = z$.

Finally, we have to show that a subset $U \subseteq V /\!\!/ \mathbf{G}$ is open if (and only if) $\pi^{-1}(U) \subseteq V$ is open: To this end, by going over to complements, let $W \subseteq V /\!\!/ \mathbf{G}$ such that $\pi^{-1}(W) \subseteq V$ is closed. Then, since π is constant on $\mathbf{G}$-orbits, $\pi^{-1}(W)$

is $\mathbf{G}$-invariant. Hence, by the closedness property and the surjectiviy of π, we infer that $W = \pi(\pi^{-1}(W)) \subseteq V/\!\!/\mathbf{G}$ is closed.

b) Let $I_j := \mathcal{I}(W_j) \trianglelefteq \mathbb{K}[V]$ be the vanishing ideal of $W_j \subseteq V$, for $j \in \mathcal{J}$. Since the W_j are $\mathbf{G}$-invariant, so are $W := \bigcap_{j\in\mathcal{J}} W_j \subseteq V$, as well as $I_j \trianglelefteq \mathbb{K}[V]$ and $I := \mathcal{I}(W) = \sum_{j\in\mathcal{J}} I_j \trianglelefteq \mathbb{K}[V]$. Thus, recalling that the preimage with respect to π^* of a radical ideal is a radical ideal again, we get $\pi(W) = \overline{\pi(W)} = \mathcal{V}\big((\pi^*)^{-1}(I)\big) = \mathcal{V}\big(I \cap \mathbb{K}[V]^{\mathbf{G}}\big) = \mathcal{V}\big((\sum_{j\in\mathcal{J}} I_j) \cap \mathbb{K}[V]^{\mathbf{G}}\big) = \mathcal{V}\big(\sum_{j\in\mathcal{J}} (I_j \cap \mathbb{K}[V]^{\mathbf{G}})\big) = \bigcap_{j\in\mathcal{J}} \mathcal{V}\big(I_j \cap \mathbb{K}[V]^{\mathbf{G}}\big) = \bigcap_{j\in\mathcal{J}} \mathcal{V}\big((\pi^*)^{-1}(I_j)\big) = \bigcap_{j\in\mathcal{J}} \overline{\pi(W_j)} = \bigcap_{j\in\mathcal{J}} \pi(W_j)$.

In particular, for $z \in V/\!\!/\mathbf{G}$ let $W := \pi^{-1}(z) \subseteq V$, a $\mathbf{G}$-invariant closed subset. Hence if $O \subseteq W$ is a $\mathbf{G}$-orbit, then we have $\overline{O} \subseteq W$ as well, thus any $\preceq$-minimal $\mathbf{G}$-orbit in $\overline{O}$ is a closed $\mathbf{G}$-orbit in W. And letting $O, O' \subseteq W$ be closed $\mathbf{G}$-orbits, then $\pi(O \cap O') = \pi(O) \cap \pi(O') = \{z\} \neq \emptyset$ implies that $O = O'$. ♯

(7.5) Quotients for reductive groups. a) Let $\mathbf{G}$ be (geometrically) reductive, and let V be an affine $\mathbf{G}$-variety. Then the invariant algebra $\mathbb{K}[V]^{\mathbf{G}}$ is a finitely generated $\mathbb{K}$-algebra (where we have only proved this here for the linear reductive case), so that the quotient $\pi\colon V \to V/\!\!/\mathbf{G}$ exists. First of all we get:

Proposition. Let $W, W' \subseteq V$ be $\mathbf{G}$-invariant closed subsets such that $W \cap W' = \emptyset$. Then there is $f \in \mathbb{K}[V]^{\mathbf{G}}$ such that $f(W) = \{0\}$ and $f(W') = \{1\}$. In particular, we have $\pi(W) \cap \pi(W') = \emptyset$.

Proof. First, there is $f_1 \in \mathbb{K}[V]$ such that $f_1(W) = \{0\}$ and $f_1(W') = \{1\}$: The embedding $W \,\dot\cup\, W' \to V$ entails the epimorphism $\mathbb{K}[V] \to \mathbb{K}[W \,\dot\cup\, W']$, hence we may assume that $V = W \,\dot\cup\, W'$. Then we have $\mathcal{I}(W) \cap \mathcal{I}(W') = \mathcal{I}(W \cup W') = \mathcal{I}(V) = \{0\}$ and $\mathcal{I}(W) + \mathcal{I}(W') = \mathcal{I}(W \cap W') = \mathcal{I}(\emptyset) = \mathbb{K}[V]$, implying $\mathbb{K}[V] \cong \mathbb{K}[V]/\mathcal{I}(W) \oplus \mathbb{K}[V]/\mathcal{I}(W') \cong \mathbb{K}[W] \oplus \mathbb{K}[W']$. Hence may take $f_1 := [0_W, 1_{W'}] \in \mathbb{K}[W] \oplus \mathbb{K}[W']$.

Let $U := \langle (f_1)^g \in \mathbb{K}[V]; g \in \mathbf{G}\rangle_{\mathbb{K}}$. Then, by local finiteness, $U \leq \mathbb{K}[V]$ is a $\mathbf{G}$-submodule, such that for all $h \in U$ we have $h(W) = \{0\}$ and $h(W')$ is a singleton set. Let $\{f_1, \ldots, f_n\} \subseteq U$ be a $\mathbb{K}$-basis, where $n := \dim_{\mathbb{K}}(U) \in \mathbb{N}_0$ and f_1 is as above. Then, by linearisation of $\mathbf{G}$-actions, the evaluation map $\varphi := [f_1^\bullet, \ldots, f_n^\bullet]\colon V \to \mathbb{K}^n\colon v \mapsto [f_1(v), \ldots, f_n(v)]$ is a homomorphism of $\mathbf{G}$-modules. Then we have $\varphi(W) = \{[0, \ldots, 0]\}$ and $\varphi(W') = \{z\}$, where $0 \neq z \in \mathbb{K}^n$. Since W' is $\mathbf{G}$-invarient, $z \in \mathbb{K}^n$ is $\mathbf{G}$-invarient as well. Hence by geometric reductivity there is a homogeneous invariant $h \in \mathbb{K}[\mathcal{X}]^{\mathbf{G}}$ of positive degree such that $h(z) = 1$, thus $h(0) = 0$ anyway. Then $f := \varphi^*(h) \in \mathbb{K}[V]^{\mathbf{G}}$ is as desired. ♯

Actually, π has all 'good' properties of quotients listed in (7.4), but (apart from the linearly reductive case, and the above separation property) we are not able to prove this here; see [2, Sect.3.3]. In particular, π is surjective, π maps closed $\mathbf{G}$-invariant subsets to closed subsets, $V/\!\!/\mathbf{G}$ carries the quotient topology afforded by π, any fibre of π contains a unique closed $\mathbf{G}$-orbit, and π induces a bijection between the closed $\mathbf{G}$-orbits and the points of $V/\!\!/\mathbf{G}$. This yields:

Corollary. Let $O \subseteq V$ be a $\mathbf{G}$-orbit; hence $\pi(O) \subseteq V/\!\!/\mathbf{G}$ is a singleton set.
i) Then the closure $\overline{O} \subseteq V$ contains a unique closed $\mathbf{G}$-orbit.

ii) Let O be closed, and let $W := \pi^{-1}(\pi(O)) \subseteq V$ be the associated fibre. Then we have $W = \{v \in V; O \subseteq \overline{v\mathbf{G}}\} = \{v \in V; O \preceq v\mathbf{G}\}$, and $W \subseteq V$ is the largest $\mathbf{G}$-invariant closed subset containing O as its unique closed $\mathbf{G}$-orbit.

Proof. i) Any $\preceq$-minimal $\mathbf{G}$-orbit in $\overline{O}$ is closed, where uniqueness follows from $O \subseteq \overline{O} \subseteq W := \pi^{-1}(\pi(O)) \subseteq V$.

ii) Let $w \in W$, then $\pi(O) = \pi(w) = \pi(\overline{w\mathbf{G}})$, hence $\pi(O \cap \overline{w\mathbf{G}}) = \pi(O) \cap \pi(\overline{w\mathbf{G}}) \neq \emptyset$, thus $O \cap \overline{w\mathbf{G}} \neq \emptyset$, hence $O \subseteq \overline{w\mathbf{G}}$. Conversely, let $v \in V$ such that $O \subseteq \overline{v\mathbf{G}}$, then $v\mathbf{G} \subseteq \overline{v\mathbf{G}} \subseteq \pi^{-1}(\pi(v)) \subseteq V$ implies $\pi(O) \cap \pi(\overline{v\mathbf{G}}) = \pi(O \cap \overline{v\mathbf{G}}) = \pi(O) = \pi(\overline{v\mathbf{G}}) = \pi(v)$, thus $v \in W$.

Moreover, W contains a unique closed $\mathbf{G}$-orbit, which hence coincides with O. Finally, let $U \subseteq V$ be a $\mathbf{G}$-invariant closed subset containing O as its unique closed $\mathbf{G}$-orbit, and assume that $U \not\subseteq W$. Then there is $\pi(O) \neq z \in V /\!\!/ \mathbf{G}$ such that $U \cap \pi^{-1}(z) \neq \emptyset$. Since the latter set is $\mathbf{G}$-invariant and closed, it contains a closed $\mathbf{G}$-orbit, which hence is distinct from O, a contradiction. ♯

(7.6) The nullcone. Let $\mathbf{G}$ be (geometrically) reductive, and let V be a $\mathbf{G}$-module. Then the fibre $\mathcal{N}(V) = \mathcal{N}_{\mathbf{G}}(V) := \pi^{-1}(\pi(0)) = \{v \in V; 0 \in \overline{v\mathbf{G}}\} = \{v \in V; \{0\} \preceq v\mathbf{G}\} \subseteq V$ is called the associated **(Hilbert) nullcone**. In can be characterised as follows:

Let $I_0 \lhd \mathbb{K}[V]$ be the maximal ideal associated with $0 \in V$. Then the maximal ideal $J_{\pi(0)} \lhd \mathbb{K}[V]^{\mathbf{G}}$ associated with $\pi(0) \in V /\!\!/ \mathbf{G}$ is given as $J_{\pi(0)} = I_0 \cap \mathbb{K}[V]^{\mathbf{G}} = \bigoplus_{d \in \mathbb{N}} \mathbb{K}[V]^{\mathbf{G}}_d \lhd \mathbb{K}[V]^{\mathbf{G}}$, where the latter is the maximal homogeneous ideal. Hence we have $\pi^{-1}(\pi(0)) = \mathcal{V}(J_{\pi(0)} \cdot \mathbb{K}[V])$, that is the zero set of the Hilbert ideal. In other words, we have $\mathcal{N}(V) = \mathcal{V}(J_{\pi(0)}) = \mathcal{V}(\sum_{d \in \mathbb{N}} \mathbb{K}[V]^{\mathbf{G}}_d) = \bigcap_{d \in \mathbb{N}} \mathcal{V}(\mathbb{K}[V]^{\mathbf{G}}_d) \subseteq V$, that is the elements of V being annihilated by all homogeneous invariants of positive degree.

The elements of the closed subset $\mathcal{N}(V)$ are called **unstable**, while the elements of the open subset $V \setminus \mathcal{N}(V)$ are called **semistable**; in other words the latter is the set of $v \in V$ such that there is $f \in \mathbb{K}[V]^{\mathbf{G}}_d$, for some $d \in \mathbb{N}$, such that $f(v) \neq 0$. In particular, geometric reductivity amounts to say that the non-zero $\mathbf{G}$-fixed points in V are semistable. Letting $V^{(m)} \subseteq V$ be the open stratum consisting of the $\mathbf{G}$-orbits of maximal dimension, the elements of the open subset $V^{(m)} \setminus \mathcal{N}(V)$ are called **stable**; in other words, since V is irreducible, these are the $\mathbf{G}$-regular semistable elements.

The relevance of the nullcone for the structure of the invariant algebra is further elucidated by the following:

Proposition: Hilbert [1893]. Let $f_1, \ldots, f_r \in \mathbb{K}[V]^{\mathbf{G}}$, for $r \in \mathbb{N}_0$, such that $\mathcal{V}(f_1, \ldots, f_r) = \mathcal{N}(V)$. Then $\mathbb{K}[f_1, \ldots, f_r] \subseteq \mathbb{K}[V]^{\mathbf{G}}$ is a finite algebra extension.

Proof. Let $J := \langle f_1, \ldots, f_r \rangle \unlhd \mathbb{K}[V]^{\mathbf{G}}$ be the ideal generated by $f_1, \ldots, f_r$. Then we have $\pi^{-1}(\mathcal{V}(J)) = \mathcal{V}(J \cdot \mathbb{K}[V]) = \mathcal{V}(f_1, \ldots, f_r) = \mathcal{N}(V) = \pi^{-1}(\pi(0)) \subseteq V$. Since π is surjective, we infer that $\mathcal{V}(J) = \pi(0)$. In other words $\sqrt{J} = J_{\pi(0)} \lhd \mathbb{K}[V]^{\mathbf{G}}$ is the maximal homogeneous ideal.

Now let $h_1, \ldots, h_s \in \mathbb{K}[V]^{\mathbf{G}}$ be homogeneous of positive degree, where $s \in \mathbb{N}_0$, such that $\mathbb{K}[V]^{\mathbf{G}} = \mathbb{K}[h_1, \ldots, h_s]$. Hence we have $h_i \in J_{\pi(0)}$, so that there is

$e \in \mathbb{N}$ such that $h_i^e \in J$, for $i \in \{1, \ldots, s\}$. Thus the finite set $\{\prod_{i=1}^s h_i^{e_i} \in \mathbb{K}[V]^{\mathbf{G}}; e_i \in \{0, \ldots, e-1\}\}$ generates $\mathbb{K}[V]^{\mathbf{G}}$ as a $\mathbb{K}[f_1, \ldots, f_r]$-module. $\sharp$

In view of **??**, if $\mathbf{G}$ is connected then it follows that $\mathbb{K}[V]^{\mathbf{G}}$ is the integral closure of $\mathbb{K}[f_1, \ldots, f_r]$ in $\mathbb{K}[V]$.

(7.7) Geometric quotients for reductive groups. Let $\mathbf{G}$ be (geometrically) reductive, let V be an affine $\mathbf{G}$-variety, and let $\pi \colon V \to V/\!\!/\,\mathbf{G}$ be the associated quotient. The 'good' properties of π yield the following characterisation:

Then π is geometric if and only if all $\mathbf{G}$-orbits are closed. Moreover, if V is irreducible, then π is geometric if and only all $\mathbf{G}$-orbits have the same dimension.

If π is geometric then the quotient variety $V/\!\!/\,\mathbf{G}$ can be identified as topological spaces with the **orbit space** of the $\mathbf{G}$-action on V, where the latter carries the quotient topology afforded by identifying points in the same $\mathbf{G}$-orbit.

Example. The $\mathbf{G}$-orbits in the open subset $V^{\mathbf{G}\text{-reg}} \subseteq V$ of $\mathbf{G}$-regular elements are closed in $V^{\mathbf{G}\text{-reg}}$. Thus if $V^{\mathbf{G}\text{-reg}}$ is affine again then it possesses a geometric quotient. But if $V^{\mathbf{G}\text{-reg}}$ contains a $\mathbf{G}$-orbit which is not closed in V, then it is not the preimage with respect to π of an (open) subset of $V/\!\!/\,\mathbf{G}$.

Similarly, letting $V^{(m)} \subseteq V^{\mathbf{G}\text{-reg}} \subseteq V$ be the open stratum consisting of the $\mathbf{G}$-orbits of maximal dimension, the latter are closed in $V^{(m)}$. Thus if $V^{(m)}$ is affine again then it possesses a geometric quotient. But if $V^{(m)}$ contains a $\mathbf{G}$-orbit which is not closed in V, then it is not the preimage with respect to π of an (open) subset of $V/\!\!/\,\mathbf{G}$. Recall that $V^{(m)} = V^{\mathbf{G}\text{-reg}}$ whenever V is irreducible.

Instead, we consider $\widetilde{V} := \{v \in V^{(m)}; v\mathbf{G} \subseteq V \text{ closed}\} \subseteq V$, that is the union of the closed $\mathbf{G}$-orbits amongst all $\mathbf{G}$-orbits of maximal dimension; hence $\widetilde{V}$ is $\mathbf{G}$-invariant, but possibly $\widetilde{V}$ is empty. Moreover, let $\widetilde{Z} := (V/\!\!/\,\mathbf{G}) \setminus \pi(V \setminus V^{(m)})$. Since $V \setminus V^{(m)} \subseteq V$ is a $\mathbf{G}$-invariant closed subset, $\pi(V \setminus V^{(m)}) \subseteq V/\!\!/\,\mathbf{G}$ is closed as well, hence $\widetilde{Z} \subseteq V/\!\!/\,\mathbf{G}$ is open. Now $\widetilde{V}$ and $\widetilde{Z}$ are related as follows:

For $v \in \widetilde{V}$ we have $v\mathbf{G} = \overline{v\mathbf{G}} \subseteq V^{(m)}$, hence $v\mathbf{G} \cap (V \setminus V^{(m)}) = \emptyset$ implies that $\pi(v\mathbf{G}) \cap \pi(V \setminus V^{(m)}) = \emptyset$, that is $\pi(v) \in \widetilde{Z}$. Next, for $v \in V \setminus V^{(m)}$ by construction we have $\pi(v) \notin \widetilde{Z}$. Finally, for $v \in V$ such that $v\mathbf{G} \neq \overline{v\mathbf{G}}$, letting $w \in \overline{v\mathbf{G}} \setminus v\mathbf{G}$ we have $\dim(w\mathbf{G}) < \dim(v\mathbf{G})$, hence $w \in V \setminus V^{(m)}$, entailing that $\pi(v) = \pi(w) \notin \widetilde{Z}$. Thus we conclude that $\pi(v) \in \widetilde{Z}$ if and only if $v \in \widetilde{V}$, that is $\pi^{-1}(\widetilde{Z}) = \widetilde{V}$. In particular, it follows that $\widetilde{V} \subseteq V^{(m)} \subseteq V$ is open.

Moreover, $\pi|_{\widetilde{V}} \colon \widetilde{V} \to \widetilde{Z}$ induces a bijection between the (closed) $\mathbf{G}$-orbits in $\widetilde{V}$ and the points of $\widetilde{Z}$. Since the affine variety $V/\!\!/\,\mathbf{G}$ carries the quotient topology induced by π, we conclude that $\widetilde{Z}$ is the orbit space of the $\mathbf{G}$-action on $\widetilde{V}$. Thus, if $\widetilde{V}$ is affine again, then it has a (geometric) quotient, which hence coincides with $\pi|_{\widetilde{V}}$; in particular, this happens if $\widetilde{V} \subseteq V$ is a principal open subset.

Example. Slightly changing notation, let $\mathbf{G}$ be an affine algebraic group, and let $\mathbf{H} \leq \mathbf{G}$ be a closed subgroup. Then $\mathbf{G}$ is an $\mathbf{H}$-variety with respect to the left translation action $\mathbf{G} \times \mathbf{H} \to \mathbf{G} \colon [g, h] \mapsto h^{-1}g$. The set of $\mathbf{H}$-orbits coincides with the set $\mathbf{H}\backslash\mathbf{G}$ of right $\mathbf{H}$-cosets in $\mathbf{G}$. Since $\mathbf{H} \leq \mathbf{G}$ is closed, and $\mathbf{G}$ acts by automorphisms of varieties on itself, we infer that the coset $\mathbf{H}g \subseteq \mathbf{G}$ is closed as well, and isomorphic as varieties to $\mathbf{H}$. Hence we conclude that all $\mathbf{H}$-orbits on $\mathbf{G}$ are closed, and have one and the same dimension.

Now let $\mathbf{H}$ additionally be (geometrically) reductive. Then we conclude that the quotient variety of the left translation action of $\mathbf{H}$ on $\mathbf{G}$ can be identified with the orbit space $\mathbf{H}\backslash\mathbf{G}$, such that the map $\pi\colon \mathbf{G} \to \mathbf{H}\backslash\mathbf{G}\colon g \mapsto \mathbf{H}g$ is the associated (geometric) quotient. In particular, the orbit space $\mathbf{H}\backslash\mathbf{G}$ carries the sructure of an affine variety. (Actually, if $\mathbf{G}$ is reductive then the converse holds as well, saying that if the orbit space $\mathbf{H}\backslash\mathbf{G}$ carries the structure of an affine variety then $\mathbf{H}$ necessarily is reductive; but we are not able to prove this here.)

If $\mathbf{H} \trianglelefteq \mathbf{G}$ is a (geometrically) reductive closed normal subgroup, the orbit space $\mathbf{G}/\mathbf{H} = \mathbf{H}\backslash\mathbf{G}$ is an affine variety as well, and thus naturally becomes an affine algebraic group such that the natural map $\pi\colon \mathbf{G} \to \mathbf{G}/\mathbf{H}$ is a homomorphism of algebraic groups. For example, this happens if $\mathbf{G}$ is reductive and $\mathbf{H} \trianglelefteq \mathbf{G}$ is a closed normal subgroup: From $R(\mathbf{H}) \trianglelefteq \mathbf{H}$ being characteristic we infer that $R(\mathbf{H}) \leq R(\mathbf{G})$, thus $\mathbf{H}$ is reductive as well. (Actually, the assumption of $\mathbf{H}$ being reductive can be dispensed of, but we are not able to prove this here.)

(7.8) Finite groups. t.b.c.

Example. Let $\mathrm{char}(\mathbb{K}) = p > 0$, the cyclic group $G = \langle g \rangle \cong C_p$ of order p acts on $V := \mathbb{K}^2$ via $g \mapsto J := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, that is g acts on V by $[x, y] \mapsto [x, y + x]$, for $x, y \in \mathbb{K}$. Thus the G-orbits in V are non-uniquely given as $[a, b] \cdot G = \{[a, b + ia] \in V; i \in \{0, \ldots, p-1\}\}$ for $a \in \mathbb{K} \setminus \{0\}$ and $b \in \mathbb{K}$, and $[0, b] \cdot G = \{[0, b]\}$ for $b \in \mathbb{K}$; recall that indeed $V^G = \langle e_2 \rangle_{\mathbb{K}}$.

Let $\mathbb{K}[V] = \mathbb{K}[X, Y]$ be the associated coordinate algebra, on which g acts by $X \mapsto X$ and $Y \mapsto Y - X$. We have already shown that the invariant algebra $\mathbb{K}[X, Y]^G = \mathbb{K}[X, Y^p - YX^{p-1}]$ is a polynomial algebra. Hence the quotient variety is $V /\!\!/ G \cong \mathbb{K}^2$, and its embedding into $\mathbb{K}[X, Y]$ shows that the (surjective) quotient morphism is given as $\pi\colon V \to \mathbb{K}^2\colon [x, y] \mapsto [x, y^p - yx^{p-1}]$.

Hence for $\beta \in \mathbb{K}$ we get $\pi^{-1}([0, \beta]) = \{[0, y] \in V; y^p = \beta\} = \{[0, \beta^{\frac{1}{p}}]\}$. Moreover, for $a \in \mathbb{K}\setminus\{0\}$ and $\beta \in \mathbb{K}$ we get $\pi^{-1}([a, \beta]) = \{[a, y] \in V; y^p - ya^{p-1} = \beta\}$, where since $Y^p - a^{p-1}Y - \beta \in \mathbb{K}[Y]$ has non-zero constant derivative we conclude that there are p pairwise distinct roots in $\mathbb{K}$, showing that $\pi^{-1}([a, \beta])$ consists of a single G-orbit; indeed for $y, z \in \mathbb{K}$, recalling that $y^p - ya^{p-1} = a^p\big((\frac{y}{a})^p - \frac{y}{a}\big) = a^p \cdot \prod_{i=0}^{p-1}(\frac{y}{a} - i) = \prod_{i=0}^{p-1}(y - ia)$, we get $y^p - ya^{p-1} = z^p - za^{p-1}$ if and only if $(y - z)^p - (y - z)a^{p-1} = 0$, if and only if $y - z = ia$ for some $i \in \{0, \ldots, p-1\}$.

(7.9) Example. We now consider a couple of reductive examples: Recall that $\mathbf{GL}_n$ and $\mathbf{SL}_n$, for $n \in \mathbb{N}$, are reductive, hence are linearly reductive if $\mathrm{char}(\mathbb{K}) = 0$.

Firstly, letting $\mathrm{char}(\mathbb{K}) \neq 2$, let $\mathcal{V} := \mathbb{K}[X_1, \ldots, X_n]_2$ be the $\mathbb{K}$-vector space of n-ary quadratic forms, being an $\mathbf{SL}_n$-variety via base change. Then by (0.3) we have $\mathbb{K}[\mathcal{V}]^{\mathbf{SL}_n} = \mathbb{K}[\Delta]$, where $\Delta \in \mathbb{K}[\mathcal{V}]$ is the discriminant.

We show that $\mathbb{K}[\Delta]$ is a univariate polynomial algebra: Let $s\colon \mathbb{K} \to \mathcal{V}\colon \delta \mapsto q_{n,\delta}$, where $q_{n,\delta} := \delta X_n^2 + \sum_{i=1}^{n-1} X_i^2 \in \mathcal{V}$, with associated comorphism $s^* \colon \mathbb{K}[\mathcal{V}] \to \mathbb{K}[T]$. Then we have $s^*(\Delta)(\delta) = \Delta(s(\delta)) = \Delta(q_{n,\delta}) = \delta$ for $\delta \in \mathbb{K}$, thus $s^*(\Delta) = T$. This shows that $\Delta \in \mathbb{K}[\mathcal{V}]$ is algebraically independent.

Hence the quotient exists (without referring to the reductivity of $\mathbf{SL}_n$), the quotient variety is $\mathcal{V}/\!\!/\,\mathbf{SL}_n \cong \mathbb{K}$, and the (surjective) quotient morphism is given as $\Delta \colon \mathcal{V} \to \mathbb{K}$; note that $\dim(\mathcal{V}) - \dim(\mathcal{V}/\!\!/\,\mathbf{SL}_n) = (n+1) - 1 = n$.

We have already seen that for $\delta \in \mathbb{K} \setminus \{0\}$ the fibre $\Delta^{-1}(\delta) = [q_{n,\delta}]$ consists of a single $\mathbf{SL}_n$-orbit, which hence is closed. But the fibre $\Delta^{-1}(0) = \coprod_{r=0}^{n-1} [q_r]$, where $q_r := \sum_{i=1}^{r} X_i^2 \in \mathcal{V}$, consists of finitely many $\mathbf{SL}_n$-orbits, where $[q_0] \preceq [q_1] \preceq \cdots \preceq [q_{n-1}]$ and thus only $[q_0]$ is closed. In particular, Δ separates the $\mathbf{SL}_n$-orbits if and only if $n = 1$, where $\mathbf{SL}_1 = \{1\}$ anyway.

(7.10) Example. Secondly, letting $\mathbb{K}$ be arbitrary, we consider $\mathcal{M} := \mathbb{K}^{n\times n}$, being a $\mathbf{GL}_n$-variety via conjugation. Then by (3.4) we have $\mathbb{K}[\mathcal{M}]^{\mathbf{GL}_n} = \mathbb{K}[\epsilon_1, \ldots, \epsilon_n]$, where the regular maps $\epsilon_i \colon \mathcal{M} \to \mathbb{K}$ are given as the coefficients of the characteristic polynomial $\chi(A) := \det(XE_n - A) = X^n + \sum_{i=1}^{n} (-1)^i \epsilon_i(A) X^{n-i} \in \mathbb{K}[X]$ of $A \in \mathcal{M}$, so that $\epsilon_i(A)$ coincides with the i-th elementary symmetric polynomial in the eigenvalues of A. We show that $\mathbb{K}[\epsilon_1, \ldots, \epsilon_n]$ is a polynomial algebra:

To this end we consider the morphisms $\epsilon \colon \mathcal{M} \to \mathbb{K}^n \colon A \mapsto [\epsilon_1(A), \ldots, \epsilon_n(A)]$ and $\sigma \colon \mathbb{K}^n \to \mathcal{M} \colon [x_1, \ldots, x_n] \mapsto \mathrm{diag}[x_1, \ldots, x_n]$. Then the morphism $\sigma\epsilon \colon \mathbb{K}^n \to \mathbb{K}^n \colon [x_1, \ldots, x_n] \mapsto [\epsilon_1(x_1, \ldots, x_n), \ldots, \epsilon_n(x_1, \ldots, x_n)]$ yields the comorphism $(\sigma\epsilon)^* \colon \mathbb{K}[\mathcal{X}] \to \mathbb{K}[\mathcal{X}] \colon X_i \mapsto \epsilon_i(\mathcal{X})$, where $\mathcal{X} := \{X_1, \ldots, X_n\}$. It is well-known that the elementary symmetric polynomials $\{\epsilon_1(\mathcal{X}), \ldots, \epsilon_n(\mathcal{X})\} \subseteq \mathbb{K}[\mathcal{X}]$ are algebraically independent, hence $(\sigma\epsilon)^*$ is injective. From $(\sigma\epsilon)^* = \epsilon^*\sigma^*$ we conclude that the comorphism $\epsilon^* \colon \mathbb{K}[\mathcal{X}] \to \mathbb{K}[\mathcal{M}] \colon X_i \mapsto \epsilon_i$ is a injective as well, hence $\{\epsilon_1, \ldots, \epsilon_n\} \subseteq \mathbb{K}[\mathcal{M}]$ is algebraically independent.

Hence the quotient exists (without referring to the reductivity of $\mathbf{GL}_n$), the quotient variety is $\mathcal{M}/\!\!/\,\mathbf{GL}_n \cong \mathbb{K}^n$, and the (surjective) quotient morphism is given as $\epsilon \colon \mathcal{M} \to \mathbb{K}^n$; note that $\dim(\mathcal{M}) - \dim(\mathcal{M}/\!\!/\,\mathbf{GL}_n) = n^2 - n = n(n-1)$.

We have already seen that for $x \in \mathbb{K}^n$ the fibre $\epsilon^{-1}(x) \subseteq \mathcal{M}$, that is the set of all matrices having one and the same characteristic polynomial with coefficients given by x, consists of finitely many $\mathbf{GL}_n$-orbits, amongst which precisely the unique semisimple $\mathbf{GL}_n$-orbit is closed; the fibre $\epsilon^{-1}(0) = \mathcal{N} \subseteq \mathcal{M}$ consists of the nilpotent matrices. In particular, ϵ separates the $\mathbf{GL}_n$-orbits if and only if $n = 1$, where $\mathbf{GL}_1 = \mathbf{G}_m$.

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