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Reliable Computation of the Complex Interval Function arcsin(z) in C-XSC

Frithjof Blomquist, Werner Hofschuster, Walter Krämer

Abstract

In **C-XSC** the complex interval standard functions $f : \mathbb{C} \to \mathbb{C}$ are implemented in `cimath.cpp`. For a function f and a given rectangular complex interval Z some inclusion interval W is computed, so that $f(z) \in W$ holds for all $z \in Z$. Thereby real interval expressions must be evaluated on the boundary of Z, and a naive interval arithmetic often leads to premature overflow or to overestimations, if for example such expressions are evaluated near their zeros. In a new version these disadvantages are avoided by using different interval expressions in different regions on the boundary of Z, and so the intervals W are now optimal inclusions of the range of $f(z)$, for $z \in Z$. As an example the strategy is presented by the inverse sine, and numerical results demonstrate the achieved improvements.

MSC Subject Classifications: 30-04, 65G30, 65K05.
Key Words: Complex interval functions, inverse sine, arcsin, reliable computations, mathematical software, C-XSC

1 Introduction

In [4, 12, 13, 6] algorithms for calculating the complex standard functions $f : \mathbb{C} \to \mathbb{C}$ are published together with guaranteed error bounds for the real and imaginary part of the function values, evaluated in floating-point arithmetic. With these bounds for a given rectangular argument interval Z the inclusion intervals W can principally be realized, so that $f(z) \in W \ \forall z \in Z$. Because of the great effort associated with this method, Neher [16] used a simpler way, which in general will enlarge the runtime of the algorithms and which is briefly described as follows:

With $z = x + i \cdot y,\ i = \sqrt{-1}$ the function $f(z) = u(x, y) + i \cdot v(x, y)$ under consideration is assumed to be analytic on the given interval Z, so that the extremal values of $u(x, y)$ and $v(x, y)$ lie on the boundary of Z. In general these extremal points are the corner points of Z, but in some cases they lie somewhere on the boundary and are possibly not representable in the IEEE-System. In these cases the extremal points must be enclosed by small machine intervals, otherwise by appropriate point intervals. If for example the real intervals enclosing the maximum point of the real part $u(x, y)$ are denoted by[1]

[1]At least one of the intervals M_x, M_y is a point interval, because the extremal points lie on the boundary of the machine interval Z.

$M_x = [x_1, x_2]$ and $M_y = [y_1, y_2]$, an upper bound of the maximum of $u(x, y)$ can be found by evaluating $u(x, y)$ with interval arguments, substituting x and y by M_x, M_y respectively. If then U is a machine inclusion of $u(M_x, M_y) \subseteq U$, a guaranteed upper bound of the maximum of the real part is given by $\texttt{Sup}(U) \geq u(x, y)\, \forall z \in Z$. In this way the inclusion of the real and imaginary part of $f(z)$ with $z \in Z$ can be realized without computing a priori error bounds as it has been presented in [4, 6, 12, 13].

The advantage of this strategy is the simple interval evaluation of the two functions $u(x, y)$ and $v(x, y)$, which in most cases are composed of the elementary interval functions already implemented in C-XSC. The disadvantage is the rather expensive interval evaluation of $u(x, y)$ and $v(x, y)$ at runtime.

In the CoStLy library [16], which is completely implemented in C-XSC, simple expressions, such as $\sqrt{x^2 + y^2}$, are evaluated in a naive interval arithmetic. Thus, for sufficiently great values of x or y, an overflow occurs and leads to significant overestimations of the calculated inclusions, although $\sqrt{x^2 + y^2}$ is still representable in the IEEE-System.

Beside the standard functions, the following functions $\ln(x^2 + y^2), \sqrt{1 + x^2}$, $\sqrt{1 - x^2}, \sqrt{x^2 - 1}, \sqrt{x^2 + y^2}, \arctan(y/x), \sqrt{1 + x} - 1, \ln(1 + x), \operatorname{arcosh}(1 + x)$ are implemented in C-XSC as well, and with these additional tools the described overestimations can mostly be avoided. As an example of the achieved improvements we consider the inverse sine.

2 The Inverse Sine

With $z = x + i \cdot y$, $T(x, y) := \sqrt{(x + 1)^2 + y^2} + \sqrt{(x - 1)^2 + y^2}$, $\alpha := T(x, y)/2$, $\beta := x/\alpha$ and $w = \arcsin(z) = \Re(w) + i \cdot \Im(w)$ it holds [12, 13]

$$\Re(w) := \arcsin(\beta), \tag{1}$$

$$\Im(w) = \begin{cases} +\operatorname{arcosh}(\alpha), & \text{if } y > 0, \\ +\operatorname{arcosh}(\alpha), & \text{if } y = 0 \text{ and } x \leq -1, \\ 0 & , \text{ if } -1 \leq x \leq +1, \\ -\operatorname{arcosh}(\alpha), & \text{if } y = 0 \text{ and } x \geq +1, \\ -\operatorname{arcosh}(\alpha), & \text{if } y < 0. \end{cases} \tag{2}$$

To calculate an inclusion W of the range $\arcsin(Z)$, the functions $\operatorname{arcosh}(\alpha)$ and $\arcsin(\beta)$ must be evaluated with intervals $\boldsymbol{\alpha}, \boldsymbol{\beta}$. For this purpose in $T(x, y)$ the variables x and y must be substituted by point intervals $\boldsymbol{x_j} = [x_j]$ and $\boldsymbol{y_j} = [y_j]$, where x_j and y_j are the coordinates of the extremal points, which, in case of the inverse sine, are appropriate corner points of the argument interval Z [13].

By calculating $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ first of all we have to fulfill the condition $T \leq \texttt{MaxReal}$, where `MaxReal` is the greatest machine number. With $M := \max\{|x|, |y|\}$ it can easily be shown by fall differentiation

$$\begin{aligned} T(x, y) = T(|x|, |y|) &\leq \sqrt{(|x| + 1)^2 + y^2} + \sqrt{(|x| + 1)^2 + y^2} \\ &= 2\sqrt{(|x| + 1)^2 + y^2} < 2\sqrt{(|x| + 1)^2 + (|y| + 1)^2} \\ &\leq 2\sqrt{2} \cdot (M + 1), \end{aligned}$$

and $T \leq \texttt{MaxReal}$ is fulfilled, if $M < \texttt{MaxReal}/(2\sqrt{2}) - 1 = 6.35580\ldots \cdot 10^{+307}$, which in practice is no significant restriction. Hence, with (1) the real part $\Re(w)$ can problem-free be included for all $z \in Z$.

The interval expressions $\sqrt{(\boldsymbol{x} \pm 1)^2 + \boldsymbol{y}^2}$ are evaluated with the C-XSC function `sqrtx1y2(x,y)`, which, in case of $T \leq \texttt{MaxReal}$, delivers tight inclusions of $\sqrt{(\boldsymbol{x} \pm 1)^2 + \boldsymbol{y}^2}$ without any internal overflow.

2.1 Inclusion of the Imaginary Part $\Im(w)$

In (2) the interval expression $\operatorname{arcosh}(\boldsymbol{\alpha})$ must be evaluated with $\boldsymbol{\alpha} = T(\boldsymbol{x}, \boldsymbol{y})/2$, where $\boldsymbol{x} = [x_j]$ and $\boldsymbol{x} = [y_j]$ are point intervals. However for $\texttt{Sup}(\boldsymbol{\alpha}) \to +1$ we get severe overestimations, because the real arcosh function is evaluated in the environment of their zero $\alpha = +1$. This disadvantage can be avoided using the identity

$$(3) \qquad \operatorname{arcosh}(\alpha) \equiv \operatorname{arcosh}(1 + \delta), \quad \delta := \alpha - 1,$$

where in C-XSC the right-hand side $\operatorname{arcosh}(1 + \delta)$ is implemented as $\texttt{acoshp1}(\delta)$. First we consider the proposition

$$|x| \geq 2 \ \wedge \ |y| \geq 2 \quad \Longrightarrow \quad \alpha(x, y) \geq 2.$$

As proof we use $\alpha(x, y) \geq \alpha(x, 0) = (|x + 1| + |x - 1|)/2 =: r(x)$, and with $|x| \geq 2$ it follows $r(x) \geq 2$. Furthermore it holds $\alpha(x, y) \geq \alpha(0, y) = |y|$, and with $|y| \geq 2$ again we have $\alpha(x, y) \geq 2$ ∎

So inside the square with the side length 4 and the centre in the origin we use the right-hand side of (3) and outside the square we use $\operatorname{arcosh}(\alpha)$. Now let us consider the interior of the square, where at first we have to calculate δ. With

$$V(x, y) := |x + 1| \cdot \left\{ \sqrt{1 + \left(\frac{y}{x + 1} \right)^2} - 1 \right\} + |x - 1| \cdot \left\{ \sqrt{1 + \left(\frac{y}{x - 1} \right)^2} - 1 \right\}$$

it holds

$$(4) \qquad \delta = \begin{cases} \dfrac{V(x, y)}{2}, & \text{if } |x| < 1, \\ \left\{ \sqrt{1 + \left(\dfrac{y}{x + 1} \right)^2} - 1 \right\} + \dfrac{|y|}{2}, & \text{if } |x| = 1, \\ \dfrac{V(x, y)}{2} + (|x| - 1), & \text{if } 1 < |x| \leq 2. \end{cases}$$

As the two summands of $V(x, y)$ are positive, no cancellation effects can arise by calculating the sum. The above expressions $\{...\}$ are evaluated with the C-XSC function $\sqrt{1 + \boldsymbol{u}} - 1$, which delivers tight inclusions of the function values, even if the machine interval $\boldsymbol{u}$ tends to zero. However there is a problem by calculating the expressions $\boldsymbol{u} := \boldsymbol{y}^2/(\boldsymbol{x} \pm 1)^2$ for $\boldsymbol{y} \to 0$. In this case the machine intervals $\boldsymbol{u}$ lie in the unnormalized region, which leads to significant overestimations of $\boldsymbol{u}$ and $\boldsymbol{\delta}$, even though the function values $\operatorname{arcosh}(1+\delta) \sim \sqrt{2\delta}$ lie in the normalized region. Hence, to avoid these overestimations, instead of (4) we use the Taylor expansion of $\operatorname{arcosh}(1+\delta)$, if δ tends to zero. This strategy will now be described in the case $|x| < 1$. With

$$Q(\delta) := \sum_{k=0}^{\infty} a_k \cdot \delta^k = 1 - \frac{1}{12}\delta + \frac{3}{160}\delta^2 - \frac{5}{896}\delta^3 \pm \ldots$$

$$a_0 = 1,\ a_{k+1} = -a_k \cdot \frac{(2k+1)^2}{4 \cdot (2k+3)(k+1)},\ k = 0, 1, 2, \ldots \quad \text{it holds}$$

$$\operatorname{arcosh}(1+\delta) = \sqrt{2\delta} \cdot Q(\delta), \quad 0 \le \delta < 2.$$

For $0 \le \delta < 2$ it can easily be shown that $Q(\delta)$ is an alternating Leibniz-series, and so, using $2\delta = V(x,y)$, we have the inclusion

$$\sqrt{V} \cdot (1 - V/24) \le \operatorname{arcosh}(1+\delta) \le \sqrt{V}, \quad V = V(x,y) < 4.$$

The requirement[2] $1 - V/24 > \texttt{pred}(1) = 1 - 2^{-53} \iff V < 3 \cdot 2^{-50}$ finally leads to

$$\sqrt{V} \cdot \texttt{pred}(1) \le \operatorname{arcosh}(1+\delta) \le \sqrt{V}, \quad \text{if } V < 3 \cdot 2^{-50}. \tag{5}$$

We now calculate lower and upper bounds of $V(x,y)$, to get more simple inclusion expressions in (5). The following estimates are valid for $0 \le t < 1$.

$$\frac{t}{2} \cdot \left(1 - \frac{t}{4}\right) \le \sqrt{1+t} - 1 \le \frac{t}{2}, \tag{6}$$

and with $V_1 := |x+1| \cdot \{\sqrt{1 + y^2/(x+1)^2} - 1\}$ we get

$$V_1 \le |x+1| \cdot \frac{1}{2} \cdot \left(\frac{y}{x+1}\right)^2 = \frac{1}{2} \cdot \frac{y^2}{|x+1|}, \quad \text{if } y^2 < (x+1)^2.$$

For $|x| < 1$ it holds $(x+1)^2 \ge (\texttt{succ}(-1) + 1)^2 = (2^{-53})^2 = 2^{-106}$, and so

$$V_1 \le \frac{1}{2} \cdot \frac{y^2}{|x+1|}, \quad \text{if } |y| < 2^{-53}, \tag{7}$$

and analogously with $V_2 := |x-1| \cdot \{\sqrt{1 + y^2/(x-1)^2} - 1\}$ it follows

$$V_2 \le |x-1| \cdot \frac{1}{2} \cdot \left(\frac{y}{x-1}\right)^2 = \frac{1}{2} \cdot \frac{y^2}{|x-1|}, \quad \text{if } y^2 < (x-1)^2.$$

For $|x| < 1$ it holds $(x-1)^2 \ge (1 - \texttt{pred}(1))^2 = (2^{-53})^2 = 2^{-106}$, and so

$$V_2 \le \frac{1}{2} \cdot \frac{y^2}{|x-1|}, \quad \text{if } |y| < 2^{-53}. \tag{8}$$

[2] $\texttt{pred}(1)$ is the greatest floating-point number smaller than 1.

Finally, with $V(x,y) = V_1 + V_2$, we get an upper bound of $V(x,y)$.

$$V(x,y) \le \frac{y^2}{2} \cdot \left[\frac{1}{|x+1|} + \frac{1}{|x-1|} \right] = \frac{y^2}{1-x^2}, \quad \text{if} \quad |y| < 2^{-53}, \ |x| < 1. \tag{9}$$

We now calculate a lower bound of $V(x,y)$. Again with (6) it holds

$$\frac{1}{2} \cdot \frac{y^2}{|x+1|} \cdot \left[1 - \frac{1}{4} \left(\frac{y}{x+1} \right)^2 \right] \le V_1, \quad \text{if} \quad |y| < 2^{-53}.$$

We require the condition

$$1 - \frac{1}{4} \left(\frac{y}{x+1} \right)^2 > \mathtt{pred}(1) = 1 - 2^{-53} \quad \Longleftrightarrow \quad \left(\frac{y}{x+1} \right)^2 < 2^{-51}.$$

For $|x| < 1$ again we use the estimate $(x+1)^2 \ge 2^{-106}$, and so the above condition is fulfilled, if

$$\frac{y^2}{2^{-106}} < 2^{-51} \quad \Longleftrightarrow \quad |y| < \sqrt{2} \cdot 2^{-79}.$$

Hence, a lower bound of V_1 is given by

$$\frac{1}{2} \cdot \frac{y^2}{|x+1|} \cdot \mathtt{pred}(1) \le V_1, \quad \text{if} \quad |y| < \sqrt{2} \cdot 2^{-79}.$$

In the same way we get the estimate

$$\frac{1}{2} \cdot \frac{y^2}{|x-1|} \cdot \mathtt{pred}(1) \le V_2, \quad \text{if} \quad |y| < \sqrt{2} \cdot 2^{-79}, \qquad \text{and so}$$

$$\frac{y^2}{2} \cdot \left[\frac{1}{|x+1|} + \frac{1}{|x-1|} \right] \cdot \mathtt{pred}(1) = \frac{y^2}{1-x^2} \cdot \mathtt{pred}(1) \le V_1 + V_2 = V(x,y).$$

Together with (9) we finally get a rather tight inclusion for $V(x,y)$.

$$\mathtt{pred}(1) \cdot \frac{y^2}{1-x^2} \le V(x,y) \le \frac{y^2}{1-x^2}, \quad \text{if} \ |y| < \sqrt{2} \cdot 2^{-79}, \tag{10}$$

and with (5) an inclusion of $\operatorname{arcosh}(1+\delta)$ is given by

$$\mathtt{pred}(1) \cdot \sqrt{\mathtt{pred}(1)} \cdot \frac{|y|}{\sqrt{1-x^2}} \le \operatorname{arcosh}(1+\delta) \le \frac{|y|}{\sqrt{1-x^2}}.$$

Using $\sqrt{\texttt{pred}(1)} > \texttt{pred}(1)$, finally we get

$$\texttt{pred}(1) \cdot \texttt{pred}(1) \cdot \frac{|y|}{\sqrt{1-x^2}} \leq \operatorname{arcosh}(1+\delta) \leq \frac{|y|}{\sqrt{1-x^2}}. \tag{11}$$

The lower bound can still be simplified using the inequality

$$\texttt{pred}(1) \cdot x \geq \texttt{pred}(x), \tag{12}$$

which can easily be shown for all machine values $x \geq 0$. Hence, for $|y| < \sqrt{2} \cdot 2^{-79}$ we have the following effective algorithm, to calculate a rather tight inclusion of $\operatorname{arcosh}(1+\delta)$.

1. Calculate with the assignment `u = abs(y)/sqrt1mx2(x)` a guaranteed inclusion of $|y|/\sqrt{1-x^2} \in \texttt{u} = [\texttt{u1}, \texttt{u2}]$, where `x` in `sqrt1mx2(x)` is a machine point interval. The C-XSC function `sqrt1mx2(x)` delivers a tight inclusion of $\sqrt{1-\mathtt{x}^2} \in$ `sqrt1mx2(x)`.
2. Hence, for $|x| < 1$, with the interval `u = [u1,u2]`, it holds
   $\texttt{pred(pred(u1))} \leq \operatorname{arcosh}(1+\delta) \leq \texttt{u2}$.

Now, after this detailed specification, we only give a short description for the case $|x| = 1$. With (4) and $t := |y| < 1$ it holds

$$\begin{aligned}
\operatorname{arcosh}(1+\delta) &= \operatorname{arcosh}(\sqrt{1+(t/2)^2} + t/2) = \sqrt{t} \cdot H(t), \\
H(t) &= 1 + \frac{1}{12}t - \frac{3}{160}t^2 - \frac{5}{896}t^3 + \frac{35}{18432}t^4 + \frac{63}{90112}t^5 - - + + \ldots \\
H(t) &= (1 + \frac{1}{12}t) - (\frac{3}{160}t^2 + \frac{5}{896}t^3) + (\frac{35}{18432}t^4 + \frac{63}{90112}t^5) - + \ldots
\end{aligned}$$

$H(t)$ is an alternating Leibniz-series, so, for $t < 1$, we have the estimates

$$\sqrt{t} \cdot \left[(1 + \frac{1}{12}t) - (\frac{3}{160}t^2 + \frac{5}{896}t^3) \right] \leq \operatorname{arcosh}(1+\delta) \leq \sqrt{t} \cdot (1 + \frac{1}{12}t).$$

Furthermore, for $t < 1$, it holds

$$\begin{aligned}
1 - \frac{3}{80}t^2 &= 1 - t^2(\frac{3}{160} + \frac{3}{160}) \leq 1 - t^2(\frac{3}{160} + \frac{5}{896} \cdot t) \\
&\leq (1 + \frac{1}{12}t) - t^2(\frac{3}{160} + \frac{5}{896} \cdot t), \quad \text{and so}
\end{aligned}$$

$$\sqrt{t} \cdot \left(1 - \frac{3}{80}t^2\right) \leq \operatorname{arcosh}(1+\delta) \leq \sqrt{t} \cdot \left(1 + \frac{t}{12}\right). \tag{13}$$

With the inclusion relation (13) we formulate the appropriate algorithm

1. Calculate with `u = sqrt(interval(t))` a guaranteed inclusion for $\sqrt{t} \subseteq$ `u = [u1,u2]`.
2. Calculate for a sufficient small t with `[pred(u1),succ(u2)]` $\supseteq \operatorname{arcosh}(1+\delta)$ a guaranteed inclusion for $\operatorname{arcosh}(1+\delta)$.

We now have to specify an upper bound of t so that the inclusion $\operatorname{arcosh}(1+\delta) \subseteq$ `[pred(u1),succ(u2)]` is valid. At first it holds $\sqrt{t} \cdot (1+t/12) \le \mathtt{u2} \cdot (1+t/12)$, and we have to desire

$$\mathtt{u2} \cdot (1+t/12) < \mathtt{succ(u2)}. \tag{14}$$

Writing the machine number $\mathtt{u2} = m \cdot 2^{ex}$, it holds

$$\mathtt{succ(u2)} = \mathtt{u2} + 2^{ex-52} = m \cdot 2^{ex} + 2^{ex-52}, \quad 1 \le m < 2,$$

and (14) is equivalent to

$$m \cdot 2^{ex} \cdot (1+t/12) < m \cdot 2^{ex} + 2^{ex-52} \quad \Longleftrightarrow \quad t < \frac{3 \cdot 2^{-50}}{m}.$$

It holds $3 \cdot 2^{-51} < 3 \cdot 2^{-50}/m$, and therefore (14) is valid, if $t = |y| < 3 \cdot 2^{-51}$. The condition $\mathtt{pred(u1)} < \mathtt{u1}(1 - 3t^2/80)$ leads to a much greater upper bound of t. Therefore, if $|x| = 1$ and $t = |y| < 6 \cdot 2^{-52}$, the inclusion $\operatorname{arcosh}(1+\delta) \subseteq$ `[pred(u1),succ(u2)]` is guaranteed.

In the remaining case $1 < |x| \le 2$ the expression $V(x,y)/2 + (|x|-1)$ can straightly be evaluated, since possible overestimations of the smaller summand $V(x,y)/2$ are of no relevance in comparison with the much greater summand $(|x|-1)$.

In the interior of the square with the side length 4 we calculate δ with (4). In the cases $|x| < 1$ or $|x| = 1$ for sufficient small values $t = |y|$ we use the guaranteed and tight inclusions:

$$\begin{aligned}
\mathtt{pred(pred(u1))} \le \operatorname{arcosh}(1+\delta) \le \mathtt{u2}, \quad &\text{if} \quad |y| < \sqrt{2} \cdot 2^{-79} \quad \text{and} \quad |x| < 1. \\
\mathtt{pred(u1)} \le \operatorname{arcosh}(1+\delta) \le \mathtt{succ(u2)}, \quad &\text{if} \quad |y| < 3 \cdot 2^{-51} \quad \text{and} \quad |x| = 1.
\end{aligned}$$

3 Numerical Examples for $f(z) = \arcsin(z)$

Point Interval $Z = [0.5] + i[2^{-1022}]$, $f(z) \in W = \Re(W) + i \cdot \Im(W)$			
$\Re(W)$	$5.23598775598^{30}_{29} \cdot 10^{-1}$	$\Im(W)$	$2.56929398235186^{6}_{5} \cdot 10^{-308}$
$\Re(W^\bullet)$	$5.23598775598^{30}_{29} \cdot 10^{-1}$	$\Im(W^\bullet)$	$[0, 2.9802322387\ldots \cdot 10^{-8}]$

Point Interval $Z = [1 - 2^{-53}] + i[2^{-1022}]$, $f(z) \in \Re(W) + i \cdot \Im(W)$			
$\Re(W)$	$1.570796311893736^{6}_{5}$	$\Im(W)$	$1.49322178960515^{5}_{4} \cdot 10^{-300}$
$\Re(W^\bullet)$	1.570796^{3}_{2}	$\Im(W^\bullet)$	$[0, 2.9802322387\ldots \cdot 10^{-8}]$

Point Interval $Z = [1.0] + i[2^{-1022}]$, $f(z) \in W = \Re(W) + i \cdot \Im(W)$			
$\Re(W)$	$1.57079632679489^{\mathbf{7}}_{\mathbf{6}}$	$\Im(W)$	$1.49166814624004\mathbf{1}^{\mathbf{3}} \cdot 10^{-154}$
$\Re(W^\bullet)$	$1.570796^{\mathbf{3}}_{\mathbf{2}}$	$\Im(W^\bullet)$	$[0, 2.9802322387\ldots \cdot 10^{-8}]$
Point Interval $Z = [1 + 2^{-52}] + i[2^{-1022}]$, $f(z) \in \Re(W) + i \cdot \Im(W)$			
$\Re(W)$	$1.57079632679489^{\mathbf{7}}_{\mathbf{6}}$	$\Im(W)$	$2.107342425544^{\mathbf{706}}_{\mathbf{699}} \cdot 10^{-8}$
$\Re(W^\bullet)$	$1.570796^{\mathbf{3}}_{\mathbf{2}}$	$\Im(W^\bullet)$	$[2.107\ldots, 4.214\ldots] \cdot 10^{-8}$
Point Interval $Z = [2^{1022}] + i[2^{1022}]$, $f(z) \in \Re(W) + i \cdot \Im(W)$			
$\Re(W)$	$7.853981633974^{\mathbf{5}}_{\mathbf{4}} \cdot 10^{-1}$	$\Im(W)$	$7.094361393031 0^{\mathbf{5}}_{\mathbf{2}} \cdot 10^{2}$
$\Re(W^\bullet)$	$[0, 1.570796326794901]$	$\Im(W^\bullet)$	$[7.090\ldots, 7.10\ldots] \cdot 10^{2}$
Complex Interval $Z = [0.5, 1] + i[2^{-1022}]$, $f(z) \in \Re(W) + i \cdot \Im(W)$			
$\Re(W)$	$[5.235987755982972 \cdot 10^{-1}, 1.570796326794897 \cdot 10^{0}]$		
$\Re(W^\bullet)$	$[5.235987755982972 \cdot 10^{-1}, 1.570796326794901 \cdot 10^{0}]$		
$\Im(W)$	$[2.569293982351859 \cdot 10^{-308}, 1.491668146240043 \cdot 10^{-154}]$		
$\Im(W^\bullet)$	$[0.000000000000000 \cdot 10^{0}, 2.980232238769538 \cdot 10^{-8}]$		

Table 1: Improved inclusions W, compared with the original values $W^\bullet$.

In all the above examples the improved enclosures W are nearly optimal compared to the original unimproved enclosures $W^\bullet = \Re(W^\bullet) + i \cdot \Im(W^\bullet)$.

References

[1] Alefeld G. and Herzberger J. *Introduction to Interval Computations.* Academic Press, New York, 1983.

[2] Bantle A., Krämer W. Automatic Forward Error Analysis for Floating Point Algorithms, *Reliable Computing*, Vol. 7, No. 4, pp. 321-340 (2001)

[3] Blomquist, F. *Anmerkungen zur Realisierung der komlexen Intervallfunktionen in der CoStLy-Bibliothek*, interner Bericht, 55 Seiten, WRSWT, Bergische Universität Wuppertal, 2005.

[4] Braune, K. *Hochgenaue Standardfunktionen für reelle und komplexe Punkte und Intervalle in beliebigen Gleitpunktrastern.* Dissertation, Universität Karlsruhe, 1987.

[5] Braune K., Krämer W. *High Accuracy Standard Functions for Real and Complex Intervals.* In Kaucher E., Kulisch U. and Ullrich Ch., editors, Computerarithmetic: Scientific Computation and Programming Languages, pages 81-114. Teubner, Stuttgart (1987)

[6] Braune K. Standard Functions for Real and Complex Point and Interval Arguments with Dynamic Accuracy. *Computing Supplementum*, 6:159-184 (1988).

[7] Hammer R., Hocks M., Kulisch U., Ratz D. *C++ Toolbox for Verified Computing.* Springer, 1994.

[8] Hofschuster W., Krämer W. Ein rechnergestützter Fehlerkalkül mit Anwendungen auf ein genaues Tabellenverfahren, Preprint Nr. 96/5, Institut für Wissenschaftliches Rechnen und Mathematische Modellbildung, Karlsruhe (1996)

[9] Hofschuster W., Krämer W. A Computer Oriented Approach to Get Sharp Reliable Error Bounds, *Reliable Computing* 3, pp. 239-248 (1997).

[10] Werner Hofschuster and Walter Krämer. FI_LIB, eine schnelle und portable Funktionsbibliothek für reelle Argumente und reelle Intervalle im IEEE-double-Format, Preprint 98/7 des Instituts für Wissenschaftliches Rechnen und Mathematische Modellbildung (IWRMM) an der Universität Karlsruhe(1998)

[11] Hofschuster W. Zur Berechnung von Funktionswerteinschließungen bei speziellen Funktionen der mathematischen Physik, Dissertation, Universität Karlsruhe (2000).

[12] Krämer, W. *Inverse Standardfunktionen für reelle und komplexe Intervallargumente mit a priori Fehlerabschätzungen für beliebige Datenformate.* Dissertation, Universität Karlsruhe, 1987.

[13] Krämer W. Inverse Standard Functions for Real and Complex Point and Interval Arguments with Dynamic Accuracy. *Computing Supplementum*, 6:185-212 (1988).

[14] Krämer W. A priori Worst Case Error Bounds for Floating-Point Computations, *IEEE Transactions on Computers*, Vol. 47, No. 7 (1998).

[15] Kulisch U. and Miranker W. L. *Computer Arithmetic in Theory and Practice.* Academic Press, New York, 1981.

[16] Neher M. Complex Standard Functions and their Implementation in the CoStLy Library. Preprint Nr. 04/18, Fakultät für Mathematik, Universität Karlsruhe, 2004.

[17] Neher M. The mean value form for complex analytic functions. *Computing*, 67:255-268, 2001.

[18] Wolff v. Gudenberg, J. *Einbettung allgemeiner Rechnerarithmetik in PASCAL mittels eines Operatorkonzeptes und Implementierung der Standardfunktionen mit optimaler Genauigkeit.* Dissertation, Universität Karlsruhe, 1980.

[19] XSC website on programming languages for scientific computing with validation. `http://www.xcs.de` [October 2005].