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Preprint
BUW-WRSWT 2006/4

Wissenschaftliches Rechnen/
Softwaretechnologie

wr
swt

Impressum

Herausgeber:	Prof. Dr. W. Krämer, Dr. W. Hofschuster Wissenschaftliches Rechnen/Softwaretechnologie Fachbereich C (Mathematik und Naturwissenschaften) Bergische Universität Wuppertal Gaußstr. 20 42097 Wuppertal, Germany

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Generalized Intervals and the Dependency Problem

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1 Abstract

Computing enclosures of the range of functions using interval arithmetic often leads to overestimations due to variables (parameters) appearing more than once within the expression to be evaluated (dependency problem). An experimental implementation of a generalized interval arithmetic which has been proposed by Hansen [2] in 1975 is now available [1, 3]. In many cases generalized intervals are well suited to reduce the dependency problem and/or the so called wrapping effect.

2 Introduction and Notation

Let us consider the following problem: Compute the range of the expression $f(x) = (10 + x)(10 - x)$ over the interval $[-r, r]$. If we replace the point x by the corresponding interval $X = [-r, r]$ we get $10 + x \in [10 - r, 10 + r]$, $10 - x \in [10 - r, 10 + r]$ and finally $[10 - r, 10 + r][10 - r, 10 + r] = [100 - 20r + r^2, 100 + 20r + r^2]$ as an enclosure of the exact range $\{f(x)|x \in [-r, r]\} = [100 - r^2, 100]$. The diameter of this interval enclosure is $40r$ whereas the diameter of the exact range is r^2. For $r = 1/10$ the overestimation factor is 400 and for $r = 1/100$ it is 4000. Replacing every occurrence of x by X (interval evaluation) leads to a significant loss of information about correlations between quantities (dependency problem).

To avoid this loss of information Hansen [2] introduced so called generalized intervals. In this paper we do not use Hansen's notation. Our notation is similar to the more convenient notation introduced in [5]. A generalized interval (we call it Hansen form) is written as $\hat{x} := x_0 + \sum_{k=1}^{n} x_k \varepsilon_k$. x_0 is called central value, x_k partial deviation, ε_k noise symbol with $|\varepsilon_k| \leq c_k$ and n being the (fixed) number of noise symbols. The central value and the partial deviations are allowed to be (ordinary) intervals. A typical Hansen form is $\hat{x} = [1, 2] + [2, 3]\varepsilon_1 + [0, 1]\varepsilon_2$ with $-c_k \leq \varepsilon_k \leq c_k, k = 1, 2$. Ordinary input intervals may be easily transformed into generalized intervals: $[a, b] = \text{mid}[a, b] + \text{rad}([a, b])\varepsilon_k$ with $|\varepsilon_k| \leq 1$. mid() denotes the midpoint and rad() the radius of an interval. Hansen forms imply interval bounds: Reduce the unevaluated Hansen form to an ordinary interval using ordinary interval arithmetic, i. e. compute the interval $[\hat{x}] := x_0 + \sum_{k=1}^{n} x_k[-c_k, c_k]$.

3 Properties of Hansen forms

3.1 Hansen forms can 'see' correlations

Let us again consider $f(x) = (10 + x)(10 - x)$ with $x \in [-r, r]$. $\hat{x} = 0 + r \cdot \varepsilon$ with $-1 \leq \varepsilon \leq 1$ yields $10 + \hat{x} = 10 + r \cdot \varepsilon$, $10 - \hat{x} = 10 - r \cdot \varepsilon$, so that $(10 + \hat{x})(10 - \hat{x}) = 100 + 0 \cdot \varepsilon - r^2 \cdot \varepsilon^2$ with corresponding Hansen form $100 + [-r^2, 0] + 0 \cdot \varepsilon$. Reducing this Hansen form results in the ordinary interval $[100 - r^2, 100]$ which is equal to the exact range of $f(x)$. In this simple example Hansen form computations do not introduce any overestimation.

3.2 Remarks on basic operations

To evaluate a formula, we must replace each of its elementary operations $z \longleftarrow f(x, y)$ for real numbers by an equivalent operation $\hat{z} \longleftarrow \hat{f}(\hat{x}, \hat{y})$ for generalized intervals, where $\hat{f}$ is a procedure that computes a generalized interval form for $z \longleftarrow f(x, y)$ that is consistent with $\hat{x}, \hat{y}$. We have $z \longleftarrow f(x, y) = f(x_0 + x_1\varepsilon_1 + \ldots + x_n\varepsilon_n, y_0 + y_1\varepsilon_1 + \ldots + y_n\varepsilon_n) = \hat{f}(\hat{x}, \hat{y}) = f^*(\varepsilon_1, \ldots, \varepsilon_n)$.

The challenge now is to replace $f^*(\varepsilon_1, \ldots, \varepsilon_n)$ by a generalized interval form $z = z_0 + z_1\varepsilon_1 + \ldots + z_n\varepsilon_n$ that preserves as much information as possible about the correlations between x, y, and z. Concrete formulae for the basic oprations $+, -, *, /$ can be found in [2]. We only note that for Hansen forms with point coefficients $\hat{x} + \hat{y}, \hat{x} - \hat{y}, \alpha \cdot \hat{x}, \alpha + \hat{x}$ are error-free (in these cases the operations have to be applied to the coefficients individually). Multiplication and division introduce some overestimation because nonlinear terms have to be linearized in some way to get a valid Hansen form for the (true) result.

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3.3 Elementary functions like exp, log, etc.

We will demonstrate two different approaches for $\sqrt{x}$. The first method (proposed by Hansen) is based on the Taylor expansion:

$$f(\hat{x}) = f(x_0) + \frac{\partial f(x_0 + \vartheta(x_1\varepsilon_1 + \ldots + x_n\varepsilon_n))}{\partial \hat{x}}(x_1\varepsilon_1 + \ldots + x_n\varepsilon_n) \text{ with } 0 \leq \vartheta \leq 1,$$

yielding

$$\sqrt{\hat{x}} = \sqrt{x_0} + \frac{x_1\varepsilon_1 + \ldots + x_n\varepsilon_n}{2\sqrt{x_0 + \vartheta(x_1\varepsilon_1 + \ldots + x_n\varepsilon_n)}} \in \sqrt{x_0} + \frac{x_1\varepsilon_1 + \ldots + x_n\varepsilon_n}{2\sqrt{[\hat{x}]}}.$$

This means that the coefficients of the resulting Hansen form are given by

$$z_0 := \sqrt{x_0}, z_i := \frac{x_i}{2\sqrt{[\hat{x}]}}, i = 1(1)n.$$

The second method is based on a linear best approximation to $\sqrt{\ }$ within the reduced interval $[\hat{x}]$. Using the Maple worksheet [4] (please feel free to contact the author) we find:

With $[a, b] := [\hat{x}] = x_0 + x_1[-c_1, c_1] + \ldots + x_n[-c_n, c_n]$, $\alpha := 1/(\sqrt{a} + \sqrt{b})$, $\beta := 0.125/\alpha + 0.5\alpha\sqrt{a}\sqrt{b}$, $\tau := 0.125\alpha(\sqrt{a} - \sqrt{b})^2$, (ordinary) interval coefficients

$$z_0 \quad := \quad \alpha x_0 + \beta + [-\tau, \tau], z_i := \alpha x_i, i = 1(1)n$$

and $\varepsilon_i \in [-c_i, c_i], i = 1(1)n$ it holds that

$$\sqrt{x_0 + x_1\varepsilon_1 + \ldots + x_n\varepsilon_n} \subseteq z_0 + z_1\varepsilon_1 + \ldots + z_n\varepsilon_n \quad \text{(this is the Hansen form we are looking for).}$$

3.4 Numerical results

$\sqrt{x}$ within $[1, 4](= [\hat{x}])$ using the Taylor ansatz:

$$\hat{x} = x_0 + x_1\varepsilon_1 := \frac{5}{2} + \frac{3}{2}\varepsilon_1, -1 \leq \varepsilon_1 \leq 1 \Longrightarrow \sqrt{\hat{x}} \in \sqrt{x_0} + \frac{x_1}{2\sqrt{[\hat{x}]}}\varepsilon_1.$$

The coefficients of the resulting Hansen form are

$$z_0 := \sqrt{x_0} = \sqrt{\frac{5}{2}}, z_1 := \frac{x_1}{2\sqrt{[\hat{x}]}} = \frac{3}{4\sqrt{[1, 4]}} = [\frac{3}{8}, \frac{3}{4}],$$

yielding the reduced Hansen form $[\hat{z}] = [\sqrt{\frac{5}{2}} - \frac{3}{4}, \sqrt{\frac{5}{2}} + \frac{3}{4}] \subset [0.831, 2.34]$.

$\sqrt{x}$ within $[1, 4]$ using the linear best approximation (again, we use [4] to obtain $\alpha = 1/3, \beta = 17/24, \tau = 1/24$): The coefficients of the resulting Hansen form are now

$$z_0 = [\frac{18}{12}, \frac{19}{12}], z_1 = \frac{1}{2}$$

yielding $[\hat{z}] = [12/12, 25/12] = [1, 2\frac{1}{12}] \subset [1, 2.084]$ which shows that the second method gives a much better enclosure of the true range $[1, 2]$.

Computing $\sqrt{\hat{x}} - \sqrt{\hat{x}}$ with the same data gives in the first case $[-18/12, 18/12]$ and in the second case $[-1/12, 1/12]$. Both results are much better than the interval $[-1, 1]$ resulting from ordinary interval evaluation. Please note that, in both cases, reducing the intermediate Hansen forms leads to ordinary intervals that are worse than the intermediate result $\sqrt{[1, 4]} = [1, 2]$ computed with ordinary interval arithmetic.

3.5 Related topics

For the reader interested in related topics (centered forms, affine arithmetic, zonotopes, Minkowski sums, Taylor models, ...) we list the names of some further authors: Comba, Stolfi, Figueiredo, Iwaarden, Gay, Kühn, Tupper, Kolev, Berz, ...

References

[1] Hassan El-Owny: A generalized interval arithmetic realized in C-XSC. Preprint WRSWT, University of Wuppertal, to appear 2006.
[2] Eldon R. Hansen: Generalized interval arithmetic, in K. L. Nickel (ed), Interval Mathematics, Lecture Notes in Computer Science, Vol. 29, Springer-Verlag, Berlin, 1975.
[3] W. Hofschuster, W. Krämer: C-XSC – a C++ class library for extended scientific computing, in R. Alt et al. (eds), Numerical Software with Result Verification, Springer Lecture Notes in Computer Science, LNCS 2991, 2004.
[4] W. Krämer: Maple worksheet to compute linear best approximations to convex/concave functions. University of Wuppertal, 2006.
[5] J. Stolfi, L. H. de Figueiredo: Self-Validated Numerical Methods and Applications. Available on the net, 1997.