DIPLOMARBEIT

Dualitäten und Adjunktionen stabiler Derivateure
(Dualities and adjunctions of stable derivators)

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Abstract

We analyse the structure of the 2-category of derivators and their pointed and stable variants. As a first main result, we prove that cofree derivators are dualizable. Moreover, we show that the stability of a derivator can be characterized by the existence of certain functors, which are adjoint to the homotopy Kan extensions. Furthermore, we prove that in the stable situation, a well-behaved Spanier-Whitehead-Duality exists, if we assume additionally some finiteness conditions. This will enable us to prove that certain duality functors describe the passage to adjoints in a particular sub-2-category of stable derivators.

Contents

0 Einleitung

Ein wesentlicher Aspekt der algebraischen Topologie ist es topologische Räume bis auf Homotopieäquivalenz zu klassifizieren. Dabei erhält man die Homotopiekategorie der topologischen Räume als Lokalisierung an der Klasse der schwachen Äquivalenzen (d.h. derjenigen stetigen Abbildungen, welche Isomorphismen auf allen Homotopiegruppen induzieren). Zwei bekannte Möglichkeiten diesen Übergang zur Homotopiekategorie zu verallgemeinern sind gegeben durch Quillens Theorie der Modellkategorien [Qui67], sowie durch die Theorie der (∞,1)-Kategorien [Lur09], [Lur11].
Jedoch gehen beim Übergang von einer Kategorie C zur Homotopiekategorie zahlreiche Informationen verloren, insbesondere existieren im Allgemeinen nur noch sehr wenige kategorielle Limiten und Kolimiten. Um dieses Problem zu beheben betrachtet zusätzlich die Homotopiekategorien von X-Diagrammen in C für kleine Kategorien X, sowie die von den Kan Erweiterungen induzierten Funktoren. Die Axiomatisierung dieser Struktur führt zum Begriff des Derivators, welcher von Jens Franke [Fra96], Alexander Grothendieck [Gro] und Alex Heller [Hel88], [Hel97] unabhängig voneinander eingeführt wurde.
Im ersten Abschnitt dieser Arbeit werden wir uns die Definition des Derivatorbegriffs, sowie der punktierten und stabilen Varianten, in Erinnerung rufen. Anschließend werden wir an Hand einiger Beispiele zeigen, dass unendlich lange Ketten adjungierter Funktor regelmäßig in der Struktur eines stabilen Derivators auftreten.
In [Gro11a] hat Groth gezeigt, dass sich punktierte Derivateure durch die Existenz gewisser zu den Homotopie Kan Erweiterungen adjungierter Funktoren charakterisieren lassen. Wir werden zeigen, dass sich die Stabilität eines Derivators auf ähnliche Weise charakterisieren lässt: Ein punktierter Derivator ist genau dann stabil, wenn Homotopie Push-outs einen rechtsadjungierten und Homotopie Pull-backs einen linksadjungierten Funktor haben (Theorem 1.33). Das Hauptproblem ist dabei die Rückrichtung, welche aus expliziten Berechnungen und der Tatsache, dass in diesem Fall Homotopie Push-outs und Pull-backs miteinander kommutieren, folgt.
Im zweiten Abschnitt werden wir die 2-Kategorien der Derivateure untersuchen. Dabei werden wir ausnutzen, dass die Evaluation an einer kleinen Kategorie X in der 2-Kategorie der Derivateure korepräsentiert durch einen Derivator $\mathbb{H}^u_{X^{op}}$ ist (Theorem 2.16). Dabei ist $\mathbb{H}^u = \mathbb{H}^u_*$ der Derivator assoziiert zur Homotopietheorie der topologischen Räume. Dieses Resultat wurde bereits von Franke [Fra96] in einigen Spezialfällen und von Cisinski [Cis08] im Allgemeinen bewiesen. Aus Theorem 2.16 folgt, dass ein Homotopie-Kolimes erhaltender Morphismus von Derivateuren $\mathbb{H}^u_{X^{op}} \longrightarrow \mathbb{D}$ bereits durch den Wert eines bestimmten Elements in $\mathbb{H}^u_{X^{op}}(X)$ bis auf Isomorphismus bestimmt ist. Dies werden wir nutzen, um zu zeigen, dass der interne $Hom-$Funktor $[\![-, \mathbb{H}^u]\!]$ eine Äquivalenz auf der Unter-2-Kategorie der kofreien Derivateure (d.h. diejenigen, die zu $\mathbb{H}^u_X$ für eine kleine Kategorie X äquivalent sind) definiert (Theorem 2.21). Dazu werden wir Theorem 2.16 benutzen um die kanonischen Evaluationsmorphismen mit den von Korollar 2.18 induzierten Äquivalenzen zu identifizieren.

Anschließend werden wir eine neue geschlossen symmetrisch monoidale Struktur auf der 2-Kategorie der kofreien Derivateure konstruieren (Proposition 2.27). Dazu werden wir Theorem 2.21 als Dualitätsaussage auffassen. Die Definition der monoidalen Struktur ist dadurch motiviert, dass es in einer geschlossen symmetrisch monoidalen Kategorie mit monoidaler Paarung $- \otimes -$, Einheit $\mathbb{S}$ und Abschluss $Hom(-,-)$, in welcher alle Objekte dualisierbar sind, einen natürlichen Isomorphismus $- \otimes - \cong Hom(Hom(-,\mathbb{S}),-)$ gibt. Der Beweis basiert ähnlich wie bei Theorem 2.21 wieder auf Theorem 2.16.
Alle Aussagen in Abschnitt 2 gelten aufgrund von Theorem 2.11 analog im punktierten und stabielen Fall
Im letzten Abschnitt werden wir uns auf stabile Derivateure, welche auf Diagramme parametrisiert durch endliche, endlich dimensionale Kategorien eingeschränkt sind, konzentrieren. In diesem Fall folgt aus Threom 2.16 direkt die Existenz einer Äquivalenz $\mathbb{H} \longrightarrow \mathbb{H}^{op}$, welche die Spanier-Whitehead-Dualität für Spektren verallgemeinert. Als Konsequenz erhalten wir, dass die 2-Kategorie der kofreien Derivateure abgeschlossen unter abstrakter Dualität (d.h. dem Funktor $\mathbb{D} \mapsto \mathbb{D}^{op}$) ist. Damit können wir zeigen, dass in einem stabilen Derivator jedes inverse Bild eines Funktors zwischen endlichen, endlich dimensionalen Kategorien eine unendliche Kette zueinander adjungierter Funktoren induziert (Theorem 3.10). Dabei benutzen wir, dass jedes inverse Bild $\mathbb{D}_Y \longrightarrow \mathbb{D}_X$ im Bild des 2-Funktors $[\![-,\mathbb{D}]\!]_L$ ist und dass 2-Funktoren unendliche Ketten zueinander adjungierter Funktoren erhalten.
Anschließend untersuchen wir das Verhalten zweier Dualitätsfunktoren auf der 2-Kategorie der stabilen kofreien Derivateure; erstens, die horizontale Dualität $\mathfrak{D}^h$, welche durch die monoidale Dualität (Theorem 2.21) induziert ist und die Richtung von 1-Morphismen umkehrt und zweitens der vertikalen Dualität $\mathfrak{D}^v$ welche durch die abstrakte Dualität (Proposition 2.9) und Spanier-Whitehead-Dualität (Proposition 3.7) induziert ist und die Richtung von 2-Morphismen umkehrt. Wir werden zeigen, dass für einen Morphisms $F : \mathbb{H}_X \longrightarrow \mathbb{H}_Y$ gilt:

$$\mathfrak{D}^h\mathfrak{D}^v(F) \dashv F \dashv \mathfrak{D}^v\mathfrak{D}^h(F).$$

Dies wird bewiesen, indem wir mit Theorem 2.16 explizit einen Rechtsadjungierten von F konstruieren, und diesen unter Nutzung der monoidalen Eigenschaften der Spanier-Whitehead-Dualität mit $\mathfrak{D}^v\mathfrak{D}^h(F)$ identifizieren.

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Basic Notation and Conventions:

The following notation will be used frequently:

- Let X be a category, then $Ob(X)$ denotes the class of objects of X. And for $x, x' \in Ob(X)$ the set $X(x, x') = Hom_X(x, x')$ is the set of morphisms from x to x'.
- By abuse of notation we will often write $x \in X$ instead of $x \in Ob(X)$, whenever there is no risk of confusion.
- The initial category $\emptyset$ is the category with no objects. It is the initial object in the category of small categories.
- For a category X the unique functor $\emptyset \longrightarrow X$ will be called ι_X.
- The terminal category $*$ is the category with one object (this is also denoted by $*$). And $Hom_*(*, *) = \{id_*\}$. It is the terminal object in the category of small categories.
- For a category X and $x \in X$, the unique functor $* \longrightarrow X$ with image $\{x\}$ will also be called x.
- The n-dimensional simplex category $\mathbf{n}$ for $n \in \mathbb{N}$ is the poset $\{0 < 1 < ... < n\}$ regarded as a category. Note that $\mathbf{0} = *$.
- The functor $d_i : \mathbf{n-1} \longrightarrow \mathbf{n}$ for $i \in \{0, 1, ..., n\}$ is induced by the unique injective monotonic map of underlying posets with i not in the image.
- The functor $s_i : \mathbf{n} \longrightarrow \mathbf{n-1}$ for $i \in \{0, 1, ..., n-1\}$ is induced by the unique surjective monotonic map of underlying posets with $s_i(i) = s_i(i + i)$.
- Whenever we draw a diagram shaped by a subcategory of $\mathbf{n} \times \mathbf{m}$, the first variable will be drawn in the horizontal direction and the second variable will be drawn in the vertical direction.

Throughout this thesis we will use the language of 2-categories and bicategories. We refer to [Ehr63], [KS05], [Lei98] for the definitions.

Note that a 2-category is a category enriched over the category of (locally) small categories. Therefore we refer to [Kel05] for the concept of enriched categories, in particular the definition and properties of tensor and cotensor functors.

1 Prerequisites and first results

In Subsection 1.1. we recall the definition of a derivator, their pointed and stable variants and some well-known results, including the triangulation theorem for stable derivators (Theorem 1.25)
In Subsection 1.2. we introduce infinite chains of adjunctions and prove a few basic properties. We discuss some examples to see that infinite chains of adjunction appear frequently in the context of stable derivators. Afterwards we can state one of our main results, Theorem 1.33 , and prove large parts of it. We close this sections with a couple of remarks.

1.1 Derivators

Definition 1.1. a) Let CAT be the 2-category of locally small categories and Cat the full sub-2-category of small categories.
b) Let $X, Y \in Cat$, $f : X \longrightarrow Y$ a functor and $y \in Ob(Y)$. The category of *f-objects under y*: $X_{y/}$ is defined as follows:
$Ob(X_{y/}) = \{(x, \alpha)|x \in Ob(X), \alpha \in Y(y, f(x))\}$
$X_{y/}((x, \alpha), (x', \alpha')) = \{\beta \in X(x, x')|\alpha' = f(\beta) \circ \alpha\}$
c) Dually one defines the category of *f-objects over y*: $X_{/y} = (X^{op}_{y/})^{op}$
d) A full sub-2-category Dia of Cat is a *category of diagrams*, if the following conditions are satisfied:
-Any finite poset (considered as a small category) is an object of Dia.
-Dia is closed under finite products and coproducts.
-If $X \in Dia$, $x \in Ob(X)$ then also $X_{x/}$ and $X_{/x}$ are objects of Dia.
-If $X \in Dia$, then so is X^{op}.
-If $p : X \longrightarrow Y$ is a Grothendieck fibration with Y and all the fibres in Dia then also $Y \in Dia$.

Important examples of categories of diagrams are Cat itself and the 2-category Fin of finite, finite dimensional (i.e. with finite dimensional nerve) categories.

Definition 1.2. Let Dia be a category of diagrams.
a) A *Dia-prederivator* is a 2-functor $\mathbb{D} : Dia^{op} \longrightarrow CAT$.
b) Let $\mathbb{D}$ be prederivator $X, Y \in Dia$ and $f : X \longrightarrow Y$ a functor. Then $f^* := \mathbb{D}(f) : \mathbb{D}(Y) \longrightarrow \mathbb{D}(X)$ is called the *inverse image* of f.
c) The 2-category of Dia-prederivators $PDer_{Dia}$ is the functor 2-category $CAT^{Dia^{op}}$.

Example 1.3. a) Let C be a locally small category and S a class of morphisms in C such that the localization $C[S^{-1}]$ is locally small. Let X be a small category, and S_X the class of morphisms in C^X that are objectwise in S. Then the assignment $X \mapsto C^X[S_X^{-1}]$ defines a Cat-prederivator.
b) Let X be a small category. The 2-functor $\underline{X} := Cat(-, X)$ is a prederivator. Then $\underline{X}$ is called the *prederivator represented by* X. A prederivator, which is

equivalent to $\underline{X}$ for some $X \in Cat$ is called a *representable prederivator*.

An important class of prederivators is given by the special case of Example 1.3 where C is a model category and S is the class of weak equivalences. The well known behavior of the homotopy Kan extension functors in this case motivates the following definitions:

Definition 1.4. Let $\mathbb{D}$ be a prederivator, $X, Y \in Dia$ and $f : X \longrightarrow Y$ a functor.
a) A left adjoint $f_! : \mathbb{D}(X) \longrightarrow \mathbb{D}(Y)$ of the inverse image f^* is called the *homotopy left Kan extension functor along* f. The prederivator $\mathbb{D}$ *admits homotopy left Kan extensions* if $f_!$ exists for every morphism f in Dia.
b) A right adjoint $f_* : \mathbb{D}(X) \longrightarrow \mathbb{D}(Y)$ of the inverse image f^* is called the *homotopy right Kan extension functor along* f. The prederivator $\mathbb{D}$ *admits homotopy right Kan extensions* if f_* exists for every morphism f in Dia.

Consider a 2-cell α in Dia:

$$\begin{tikzcd}
W & X \arrow[l, "g"'] \\
Y \arrow[u, "f"] \arrow[ur, phantom, "\alpha\Swarrow"] & Z \arrow[l, "h"] \arrow[u, "f'"']
\end{tikzcd}$$

Suppose furthermore that f and f' are part of adjunctions $\{f_L, f, \eta, \epsilon\}$ and $\{f'_L, f', \eta', \epsilon'\}$. Then one can construct a new 2-cell α_L, the left Beck-Chevalley-transform of α as follows:
$\alpha_L = \epsilon \circ \alpha \circ \eta' : f_L \circ g \longrightarrow f_L \circ g \circ f' \circ f'_L \longrightarrow f_L \circ f \circ h \circ f'_L \longrightarrow h \circ f'_L$
Or visualized as a diagram:

$$\begin{tikzcd}
Y \arrow[dr, equal, "id"'] & W \arrow[l, "f_L"'] \arrow[d, phantom, "\epsilon\Swarrow"' near start] & X \arrow[l, "g"'] \arrow[dr, equal, "id"] & \\
& Y \arrow[u, "f"'] \arrow[ur, phantom, "\alpha\Swarrow"] & Z \arrow[l, "h"] \arrow[u, "f'"] \arrow[ur, phantom, "\eta'\Swarrow" near end] & X \arrow[l, "f'_L"]
\end{tikzcd}$$

Dually for a 2-cell β:

$$\begin{tikzcd}
W & X \arrow[l, "g"'] \\
Y \arrow[u, "f"] \arrow[ur, phantom, "\beta\Nearrow"] & Z \arrow[l, "h"] \arrow[u, "f'"']
\end{tikzcd}$$

such that f and f' are part of adjunctions $\{f, f_R, \eta, \epsilon\}$ and $\{f', f'_R, \eta', \epsilon'\}$. Then one can construct a new 2-cell β_R, the right Beck-Chevalley-transform of β as follows:
$\beta_R = \epsilon' \circ \beta \circ \eta : h \circ f'_R \longrightarrow \circ f_R \circ f \circ h \circ f'_R \longrightarrow \circ f_R \circ g \circ f' \circ f'_R \longrightarrow f_R \circ g$
Again visualized as a diagram:

$$\begin{tikzcd}
Y & W \arrow[l, "f_R"'] & X \arrow[l, "g"'] \arrow[dr, equal, "id"] & \\
& Y \arrow[u, "f"'] \arrow[ul, equal, "id"] \arrow[u, phantom, shift left=8, "\eta\Nearrow"] & Z \arrow[l, "h"] \arrow[u, "f'"'] \arrow[ul, phantom, "\beta\Nearrow"] & X \arrow[l, "f'_R"] \arrow[ul, phantom, "\epsilon'\Nearrow"]
\end{tikzcd}$$

In the following, we will consider two examples of 2-cells, which will be important for the definition of a derivator:

Let $f : X \longrightarrow Y$ be a functor in Dia, $y \in Ob(Y)$, $p_{y/} : X_{y/} \longrightarrow X$ the natural projection, $q_{y/} : X_{y/} \longrightarrow *$ the projection to the terminal category and $y : * \longrightarrow Y$ the unique functor with image $y \in Y$.
Then the 2-cell $\alpha_y : y \circ q_{y/} \longrightarrow f \circ p_{y/}$ is defined by: $\alpha_{y,(x,\phi)} = (\phi : y \longrightarrow f(x))$

Dually, let additionally $p_{/y} : X_{/y} \longrightarrow X$ the natural projection and $q_{/y} : X_{/y} \longrightarrow *$ the projection to the terminal category.
Then the 2-cell $\beta_y : f \circ p_{/y} \longrightarrow y \circ q_{/y}$ is defined by: $\beta_{y,(x,\phi)} = (\phi : f(x) \longrightarrow y)$

In diagrams:

$$\begin{tikzcd}
X_{y/} \arrow[r, "p_{y/}"] \arrow[d, "q_{y/}"'] \arrow[dr, phantom, "\alpha_y\Nearrow"] & X \arrow[d, "f"] \\
* \arrow[r, "y"] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X_{/y} \arrow[r, "p_{/y}"] \arrow[d, "q_{/y}"'] \arrow[dr, phantom, "\beta_y\Swarrow"] & X \arrow[d, "f"] \\
* \arrow[r, "y"] & Y
\end{tikzcd}$$

Definition 1.5. a) Let $\mathbb{D}$ be a Dia-prederivator admitting homotopy left Kan extensions.
Then $\mathbb{D}$ *satisfies base change for homotopy left Kan extensions*, if for all $f : X \longrightarrow Y$ in Dia and $y \in Ob(Y)$ the Beck-Chevalley-transform $(\alpha_y^*)_L$ is a natural isomorphism.

$$\begin{tikzcd}
\mathbb{D}(X_{y/}) \arrow[d, "q_{y/!}"'] \arrow[dr, phantom, "(\alpha_y^*)_L \Swarrow"] & \mathbb{D}(X) \arrow[l, "p_{y/}^*"'] \arrow[d, "f_!"] \\
\mathbb{D}(*) & \mathbb{D}(Y) \arrow[l, "y^*"']
\end{tikzcd}$$

b) Let $\mathbb{D}$ be a *Dia*-prederivator admitting homotopy right Kan extensions. Then $\mathbb{D}$ *satisfies base change for homotopy right Kan extensions*, if for all $f : X \longrightarrow Y$ in *Dia* and $y \in Ob(Y)$ the Beck-Chevalley-transform $(\beta_y^*)_R$ is a natural isomorphism.

$$\begin{tikzcd}
\mathbb{D}(X_{/y}) \arrow[d, "q_{/y*}"'] \arrow[dr, phantom, "(\beta_y^*)_R \Nwarrow"] & \mathbb{D}(X) \arrow[l, "p_{/y}^*"'] \arrow[d, "f_*"] \\
\mathbb{D}(*) & \mathbb{D}(Y) \arrow[l, "y^*"']
\end{tikzcd}$$

Given a *Dia*-prederivator $\mathbb{D}$, one can assign to each object of $\mathbb{D}(X \times Y)$ a X-shaped diagram in $\mathbb{D}(Y)$ as follows:
Let $X, Y \in Dia$ and $k : X \longrightarrow Cat(Y, X \times Y)$ the functor that corresponds to the identity on $X \times Y$ under the exponential adjunction in *Cat*. Since $\mathbb{D}$ is a 2-functor, one gets a functor $Cat(Y, X \times Y) \longrightarrow CAT(\mathbb{D}(X \times Y), \mathbb{D}(Y))$. The composition
$X \longrightarrow CAT(\mathbb{D}(X \times Y), \mathbb{D}(Y)) \in CAT(X, CAT(\mathbb{D}(X \times Y), \mathbb{D}(Y)))$
$\cong CAT(X \times \mathbb{D}(X \times Y), \mathbb{D}(Y)) \cong CAT(\mathbb{D}(X \times Y), CAT(X, \mathbb{D}(Y)))$
corresponds to a functor $dia_{X,Y} : \mathbb{D}(X \times Y) \longrightarrow CAT(X, \mathbb{D}(Y))$ using the exponential adjunction in CAT twice.
Let $A \in \mathbb{D}(X \times Y)$, $x \in Ob(X)$ and $x_Y : Y \longrightarrow X \times Y, y \mapsto (x, y)$, then $dia_{X,Y}$ satisfies $dia_{X,Y}(A)(x) = x_Y^*(A)$.

Now we are ready to give the definition of a derivator:

Definition 1.6. A *Dia-derivator* $\mathbb{D}$ is a *Dia*-prederivator, satisfying the following conditions:
(Der1) Let $X_1, X_2 \in Dia$, $i_k : X_k \longrightarrow X_1 \coprod X_2$ the natural inclusions, then $i_1^* \times i_2^* : \mathbb{D}(X_1 \coprod X_2) \longrightarrow \mathbb{D}(X_1) \times \mathbb{D}(X_2)$ is an equivalence of categories. Moreover $\mathbb{D}(\emptyset) \cong *$.
(Der2) Let $A, B \in \mathbb{D}(X)$. A morphism $T : A \longrightarrow B$ is an isomorphism if and only if $dia_{X,*}(T)$ is an isomorphism.
(Der3) $\mathbb{D}$ admits homotopy left Kan extensions and homotopy right Kan extensions.

(Der4) $\mathbb{D}$ satisfies base change for homotopy left Kan extensions and homotopy right Kan extensions.
(Der5) The functor $dia_{\mathbf{1},X} : \mathbb{D}(\mathbf{1} \times X) \longrightarrow CAT(\mathbf{1}, \mathbb{D}(X))$ is an epivalence (full and essentially surjective) for all $x \in Dia$.

Remark 1.7. a) (Der1) implies that for $X \in Dia$ discrete $dia_{X,*}$ is an equivalence, and that for the projection $X \longrightarrow *$ the associated homotopy Kan extension functors are given by the usual (co)products in $\mathbb{D}$, in particular they exist. Moreover $\mathbb{D}(Y)$ has initial and final objects for any $Y \in Dia$ and they are given by $\iota_{X,!}(*)$ resp.$\iota_{X,*}(*)$ (where $\iota_X : \emptyset \longrightarrow X$).
b) (Der3) and (Der4) axiomize the existence of homotopy Kan extensions and their computability in terms of homotopy (co)limits by Kan's well known formulas ([Kan58]).
c) (Der5) will be used in the proof of the triangulation Theorem 1.25. Note that some authors omit (Der5) from the definition of a derivator and call a derivator satisfying it a strong derivator. However, since all interesting examples of derivators satisfy (Der5) ([Tab] 3.11), we will keep it in the definition.

The following Proposition is immediate by applying the 2-functoriality of derivators to (co)units of adjunctions.

Proposition 1.8. *Let $\mathbb{D}$ be a derivator and $f \dashv g$ an adjunction in Dia. Then the inverse images also form an adjoint pair: $g^* \dashv f^*$*

Proof. [CN08] Lemma 6.1. □

Proposition 1.9. *Let $\mathbb{D}$ be a derivator and f a fully faithful functor in Dia. Then $f_!$ and f_* are fully faithful.*

Proof. [CN08] Proposition 7.1. □

Definition 1.10. A *pointed Dia-derivator* $\mathbb{D}$ is a *Dia*-derivator satisfying the following condition:
(Der6) The category $\mathbb{D}(X)$ is pointed (initial and final object are isomorphic) for all $X \in Dia$.

In [Gro11a] Groth gives an equivalent characterization of pointed derivators. To describe it, we need some more notation.

Definition 1.11. Let $\mathbb{D}$ be a derivator, $X, Y \in Dia$ and $f : X \longrightarrow Y$ a functor.
a) $\mathbb{D}$ has an *coexceptional inverse image* $f^?$ along f, if there is an adjunction $f^? \dashv f_!$.
b) $\mathbb{D}$ has an *exceptional inverse image* $f^!$ along f, if there is an adjunction $f_* \dashv f^!$.

Definition 1.12. Let $f : X \longrightarrow Y$ be a fully faithful functor, that is injective on objects.
a)f is called a *cosieve* if for all morphisms $y \longrightarrow y'$ with y in the image of f, also y' is in the image of f.
b)f is called a *sieve* if for all morphisms $y \longrightarrow y'$ with y' in the image of f, also y is in the image of f.

Proposition 1.13. *Let $\mathbb{D}$ be a derivator. then the following condition are equivalent:*
a) $\mathbb{D}$ is pointed.
b) $\mathbb{D}$ has (co)exceptional inverse images along inclusions of the empty category.
c) $\mathbb{D}$ has coexceptional inverse images along cosieves and exceptional inverse images along sieves.

Proof. c) $\Rightarrow$ b) Inclusions of the empty category are always sieves and cosieves.
b) $\Rightarrow$ a) The functors $\iota_X^?$ and ι_X^* are both the projection to the terminal category. Hence $\iota_{X,!}$ is both, left and right adjoint to the projection to the terminal category, thus its image is a zero object.
a) $\Rightarrow$ c) [Gro11a] Corollary 3.8. $\square$

Moreover (co)exceptional inverse image functors can be described in terms of homotopy Kan extensions and (usual) inverse image functors:

Definition 1.14. Let $f : X \longrightarrow Y$ be a sieve in *Dia*. The *mapping cylinder category* $Cyl(f)$ is defined by the pushout:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_0"'] & Y \arrow[d] \\ X \times \mathbf{1} \arrow[r] & Cyl(f) \end{tikzcd}$$

More explicitely, one can describe $Cyl(f)$ as the full subcategory of $Y \times \mathbf{1}$ on the objects of the form $(y, 0)$ and $(f(x), 1)$.
We have functors $i : X \longrightarrow Cyl(f), x \mapsto (f(x), 1)$, $j : Y \longrightarrow Cyl(f), y \mapsto (y, 0)$ and $q : Cyl(f) \longrightarrow Y$ defined by $qi = f$ and $qj = id_Y$.

Dually for a cosieve $f' : X \longrightarrow Y$, the category $Cyl'(f')$ is defined by the pushout:

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "i_1"'] & Y \arrow[d] \\ X \times \mathbf{1} \arrow[r] & Cyl'(f') \end{tikzcd}$$

and the maps i', j' and q' are defined in the analogous way.

Definition 1.15. Let $\mathbb{D}$ be a pointed derivator, and $f : X \longrightarrow Y$ be the inclusion of a full subcategory.
The category $\mathbb{D}(Y, X)$ is the full subcategory of $\mathbb{D}(Y)$ on those objects $A \in \mathbb{D}(Y)$ with $x^*(A) \cong 0$ for all $x \in X$.

Proposition 1.16. *Let $\mathbb{D}$ be a pointed derivator.*
a) Let $f : X \longrightarrow Y$ be a cosieve and $g : X' = Y - f(X) \longrightarrow Y$ the associated sieve. Then the coexceptional inverse image $f^?$ is given by the composition:

$$\mathbb{D}(Y) \xrightarrow{j_*} \mathbb{D}(Cyl(g), i(X')) \xrightarrow{q_!} \mathbb{D}(Y, g(X')) \xrightarrow{f^*} \mathbb{D}(X)$$

b) Let $f' : X \longrightarrow Y$ be a sieve and $g' : X' = Y - f(X) \longrightarrow Y$ the associated cosieve. Then the exceptional inverse image $f'^!$ is given by the composition:

$$\mathbb{D}(Y) \xrightarrow{j'_!} \mathbb{D}(Cyl'(g'), i'(X')) \xrightarrow{q'_*} \mathbb{D}(Y, g'(X')) \xrightarrow{f'^*} \mathbb{D}(X)$$

Proof. [Gro11a] Corollary 3.8. $\square$

Now we can head towards stable derivators:
Notation: Let $\square := \mathbf{1} \times \mathbf{1}, \ulcorner := \square - (1,1), \lrcorner := \square - (0,0)$
and $i_\ulcorner : \ulcorner \longrightarrow \square, i_\lrcorner : \lrcorner \longrightarrow \square$ the inclusions.

Definition 1.17. Let $\mathbb{D}$ be a pointed *Dia*-derivator.
a) A *square* in $\mathbb{D}$ is an object of $\mathbb{D}(\square)$.
b) A square in $\mathbb{D}$ is called *cocartesian*, if it is in the essential image of $i_{\ulcorner !}$.
c) A square in $\mathbb{D}$ is called *cartesian*, if it is in the essential image of $i_{\lrcorner *}$.
d) A square in $\mathbb{D}$ is called *bicartesian*, if it is cartesian and cocartesian.

Definition 1.18. A *stable Dia-derivator* $\mathbb{D}$ is a pointed *Dia*-derivator satisfying the following condition:
(Der7) A square in $\mathbb{D}$ is cartesian if and only if it is cocartesian.

Proposition 1.19. *Let $\mathbb{D}$ be a stable Dia-derivator and $A \in \mathbb{D}(\mathbf{2} \times \mathbf{1})$.*
If two of the squares $d_k^*(A), 0 \leqq k \leqq 2$ are bicartesian, then also the third is bicartesian.

Proof. [Fra96] Proposition 6b. ∎

Proposition 1.20. *Let $\mathbb{D}$ be a stable Dia-derivator, $f : X \longrightarrow Y, i : \square \longrightarrow Y$ be functors, i injective on objects.*
a) If the composition $\ulcorner \xrightarrow{i|_\ulcorner} Y - i(1,1) \longrightarrow Y - i(1,1)_{/i(1,1)}$ has an left adjoint and $i(1,1)$ is not in the image of f. Then for all $A \in \mathbb{D}(X)$ the square $i^* f_!(X)$ is bicartesian.
b) If the composition $\lrcorner \xrightarrow{i|_\lrcorner} Y - i(0,0) \longrightarrow Y - i(0,0)_{i(0,0)/}$ has an right adjoint and $i(0,0)$ is not in the image of f. Then for all $A \in \mathbb{D}(X)$ the square $i^* f_*(X)$ is bicartesian.

Proof. [Fra96] Proposition 5. ∎

Proposition 1.21. *Let $\mathbb{D}$ be a stable derivator and $A \in \mathbb{D}$ a bicartesian square. The following properties are equivalent:*
a) The underlying morphism of $(0 \times \mathbf{1})^*(A)$ is an isomorphism.
b) The underlying morphism of $(1 \times \mathbf{1})^*(A)$ is an isomorphism.

Proof. [Gro11a] Proposition 3.13.ii. ∎

In the following we will prepare to state the triangulation theorem for stable derivators. Therefore we have to specify the shift functor and the class of distinguished triangles.

Definition 1.22. Let $\mathbb{D}$ be a stable derivator.
a) The *suspension functor* $\Sigma : \mathbb{D}(*) \longrightarrow \mathbb{D}(*)$ is the composition:
$\mathbb{D}(*) \xrightarrow{(0,0)_*} \mathbb{D}(\ulcorner) \xrightarrow{i_{\ulcorner !}} \mathbb{D}(\square) \xrightarrow{(1,1)^*} \mathbb{D}(*)$
b) The *loop functor* $\Omega : \mathbb{D}(*) \longrightarrow \mathbb{D}(*)$ is the composition:
$\mathbb{D}(*) \xrightarrow{(1,1)_!} \mathbb{D}(\lrcorner) \xrightarrow{i_{\lrcorner *}} \mathbb{D}(\square) \xrightarrow{(0,0)^*} \mathbb{D}(*)$
c) The *cone functor* $C : \mathbb{D}(\mathbf{1}) \longrightarrow \mathbb{D}(*)$ is the composition:
$\mathbb{D}(\mathbf{1}) \xrightarrow{(i_\ulcorner \circ (\mathbf{1} \times 0))_*} \mathbb{D}(\ulcorner) \xrightarrow{i_{\ulcorner !}} \mathbb{D}(\square) \xrightarrow{(1,1)^*} \mathbb{D}(*)$
d) The *fiber functor* $F : \mathbb{D}(\mathbf{1}) \longrightarrow \mathbb{D}(*)$ is the composition:
$\mathbb{D}(\mathbf{1}) \xrightarrow{(i_\lrcorner \circ (1 \times \mathbf{1}))_!} \mathbb{D}(\lrcorner) \xrightarrow{i_{\lrcorner *}} \mathbb{D}(\square) \xrightarrow{(0,0)^*} \mathbb{D}(*)$
e) The *full cone functor* $Cone : \mathbb{D}(\mathbf{1}) \longrightarrow \mathbb{D}(\mathbf{1})$ is the composition:
$\mathbb{D}(\mathbf{1}) \xrightarrow{(i_\ulcorner \circ (\mathbf{1} \times 0))_*} \mathbb{D}(\ulcorner) \xrightarrow{i_{\ulcorner !}} \mathbb{D}(\square) \xrightarrow{(1 \times \mathbf{1})^*} \mathbb{D}(\mathbf{1})$
f) The *full fiber functor* $Fiber : \mathbb{D}(\mathbf{1}) \longrightarrow \mathbb{D}(\mathbf{1})$ is the composition:
$\mathbb{D}(\mathbf{1}) \xrightarrow{(i_\lrcorner \circ (1 \times \mathbf{1}))_!} \mathbb{D}(\lrcorner) \xrightarrow{i_{\lrcorner *}} \mathbb{D}(\square) \xrightarrow{(\mathbf{1} \times 0)^*} \mathbb{D}(\mathbf{1})$

Remark 1.23. a) It is clear by the construction of the above functors, that there are adjunctions: $\Sigma \dashv \Omega$ and $Cone \dashv Fiber$. Moreover, the stability condition

(Der7) ensures that both adjunctions are adjoint equivalences of categories.
b) Using the explicit description of the (co)exceptional inverse images of the cosieve $1 : * \longrightarrow \mathbf{1}$ and the sieve $0 : * \longrightarrow \mathbf{1}$ one obtains natural isomorphisms: $C \cong 1^?$ and $F \cong 0^!$, and thus: $\Sigma \cong 1^? 0_*$ and $\Omega \cong 0^! 1_!$ ([Gro11a] Proposition 3.24.)

Let S be the category $(\mathbf{2} \times \mathbf{1}) - \{(1,1),(2,1)\}$, $i_2 : S \longrightarrow \mathbf{2} \times \mathbf{1}$ the inclusion, $i_1 : \mathbf{1} \longrightarrow S$ the functor defined by $i_1(x) = (x,0)$, $i := i_2 \circ i_1$ and $j : \mathbf{3} \longrightarrow \mathbf{2} \times \mathbf{1}$ the functor defined by $j(0) = (0,0), j(1) = (1,0), j(2) = (1,1), j(3) = (2,1)$.
Consider the functor $F := i_{2!} i_{1*}$ and let A be in the essential image. By Proposition 1.19 and 1.20 the three squares $d_k^*(A), 0 \leqq k \leqq 2$ are bicartesian and by (Der4): $(2,0)^*(A) \cong 0 \cong (0,1)^*(A)$.
Thus we can conclude: $(2,1)^*(A) \cong \Sigma((0,0)^*(A))$ and $(1,1)^*(A) \cong C(i^*(A))$

Definition 1.24. Let $\mathbb{D}$ be a stable *Dia*-derivator. An object $T \in \mathbb{D}(*)^{\mathbf{3}}$ is called a *distinguished triangle* if it is in the essential image of the functor $dia_{\mathbf{3},*} \circ j^* \circ F : \mathbb{D}(\mathbf{1}) \longrightarrow \mathbb{D}(*)^{\mathbf{3}}$

Theorem 1.25. *Let $\mathbb{D}$ be a stable Dia-derivator. Then $\mathbb{D}(*)$ is a triangulated category with the suspension Σ as shift functor and the class of distinguished triangles as defined in 1.24.*

Proof. [Fra96] Theorem 1. □

Definition 1.26. Let $\mathbb{D}$ be a *Dia*-prederivator and $X \in Dia$
The *Dia*-prederivator $\mathbb{D}_X$ is defined by the composition

$$Dia^{op} \xrightarrow{X \times -} Dia^{op} \xrightarrow{\mathbb{D}} CAT$$

In particular: $\mathbb{D}_X(Y) = \mathbb{D}(X \times Y)$

Proposition 1.27. *Let $\mathbb{D}$ be a (pointed, resp. stable) Dia-derivator and $X \in Dia$. Then also $\mathbb{D}_X$ is a (pointed, resp. stable) Dia-derivator.*

Proof. [Gro11a] Theorem 1.31 and Proposition 4.3. □

Corollary 1.28. *Let $\mathbb{D}$ be a stable Dia-derivator and $X \in Dia$. Then $\mathbb{D}(X)$ is canonically endowed with a triangulated structure. Moreover all inverse image and homotopy Kan extension functors are exact functors of triangulated categories.*

Proof. Theorem 1.25 applied to the stable derivator $\mathbb{D}_X$ implies that $\mathbb{D}(X) = \mathbb{D}_X(*)$ is triangulated.

Inverse images are exact, because they commute with other inverse images and homotopy Kan extensions, thus they commute also with suspensions and cones. Homotopy Kan extensions are exact, because they are adjoint to inverse images, which are exact.

□

Example 1.29. a) Let $\mathcal{M}$ be a Quillen model category. Then the prederivator $\mathbb{D}^{\mathcal{M}}$ defined by $\mathbb{D}^{\mathcal{M}}(X) = Ho(\mathcal{M}^X)$ is a derivator. Moreover, if $\mathcal{M}$ is pointed resp. stable, then so is $\mathbb{D}^{\mathcal{M}}$ [Cis03].
b) Let $\mathcal{C}$ be an ∞-category. Then $\mathbb{D}^{\mathcal{C}}$ defined by $\mathbb{D}^{\mathcal{C}}(X) = Ho(\mathcal{C}^{N}(X))$ is a prederivator ([Joy08]). We can expect that $\mathbb{D}^{\mathcal{C}}$ is a derivator if $\mathcal{C}$ is presentable (see also Remark 1.34.a)), although there seems to be no proof available in literature.

1.2 Infinite chains of adjunctions

Definition 1.30. Let C be a 2-category, x, y objects in C and $f : x \longrightarrow y$ a 1-morphism.
a) We say f has an *n-th right adjoint* $f^{[n]}$, if there are adjunctions:
$f =: f^{[0]} \dashv f^{[1]} \dashv f^{[2]} \dashv .... \dashv f^{[n-1]} \dashv f^{[n]}$
b) We say f has an *n-th left adjoint* $f^{[-n]}$, if there are adjunctions:
$f^{[-n]} \dashv f^{[-(n-1)]} \dashv ..... \dashv f^{[-2]} \dashv f^{[-1]} \dashv f^{[0]}$
c) We say f *generates an infinite chain of adjunctions*, if it has an n-th right adjoint and an n-th left adjoint for every $n \in \mathbb{N}$
d) Let f generate an infinite chain of adjunctions and $e : x \longrightarrow x$ be an equivalence. Then the infinite chain of adjunctions generated by f is said to be *periodic of order $n \in 2\mathbb{Z}$ with respect to e*, if there is an invertible 2-morphism $f \circ e \xrightarrow{\cong} f^{[n]}$.

Proposition 1.31. *a) Let $f : x \longrightarrow y$ be an equivalence. Then f generates an infinite chain of adjunctions, which is periodic of order 2 with respect to id_x.*
b) Let $f : x \longrightarrow y$ and $g : y \longrightarrow z$ generate infinite chains of adjunctions. Then also $g \circ f : x \longrightarrow y$ generates an infinite chain of adjunctions.
c) Let $f : x \longrightarrow y$ be a 1-morphism, such that $f^{[k]}$ exists for $0 \leqq k \leqq n$ and n even. Let $h, h' : x \longrightarrow x$ generate an infinite chain of adjunctions. If there are an invertible 2-morphisms $f \circ h \xrightarrow{\cong} f^{[n]}$ and $f^{[n]} \circ h' \xrightarrow{\cong} f$, then f generates an infinite chain of adjunctions.
d) Let f generate an infinite chain of adjunctions and $F : C \longrightarrow D$ a pseudo-functor. The also $F(f)$ generates an infinite chain of adjunctions.

Proof. a) Let f^{-1} be an inverse of f. Then we can choose: $f^{[n]} = f$ if n is even and $f^{[n]} = f^{-1}$ if n is odd.
b) We can choose $(g \circ f)^{[n]} = g^{[n]} \circ f^{[n]}$ if n is even and $(g \circ f)^{[n]} = f^{[n]} \circ g^{[n]}$ if n is odd, since the composition of two left (resp. right) adjoints is the left (resp. right) adjoint of the composition.

c) Again one can describe $f^{[l]}$ explicitly:
The case $l > n$:
In this case one can write $l = mn+k$ in a unique way with $m \geqq 1, 0 \leqq k \leqq n-1$. The following formulas give us an l-th right adjoint of f:
$f^{[l]} = f^{[k]} \circ h^{[k]} \circ h^{[n+k]} \circ h^{[2n+k]} \circ ... \circ h^{[(m-1)n+k]}$ if l is even and
$f^{[l]} = h^{[(m-1)n+k]} \circ ... \circ h^{[2n+k]} \circ h^{[n+k]} \circ h^{[k]} \circ f^{[k]}$ if l is odd
The case $l < 0$:
Here we write $l = -mn + k$ with $m < 0, 0 \leqq k \leqq n-1$. There are similar formulas to describe the l-th left adjoint of f:
$f^{[l]} = f^{[k]} \circ h'^{[-n+k]} \circ h'^{[-2n+k]} \circ ... \circ h'^{[-mn+k]}$ if l is even and
$f^{[l]} = h'^{[-mn+k]} \circ ... \circ h'^{[-2n+k]} \circ h'^{[-n+k]} \circ f^{[k]}$ if l is odd
d) Since pseudofunctors preserve adjunctions, we have $F(f^{[n]}) \dashv F(f^{[n+1]})$ for all $n \in \mathbb{Z}$.Thus we can choose $(F(f))^{[n]} = F(f^{[n]})$.

□

Example 1.32. a) Let $\mathbb{D}$ be a pointed derivator, $X \in Cat$ and $\iota_X : \emptyset \longrightarrow X$ the inclusion of the empty category. By (Der1) and (Der6) $\iota_{X!}$ and $\iota_{X*} : \mathbb{D}(\emptyset) \longrightarrow \mathbb{D}(X)$ are both the inclusion of the zero object. Thus $\iota_{X!}$ generates an infinite chain of adjunctions which is periodic of order 2 with respect to $id_{\mathbb{D}(\emptyset)}$.

b) Let $\mathbb{D}$ be a stable derivator, X a finite discrete category and $\rho_X : X \longrightarrow *$ the projection to the terminal category. By (Der2) $\rho_{X!}$ is the coproduct and ρ_{X*} is the product indexed over the objects of X in the category $\mathbb{D}(*)$.
But by Theorem 1.25 $\mathbb{D}$ is additive, thus coproducts and products coincide. Hence $\rho_{X!}$ generates an infinite chain of adjunction, periodic of order 2 with respect to $id_{\mathbb{D}(*)}$.

c) Let $p : \mathbf{1} \longrightarrow *$ be the unique functor and $x : * \longrightarrow \mathbf{1}$ the functor with image $x \in Ob(\mathbf{1})$.
Since 1 is a cosieve, 0 is a sieve and there are adjunctions $0 \dashv p \dashv 1$, Proposition 1.8 and 1.13 yield a chain of adjunctions:
$1^? \dashv 1_! \dashv 1^* \dashv p^* \dashv 0^* \dashv 0_* \dashv 0^!$
By Remark 1.23.b) we can identify $1^?$ with the cone functor C and $0^!$ with the fiber functor F.
Let $S := (\mathbf{1} \times \mathbf{2}) - \{(1,1),(1,2)\}$, $s : \mathbf{1} \longrightarrow S, x \mapsto (0,x)$ and $t : S \longrightarrow \mathbf{1} \times \mathbf{2}$ the inclusion.
Let $A \in \mathbb{D}(\mathbf{1})$ with underlying diagram $A_0 \xrightarrow{a} A_1$. Then the diagram underlying $t_! \circ s_* \circ Fiber(A)$ looks like:

$$\begin{tikzcd}
FA \arrow[r] \arrow[d] & 0 \arrow[d] \\
A_0 \arrow[r, "a"] \arrow[d] & A_1 \arrow[d] \\
0 \arrow[r] & CA
\end{tikzcd}$$

By Proposition 1.19 and 1.20 the outer square is bicartesian, hence $CA = \Sigma FA$. But since $\mathbb{D}(*)$ is triangulated, Σ is an equivalence, thus $p^* : \mathbb{D}(*) \longrightarrow \mathbb{D}(\mathbf{1})$ generates an infinite chain of adjuntions, periodic of order 6 with respect to Σ. Moreover, the explicit formulas in the proof of Proposition 1.31.c) give us new characterizations of iterated suspension and loop functors:

From Remark 1.23.b) we know that $\Sigma \cong p^{[-3]} \circ p^{[2]}$ and $\Omega \cong p^{[3]} \circ p^{[-2]}$.
But Σ and Ω are equivalences, hence by Proposition 1.31.a) :
$\Sigma \cong p^{[2n-5]} \circ p^{[2n]}$ and $\Omega \cong p^{[2n+5]} \circ p^{[2n]}$ for all $n \in \mathbb{Z}$.

More generally, by Proposition 1.31.c) :
$p^{[6k+l]} = \Sigma^{-k} \circ p^{[l]}$ for $k \in \mathbb{Z}$, $l \in \{-1, 1, 3\}$.
But $p^{[-1]} \circ p^{[0]} \cong id_{\mathbb{D}(*)} \cong p^{[1]} \circ p^{[0]}$ and $p^{[3]} \circ p^{[0]} \cong 0$. Hence:

$$p^{[6k+l]} \circ p^{[0]} \cong \begin{cases} \Sigma^{-k} & \text{if } l = \text{-1 or 1} \\ 0 & \text{if } l = 3 \end{cases}$$

d) Let $\mathbf{n}_+$ be the full subcategory of $\mathbf{n} \times \mathbf{1}$ with $Ob(\mathbf{n}_+) = \{(x, y) | y = 0 \text{ or } x = 0\}$ and $\mathbf{n}^+$ the full subcategory of $\mathbf{n} \times \mathbf{1}$ with $Ob(\mathbf{n}^+) = \{(x, y) | y = 1 \text{ or } x = n\}$. Let $k_+ : \mathbf{n}_+ \longrightarrow \mathbf{n} \times \mathbf{1}$ and $k^+ : \mathbf{n}^+ \longrightarrow \mathbf{n} \times \mathbf{1}$ the inclusions. Consider functors $i_+ : \mathbf{n-1} \longrightarrow \mathbf{n}_+, x \mapsto (x, 0)$, $\tilde{i}_+ : \mathbf{n} \longrightarrow \mathbf{n}_+, x \mapsto (x, 0)$ and $i^+ : \mathbf{n-1} \longrightarrow \mathbf{n}^+, x \mapsto (x+1, 1)$, $\tilde{i}^+ : \mathbf{n} \longrightarrow \mathbf{n}^+, x \mapsto (x, 1)$ and the compositions $j_+ := k_+ \circ i_+$, $\tilde{j}_+ := k_+ \circ \tilde{i}_+$ and $j^+ := k^+ \circ i^+$, $\tilde{j}^+ := k^+ \circ \tilde{i}^+$. Moreover let R be the full sub-category of $(\mathbf{2n-1}) \times (\mathbf{n+1})$ with $Ob(R) = \{(x, y) | -1 \leqq x - y \leqq n\}$ and P the full subcategory of R with $Ob(P) = \{y = 0 \text{ or } x - y = -1 \text{ or } x - y = n\}$. Consider $i_P : \mathbf{n-1} \longrightarrow P, x \mapsto (x, 0)$, the inclusion $j_R : P \longrightarrow R$ and $i_{R,k} : \mathbf{n-1} \longrightarrow R, x \mapsto (x + k, k)$

There is a chain of adjoint functors between $\mathbf{n-1}$ and $\mathbf{n}$:
$d_n \dashv s_{n-1} \dashv d_{n-1} \dashv .... \dashv d_1 \dashv s_0 \dashv d_0$
Moreover d_0 is a cosieve and d_n is a sieve. So by Proposition 1.8 there is a chain of adjoint functors between $\mathbb{D}(\mathbf{n-1})$ and $\mathbb{D}(\mathbf{n})$:
$d_0^? \dashv d_{0!} \dashv d_0^* \dashv s_0^* \dashv d_1^* \dashv .... \dashv s_{n-1}^* \dashv d_n^* \dashv d_{n*} \dashv d_n^!$
With Proposition 1.16 we can describe $d_0^?$ and $d_n^!$ explicitly:

$d_0^{?} = j^{+*} \circ k_{+!} \circ \tilde{i}_{+*} =: C_n$ and $d_n^{!} = j_+^{*} \circ k_*^{+} \circ \tilde{i}_!^{+} =: F_n$

Analogously to Example c) there is an isomorphism $C_n \cong \Sigma_n \circ F_n$, where $\Sigma_n = j^{+*} \circ k_{+!} \circ i_{+*}$.

But there is an isomorphism: $(\Sigma_n)^{n+1} \cong (\Sigma)^2$. To see this, consider an element $A \in \mathbb{D}(\mathbf{n-1})$ with underlying diagram: $A_0 \xrightarrow{a_1} A_1 \xrightarrow{a_2} \ldots \xrightarrow{a_{n-1}} A_{n-1}$
We will analyse $j_{R!} \circ i_{P*}(A)$. By Proposition 1.19 and Proposition 1.20 all squares with vertices $\{(x,y),(x+x',y),(x,y+y'),(x+x',y+y')\}$ (with x,y,x',y' natural numbers such that the corresponding functor $\square \longrightarrow R$ exist) are bicartesian. Therefore the underlying diagram of $B := j_{R!} \circ i_{P*}(A)$ looks as follows (with the notation $C_{\alpha\beta}$ = Cone of the composition $a_\alpha \circ a_\beta$)

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & A_{n-1} \arrow[r] & 0 \arrow[d] & & \\
0 \arrow[r] & C_1 \arrow[r] \arrow[d] & C_{21} \arrow[r] \arrow[d] & \cdots \arrow[rr] & & \Sigma A_0 \arrow[r] \arrow[d] & 0 \arrow[d] & \\
& 0 \arrow[r] & C_2 \arrow[r] \arrow[d] & \cdots \arrow[rr] & & \Sigma A_1 \arrow[r] \arrow[d] & \Sigma C_1 \arrow[r] \arrow[d] & 0 \\
& & 0 \arrow[r] & \cdots \arrow[rr] & & \Sigma A_2 \arrow[r] \arrow[d] & {} & \ddots \\
& & & \ddots & & \vdots \arrow[d] & & \ddots \\
& & & & \ddots & {} & & \ddots \\
& & & & 0 \arrow[r] & \Sigma A_{n-1} \arrow[r] \arrow[d] & \cdots & 0 \arrow[d] \\
& & & & & 0 \arrow[r] & \Sigma^2 A_0 \arrow[r] & \cdots \; \Sigma^2 A_{n-2} \arrow[r] & \Sigma^2 A_{n-1}
\end{tikzcd}$$

We use that the squares $\{(k,0),(n,0),(k,k+1),(n,k+1)\}$ are bicartesian to conclude that $(n,k+1)^*(B) \cong \Sigma A_k$, the squares $\{(n,k+1),(n+k+1,k+1),(n,n),(n+k+1,n)\}$ to conclude further that $(n+k+1,n)^*(B) \cong \Sigma^2 A_k$. Thus (also use the obvious natural transformations between the above squares and (Der2)) $i_{R,n}^*(B) \cong \Sigma^2 A$.
On the other hand we use bicartesianess of the squares $\{(k,l),(k',l),(k,l+1),(k',l+1)\}$ to show that $i_{R,l+1}^*(B) \cong \Sigma_n(i_{R,l}^*(B))$.
Hence $\Sigma^2 A \cong i_{R,n}^*(B) \cong (\Sigma_n)^{n+1} A$

In particular Σ_n is an equivalence and thus d_0^* generates an infinite chain of adjunctions, which is periodic of order $4n+2$.

Moreover, this shows that all inverse images of functors between simplex categories generate infinite chains of adjunctions.

However, the minimal periodicity numbers for the inverse images of compositions behave nontrivially:
Consider the functors between $\mathbf{0}$ and $\mathbf{2}$.
There are four functors:
The projection $p : \mathbf{2} \longrightarrow \mathbf{0}$ and the inclusions $x : \mathbf{0} \longrightarrow \mathbf{2}, 0 \leqq x \leqq 2$.
With the explicit description of the infinite chains of adjunctions above, we can calculate all the adjoints of p^* and x^*. It turns out that:
- $p^*, 0^*$ and 2^* are part of the same infinite chain of adjuntions, which is periodic of order 8 with respect to Σ^2.
-1^* generates an infinite chain of adjunction, which is periodic of order 4 with respect to Σ.

e) Let C be a symmetric monoidal category with monoidal pairing $- \otimes -$, and $c \in C$ a dualizable object. Then the functor $c \otimes - : C \longrightarrow C$ generates an infinite chain of adjunctions, which is periodic of order 2 with respect to id_C.
A well known example is the homotopy category of finite spectra together with the smash product. In this case every object is dualizable and the dual is given by the Spanier-Whitehead dual ([Ada74] III.5).
We will use a generalized version of Spanier-Whitehead-duality to prove Theorem 3.12.

Theorem 1.33. *Let $\mathbb{D}$ be a pointed derivator. Then the following conditions are equivalent:*
a) $\mathbb{D}$ is stable.
b) $\mathbb{D}$ has coexceptional inverse images along $(i_\ulcorner \times X)$ and exceptional inverse images along $(i_\lrcorner \times X)$ for all $X \in Fin$.
c) For all functors $f : X \longrightarrow Y$, with X and Y finite, finite dimensional categories, f^* generates an infinite chain of adjunctions.

Proof. a)$\Rightarrow$b) The functors $i_\ulcorner$ and $i_\lrcorner$ are fully faithful, hence so are $i_{\ulcorner!}$ and $i_{\lrcorner*}$ by Proposition 1.9. Since $\mathbb{D}$ is stable, their essential images are equal. Moreover we have natural isomorphisms $i_\ulcorner^* i_{\ulcorner!} \cong id_{\mathbb{D}(\ulcorner)}$ and $i_\lrcorner^* i_{\lrcorner*} \cong id_{\mathbb{D}(\lrcorner)}$. Hence $i_\lrcorner^* i_{\ulcorner!}$ and $i_\ulcorner^* i_{\lrcorner*}$ are mutually inverse equivalences. In particular we have $i_{\ulcorner!} \cong (i_{\lrcorner*} \circ i_\lrcorner^*) \circ i_{\ulcorner!} \cong i_{\lrcorner*} \circ (i_\lrcorner^* \circ i_{\ulcorner!})$. Thus $i_{\ulcorner!}$ is a right adjoint, because $i_{\lrcorner*}$ and $i_\lrcorner^* \circ i_{\ulcorner!}$ are.
Dually $i_{\lrcorner*}$ is a left adjoint since $i_{\lrcorner*} \cong (i_{\ulcorner!} \circ i_\ulcorner^*) \circ i_{\lrcorner*} \cong i_{\ulcorner!} \circ (i_\ulcorner^* \circ i_{\lrcorner*})$.
This proves b) for $X = *$.
For general X one applies the proved case to the stable derivator $\mathbb{D}_X$.

b)$\Rightarrow$a) Since $i_{\lrcorner*}$ is a left adjoint, it commutes with arbitrary homotopy left Kan extensions ([Gro11a] Corollary 2.12). In particular we have:
$i_{\lrcorner*} \circ i_{\ulcorner!} \cong i_{\ulcorner!} \circ i_{\lrcorner*} : \mathbb{D}(\ulcorner \times \lrcorner) \longrightarrow \mathbb{D}(\square \times \square)$
To simplify the notation, we rename the objects of $\square$ and its subcategories:

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & 2 \arrow[d] \\
2' \arrow[r] & 3
\end{tikzcd}$$

Consider the functor $p : \ulcorner\times\lrcorner \longrightarrow \ulcorner$ with $p(x,3) = x, p(2,2) = 2, p(2',2') = 2', p(1,2) = p(2',2) = p(1,2') = p(2,2') = 1$

Now let $A \in \mathbb{D}(\square)$ be a cocartesian square with underlying diagram:

$$\begin{tikzcd}
a \arrow[r] \arrow[d] & b \arrow[d] \\
c \arrow[r] & d
\end{tikzcd}$$

Then the underlying diagram of $B := p^*(i_\ulcorner^*(A))$ looks as follows:

$$\begin{tikzcd}
 & & & a \arrow[r] \arrow[d] \arrow[ddd, bend left] & b \arrow[ddd, bend left] \\
 & & & a \arrow[dddd, bend left] & \\
 & & & & \\
a \arrow[r] \arrow[d] \arrow[rrr, bend right] & a \arrow[rrr, bend right] & & a \arrow[r] \arrow[d] & b \\
c \arrow[rrr, bend right] & & & c &
\end{tikzcd}$$

In the next step we analyse $C := i_{\ulcorner !}(B)$.

Since homotopy Kan extensions commute with inverse images ([Gro11a] Proposition 2.6), in particular the inclusions $x : * \longrightarrow \square$, and using [Gro11a] Proposition 3.13.ii), we see that $(3,2)^*(C) \cong b, (3,2')^*(C) \cong c$ and $(3,3)^*(C) \cong d$.

Using the same arguments for $D := i_{\lrcorner *}(C)$ we can conclude that $(1,1)^*(D) \cong a$,

$(2,1)^*(D) \cong a$ and $(2',1)^*(D) \cong a$. Because $i_{\lrcorner *}$ and $i_{\ulcorner !}$ commute (and using again, that both of them commute with inverse images) every square $(x \times \square)^*(D)$ is cartesian and every square $(\square \times x)^*$ is cocartesian for $x \in \square$.
Consider the diagonal $\delta : \square \longrightarrow \square \times \square, x \mapsto (x,x)$ and the unique natural transformations $\alpha : \delta \Rightarrow (3 \times \square)$ and $\beta : \delta \Rightarrow (\square \times 3)$. The induced morphisms $\alpha^* : \delta^*(D) \longrightarrow (3 \times \square)^*(D)$ and $\beta^* : \delta^*(D) \longrightarrow (\square \times 3)^*(D)$ are isomorphisms because of (Der2) and the calculations above.
Therefore $A \cong (\square \times 3)^*(D) \cong \delta^*(D) \cong (3 \times \square)^*(D)$. Thus A is cartesian.

The implication "cartesian" $\Rightarrow$ "cocartesian" is proved dually.

$$\begin{tikzcd}
a \arrow[r] \arrow[d] \arrow[rr, bend left] \arrow[dd, bend right] & a \arrow[d] \arrow[rr, bend left] \arrow[dd, bend right] & a \arrow[r] \arrow[d] \arrow[dd, bend left] & b \arrow[d] \arrow[dd, bend left] \\
a \arrow[r] \arrow[rr, bend left] \arrow[dd, bend right] & a \arrow[rr, bend left] \arrow[dd, bend right] & a \arrow[r] \arrow[dd, bend left] & b \arrow[dd, bend left] \\
a \arrow[r] \arrow[d] \arrow[rr, bend right] & a \arrow[d] \arrow[rr, bend right] & a \arrow[r] \arrow[d] & b \arrow[d] \\
c \arrow[r] \arrow[rr, bend right] & c \arrow[rr, bend right] & c \arrow[r] & d
\end{tikzcd}$$

The underlying diagram of D:
-The "big circles" are cartesian
-The "small squares" are cocartesian
-δ is defined by the most outer 4 objects

c)$\Rightarrow$b) is obvious.
a)$\Rightarrow$c) is the content of Theorem 3.10

$\square$

1.3 Some Remarks and Interpretation, Part 1

Remark 1.34. a) We have seen that many interesting examples of derivators are associated to Quillen model categories or ∞-categories. In fact one can show that these three concepts are (in some sence) equivalent:
Renaudin has proven in [Ren09] that the pseudolocalization of the 2-category of combinatorial model categories at the class of Quillen equivalences maps fully faithfully into the 2-category of derivators. More precisely, the natural pseudofunctor which is defined by the construction described in Example 1.29.a) on objects induces a local equivalence on the pseudolocalization.
On the other hand, one can assign to each combinatorial model category ([Dug01]) a presentable ∞-category via the homotopy coherent nerve construction on the full subcategory of bifibrant objects. Moreover, every presentable ∞-category is equivalent to the homotopy coherent nerve of some combinatorial model category([Lur09] Proposition A.3.7.6).
Now, a method linking presentable ∞-categories and derivators directly, in the best case proving that the homotopy bicategory of the $(\infty, 2)$-category of presentable ∞-categories maps fully faithfully into the 2-category of derivators, would be a great advantage for the theory of derivators: Since the $(\infty, 2)$-category of presentable ∞-categories is highly structured (bicomplete [Lur09] 5.5.3, symmetric monoidal [Lur11] 6.3) one would immediately get similar structures on the full sub-2-category of the 2-category of derivators, consisting of those derivators which (up to equivalence) come from ∞-categories, by passing to derived pseudofunctors.
b) Since there are two well-developed alternatives, one could of course ask, why one should consider derivators at all. All three concepts have some advantages and disadvantages depending on the purpose.
Model categories build the most fundamental concept. Both the theory of ∞-categories ([Lur09] A.2) and of derivators ([Cis08]) depend on it. Unfortunately many interesting properties of and structures on model categories are not invariant under Quillen equivalences (e.g. symmetric monoidal structures), so one often has to consider many different models for the same homotopy theory. On the other hand in many applications there seems to be no way to avoid good point-set level models (e.g. in global stable homotopy theory [Sch13]).
Presentable ∞-categories are perhaps the most natural model for the "homotopy theory of homotopy theories", since the homotopy theories of spaces resp. spectra are characterised by universal properties, and much more structures (monoidal structures, enrichments, Kan extensions) can be described very naturally.
Derivators share many advantages with ∞-categories, but with part a) of this remark in mind, one should consider derivators still as a derived theory, i.e. one should expect that many informations, that are important for the global understanding of the 2-category of derivators, won't be available, if one considers derivators without any higher models. On the other hand derivators are much closer to the actual homotopy categories, so they may provide us with a lot of techniques of analysing homotopy categories in a very structured way.

2 The structure of the 2-categories of Derivators

In this section we specialize to $Dia = Cat$.
In Subsection 2.1 we define morphisms of derivators and discuss in details the abstract duality. In Subsection 2.2 we explain how we can pass from derivators to pointed and stable ones. In Subsection 2.3 we use that the evaluation at $X \in Cat$ is corepresentable (Theorem 2.16), to prove the dualizability theorem for cofree derivators.
In Subsection 2.4 we use that in a closed symmetric monoidal category with full duality the duality relates the monoidal pairing with the closure, to define a completely new symmetric monoidal structure on the 2-category of cofree derivators.
Again we close the section with some remarks and ideas for possible future research.

2.1 Morphisms of Derivators and abstract Duality

Definition 2.1. Let $\mathbb{D}, \mathbb{D}'$ be derivators and $F : \mathbb{D} \longrightarrow \mathbb{D}'$ a morphism of prederivators.
a) *F commutes with homotopy left Kan extensions*, if for all functors $f : X \longrightarrow Y$ the composition of 2-cells:

$$\begin{tikzcd}
\mathbb{D}(X) \arrow[r, "f_{\mathbb{D}!}"] \arrow[dr, equal, "id"'] \arrow[r, phantom, shift right=6, "\eta\Nearrow"] & \mathbb{D}(Y) \arrow[r, "F_Y"] \arrow[d, "f^*_{\mathbb{D}}"] \arrow[dr, phantom, "\Nearrow"] & \mathbb{D}'(Y) \arrow[d, "f^*_{\mathbb{D}'}"] \arrow[dr, equal, "id"] & \\
& \mathbb{D}(X) \arrow[r, "F_X"'] & \mathbb{D}'(X) \arrow[r, "f_{\mathbb{D}'!}"'] \arrow[r, phantom, shift left=6, "\epsilon'\Nearrow"] & \mathbb{D}'(Y)
\end{tikzcd}$$

is invertible.
Dually, *F commutes with homotopy right Kan extensions*, if for all functors $f : X \longrightarrow Y$ the composition of 2-cells:

$$\begin{tikzcd}
\mathbb{D}(X) \arrow[r, "f_{\mathbb{D}*}"] \arrow[dr, equal, "id"'] \arrow[r, phantom, shift right=6, "\epsilon\Swarrow"] & \mathbb{D}(Y) \arrow[r, "F_Y"] \arrow[d, "f^*_{\mathbb{D}}"] \arrow[dr, phantom, "\Swarrow"] & \mathbb{D}'(Y) \arrow[d, "f^*_{\mathbb{D}'}"] \arrow[dr, equal, "id"] & \\
& \mathbb{D}(X) \arrow[r, "F_X"'] & \mathbb{D}'(X) \arrow[r, "f_{\mathbb{D}'*}"'] \arrow[r, phantom, shift left=6, "\eta'\Swarrow"] & \mathbb{D}'(Y)
\end{tikzcd}$$

is invertible.
b) Let $\mathbb{D}$, $\mathbb{D}$' be derivators. Then the category $[\mathbb{D},\mathbb{D}']_L$ (resp. $[\mathbb{D},\mathbb{D}']_R$) of colimit preserving (resp.limit preserving) morphisms from $\mathbb{D}$ to $\mathbb{D}'$ is defined to be the full subcategory of $PDer(\mathbb{D},\mathbb{D}')$ on those morphisms which commute with the homotopy left Kan extensions (resp. homotopy right Kan extensions).
c) The prederivator of colimit preserving (resp. limit preserving) morphisms $[\![\mathbb{D},\mathbb{D}']\!]_L$ (resp. $[\![\mathbb{D},\mathbb{D}']\!]_R$) is defined by

$$[\![\mathbb{D},\mathbb{D}']\!]_L(X) := [\mathbb{D},\mathbb{D}'_X]_L \text{ (resp. } [\![\mathbb{D},\mathbb{D}']\!]_L(X) := [\mathbb{D},\mathbb{D}'_X]_R)$$

for $X \in Cat$.
d) The 2-category of derivators Der^u_L (resp. Der^u_R) is defined by:
$Ob(Der^u_L) = Ob(Der^u_R) = \{Derivators\}$
$Der^u_L(\mathbb{D},\mathbb{D}') = [\mathbb{D},\mathbb{D}']_L$ (resp. $Der^u_R(\mathbb{D},\mathbb{D}') = [\mathbb{D},\mathbb{D}']_R$)

Example 2.2. Let $\mathbb{D}$ be a derivator and $f : X \longrightarrow Y$ a functor with $X, Y \in Cat$
a) The functors $(f \times Z)^* : \mathbb{D}(Y \times Z) \longrightarrow \mathbb{D}(X \times Z)$ assemble into a morphism of prederivators $f^* : \mathbb{D}_Y \longrightarrow \mathbb{D}_X$.
Moreover $f^* \in [\mathbb{D}_Y, \mathbb{D}_X]_L \cap [\mathbb{D}_Y, \mathbb{D}_X]_R$ ([Gro11a] Proposition 2.6)
b) The functors $(f \times Z)_! : \mathbb{D}(X \times Z) \longrightarrow \mathbb{D}(Y \times Z)$ assemble into a morphism of prederivators $f_! : \mathbb{D}_X \longrightarrow \mathbb{D}_Y$.
Moreover $f_! \in [\mathbb{D}_X, \mathbb{D}_Y]_L$ ([Gro11b] Lemma 2.6)
c) The functors $(f \times Z)_* : \mathbb{D}(X \times Z) \longrightarrow \mathbb{D}(Y \times Z)$ assemble into a morphism of prederivators $f_* : \mathbb{D}_X \longrightarrow \mathbb{D}_Y$.
Moreover $f_* \in [\mathbb{D}_X, \mathbb{D}_Y]_R$

Remark 2.3. By [CT12] Theorem A.3 $[\![\mathbb{D},\mathbb{D}']\!]_L$ and $[\![\mathbb{D},\mathbb{D}']\!]_R$ are in fact derivators. We will show this in some special cases, i.e. when $\mathbb{D}$ is cofree derivator in the case of $[\![\mathbb{D},\mathbb{D}']\!]_L$ or a free derivator in the case of $[\![\mathbb{D},\mathbb{D}']\!]_R$.

Proposition 2.4. *a) Let $\mathbb{D}$ and $\mathbb{D}'$ be derivators. Then $F \in PDer(\mathbb{D},\mathbb{D}')$ is a left adjoint if and only if there is $G \in PDer(\mathbb{D}',\mathbb{D})$ such that for all $X \in Cat$ there are adjunctions of functors: $F_X \dashv G_X$.*
b) If $F \in PDer(\mathbb{D},\mathbb{D}')$ is a left adjoint, then $F \in [\mathbb{D},\mathbb{D}']_L$.

Proof. a) [Gro11a] Proposition 2.11
b) [Gro11a] Corollary 2.12

□

Definition 2.5. Let $\mathbb{D}$ and $\mathbb{D}'$ be derivators.
a) Then their cartesian product $\mathbb{D} \times \mathbb{D}'$ is the composition:
$Cat^{op} \xrightarrow{diagonal} Cat^{op} \times Cat^{op} \xrightarrow{\mathbb{D}\times\mathbb{D}'} CAT \times CAT \xrightarrow{-\times-} CAT$
b) And their exterior product $\mathbb{D} \boxtimes \mathbb{D}'$ is the composition:
$Cat^{op} \times Cat^{op} \xrightarrow{\mathbb{D}\times\mathbb{D}'} CAT \times CAT \xrightarrow{-\times-} CAT$

Proposition 2.6. *Let* $\mathbb{D}, \mathbb{D}'$ *and* $\mathbb{D}''$ *be derivators. Then are natural equivalences of catergories:*

$$PsNat(\mathbb{D} \boxtimes \mathbb{D}', \mathbb{D}'' \circ (- \times -)) \cong PDer(\mathbb{D} \times \mathbb{D}', \mathbb{D}'')$$

where $PsNat$ *denotes the category of pseudonatural transformations.*

Proof. [Gro12] Proposition 1.4. □

Example 2.7. Let $\mathbb{D}, \mathbb{D}'$ and $\mathbb{D}''$ be derivators. The composition of morphisms of derivators defines a pseudonatural transformation:
$[\![\mathbb{D}, \mathbb{D}']\!]_L \boxtimes [\![\mathbb{D}', \mathbb{D}'']\!]_L \longrightarrow [\![\mathbb{D}, \mathbb{D}'']\!]_L \circ (- \times -)$
which is defined by:
$(F \in [\mathbb{D}, \mathbb{D}'_X]_L, G \in [\mathbb{D}', \mathbb{D}''_Y]_L) \mapsto G_X \circ F \in [\mathbb{D}, \mathbb{D}''_{X \times Y}]_L$
for $(X, Y) \in Cat \times Cat$.
Thus by Proposition 2.6 there is a well defined composition morphism:
$[\![\mathbb{D}, \mathbb{D}']\!]_L \times [\![\mathbb{D}', \mathbb{D}'']\!]_L \longrightarrow [\![\mathbb{D}, \mathbb{D}'']\!]_L$

The 2-categories Der^u_L and Der^u_R are related to each other by the abstract duality in the 2-category $PDer$:

Definition 2.8. Let $\mathbb{D}$ be a prederivator. Then the *opposite prederivator* $\mathbb{D}^{op}$ is defined by:

$$\mathbb{D}^{op}(X) = \mathbb{D}(X^{op})^{op}$$

In other words $\mathbb{D}^{op}$ is given by the following composition:

$$Cat^{op} \xrightarrow{(-)^{op}} Cat^{coop} \xrightarrow{\mathbb{D}^{co}} CAT^{co} \xrightarrow{(-)^{op}} CAT$$

The following result is well-known. Nevertheless we will give the details as it is one of the main ingredients for Theorem 3.10.

Proposition 2.9. *a)* $(-)^{op}$ *defines an self-inverse equivalence* $PDer^{co} \longrightarrow PDer$.
b) $(-)^{op}$ *restricts to an equivalence* $Der^{u,co}_L \longrightarrow Der^u_R$.
In particular we have equivalences of categories $[\mathbb{D}, \mathbb{D}']^{op}_L \cong [\mathbb{D}^{op}, \mathbb{D}'^{op}]_R$
c) The equivalences described in part b) assemble to equivalences of derivators:

$$[\![\mathbb{D}, \mathbb{D}']\!]^{op}_L \cong [\![\mathbb{D}^{op}, \mathbb{D}'^{op}]\!]_R$$

Proof. Part a) is obvious.
b) First note that the opposite of a derivator is again a derivator.
Now let $f : X \longrightarrow Y$ be a functor in Cat. Then by definition $f^*_{\mathbb{D}^{op}} = (f^{op})^{*\,op}_{\mathbb{D}}$.
Since $(-)^{op}$ is a 2-functor, it maps adjunctions to adjunctions but it interchanges the role of left and right adjoint, because the directions of unit and counit get

reversed. Therefore we have: $f_{*\mathbb{D}^{op}} = (f^{op})_{!\mathbb{D}}{}^{op}$. Let $F \in [\mathbb{D}, \mathbb{D}']_L$, then for any functor $f^{op} \in Cat(X^{op}, Y^{op})$ we have:
$(f^{op})_{!\mathbb{D}'} \circ F_{X^{op}} \cong F_{Y^{op}} \circ (f^{op})_{!\mathbb{D}}$
After applying $(-)^{op}$ to this equivalence, this turns into:
$f_{*\mathbb{D}'^{op}} \circ F_X^{op} \cong (f^{op})_{!\mathbb{D}'}{}^{op} \circ F_X^{op} \cong F_Y^{op} \circ (f^{op})_{!\mathbb{D}}{}^{op} \cong F_Y^{op} \circ f_{*\mathbb{D}^{op}}$
Hence $F^{op} \in [\mathbb{D}^{op}, \mathbb{D}'^{op}]_R$. Moreover $(F^{op})^{op} = F$, so $(-)^{op}$ defines equivalences on morphism categories.
This proves b).
c)Let $X \in Cat$. Then for all $Y \in Cat$ we have natural equivalences:
$(\mathbb{D}_X)^{op}(Y) = \mathbb{D}(X \times Y^{op})^{op} \cong \mathbb{D}^{op}((X^{op} \times Y)^{op})^{op} \cong \mathbb{D}^{op}(X^{op} \times Y) \cong (\mathbb{D}^{op})_{X^{op}}(Y)$
Clearly these assemble into an equivalence $(\mathbb{D}_X)^{op} \cong (\mathbb{D}^{op})_{X^{op}}$.

Applying this to $\mathbb{D}'^{op}$ in $[\![\mathbb{D}^{op}, \mathbb{D}'^{op}]\!]_R$ yields:
$[\![\mathbb{D}^{op}, \mathbb{D}'^{op}]\!]_R(X) = [\mathbb{D}^{op}, (\mathbb{D}'^{op})_X]_R \cong [\mathbb{D}^{op}, (\mathbb{D}'_{X^{op}})^{op}]_R$
Since we also have:
$[\![\mathbb{D}, \mathbb{D}']\!]_L^{op}(X) = [\![\mathbb{D}, \mathbb{D}']\!]_L(X^{op})^{op} = [\mathbb{D}, \mathbb{D}'_{X^{op}}]_L^{op}$
we can now apply b) to get equivalences $[\![\mathbb{D}^{op}, \mathbb{D}'^{op}]\!]_R(X) \cong [\![\mathbb{D}, \mathbb{D}']\!]_L^{op}(X)$. These equivalences assemble into the desired equivalence of (pre)derivators. □

2.2 The 2-categories of pointed and stable derivators

Definition 2.10. a) The 2-categories Der_L^p resp. Der_R^p of pointed derivators, are defined to be the full sub-2-categories of Der_L^u resp. Der_R^u with pointed derivators as objects.
b) The 2-categories Der_L^s resp. Der_R^s of stable derivators, are defined to be the full sub-2-categories of Der_L^u resp. Der_R^u with stable derivators as objects.

The following theorem describes how we can pass from derivators to pointed and stable derivators in a functorial way.

Theorem 2.11. *a)Let $\mathbb{D} \in Der_L^u$. Then there exists a pointed derivator $\mathbb{D}^p$ together with a colimit preserving morphism $\mathbb{D} \xrightarrow{P_{\mathbb{D}}} \mathbb{D}^p$, such that $P_{\mathbb{D}}$ induces for every $\mathbb{D}' \in Der_L^p$ an equivalence of categories $[\mathbb{D}_p, \mathbb{D}']_L \xrightarrow{\cong} [\mathbb{D}, \mathbb{D}']_L$.*
b)Let $\mathbb{D} \in Der_L^p$ be regular (i.e. sequential homotopy colimits commute with products and homotopy pull-backs). Then there exists a stable derivator $\mathbb{D}^s$ together with a colimit preserving morphism $\mathbb{D} \xrightarrow{S_{\mathbb{D}}} \mathbb{D}^s$, such that $S_{\mathbb{D}}$ induces for every $\mathbb{D}' \in Der_L^s$ an equivalence of categories $[\mathbb{D}_s, \mathbb{D}']_L \xrightarrow{\cong} [\mathbb{D}, \mathbb{D}']_L$.

Proof. a) [Cis08] Proposition 4.17
b) [Hel97]

□

2.3 Universal derivators

The following is an immediate consequence of the biYoneda Lemma [Bak]:

Proposition 2.12. *Let $X \in Cat$ and $\underline{X}$ the associated representable prederivator, i.e. $\underline{X}(Y) = Cat(Y, X)$. Then for every prederivator $\mathbb{D}$ we have a natural equivalence of categories:*

$$PDer(\underline{X}, \mathbb{D}) \cong \mathbb{D}(X)$$

Now the derivator associated to the homotopy theory of simplicial sets plays a crucial role in Der^u_L. It can be regarded as a cofree object on one generator.

Definition 2.13. a) $\mathbb{H}^u$ is defined to be the derivator associated to the Quillen model category of simplicial sets (i.e. the weak equivalences are those maps which become weak equivalences of topological spaces after applying geometric realization) [Qui67].
b) The stabilisation of the pointed derivator $\mathbb{H}^p := (\mathbb{H}^u)^p$ exists and will be denoted by $\mathbb{H}^s$.

Remark 2.14. $\mathbb{H}^s$ is equivalent to any derivator associated to any combinatorial model category of spectra [CT11] Theorem A.14.
In particular $\mathbb{H}^s(*)$ is equivalent to the stable homotopy category.

Notation: In the following, let $\star \in \{u, p, s\}$
A $\star$-derivator is a derivator if $\star = u$, a pointed derivator if $\star = p$ and a stable derivator if $\star = s$.

Definition 2.15. a) A $\star$-derivator is *cofree* in $Der^\star_L$, if it is equivalent to $\mathbb{H}^\star_X$ for some $X \in Cat$. And $FDer^\star_L$ is the full sub-2-category of $Der^\star_L$ on the cofree $\star$-derivators.
b) A $\star$-derivator is *free* in $Der^\star_R$, if it is equivalent to $\mathbb{H}^{\star op}_X$ for some $X \in Cat$. And $FDer^\star_R$ is the full sub-2-category of $Der^\star_R$ on the free $\star$-derivators.

Let $X \in Cat$, then by applying $CAT(Y, -)$ to the Yoneda embedding $X \longrightarrow CAT(X^{op}, sSet)$, where we regard Hom-sets as constant simplicial sets, before passing to homotopy categories, we get functors

$$\underline{X}(Y) = Cat(Y, X) \longrightarrow CAT(Y, CAT(X^{op}, sSet)) \cong CAT(X^{op} \times Y, sSet) \longrightarrow \mathbb{H}^u_{X^{op}}(Y)$$

and hence a morphism $\underline{X} \xrightarrow{h_X} \mathbb{H}_{X^{op}}$.

Theorem 2.16. *Let $X \in Cat, \mathbb{D} \in Der^u_L$, then h_X induces an equivalence of categories*

$$[\mathbb{H}^u_{X^{op}}, \mathbb{D}]_L \longrightarrow PDer(\underline{X}, \mathbb{D}).$$

Proof. [Cis08] Corollary 3.28, see also [Fra96] Theorem 4. □

Corollary 2.17. *a) Let $X \in Cat, \mathbb{D} \in Der^p_L$. Then composition with $P_{\mathbb{H}^u_{X^{op}}}$ induces an equivalence*

$$[\mathbb{H}^p_{X^{op}}, \mathbb{D}]_L \longrightarrow PDer(\underline{X}, \mathbb{D}).$$

b) Let $X \in Cat, \mathbb{D} \in Der^s_L$. Then composition with $S_{\mathbb{H}^p_{X^{op}}}$ induces an equivalence

$$[\mathbb{H}^s_{X^{op}}, \mathbb{D}]_L \longrightarrow PDer(\underline{X}, \mathbb{D}).$$

Proof. Just apply Theorem 2.11. □

Corollary 2.18. *Let $X, Y \in Cat, \mathbb{D} \in Der^\star_L$. Composing the equivalences of Proposition 2.12, Theorem 2.16 and Corollary 2.17 yields equivalences*

$$[\mathbb{H}^\star_{X^{op}}, \mathbb{D}_Y]_L \longrightarrow \mathbb{D}_X(Y).$$

These assemble into an equivalence of prederivators:

$$d^h_{\mathbb{D},X} : [\![\mathbb{H}^\star_{X^{op}}, \mathbb{D}]\!]_L \longrightarrow \mathbb{D}_X.$$

Corollary 2.19. *Let $X \in Cat, \mathbb{D} \in Der^\star_R$. Then there are natural equivalences of prederivators*

$$[\![(\mathbb{H}^{\star op})_{X^{op}}, \mathbb{D}]\!]_R \longrightarrow \mathbb{D}_X.$$

Proof. By Proposition 2.9 and Corollary 2.18 there are equivalences:
$[\![(\mathbb{H}^{\star op})_{X^{op}}, \mathbb{D}]\!]_R$
$\cong [\![((\mathbb{H}^{\star op})_{X^{op}})^{op}, \mathbb{D}^{op}]\!]^{op}_L$
$= [\![((\mathbb{H}^{\star op})^{op})_X, \mathbb{D}^{op}]\!]^{op}_L$
$= [\![\mathbb{H}^\star_X, \mathbb{D}^{op}]\!]^{op}_L$
$\cong ((\mathbb{D}^{op})_{X^{op}})^{op} = (\mathbb{D}_X)^{opop} = \mathbb{D}_X$ □

Note that, since $\mathbb{D}_X$ is a $\star$-derivator, the same is true for $[\![\mathbb{H}^\star_{X^{op}}, \mathbb{D}]\!]_L$ and $[\![(\mathbb{H}^{\star op})_{X^{op}}, \mathbb{D}]\!]_R$. This proves the statement of Remark 2.3 in the cofree (resp. free) case and gives us well-behaving internal *Hom*-functors on $FDer^\star_L$ and $FDer^\star_R$.

Remark 2.20. a)Let $\mathfrak{Y}_X \in \mathbb{H}^\star_{X^{op}}(X)$ be the image of $id_X \in \underline{X}(X)$ under the morphism h_X, then the equivalence $d^h_{\mathbb{D},X^{op}}$ maps a morphism $F : \mathbb{H}^\star_X \longrightarrow \mathbb{D}$ to $F_{X^{op}}(\mathfrak{Y}_X) \in \mathbb{D}_X(*)$, in particular $\mathfrak{Y}_X$ corresponds to the identity on $\mathbb{H}^\star_X$.
By abstract duality, we can describe the equivalence of Corollary 2.19 by $(F : (\mathbb{H}^{\star op})_X \longrightarrow \mathbb{D}) \mapsto F_{X^{op}}(\mathfrak{Y}_{X^{op}})$ with $\mathfrak{Y}_{X^{op}} \in \mathbb{H}^\star_X(X^{op})^{op} = (\mathbb{H}^{\star op})_{X^{op}}(X)$

Moreover, by definition of the morphism h_X, we see that $\mathfrak{Y}_X$ is given by the homotopy class of the element $Hom_X(-,-) \in sSet^{X^{op} \times X}$. The twist functor

$t_X : X^{op} \times X \xrightarrow{\cong} X \times X^{op}$ identifies $Hom_X(-,-)$ with $Hom_{X^{op}}(-,-)$, thus we get an isomorphism $\mathfrak{Y}_X \cong t_X^* \mathfrak{Y}_{X^{op}}$.

$\mathfrak{Y}_X$ will be called the *Yoneda element* in the following.

b) Let $W, X, Y, Z \in Cat$.
We can use the equivalences of Corollaries 2.18 and 2.19 to define relative tensor and cotensor functors in $\mathbb{H}^\star$:

The composition map: $[\![\mathbb{H}^\star_X, \mathbb{H}^\star_Y]\!]_L \times [\![\mathbb{H}^\star_Y, \mathbb{H}^\star_Z]\!]_L \longrightarrow [\![\mathbb{H}^\star_X, \mathbb{H}^\star_Z]\!]_L$
gives rise to: $- \otimes_{[X,Y,Z]} - : \mathbb{H}^\star_{X^{op} \times Y} \times \mathbb{H}^\star_{Y^{op} \times Z} \longrightarrow \mathbb{H}^\star_{X^{op} \times Z}$

The composition map: $[\![(\mathbb{H}^\star_X)^{op}, (\mathbb{H}^\star_Y)^{op}]\!]_R \times [\![(\mathbb{H}^\star_Y)^{op}, \mathbb{H}^\star_Z]\!]_R \longrightarrow [\![(\mathbb{H}^\star_X)^{op}, \mathbb{H}^\star_Z]\!]_R$
gives rise to: $Hom_{[X,Y,Z]}(-,-) : (\mathbb{H}^{\star op})_{X \times Y^{op}} \times \mathbb{H}^\star_{Y \times Z} \longrightarrow \mathbb{H}^\star_{X \times Z}$

And dually:

The composition map: $[\![(\mathbb{H}^\star_X)^{op}, (\mathbb{H}^\star_Y)^{op}]\!]_R \times [\![(\mathbb{H}^\star_Y)^{op}, (\mathbb{H}^\star_Z)^{op}]\!]_R \longrightarrow [\![(\mathbb{H}^\star_X)^{op}, (\mathbb{H}^\star_Z)^{op}]\!]_R$
gives rise to: $-\check{\otimes}_{[X,Y,Z]}- : (\mathbb{H}^{\star op})_{X \times Y^{op}} \times (\mathbb{H}^{\star op})_{Y \times Z^{op}} \longrightarrow (\mathbb{H}^{\star op})_{X \times Z^{op}}$

The composition map: $[\![\mathbb{H}^\star_X, \mathbb{H}^\star_Y]\!]_L \times [\![\mathbb{H}^\star_Y, (\mathbb{H}^\star_Z)^{op}]\!]_L \longrightarrow [\![\mathbb{H}^\star_X, (\mathbb{H}^\star_Z)^{op}]\!]_L$
gives rise to: $\check{H}om_{[X,Y,Z]}(-,-) : \mathbb{H}^\star_{X^{op} \times Y} \times (\mathbb{H}^{\star op})_{Y^{op} \times Z^{op}} \longrightarrow (\mathbb{H}^{\star op})_{X^{op} \times Z^{op}}$

Up to the equivalences of Proposition 2.9 $\check{\otimes}$ is the opposite of $\otimes$ and $\check{H}om$ is the opposite of Hom.

One can use the well known properties of the composition to deduce some properties of the tensor and cotensor functors:

(unitality) Since the Yoneda-element $\mathfrak{Y}_X$ corresponds to the identity on $\mathbb{H}^\star_X$ we have:
$\mathfrak{Y}_Y \otimes_{[Y,Y,Z]} - \cong id_{\mathbb{H}^\star_{Y \times Z^{op}}}$ and $- \otimes_{[X,Y,Y]} \mathfrak{Y}_Y \cong id_{\mathbb{H}^\star_{X \times Y^{op}}}$
$Hom_{[Y,Y,Z]}(\mathfrak{Y}_Y, -) \cong id_{\mathbb{H}^\star_{Y \times Z}}$

And dually:
$\mathfrak{Y}_Y \check{\otimes}_{[Y,Y,Z]} - \cong id_{(\mathbb{H}^{\star op})_{Y \times Z^{op}}}$ and $-\check{\otimes}_{[X,Y,Y]} \mathfrak{Y}_Y \cong id_{(\mathbb{H}^{\star op})_{X \times Y^{op}}}$
$\check{H}om_{[Y,Y,Z]}(\mathfrak{Y}_Y, -) \cong id_{(\mathbb{H}^{\star op})_{Y^{op} \times Z^{op}}}$

(associativity) The associativity of the composition:
$[\![\mathbb{H}^\star_W, \mathbb{H}^\star_X]\!]_L \times [\![\mathbb{H}^\star_X, \mathbb{H}^\star_Y]\!]_L \times [\![\mathbb{H}^\star_Y, \mathbb{H}^\star_Z]\!]_L \longrightarrow [\![\mathbb{H}^\star_W, \mathbb{H}^\star_Z]\!]_L$
and its abstract dual yield associtivity for $\otimes$ and $\check{\otimes}$:
$(A \otimes_{[W,X,Y]} B) \otimes_{[W,Y,Z]} C \cong A \otimes_{[W,X,Z]} (B \otimes_{[X,Y,Z]} C)$ and
$(A \check{\otimes}_{[W,X,Y]} B) \check{\otimes}_{[W,Y,Z]} C \cong A \check{\otimes}_{[W,X,Z]} (B \check{\otimes}_{[X,Y,Z]} C)$

(exponential law) The associativity of the composition:

$[\![(\mathbb{H}^\star_W)^{op}, (\mathbb{H}^\star_X)^{op}]\!]_R \times [\![(\mathbb{H}^\star_X)^{op}, (\mathbb{H}^\star_Y)^{op}]\!]_R \times [\![(\mathbb{H}^\star_Y)^{op}, \mathbb{H}^\star_Z]\!]_R \longrightarrow [\![(\mathbb{H}^\star_W)^{op}, \mathbb{H}^\star_Z]\!]_R$
and its abstract dual yield the following variants of the exponential law:
$Hom_{[W,Y,Z]}(A \check{\otimes}_{[W,X,Y]} B, C) \cong Hom_{[W,X,Z]}(A, Hom_{[X,Y,Z]}(B,C))$ and
$\check{H}om_{[W,Y,Z]}(A \otimes_{[W,X,Y]} B, C) \cong \check{H}om_{[W,X,Z]}(A, \check{H}om_{[X,Y,Z]}(B,C))$

c) Furthermore, one can identify a few special cases of those relative tensor and cotensor functors with some well-known derived functors. To simplify the notation we'll only discuss the stable case, but there are obvious analogs in the unpointed and pointed case.
$\mathbb{H}^s$ is equivalent to the derivator associated to the model category of simplicial symmetric spectra Sp_Σ and $\mathbb{H}^s_X$ to the derivator associated to the model category of X-diagrams in symmetric spectra Sp^X_Σ. It is well known that Sp_Σ is symmetric monoidal and Sp^X_Σ is enriched, tensored and cotensored over Sp_Σ.
The tensored structure is a left Quillen functor in two variables and its derived functor has the following property:
For $A \in Sp^X_\Sigma, \underline{A}$ its class in the homotopy category and $\mathbb{S}$ the class of the sphere spectrum we have:
$\underline{A} \otimes \mathbb{S} \cong \underline{A}$
But this is the property which characterises the functors $\underline{A} \otimes_{[X,*,*]} -$ and $- \otimes_{[*,*,X]} \underline{A}$ up to unique isomorphism. Hence we can identify both, $- \otimes_{[X,*,*]} -$ and $- \otimes_{[*,*,X]} -$, with the tensored structure of Sp^X_Σ.
The partial right Quillen adjoints of the tensored structure are given by the enrichment and by the cotensored structure. Thus, using the uniqueness of right adjoints and the exponential law in part b) of this remark, we can identify $Hom_{[*,X,*]}(-,-)$ with the derived of the enriched structure on Sp^X_Σ, and $Hom_{[*,*,X]}(-,-)$ with the derived of the cotensored structure on Sp^X_Σ.

d) One should note, that in [Gro12] Section 2.3 Groth defines the analogs of the relative tensor functors for arbitrary monoidal derivators with an completely independent method. One can construct a bicategory associated to a monoidal derivator, the bicategory of distributors, and define the analog of the composition pairing by homotopy coend functors. In particular, the relative tensor and cotensor functors capture a large part of the closed monoidal structure of $\mathbb{H}^\star$.

Theorem 2.21. *The 2-functor* $[\![-, \mathbb{H}^\star]\!]_L : FDer^{\star op}_L \longrightarrow FDer^\star_L$ *is an self-inverse equivalence of 2-categories.*

Proof. To prove the theorem one has to construct a pseudonatural equivalence $id \Rightarrow [\![[\![-, \mathbb{H}^\star]\!]_L, \mathbb{H}^\star]\!]_L$

There is a pseudonatural transformation $ev : id \Rightarrow [\![[\![-, \mathbb{H}^\star]\!]_L, \mathbb{H}^\star]\!]_L$ defined on $\mathbb{D}(X)$ by: $ev_{\mathbb{D}}(A) = [F \mapsto F_X(A)]$ for $\mathbb{D} \in FDer^{\star op}_L, A \in \mathbb{D}(X)$ and $F \in [\mathbb{D}, \mathbb{H}^\star]_L$.

$ev_{\mathbb{D}}$ is colimit preserving, because for every $f : X \longrightarrow Y$ there is a commutative diagram:

$$\begin{tikzcd}
A \arrow[rrr, mapsto] \arrow[dddd, mapsto] \arrow[dr, phantom, "\in" sloped] & & & {[F \mapsto F_X(A)]} \arrow[ddd, mapsto] \arrow[dl, phantom, "\ni" sloped] \\
& \mathbb{D}(X) \arrow[r] \arrow[d] & {[\![[\![\mathbb{D}, \mathbb{H}^\star]\!]_L, \mathbb{H}^\star_X]\!]_L} \arrow[d] & \\
& \mathbb{D}(Y) \arrow[r] & {[\![[\![\mathbb{D}, \mathbb{H}^\star]\!]_L, \mathbb{H}^\star_Y]\!]_L} \arrow[r, phantom, "\ni"] & {[F \mapsto f_! \circ F_X(A)]} \\
& & & \\
f_!(A) \arrow[rr, mapsto] \arrow[uur, phantom, "\in" sloped] & & {[F \mapsto F_Y(f_!(A))]} \arrow[uu, phantom, "\in" sloped] &
\end{tikzcd}$$

To show that ev is a pseudonatural equivalence, it suffices to show that $ev_{\mathbb{D}}$ is an equivalence of derivators for every $\mathbb{D} \in FDer_L^{\star op}$.

For $\mathbb{D}$ cofree, there exist $X \in Cat$ and an equivalence $G : \mathbb{D} \longrightarrow \mathbb{H}^\star_X$
By the pseudonaturality of ev we get an natural isomorphism $ev_{\mathbb{D}} \cong [\![[\![G^{-1}, \mathbb{H}^\star]\!]_L, \mathbb{H}^\star]\!]_L \circ ev_{\mathbb{H}^\star_X} \circ G$
Hence $ev_{\mathbb{D}}$ is an equivalence if and only if $ev_{\mathbb{H}^\star_X}$ is. Therefore it suffices to show that $ev_{\mathbb{H}^\star_X}$ is an equivalence of derivators for every $X \in Cat$.

So $ev_{\mathbb{H}^\star_X}$ is a morphism $\mathbb{H}^\star_X \longrightarrow [\![[\![\mathbb{H}^\star_X, \mathbb{H}^\star]\!]_L, \mathbb{H}^\star]\!]_L$
But also $\widetilde{ev}_{\mathbb{H}^\star_X} := [\![d^h_{\mathbb{H}^\star, X^{op}}, \mathbb{H}^\star]\!]_L \circ {d^h_{\mathbb{H}^\star, X}}^{-1}$ defines an morphism $\mathbb{H}^\star_X \longrightarrow [\![[\![\mathbb{H}^\star_X, \mathbb{H}^\star]\!]_L, \mathbb{H}^\star]\!]_L$ which is an equivalence by Corollary 2.18

We will show that $ev_{\mathbb{H}^\star_X}$ and $\widetilde{ev}_{\mathbb{H}^\star_X}$ are naturally isomorphic. This will finish the proof of the theorem, since the latter one is an equivalence. By Corollary 2.18 we only have to check that $\widetilde{ev}_{\mathbb{H}^\star_X}(\mathfrak{Y}_{X^{op}}) \cong ev_{\mathbb{H}^\star_X}(\mathfrak{Y}_{X^{op}})$

$\widetilde{ev}$ is the composition: $\mathbb{H}^\star_X \xrightarrow{{d^h_{\mathbb{H}^\star, X}}^{-1}} [\![\mathbb{H}^\star_{X^{op}}, \mathbb{H}^\star]\!]_L \xrightarrow{[\![d^h_{\mathbb{H}^\star, X^{op}}, \mathbb{H}^\star]\!]_L} [\![[\![\mathbb{H}^\star_X, \mathbb{H}^\star]\!]_L, \mathbb{H}^\star]\!]_L$

By definition of $\mathfrak{Y}_{X^{op}}$: ${d^h_{\mathbb{H}^\star, X}}^{-1}(\mathfrak{Y}_{X^{op}}) = id_{\mathbb{H}^\star_{X^{op}}} \in [\![\mathbb{H}^\star_{X^{op}}, \mathbb{H}^\star]\!]_L(X^{op})$

Thus $\widetilde{ev}_{\mathbb{H}^\star_X}(\mathfrak{Y}_{X^{op}}) = [\![d^h_{\mathbb{H}^\star, X^{op}}, \mathbb{H}^\star]\!]_L(id_{\mathbb{H}^\star_{X^{op}}})$ but the latter one is the composition:
$[\![\mathbb{H}^\star_X, \mathbb{H}^\star]\!]_L \xrightarrow{d^h_{\mathbb{H}^\star, X^{op}}} \mathbb{H}^\star_{X^{op}} \xrightarrow{id} \mathbb{H}^\star_{X^{op}}$
But this is the evaluation at $\mathfrak{Y}_{X^{op}}$ by remark 2.20

$\square$

The following Proposition will be one of the main ingredients for the proof of Theorem 3.10:

Proposition 2.22. *Let $f : X \longrightarrow Y$ be a functor in Cat and $\mathbb{D} \in Der^{\star}_{L}$.*
a) The composition $d^h_{\mathbb{D},X} \circ \llbracket f^{op}_{\mathbb{H}^\star !}, \mathbb{D} \rrbracket_L \circ (d^h_{\mathbb{D},Y})^{-1}$ and $f^*_{\mathbb{D}}$ are equivalent as morphisms $\mathbb{D}_Y \longrightarrow \mathbb{D}_X$.
b) The composition $d^h_{\mathbb{D},Y} \circ \llbracket f^{op*}_{\mathbb{H}^\star}, \mathbb{D} \rrbracket_L \circ (d^h_{\mathbb{D},X})^{-1}$ and $f_{\mathbb{D}!}$ are equivalent as morphisms $\mathbb{D}_X \longrightarrow \mathbb{D}_Y$.

Proof. Since $\llbracket -, \mathbb{D} \rrbracket_L$ is a contravariant 2-functor, the statements a) and b) are equivalent, hence it suffices to prove b).
Consider the diagram:

$$
\begin{tikzcd}
F \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & F_X(\mathfrak{Y}_X) \arrow[dd, mapsto] \\
& {[\mathbb{H}^\star_{X^{op}}, \mathbb{D}]_L} \arrow[r] \arrow[d] \arrow[ul, phantom, "\in"] & \mathbb{D}(X) \arrow[d] \arrow[ur, phantom, "\ni"] & \\
& {[\mathbb{H}^\star_{Y^{op}}, \mathbb{D}]_L} \arrow[r] \arrow[dl, phantom, "\in"] & \mathbb{D}(Y) \arrow[d, phantom, "\cup"] & f_! \circ F_X(\mathfrak{Y}_X) = F_Y(f_!(\mathfrak{Y}_X)) \arrow[l, phantom, "\ni"] \\
F \circ f^{op*} \arrow[rr, mapsto] & & (F \circ f^{op*})_Y(\mathfrak{Y}_Y) = F_Y(f^{op*}(\mathfrak{Y}_Y)) &
\end{tikzcd}
$$

Hence it suffices to show that $f_!(\mathfrak{Y}_X) \cong f^{op*}(\mathfrak{Y}_Y)$ to conclude its commutativity.
Then one can apply this to $\mathbb{D}' = \mathbb{D}_Z$ for $Z \in Cat$. By 2-functoriality in Z, this will prove the proposition.

Let $Yon_Z : Z \longrightarrow Set^{Z^{op}}$ be the Yoneda embedding.
Since $Set^{Y^{op}}$ is cocomplete, the composition $Yon_Y \circ f : X \longrightarrow Set^{Y^{op}}$ has an Yoneda extension, i.e. there is a unique colimit preserving functor $\tilde{f} : Set^{X^{op}} \longrightarrow Set^{Y^{op}}$ making the following diagram commutative:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
Set^{X^{op}} \arrow[r, "\tilde{f}"] & Set^{Y^{op}}
\end{tikzcd}$$

$\tilde{f}$ is by definition the left Kan extension of f^{op}, so $\tilde{f} = f_!^{op}$.
By applying $(-)^{\Delta^{op}}$ to $f_!^{op}$, and using the exponential adjunction in CAT twice, we get the left Kan extension of f^{op} in simplicial sets:
$f_!^{op} : sSet^{X^{op}} \longrightarrow sSet^{Y^{op}}$
Therefore applying $(-)^Z$ and using the exponential adjunction again, yields commutative diagrams, which are 2-functorial in $Z \in Cat$:

$$\begin{tikzcd}
X^Z \arrow[r] \arrow[d] & Y^Z \arrow[d] \\
sSet^{X^{op} \times Z} \arrow[r, "f^{op}_{!Z}"] & sSet^{Y^{op} \times Z}
\end{tikzcd}$$

Since $f^{op}_{!Z}$ is a left Quillen functor with respect to the projective model structure [Lur09] Proposition A.2.8.2, it gives rise to an derived functor, which is by Definition 2.13

$$f^{op}_{!Z} : \mathbb{H}^u_{X^{op}}(Z) \longrightarrow \mathbb{H}^u_{Y^{op}}(Z).$$

Hence we have a commutative diagram of prederivators:

$$\begin{tikzcd}
\underline{X} \arrow[r, "f"] \arrow[d, "h_X"'] & \underline{Y} \arrow[d, "h_Y"] \\
\mathbb{H}^u_{X^{op}} \arrow[r, "f^{op}_!"] & \mathbb{H}^u_{Y^{op}}
\end{tikzcd}$$

Note that by Theorem 2.16 $f_!^{op}$ is up to isomorphism the only colimit preserving morphism which makes this diagram commutative.
Moreover Theorem 2.16 implies that both compositions define isomorphic elements in $\mathbb{H}^u_{Y^{op}}(X)$, but the upper one gives rise to $f^*(\mathfrak{Y}_Y)$ and the lower one to $f_!^{op}(\mathfrak{Y}_X)$.

This completes the proof in the unpointed case. For the pointed (resp. stable) case one has to add base points (resp. pass to suspension spectra) before localising at weak equivalences.

□

2.4 Monoidal Structures

In the last subsection we have seen that there is a nice internal *Hom*-functor $[\![-,-]\!]_L$ and proved a duality result (Theorem 2.21) for cofree derivators. Now we'll use this to define a symmetric monoidal structure on $FDer^\star_L$ such that the duality of Theorem 2.21 is the monoidal duality and $[\![-,-]\!]_L$ is the closure of this monoidal structure.
Assume, such a symmetric monoidal structure exists, then one could express products via the duality in terms of internal *Hom*s. This we'll use as a definition:

Definition 2.23. Let $\mathbb{D}$, $\mathbb{D}$' be cofree $\star$-derivators.
The pairing $-[\otimes]_L- : FDer^\star_L \times FDer^\star_L \longrightarrow FDer^\star_L$ is defined by

$$\mathbb{D}[\otimes]_L\mathbb{D}' := [\![[\![\mathbb{D},\mathbb{H}^\star]\!]_L,\mathbb{D}']\!]_L.$$

For the proof of the desired properties of $-[\otimes]_L-$ we have to work with "multilinear" morphisms of derivators. Moreover we'll need another result of Cisinski [Cis08]:

Definition 2.24. a) Let $\mathbb{D}_k$, $\mathbb{D}'$ be cofree $\star$-derivators, $1 \leqq k \leqq n$. The category of multi-cocontinuous morphism $[\{\mathbb{D}_1,...,\mathbb{D}_n\},\mathbb{D}']_L$ is the full subcategory of $PDer(\mathbb{D}_1 \times ... \times \mathbb{D}_n, \mathbb{D}')$ spanned by those morphism which preserve homotopy colimits in each variable seperately.
b) The prederivator $[\![\{\mathbb{D}_1,...,\mathbb{D}_n\},\mathbb{D}']\!]_L$ is defined by:

$$[\![\{\mathbb{D}_1,...,\mathbb{D}_n\},\mathbb{D}']\!]_L(X) = [\{\mathbb{D}_1,...,\mathbb{D}_n\},\mathbb{D}'_X]_L.$$

Example 2.25. Let $\mathbb{D}_k$, $\mathbb{D}'$ be cofree $\star$-derivators. The morphism $T_{\mathbb{D},\mathbb{D}'}$

$$\mathbb{D}\times\mathbb{D}' \xrightarrow{ev_{\mathbb{D}}\times d^h_{\mathbb{D}'}} [\![[\![\mathbb{D},\mathbb{H}^\star]\!]_L,\mathbb{H}^\star]\!]_L \times [\![\mathbb{H}^\star,\mathbb{D}']\!]_L \longrightarrow [\![[\![\mathbb{D},\mathbb{H}^\star]\!]_L,\mathbb{D}']\!]_L = \mathbb{D}[\otimes]_L\mathbb{D}'$$

where the second map is the composition, is bi-cocontinuous.

Proposition 2.26. *Let $\mathbb{D}_k$, $\mathbb{D}'$ be cofree $\star$-derivators, $1 \leqq k \leqq n$. There are pseudonatural equivalences of categories:*

$$[\{\mathbb{D}_1,...,\mathbb{D}_n\},\mathbb{D}']_L \cong [\{\mathbb{D}_1,...,\hat{\mathbb{D}}_i,...,\mathbb{D}_n\},[\![\mathbb{D}_i,\mathbb{D}']\!]_L]_L$$

Proof. [Cis08] Lemma 5.18

□

Proposition 2.27. *a) There are pseudonatural isomorphisms:*
$\mathbb{H}^\star[\otimes]_L\mathbb{D} \cong \mathbb{D}$ *and* $\mathbb{D}[\otimes]_L\mathbb{H}^\star \cong \mathbb{D}$
b) There is a pseudonatural isomorphism:
$\mathbb{D}[\otimes]_L\mathbb{D}' \cong \mathbb{D}'[\otimes]_L\mathbb{D}$
c) There is a pseudonatural isomorphism:
$(\mathbb{D}[\otimes]_L\mathbb{D}')[\otimes]_L\mathbb{D}'' \cong \mathbb{D}[\otimes]_L(\mathbb{D}'[\otimes]_L\mathbb{D}'')$
d) The morphism $T_{\mathbb{D},\mathbb{D}'}$ *induces a pseudonatural isomorphism:*
$[\mathbb{D}[\otimes]_L\mathbb{D}', \mathbb{D}'']_L \xrightarrow{\cong} [\{\mathbb{D}, \mathbb{D}'\}, \mathbb{D}'']_L$
e) $-[\otimes]_L\mathbb{D}$ *is left biadjoint to* $[\![\mathbb{D}, -]\!]_L$
f) $FDer^\star_L$ *admits a closed symmetric monoidal structure with* $-[\otimes]_L-$ *as monoidal pairing.*
g) $\mathbb{D}[\otimes]_L-$ *is left and right biadjoint to* $[\![\mathbb{D}, \mathbb{H}^\star]\!]_L[\otimes]_L-$
h) There is a pseudonatural isomorphism:
$[\![\mathbb{D}[\otimes]_L\mathbb{D}', \mathbb{H}^\star]\!]_L \cong [\![\mathbb{D}', \mathbb{H}^\star]\!]_L[\otimes]_L[\![\mathbb{D}, \mathbb{H}^\star]\!]_L$

Proof. a) $\mathbb{H}^\star[\otimes]_L\mathbb{D} = [\![[\![\mathbb{H}^\star, \mathbb{H}^\star]\!]_L, \mathbb{D}]\!]_L \cong [\![\mathbb{H}^\star, \mathbb{D}]\!]_L \cong \mathbb{D}$
Both equivalences are induced by Corollary 2.18.

$\mathbb{D}[\otimes]_L\mathbb{H}^\star = [\![[\![\mathbb{D}, \mathbb{H}^\star]\!]_L, \mathbb{H}^\star]\!]_L \cong \mathbb{D}$
Here the equivalence is the inverse of the evaluation.

b)$\mathbb{D}[\otimes]_L\mathbb{D}' = [\![[\![\mathbb{D}, \mathbb{H}^\star]\!]_L, \mathbb{D}']\!]_L \cong [\![[\![\mathbb{D}', \mathbb{H}^\star]\!]_L, [\![[\![\mathbb{D}, \mathbb{H}^\star]\!]_L, \mathbb{H}^\star]\!]_L]\!]_L \cong [\![[\![\mathbb{D}', \mathbb{H}^\star]\!]_L, \mathbb{D}]\!]_L = \mathbb{D}'[\otimes]_L\mathbb{D}$
The first equivalence is induced by applying the contravariant equivalence $[\![-, \mathbb{H}^\star]\!]_L$, and the second one by the inverse of the evaluation.

c)$(\mathbb{D}[\otimes]_L\mathbb{D}')[\otimes]_L\mathbb{D}''$
$\cong \mathbb{D}''[\otimes]_L(\mathbb{D}[\otimes]_L\mathbb{D}')$
$= [\![[\![\mathbb{D}'', \mathbb{H}^\star]\!]_L, [\![[\![\mathbb{D}, \mathbb{H}^\star]\!]_L, \mathbb{D}']\!]_L]\!]_L$
$\cong [\![\{[\![\mathbb{D}'', \mathbb{H}^\star]\!]_L, [\![\mathbb{D}, \mathbb{H}^\star]\!]_L\}, \mathbb{D}']\!]_L$
$\cong [\![[\![\mathbb{D}, \mathbb{H}^\star]\!]_L, [\![[\![\mathbb{D}'', \mathbb{H}^\star]\!]_L, \mathbb{D}']\!]_L]\!]_L$
$= \mathbb{D}[\otimes]_L(\mathbb{D}''[\otimes]_L\mathbb{D}')$
$\cong \mathbb{D}[\otimes]_L(\mathbb{D}'[\otimes]_L\mathbb{D}'')$
The first and fourth equivalence are induced by b) and the second and third equivalence are induced by Proposition 2.26.

d) We will use a similar strategy as in the proof of theorem 2.21. $T_{\mathbb{D},\mathbb{D}'}$ is clearly pseudonatural. Thus w.l.o.g. we can assume $\mathbb{D} = \mathbb{H}^\star_X, \mathbb{D}' = \mathbb{H}^\star_Y$ and $\mathbb{D}'' = \mathbb{H}^\star_Z$.
We'll show that the composition (1) to (9) below is isomorphic to the identity. It is clear that all morphisms, with exception of (4) to (5), are equivalences. Thus also (4) to (5) will be an equivalence. This will complete the proof of d), since the morphism induced by $T_{\mathbb{H}_X,\mathbb{H}_Y}$ is the composition (4) to (6).

In the diagram below, we use a shortened notation. Therefore we discuss the morphism (1) to (2) in detail:

The Yoneda-element $\mathfrak{Y}_{X\times Y\times Z^{op}} \in \mathbb{H}^\star_{X^{op}\times Y^{op}\times Z}(X\times Y\times Z^{op})$ corresponds to a morphism in $\llbracket \mathbb{H}^\star_{X\times Y}, \mathbb{H}^\star_Z\rrbracket_L(X\times Y\times Z^{op}) = [\mathbb{H}^\star_{X\times Y}, \mathbb{H}^\star_{Z\times X\times Y\times Z^{op}}]_L$. By Corollary 2.18 this morphism is defined by mapping $\mathfrak{Y}_{X^{op}\times Y^{op}} \in \mathbb{H}^\star_{X\times Y}(X^{op}\times Y^{op})$ to $\mathfrak{Y}_{X\times Y\times Z^{op}} \in \mathbb{H}^\star_{Z\times X\times Y\times Z^{op}}(X^{op}\times Y^{op}) = \mathbb{H}^\star_{X^{op}\times Y^{op}\times Z}(X\times Y\times Z^{op})$.
In the diagram we suppress the notation of the components of derivators and we indicate morphisms only by their action on Yoneda-elements.

$$
\begin{tikzcd}
 & & (\text{eval. at } \mathfrak{Y}_{X^{op}}, \mathfrak{Y}_* \mapsto \mathfrak{Y}_{Y^{op}}) \mapsto \mathfrak{Y}_{X\times Y\times Z^{op}} \arrow[d, phantom, "\in"] \arrow[ddrr, bend left] & & \\
 & & \llbracket \{ \llbracket \llbracket \mathbb{H}^\star_X, \mathbb{H}^\star\rrbracket_L, \mathbb{H}^\star\rrbracket_L, \llbracket \mathbb{H}^\star, \mathbb{H}^\star_Y\rrbracket_L\}, \mathbb{H}^\star_Z\rrbracket_L \ (5) \arrow[ddr, "\cong"] & & \\
(id_{\mathbb{H}^\star_X} \mapsto \mathfrak{Y}_{X^{op}\times Y^{op}}) \mapsto \mathfrak{Y}_{X\times Y\times Z^{op}} \arrow[dr, phantom, "\in"] & & & & (\mathfrak{Y}_{X^{op}}, \mathfrak{Y}_{Y^{op}}) \mapsto \mathfrak{Y}_{X\times Y\times Z^{op}} \arrow[dl, phantom, "\in"] \arrow[dd, mapsto] \\
 & (4)\ \llbracket \llbracket \llbracket \mathbb{H}^\star_X, \mathbb{H}^\star\rrbracket_L, \mathbb{H}^\star_Y\rrbracket_L, \mathbb{H}^\star_Z\rrbracket_L \arrow[uur] & & (6)\ \llbracket \{\mathbb{H}^\star_X, \mathbb{H}^\star_Y\}, \mathbb{H}^\star_Z\rrbracket_L \arrow[dd, "\cong"] & \\
(\mathfrak{Y}_X \mapsto \mathfrak{Y}_{X^{op}\times Y^{op}}) \mapsto \mathfrak{Y}_{X\times Y\times Z^{op}} \arrow[uu, mapsto] \arrow[dr, phantom, "\in"] & & & & \mathfrak{Y}_{X^{op}}, \mapsto (\mathfrak{Y}_{Y^{op}} \mapsto \mathfrak{Y}_{X\times Y\times Z^{op}}) \arrow[dl, phantom, "\in"] \arrow[dd, mapsto] \\
 & (3)\ \llbracket \llbracket \mathbb{H}^\star_{X^{op}}, \mathbb{H}^\star_Y\rrbracket_L, \mathbb{H}^\star_Z\rrbracket_L \arrow[uu, "\cong"] & & (7)\ \llbracket \mathbb{H}^\star_X, \llbracket \mathbb{H}^\star_Y, \mathbb{H}^\star_Z\rrbracket_L\rrbracket_L \arrow[dd, "\cong"] & \\
\mathfrak{Y}_{X^{op}\times Y^{op}} \mapsto \mathfrak{Y}_{X\times Y\times Z^{op}} \arrow[uu, mapsto] \arrow[dr, phantom, "\in"] & & & & \mathfrak{Y}_{X^{op}} \mapsto \mathfrak{Y}_{X\times Y\times Z^{op}} \arrow[dl, phantom, "\in"] \arrow[dd, mapsto] \\
 & (2)\ \llbracket \mathbb{H}^\star_{X\times Y}, \mathbb{H}^\star_Z\rrbracket_L \arrow[uu, "\cong"] & & (8)\ \llbracket \mathbb{H}^\star_X, \mathbb{H}^\star_{Y^{op}\times Z}\rrbracket_L \arrow[dd, "\cong"] & \\
\mathfrak{Y}_{X\times Y\times Z^{op}} \arrow[uu, mapsto] \arrow[dr, phantom, "\in"] & & & & \mathfrak{Y}_{X\times Y\times Z^{op}} \arrow[dl, phantom, "\in"] \\
 & (1)\ \mathbb{H}^\star_{X^{op}\times Y^{op}\times Z} \arrow[uu, "\cong"] & & (9)\ \mathbb{H}^\star_{X^{op}\times Y^{op}\times Z} &
\end{tikzcd}
$$

The equivalences (1) to (4) and(7) to (9) are induced by Corollary 2.18, the equivalence (6) to (7) by Proposition 2.26 and the equivalence (5) to (6) by Corollary 2.18 and Theorem 2.21
By checking at $id_{\mathbb{H}^\star_X}$ we see, that $(id_{\mathbb{H}^\star_X} \mapsto \mathfrak{Y}_{X^{op}} \otimes_{[X\times X^{op},*,Y\times Y^{op}]} \mathfrak{Y}_{Y^{op}}) \mapsto \mathfrak{Y}_{X\times Y\times Z^{op}} \in$(4) maps to the right element in (5).

Finally the canonical equivalence of categories:
$(X \times X^{op}) \times (Y \times Y^{op}) \cong (X \times Y) \times (X \times Y)^{op}$
identifies $Hom_{X^{op}}(-,-) \times Hom_{Y^{op}}(-,-)$ with $Hom_{(X\times Y)^{op}}((-,-),(-,-))$, thus giving the desired isomorphism:
$\mathfrak{Y}_{X^{op}} \otimes_{[X\times X^{op},*,Y\times Y^{op}]} \mathfrak{Y}_{Y^{op}} \cong \mathfrak{Y}_{X^{op}\times Y^{op}}$

e) By d) and Proposition 2.26 there are equivalences, which are pseudonatural in $\mathbb{D}, \mathbb{D}'$ and $\mathbb{D}''$:
$[\mathbb{D}[\otimes]_L\mathbb{D}', \mathbb{D}'']_L \xrightarrow{\cong} [\{\mathbb{D}, \mathbb{D}'\}, \mathbb{D}'']_L \xrightarrow{\cong} [\mathbb{D}, [\![\mathbb{D}', \mathbb{D}'']\!]_L]_L$

f) The full definition of a symmetric monoidal 2-category is spread over [Gur07], [MC00] and [SchP03]. Fortunately we don't have to check all the coherence diagrams one by one.

By [Gro12] Lemma 1.2 the category $PDer$ is a cartesian closed symmetric monoidal 2-category.
In particular there are biadjunctions $-\times\mathbb{D} \dashv Hom_{PDer}(\mathbb{D}, -)$, which are pseudonatural in $\mathbb{D} \in PDer$.
This symmetric monoidal structure restricts to a closed symmetric monoidal structure on $Der_L^\star$ by [Gro11b] Proposition 2.8 , and furthermore to a symmetric monoidal structure on $FDer_L^\star$ by (Der1).
All the coherence diagrams for $- \times -$ correspond via the above biadjunctions to adjoint coherence diagrams for $Hom_{PDer}(-,-)$. Since the internal Hom-objects in $FDer_L^\star$ are strict subobjects of the internal Hom-objects in $PDer$, all the adjoint coherence diagrams for $Hom_{PDer}(-,-)$ restrict to coherence diagrams for $[\![-,-]\!]_L$.
Now the biadjunctions of part e) yield the coherence diagrams for $-[\otimes]_L-$.

Note that we get those structure morphisms, which we have not defined explicitly (e.g. the syllapsy, the associativity and unitality morphisms α, λ, ρ ...) also via the above biadjunctions.

Although $- \times -$ defines a symmetric monoidal structure in a very strict sense (i.e. some of the structure morphisms are in fact equalities), the above argument only shows that $-[\otimes]_L-$ defines a symmetric monoidal structure in the weaker sense of [Gur07], [MC00] and [SchP03], since the passage to the adjoint, as described above, preserves only equalities of 2-morphisms, and not of 1-morphisms or objects.

g) For the "left biadjoint" statement, we apply b), e) and Theorem 2.21:
$\mathbb{D}[\otimes]_L- \cong -[\otimes]_L\mathbb{D} \dashv [\![\mathbb{D}, -]\!]_L \cong [\![[\![[\![\mathbb{D}, \mathbb{H}^\star]\!]_L, \mathbb{H}^\star]\!]_L, -]\!]_L = [\![\mathbb{D}, \mathbb{H}^\star]\!]_L[\otimes]_L-$

For the "right biadjoint" statement, we apply the "left biadjoint" statement to $[\![\mathbb{D}, \mathbb{H}^\star]\!]_L$ and use Theorem 2.21:
$[\![\mathbb{D}, \mathbb{H}^\star]\!]_L[\otimes]_L- \dashv [\![[\![\mathbb{D}, \mathbb{H}^\star]\!]_L, \mathbb{H}^\star]\!]_L[\otimes]_L- \cong \mathbb{D}[\otimes]_L-$

h) We use $(\mathbb{D}[\otimes]_L\mathbb{D}')[\otimes]_L- \cong (\mathbb{D}[\otimes]_L-)\circ(\mathbb{D}'[\otimes]_L-)$, g) and that biadjunctions behave well with compositions:

$$\begin{tikzcd}
(\mathbb{D}[\otimes]_L\mathbb{D}')[\otimes]_L- \arrow[d, "\cong"'] \arrow[r, phantom, "\dashv"] & \llbracket \mathbb{D}[\otimes]_L\mathbb{D}', \mathbb{H}^\star \rrbracket_L[\otimes]_L- \\
(\mathbb{D}[\otimes]_L-)\circ(\mathbb{D}'[\otimes]_L-) \arrow[r, phantom, "\dashv"] & (\llbracket \mathbb{D}', \mathbb{H}^\star \rrbracket_L[\otimes]_L-)\circ(\llbracket \mathbb{D}, \mathbb{H}^\star \rrbracket_L[\otimes]_L-)
\end{tikzcd}$$

To finish the proof, we apply the right biadjoints to $\mathbb{H}^\star$ and use a).

□

Definition 2.28. A *(symmetric) monoidal cofree $\star$-derivator* is a (symmetric) monoid object in $FDer^\star_L$ with respect to the $[\otimes]_L$-monoidal structure.

Remark 2.29. a) Proposition 2.27.d) implies that giving a (symmetric) monoidal structure on a cofree $\star$-derivator $\mathbb{D}$ in the sense of definition 2.28, is equivalent to give a (symmetric) monoidal structure on $\mathbb{D}$ in the sense of Groth [Gro12] Definition 2.4 or a left exact (symmetric) monoidal structure on $\mathbb{D}$ in the sense of Cisinski [Cis08] 5.4.
b) Thus for a (symmetric) monoidal cofree $\star$-derivator $\mathbb{D}$ and $X \in Cat$, the category $\mathbb{D}(X)$ has a canonical (symmetric) monoidal structure [Gro12] Example 2.5. And moreover, if $\mathbb{D}$ is stable, the (symmetric) monoidal structure is compatible with the triangulation (Theorem 1.25) in the sense of [HPS97] and [May01] (see [GPS12]).

Example 2.30. a) $\mathbb{H}^\star$ has a unique structure as a symmetric monoidal $\star$-derivator ([Cis08] Theorem 5.22) via:
- $id_{\mathbb{H}^\star}$ as unit.
- The multiplication $\mathbb{H}^\star[\otimes]_L\mathbb{H}^\star = \llbracket\llbracket \mathbb{H}^\star, \mathbb{H}^\star\rrbracket_L, \mathbb{H}^\star\rrbracket_L \longrightarrow \mathbb{H}^\star$ is defined by the evaluation.
b) Let $\mathbb{D}$ be a cofree $\star$-derivator. Then $End(\mathbb{D}) := \llbracket \mathbb{D}, \mathbb{D}\rrbracket_L = \llbracket \mathbb{D}, \mathbb{H}^\star\rrbracket_L[\otimes]_L\mathbb{D}$ is a monoidal cofree $\star$-derivator ([Cis08] Corollary 5.10) via:
- The unit is the unique homotopy colimit preserving map $\mathbb{H}^\star \longrightarrow End(\mathbb{D})$, defined by $id_{\mathbb{D}}$.
- The multiplicative structure is induced by the composition of endomophisms and Proposition 2.27.d)

Definition 2.31. a) Let $\mathbb{A}, \mathbb{A}'$ be monoidal prederivators with respect to the cartesian monoidal structure, then $PDer^m(\mathbb{A}, \mathbb{A}')$ is the category of monoidal morphisms from $\mathbb{A}$ to $\mathbb{A}'$.
b) Let $\mathbb{A}, \mathbb{A}'$ be monoidal $\star$-derivators, then $[\mathbb{A}, \mathbb{A}']^m_L$ is the category of homotopy colimit preserving monoidal morphism from $\mathbb{A}$ to $\mathbb{A}'$.

The following is a monoidal variant of Theorem 2.16.

Theorem 2.32. *a) Let X be a monoidal category, then $\underline{X}$ is a monoidal prederivator and there is a unique monoidal structure on $\mathbb{H}^\star_{X^{op}}$ such that h_X is monoidal.*
b) Let $\mathbb{A}$ be a monoidal $\star$-derivator. The monoidal morphism h_X induces an equivalence of categories:

$$[\mathbb{H}^\star_{X^{op}}, \mathbb{A}]^m_L \cong PDer^m(\underline{X}, \mathbb{A})$$

Proof. [CT12] Theorem A.3 □

2.5 Some Remarks and Interpretation, Part 2

Remark 2.33. a) Note that there are no finiteness conditions for the duality theorem 2.21. This shows that the role of finite sets in the uncategorified theory is replaced by small categories in this case.
b) Since the proof of Theorem 2.21 is formal, it will generalize to other situations:
Let us discuss informally the following example:
Let $\mathbb{A}$ be a symmetric monoidal derivator in the sense of [Gro12] Definition 2.4 which also a $\star$-derivator.
Now an $\mathbb{A}$-module is a $\star$- derivator $\mathbb{D}$ together with a morphism $\mathbb{A} \times \mathbb{D} \longrightarrow \mathbb{D}$ such that the usual diagrams commute up to some appropriate coherence.
For two $\mathbb{A}$-modules $\mathbb{D}, \mathbb{D}'$, let $[\mathbb{D}, \mathbb{D}']^{\mathbb{A}}_L$ be the full subcategory of $[\mathbb{D}, \mathbb{D}']_L$ on those morphisms which preserve the module structures.
The unit morphism $\underline{*} \longrightarrow \mathbb{A}$ corresponds by Theorem 2.16/Corollary 2.17 to a homotopy colimit preserving morphism $E : \mathbb{H}^\star \longrightarrow \mathbb{A}$. It should not be hard to show that composition with E induces equivalences $[\mathbb{A}_{X^{op}}, \mathbb{D}]^{\mathbb{A}}_L \longrightarrow \mathbb{D}(X)$ for $X \in Cat, \mathbb{D}$ an $\mathbb{A}$-module.
In this situation the proof of Theorem 2.21 will go through to show that $[\![-, \mathbb{A}]\!]^{\mathbb{A}}_L$ is an equivalence on cofree $\mathbb{A}$-modules.
c) There is a far reaching analogy to elementary linear algebra:
-$\mathbb{H}^\star_X$ is the analog of the abelian group $\mathbb{Z}^n$
-Corollary 2.18 is the analog of $Hom_{\mathbb{Z}}(\mathbb{Z}^n, A) \cong A^n$ for $A \in \mathbb{Z} - Mod$.
-Theorem 2.21 is the analog of the fact, that every free abelian group of finite rank is dualizable.
-Let $\mathbb{D}$ be cofree, then an element of $\mathbb{D}(X)$ which induces an equivalence $\mathbb{H}_{X^{op}} \longrightarrow \mathbb{D}$ is the analog of the choice of a basis of a free abelian group of finite rank.
-The Yoneda-element $\mathfrak{Y}_X \in \mathbb{H}^\star_{X^{op}}(X)$ is the analog of the canonical basis of $\mathbb{Z}^n$
-The cartesian monoidal structure on $Der^\star_L$ described in [Gro12] is the analog of the cartesian monoidal structure $- \oplus -$ on $\mathbb{Z}$-mod.
Its unit is $\mathbb{H}^\star_\emptyset \cong *$ and there are natural equivalences: $\mathbb{H}^\star_X \times \mathbb{H}^\star_Y \cong \mathbb{H}^\star_{X \coprod Y}$.
The latter equivalence is a direct consequence of (Der1).
-The monoidal pairing $-[\otimes]_L-$ on $FDer^\star_L$ is the analog of the tensor product of abelian groups, because it has $[\![-, -]\!]_L$ as its closure, $\mathbb{H}^\star$ as its unit, and it satisfies $\mathbb{H}^\star_X[\otimes]_L\mathbb{H}^\star_Y \cong \mathbb{H}^\star_{X \times Y}$

-We can even enlarge this analogy by defining $Gl(X, \mathbb{H}^\star)$ to be the full subcategory of $\mathbb{H}^\star_X(X^{op})$ on those objects corresponding to autoequivalences of $\mathbb{H}^\star_X$.
The category $Gl(X, \mathbb{H}^\star)$ comes with a natural pairing:
$-\otimes - : Gl(X, \mathbb{H}^\star) \times Gl(X, \mathbb{H}^\star) \longrightarrow Gl(X, \mathbb{H}^\star)$
which corresponds to the composition of autoequivalences and has the property that $F \otimes -$ and $- \otimes F$ are equivalences for every $F \in Gl(X, \mathbb{H}^\star)$
Hence it induces a group structure on the set of equivalence classes of objects denoted by $gl(X, \mathbb{H}^\star)$.
-One can describe $Gl(*, \mathbb{H}^s)$ as follows:
$Ob(Gl(*, \mathbb{H}^s)) = \{\Sigma^n | n \in \mathbb{Z}\}$,
$Gl(*, \mathbb{H}^s)(\Sigma^n, \Sigma^m) = \pi_{n-m}(\mathbb{S})$ because suspensions of the sphere spectrum are known to be the only spectra, which are smash invertible. Composition is given by the multiplication in $\pi_*(\mathbb{S})$.
-Note that for every $X \in Cat$ the suspension functor defines an group homomorphism $\mathbb{Z} \longrightarrow gl(X, \mathbb{H}^s), 1 \mapsto [\Sigma]$.
The calculations of Example 1.32.d) show that there is $\xi_{n+2} \in gl([n], \mathbb{H}s)$ with $\xi_{n+2}^{n+2} = [\Sigma]^2$
-With part b) of this remark in mind, one can also define $Gl(X, \mathbb{A})$, by replacing $\mathbb{H}^\star$ by $\mathbb{A}$ everywhere in the definition.
d) The extension of the monoidal structure 2.23 to the entire 2-category of derivators is still a mayor problem in the theory, since the definition we gave won't generalize well, because we cannot expect arbitrary derivators to be dualizable. Even a good candidate for a more general definition is not in sight.
One way of attacking this problem would be to mimic the construction of the usual tensor product in linear algebra step by step, to define $\mathbb{D}[\otimes]_L\mathbb{D}'$ as a certain quotient (coequalizer) of the free derivator associated to the prederivator $\mathbb{D} \times \mathbb{D}'$.
But as explained in Remark 1.34.a), the 2-category of derivators is still a derived category. Therefore one should use a homotopical version of the coequalizer above.
With this in mind, a better way for constructing the general monoidal structure would be to develop the theory outlined in Remark 1.34.a) first, to obtain the desired monoidal structure by deriving the one on presentable $(\infty, 1)$-categories.

3 Spanier-Whitehead-Duality for stable Derivators

In subsection 3.1 we introduce the Spanier-Whitehead-Duality for the stable derivator $\mathbb{H}$ and in subsection 3.2 we prove our main results Theorem 3.10 and 3.12.

3.1 Spanier-Whitehead-Duality

We can define Spanier-Whitehead-Duality functors in very general situation, but they will often be far away from being equivalences:

Definition 3.1. The *Spainier-Whitehead-Duality* $D : \mathbb{H}^\star \longrightarrow \mathbb{H}^{\star op}$ is the unique colimit preserving morphism with $D(\mathbb{S}) = \mathbb{S}$.

Remark 3.2. a) Note the by Proposition 2.9 ,Corollary 2.18 $D^{op} : \mathbb{H}^{\star op} \longrightarrow \mathbb{H}^\star$ is the unique limit preserving morphism with $D^{op}(\mathbb{S}) = \mathbb{S}$.
b) Note that the Spanier-Whitehead-Duality functor for the stable $(\infty,1)$-category of spectra, can be constructed in a similar way, as described in [Lur11] 7.3.3.2

D will be an equivalence in situations, where "homotopy colimit preserving" and "homotopy limit preserving" are equivalent. To ensure this we have to impose stability and finiteness conditions:

Definition 3.3. Fin is the diagram category of finite, finite dimensional categories

Proposition 3.4. *Let $\mathbb{D}, \mathbb{D}'$ be stable Fin-derivators and $F : \mathbb{D} \longrightarrow \mathbb{D}'$ be a morphism of prederivators. Then the following conditions are equivalent:*
a) F preserves homotopy colimits.
b) F preserves homotopy limits.
Whenever one of the conditions is satisfied F will be called an exact morphism.

Proof. [Fra96] Theorem 2. □

Definition 3.5. a) Let Der be the 2-category of stable Fin-derivators and exact morphisms.
b) $\mathbb{H}$ is the stable Fin-derivator associated to the homotopy theory of finite spectra, i.e. $\mathbb{H}(X)$ is the homotopy category of X-diagrams in finite spectra.
c) $FDer$ is the full sub-2-category on the cofree stable Fin-derivators, i.e. those who are equivalent to $\mathbb{H}_X$ for some $X \in Fin$.
d) $SFDer$ is the full sub-2-category on the $\mathbb{H}_X$ for $X \in Fin$.
e) For $\mathbb{D}, \mathbb{D}' \in Der$ let $[\mathbb{D}, \mathbb{D}'] := Der(\mathbb{D}, \mathbb{D}')$ and $[\![\mathbb{D}, \mathbb{D}']\!]$ be the prederivator

given by $[\![\mathbb{D}, \mathbb{D}']\!](X) = [\mathbb{D}, \mathbb{D}'_X]$.

Theorem 3.6. *Let $X \in Fin, \mathbb{D} \in Der$, then are equivalences of categories $[\mathbb{H}_{X^{op}}, \mathbb{D}] \longrightarrow \mathbb{D}(X)$.*

Proof. In [Fra96] Theorem 4, Franke proves the theorem in the case $X = *$. One can deduce the general case by modifying Cisinski's proof [Cis08] with finiteness conditions

□

Proposition 3.7. *$D : \mathbb{H} \longrightarrow \mathbb{H}^{op}$ is an self-inverse equivalence of derivators.*

Proof. $D^{op} \circ D$ is by Proposition 3.4 an exact endomorphism of $\mathbb{H}$ which maps $\mathbb{S}$ to itself.
Hence it is isomorphic to the identity by Theorem 2.16.

□

Remark 3.8. a) D induces equivalences $\mathbb{H}_X(Y) = \mathbb{H}(X \times Y) \cong \mathbb{H}^{op}(X \times Y) = (\mathbb{H}^{\star}_{X^{op}})^{op}(Y)$ for $X, Y \in Cat$ and hence $d^v_X := D_X : \mathbb{H}_X \xrightarrow{\cong} (\mathbb{H}_{X^{op}})^{op}$
b) D is clearly a symmetric monoidal morphism, since by Example 2.30.a) the units of $\mathbb{H}$ and $\mathbb{H}^{op}$ the identity morphisms and the diagram:

$$\begin{tikzcd} {[\![[\![\mathbb{H}, \mathbb{H}]\!]_L, \mathbb{H}]\!]_L} \arrow[d, "{[\![[\![D, D]\!], D]\!]}"] & = \mathbb{H}[\otimes]_L\mathbb{H} \arrow[r] & \mathbb{H} \arrow[d, "D"] \\ {[\![[\![\mathbb{H}^{op}, \mathbb{H}^{op}]\!]_R, \mathbb{H}^{op}]\!]_R} & = \mathbb{H}[\otimes]_L\mathbb{H} \arrow[r] & \mathbb{H} \end{tikzcd}$$

commutes since D is self-inverse.
Alternatively, one could also use Theorem 2.32.b) to see the D corresponds to the morphism of prederivators $\underline{*} \longrightarrow \mathbb{H}^{op}$, which is clearly monoidal. This also shows that D is up to isomorphism the unique monoidal morphism $\mathbb{H} \longrightarrow \mathbb{H}^{op}$. In particular D preserves the relative tensor and cotensor structures. In the notation of Remark 2.20 this precisely means:

$$D(- \otimes_{[X,Y,Z]} -) \cong D(-)\check{\otimes}_{[X^{op},Y^{op},Z^{op}]}D(-),$$
$$DHom_{[X,Y,Z]}(-,-) \cong \check{H}om_{[X^{op},Y^{op},Z^{op}]}(D(-), D(-))$$
$$\text{and } D\mathfrak{Y}_X \cong \mathfrak{Y}_{X^{op}}.$$

3.2 The relation between dualities and adjunctions

Another advantage of the Spanier-Whitehead-Duality is, that $(-)^{op}$ restricts to an equivalence on $FDer$. Furthermore, since the inclusion of $SFDer$ into $FDer$ is an equivalence, we can define two duality functors on the 2-category $SFDer$:

Definition 3.9. a) We define the *vertical duality* $\mathfrak{D}^v : SFDer^{co} \longrightarrow SFDer$
on objects: $\mathbb{H}_X \mapsto \mathbb{H}_{X^{op}}$
on morphisms: $(F : \mathbb{H}_X \longrightarrow \mathbb{H}_Y) \mapsto (d^v_{Y^{op}})^{-1} \circ F^{op} \circ d^v_{X^{op}}$
b) We define the *horizontal duality* $\mathfrak{D}^h : SFDer^{op} \longrightarrow SFDer$
on objects: $\mathbb{H}_X \mapsto \mathbb{H}_{X^{op}}$
on morphisms: $(F : \mathbb{H}_X \longrightarrow \mathbb{H}_Y) \mapsto (d^h_{Y^{op}})^{-1} \circ [\![F, \mathbb{H}]\!] \circ d^h_{X^{op}}$

Note that $\mathfrak{D}^v$ and $\mathfrak{D}^h$ are only pseudofunctors, since they preserve composition and identities only up to coherent isomorphisms. But this is the price we have to pay, because since we want to construct adjoints, and adjoints have by definition the same domain and codomain (only reversed), we are forced to use $SFDer$ and functors which are strictly the identity on objects. Fortunately pseudofunctors still preserve adjunctions.

Finally we have everything we need to finish the proof of Theorem 1.33:

Theorem 3.10. *Let $\mathbb{D}$ be a stable Cat-derivator and $f : X \longrightarrow Y$ a functor in Fin. Then $f^* : \mathbb{D}_Y \longrightarrow \mathbb{D}_X$ induces in infinite chain of adjunctions.*

Proof. Step 1: We prove the theorem for the Fin-derivator $\mathbb{H}$
Consider $f^* : \mathbb{H}_Y \longrightarrow \mathbb{H}_X$
By 2.2 and Proposition 3.4 we have adjunctions $f_! \dashv f^* \dashv f_*$ in Der, so $f^{*[m]}$ exists for $|m| \leqq 1$.
Assume by induction that $f^{*[m]}$ exists for $|m| \leqq n$.
By Proposition 2.22 $\mathfrak{D}^h(f^*) \cong f_!^{op}$.
By the Proof of Proposition 2.9 and because D is a morphism of prederivators we have: $\mathfrak{D}^v(f^*) \cong f^{op*}$.
Therefore $\mathfrak{D}^h\mathfrak{D}^v(f^*) \cong f_!$ and $\mathfrak{D}^v\mathfrak{D}^h(f_!) \cong f^*$.
Since $\mathfrak{D}^h\mathfrak{D}^v$ and $\mathfrak{D}^v\mathfrak{D}^h$ preserve adjunctions, we have:
$\mathfrak{D}^h\mathfrak{D}^v(f^{*[-n]}) \dashv f^{*[-n]}$ and $f^{*[n]} \dashv \mathfrak{D}^v\mathfrak{D}^h(f^{*[n]})$.
Hence $f^{*[m]}$ exists for $|m| \leqq n+1$ which shows the inductive step.

Step 2: We prove the theorem for the Cat-derivator $\mathbb{H}^s$
Let $\mathbb{H}^s|_{Fin}$ be the restriction to the diagram category Fin.
For any $Z \in Cat$ the 2-functor $[\![-, (\mathbb{H}^s|_{Fin})_Z]\!]$ preserves infinite chains of adjunctions and by Proposition 2.22 the image of $f_!^{op} : \mathbb{H}_{X^{op}} \longrightarrow \mathbb{H}_{Y^{op}}$ (which induces an infinite chain of adjunctions by Step 1) can be identified (to be more precise: is up to composition with equivalences equal to) with $f^* : (\mathbb{H}^s|_{Fin})_{Z\times Y} \longrightarrow (\mathbb{H}^s|_{Fin})_{Z\times X}$.
So in particular its value on the terminal category $*$:
$f^* : \mathbb{H}^s_Z(Y) \longrightarrow \mathbb{H}^s_Z(X)$
induces in infinite chain of adjunctions by Proposition 1.31.a) and b). Since $[\![-, (\mathbb{H}^s|_{Fin})_Z]\!]$ is 2-functorial in $Z \in Cat$, all the adjoints assemble into morphisms of prederivators, which is by Proposition 2.4 a morphism of derivators.

Step 3: The general case:
Let $\mathbb{D}$ be a stable Cat-derivator.
We apply $[\![-, \mathbb{D}]\!]$ to $f_!^{op} : \mathbb{H}^s_{X^{op}} \longrightarrow \mathbb{H}^s_{Y^{op}}$ (which induces an infinite chain of adjunctions by Step 2), hence its image will also induce an infinite chain of adjunctions. But again by Proposition 2.22 the image can be identified with $f^* : \mathbb{D}_Y \longrightarrow \mathbb{D}_X$.

$\square$

The behavior of the functors $\mathfrak{D}^h$ and $\mathfrak{D}^v$ in Step one of the proof motivates the following definition:

Definition 3.11. a) The *left translation functor* $\mathfrak{L} : SFDer^{coop} \longrightarrow SFDer$ is the composition $\mathfrak{D}^h \circ \mathfrak{D}^v$.
b) The *right translation functor* $\mathfrak{R} : SFDer^{coop} \longrightarrow SFDer$ is the composition $\mathfrak{D}^v \circ \mathfrak{D}^h$.

Theorem 3.12. *Let $F : \mathbb{H}_X \longrightarrow \mathbb{H}_Y$ be a morphism in $SFDer$. Then there are adjunctions:*
$\mathfrak{L}(F) \dashv F \dashv \mathfrak{R}(F)$

Proof. Since $\mathfrak{L}$ and $\mathfrak{R}$ are mutually inverse, it suffices to proof: $F \dashv \mathfrak{R}(F)$

Step 1: The identification of $\mathfrak{D}^h(F)$:

$\mathfrak{D}^h(F)$ is an exact morphism $\mathbb{H}_{Y^{op}} \longrightarrow \mathbb{H}_{X^{op}}$, hence by Remark 2.20.a) it is determined by its value on $\mathfrak{Y}_Y$:

$$\begin{tikzcd}
\mathbb{H}_{Y^{op}}(Y) \arrow[r] \arrow[rrr, bend left, "\mathfrak{D}^h(F)_Y"] & {[\![\mathbb{H}_Y, \mathbb{H}]\!](Y)} \arrow[r] & {[\![\mathbb{H}_X, \mathbb{H}]\!](Y)} \arrow[r] & \mathbb{H}_{X^{op}}(Y) \\
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\
\mathfrak{Y}_Y \arrow[r, mapsto] & id_{\mathbb{H}_Y} \arrow[r, mapsto] & F \arrow[r, mapsto] & F_{X^{op}}(\mathfrak{Y}_X)
\end{tikzcd}$$

By Proposition 3.4 the morphism $Hom_{[X^{op}, Y^{op}, *]}(-,-)$ is exact in both variables. By the unitality of Hom (Remark 2.20.b)): $Hom_{[Y^{op}, Y^{op}, Y]}(\mathfrak{Y}_{Y^{op}}, \mathfrak{Y}_Y) \cong \mathfrak{Y}_Y$
In particular the morphism $Hom_{[*, Y^{op}, Y]}(-, \mathfrak{Y}_Y) : (\mathbb{H}^{op})_Y \longrightarrow \mathbb{H}_Y$ maps $\mathfrak{Y}_{Y^{op}}$ to $\mathfrak{Y}_Y$. But since D is monoidal (Remark 3.8) this is the characterising property of $D^{op}_{Y^{op}}$.
Moreover, since the Z-component of $Hom_{[*, Y^{op}, Y]}(-, \mathfrak{Y}_Y)$ is the $*$-component of $Hom_{[Z, Y^{op}, Y]}(-, \mathfrak{Y}_Y)$, there are equivalences:

$Hom_{[Z,Y^{op},Y]}(-,\mathfrak{Y}_Y) \cong D^{op}_{Z\times Y}$ for all $Z \in Fin$.

Thus:
$Hom_{[X^{op},Y^{op},Y]}(D_{X^{op}\times Y}(F_{X^{op}}(\mathfrak{Y}_X)),\mathfrak{Y}_Y)$
$\cong D^{op}_{X\times Y^{op}} \circ D_{X^{op}\times Y}(F_{X^{op}}(\mathfrak{Y}_X))$
$\cong F_{X^{op}}(\mathfrak{Y}_X)$

Therefore the morphism $Hom_{[X^{op},Y^{op},*]}(D_{X^{op}\times Y}(F_{X^{op}}(\mathfrak{Y}_X)),-)$ maps $\mathfrak{Y}_Y$ to $F_X(\mathfrak{Y}_X)$. But as we have seen above, this is the characterising property of $\mathfrak{D}^h(F)$.

Hence there is a natural isomorphism:
$\mathfrak{D}^h(F) \cong Hom_{[X^{op},Y^{op},*]}(D_{X^{op}\times Y}(F_{X^{op}}(\mathfrak{Y}_X)),-)$.

Step 2: The identification of $\mathfrak{R}(F)$:

Since D is monoidal it behaves well with respect to Hom (Remark 3.8). Hence:
$\mathfrak{D}^h(F)$
$\cong Hom_{[X^{op},Y^{op},*]}(D_{X^{op}\times Y}(F_{X^{op}}(\mathfrak{Y}_X)),-)$
$\cong D^{op}_X \check{H}om_{[X,Y,*]}(D^{op}_{X\times Y^{op}} \circ D_{X^{op}\times Y}(F_{X^{op}}(\mathfrak{Y}_X)), D_{Y^{op}}(-))$
$\cong D^{op}_X \check{H}om_{[X,Y,*]}(F_{X^{op}}(\mathfrak{Y}_X), D_{Y^{op}}(-))$

But as we have seen in 2.20 $\check{H}om_{[X,Y,*]}$ is just the opposite of $Hom_{[X,Y,*]}$. Hence:
$\mathfrak{R}(F)$
$\cong D^{op}_X \circ (\mathfrak{D}^h(F))^{op} \circ D_Y$
$\cong D^{op}_X D_{X^{op}} Hom_{[X,Y,*]}(F_{X^{op}}(\mathfrak{Y}_X), D^{op}_{Y^{op}} D_Y(-))$
$\cong Hom_{[X,Y,*]}(F_{X^{op}}(\mathfrak{Y}_X),-)$

Step 3: Final Step

By Corollary 2.18 there is an natural isomorphism $F \cong -\otimes_{[*,X,Y]} F_{X^{op}}(\mathfrak{Y}_X)$. And by the exponential law described in 2.20 there is an natural isomorphism:
$Hom_{[*,Y,*]}(-\check{\otimes}_{[*,X,Y]}F_{X^{op}}(\mathfrak{Y}_X),-) \cong Hom_{[*,X,*]}(-,Hom_{[X,Y,*]}(F_{X^{op}}(\mathfrak{Y}_X),-))$

But $\check{\otimes}_{[*,X,Y]}$ is simply the opposite of $\otimes_{[*,X,Y]}$, and, as we have seen, $Hom_{[*,Z,*]}$ describes the canonical enrichment of $\mathbb{H}_Z$ over $\mathbb{H}$. Therefore the above exponential formula yields:
$-\otimes_{[*,X,Y]} F_{X^{op}}(\mathfrak{Y}_X) \dashv Hom_{[X,Y,*]}(F_{X^{op}}(\mathfrak{Y}_X),-)$

Combining with Step 2, we have shown:
$F \cong -\otimes_{[*,X,Y]} F_{X^{op}}(\mathfrak{Y}_X) \dashv Hom_{[X,Y,*]}(F_{X^{op}}(\mathfrak{Y}_X),-) \cong \mathfrak{R}(F)$ □

Remark 3.13. Note that for proving only the existence of adjoints (without identification), we used only 2.20 and 3.4.

3.3 Some Remarks and Interpretation, Part 3

Remark 3.14. a) Again one should expect that an analog of Theorem 3.12. holds true for the 2-category of cofree modules over an arbitrary stable symmetric monoidal Fin-derivator.
b) The following remark is of highly conjectural nature, and shouldn't be regarded as more than a first idea for hopefully interesting future research.
One could of course ask whether the monoidal duality (Theorem 2.21) extends to a larger subcategory of $Der^{\star}$, or how we have to modify the theory the obtain a full duality.
As we have seen in section 3.1 it was sufficient to make the conditions "homotopy colimit preserving" and "homotopy limit preserving" equivalent, to obtain a good Spanier-Whitehead-Duality. Therefore we can expect that an analog of the monoidal duality functor above is an equivalence whenever the properties "homotopy bicolimit preserving" and "homotopy bilimit preserving" are equivalent. To make those properties more precise, we remember that by Renaudin's Theorem a full sub-2-catetegory of $Der^{\star}_L$, let's call it $\mathscr{H}$, describes the homotopy theory of homotopy theories. Assume further that an equivalence $Ho(\mathcal{P}r^L) \overset{\cong}{\longrightarrow} \mathscr{H}$, where $\mathcal{P}r^L$ denotes the $(\infty, 2)$-catgeory of presentable $(\infty, 1)$-categories [Lur09] 5.5.3.1, holds, as described in Remark 1.34.a). Let X be a bicategory, then we define $\mathscr{H}(X)$ to be the homotopy bicategory $Ho((\mathcal{P}r^L)^X)$. Note that $\mathscr{H}(*) \cong \mathscr{H}$. Now a pseudofunctor between bicategories $f : X \longrightarrow Y$ induces a functor $(\mathcal{P}r^L)^Y \longrightarrow (\mathcal{P}r^L)^X$ and thus a derived pseudofunctor $f^* : \mathscr{H}(Y) \longrightarrow \mathscr{H}(X)$. Hence we get a trifunctor $\mathscr{H} : Bicat^{op} \longrightarrow BICAT$ ([Gur07]). Since $\mathcal{P}r^L$ is bicomplete, all the pseudofunctors f^* will have left and right biadjoints, the homotopy Kan biextensions $f_!$ and f_*.
The aim is now to approximate $\mathscr{H}$ by a similar trifunctor $\mathscr{H}^s$, which is in the largest full subtricategory of those trifunctors $Bicat^{op} \longrightarrow BICAT$, that has the property, that a 1-morphism commutes with the $f_!$ if and only if it commutes with the f_*, in a similar way as we approximated $\mathbb{H}^u$ by $\mathbb{H}^s$.
A possible strategy for the construction of $\mathscr{H}^s$, would be to impose the minimal relations on homotopy Kan biextensions (i.e. higher analogs of (Der6) and (Der7)), which ensure that homotopy left Kan biextensions can be expressed in terms of homotopy right Kan biextensions and vice versa. Thus we can regard this approximation procedure as a higher stabilization.
Note that the 2-categories $Der^{\star}_L$ already satisfy some of these relations:
-The $\star$-derivator $\mathbb{H}^{\star}_{\emptyset}$ is a zero-object in $Der^{\star}_L$.
-Let $\mathbb{D}, \mathbb{D}' \in Der^{\star}_L$ and $X \in Cat$. We apply Proposition 2.12 to the $\star$-derivator $[\![\mathbb{D}, \mathbb{D}']\!]_L$ to obtain an equivalence:

$$PDer(\underline{X}, [\![\mathbb{D}, \mathbb{D}']\!]_L) \cong [\![\mathbb{D}, \mathbb{D}']\!]_L(X) \cong [\![\mathbb{D}, \mathbb{D}'_X]\!]_L$$

Thus the cotensor functor associated to the prederivator $\underline{X}$ exists and is given by $(-)_X$. But by Theorem 2.16 there are natural equivalence:

$$\mathbb{H}^{\star}_X[\otimes]_L - = [\![[\![\mathbb{H}^{\star}_X, \mathbb{H}^{\star}]\!]_L, -]\!]_L \cong [\![\mathbb{H}^{\star}_{X^{op}}, -]\!]_L \cong (-)_X$$

and by Proposition 2.27.g) :

$$(-)_{X^{op}} \cong \mathbb{H}^\star_{X^{op}}[\otimes]_L - \cong [\![\mathbb{H}^\star_X, \mathbb{H}^\star]\!]_L[\otimes]_L - \dashv \mathbb{H}^\star_X[\otimes]_L -$$

Thus there are natural equivalences:

$$[\![\mathbb{D}_{X^{op}}, \mathbb{D}']\!]_L \cong PDer(\underline{X}, [\![\mathbb{D}, \mathbb{D}']\!]_L) \cong [\![\mathbb{D}, \mathbb{D}'_X]\!]_L$$

Hence there are non-trivial relations between the tensor and cotensor functors associated to representable prederivators. This is a higher analog of the natural isomorphism between coproducts and products in additive categories.

Thus we can expect that $\mathscr{H}^s$ on the one hand will still capture a lot of information of $\mathscr{H}$ and on the other hand will have more structure (i.e. monoidal duality and a higher analog of a triangulated structure) and therefore better computability properties.

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