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Visualization of Parametric Solution Sets

Evgenija D. Popova, Walter Krämer

Abstract. We characterize the boundary $\partial\Sigma^p$ of the solution set Σ^p of a parametric linear system $A(p)x = b(p)$ where the elements of the $n \times n$ matrix and the right-hand side vector depend on a number of parameters p varying within interval bounds. The characterization of $\partial\Sigma^p$ is by means of pieces of parametric hypersurfaces, the latter represented by their coordinate functions depending on corresponding subsets of $n-1$ parameters. The presented approach has a direct application for efficient visualization of parametric solution sets by utilizing some plotting functions supported by *Mathematica* and Maple.

Keywords: Linear systems with depended data, interval uncertainties, parametric solution set, visualization.

AMS subject classification (2000): 15A06, 65G99, 65S05, 68U05.

1 Introduction

Consider the linear algebraic system

$$A(p) \cdot x \;=\; b(p), \tag{1}$$

where the elements of the $n \times n$ matrix $A(p)$ and the vector $b(p)$ are either nonlinear functions

$$a_{ij}(p) = a_{ij}(p_1, \ldots, p_k), \qquad b_i(p) \;=\; b_i(p_1, \ldots, p_k), \qquad i, j = 1, \ldots, n, \tag{2}$$

or affine-linear functions

$$\begin{array}{ll} a_{ij}(p) := a_{ij,0} + \sum_{\nu=1}^{k} a_{ij,\nu} p_\nu, & b_i(p) := b_{i,0} + \sum_{\nu=1}^{k} b_{i,\nu} p_\nu, \\ a_{ij,\nu}, b_{i,\nu} \in \mathbb{R}, & \nu = 0, \ldots, k,\; i, j = 1, \ldots, n \end{array} \tag{3}$$

of k parameters. The parameters are considered to be uncertain and varying within given intervals

$$p \;\in\; [p] = ([p_1], \ldots, [p_k])^\top. \tag{4}$$

Such systems are common in many engineering analysis or design problems, models in operational research, linear prediction problems, etc., where there are complicated dependencies between the coefficients of the system [1], [2], [10]. The uncertainties in the model parameters could originate from an inexact knowledge of these parameters, measurement imprecision, or round-off errors. Linear systems with interval input data are applicable also to uncertainty theories which rely on interval arithmetic for computations, such as fuzzy set theory, random set theory, or probability bounds theory.

The set of solutions to (1–4), called parametric solution set, is

$$\Sigma^p = \Sigma\left(A(p), b(p), [p]\right) \ := \ \{x \in \mathbb{R}^n \mid A(p) \cdot x = b(p) \text{ for some } p \in [p]\}. \qquad (5)$$

The well-known non-parametric interval linear system $[A]x = [b]$, which is the most studied in the interval literature, can be considered as a special case of the parametric linear system with $n^2 + n$ independent parameters $a_{ij} \in [a_{ij}]$, $b_i \in [b_i]$, $i, j = 1, \ldots, n$. For a parametric system (1–4), the corresponding non-parametric one with $[A] = ([a_{ij}]) \in \mathbb{IR}^{n\times n}$, $[b] \in \mathbb{IR}^n$ can be obtained as

$$[a_{ij}] = \square\{a_{ij}(p) \mid p \in [p]\}, \quad [b_i] = \square\{b_i(p) \mid p \in [p]\}, \quad i, j = 1, \ldots, n,$$

where $\square$ denotes the interval hull, defined by $\square S \ := \ [\inf S, \ \sup S]$ for a nonempty bounded set $S \subseteq \mathbb{R}^n$. The non-parametric solution set, called also united solution set, is defined as

$$\Sigma\left([A], [b]\right) \ := \ \{x \in \mathbb{R}^n \mid \exists A \in [A], \exists b \in [b], A \cdot x = b\}.$$

The parametric solution set Σ^p is much more complicated than the corresponding non-parametric solution set. For example, Σ^p is generally not convex even in a single orthant. Therefore, it would be interesting and helpful to see how some parametric solution sets look like. The visualization of Σ^p (even only in 2D or 3D) would be helpful not only for graphical illustration but also for exploration of some properties and for comparison of some numerical results.

In a series of papers (see e.g. [3], [4], [5] and the references therein) Alefeld, Kreinovich, and Mayer give various descriptions of the solution sets for systems of interval linear equations with dependent coefficients, paying particular attention to the symmetric solution set. For example, in [4] the parametric solution set is described as a semialgebraic set. The latter is a subset of $\mathbb{R}^n$ which is a finite Boolean combination of sets of the form $\{x \in \mathbb{R}^n : f(x) > 0\}$ and $\{x \in \mathbb{R}^n : g(x) = 0\}$, where f and g are polynomials in $x_1, \ldots, x_n$ over the reals [14]. Unfortunately, most of these descriptions do not contain a constructive process which could be used for drawing the parametric solution set. This gap was fulfilled in [6] where the solution sets of parametric linear systems involving affine-linear dependencies were characterized by systems of inequalities obtained by a Fourier-Motzkin like elimination process. Provided that the proposed Fourier-Motzkin like elimination process is implemented in suitable software tools, this approach can be applied for drawing the parametric solution sets in environments supporting tools for inequalities plotting. The algorithms for plotting inequalities are usually based on cylindrical algebraic decomposition (CAD) [7]. Although CAD is an algorithmic process, it becomes computationally infeasible for complicated inequalities.

Utilizing the set of inequalities by which the famous Oettli-Prager theorem [11] characterizes the non-parametric solution set and the tools for inequalities plotting supported in *Mathematica* [15], corresponding functions for drawing non-parametric solution sets in 2D and 3D were developed and a suitable web interface to these functions was provided [12]. Recently some other Java-based tools for drawing non-parametric solution sets were also reported [9]. In

[12] tools for drawing connected parametric solution sets were also reported. The corresponding visualization function utilizes the available *Mathematica* functions for plotting parametric curves but the quality of the produced solution set image has some deficiencies.

In this paper we present an approach for characterizing the parametric solution set which is alternative to that based on systems of inequalities [6]. Our approach is designed particularly for visualization of the parametric solution set boundary and can be easily implemented in the environments of *Mathematica* [15] and Maple [8] which support functions for plotting parametric curves and surfaces. Section 3 discusses how to utilize the plotting functions in *Mathematica* and Maple for visualization the solution sets of some special cases of parametric linear systems, in particular for visualization the solution sets of 2D linear systems involving nonlinear dependencies. In Section 4 we derive our approach and characterize the boundary $\partial\Sigma^p$ of the solution set Σ^p to a system involving affine-linear dependencies by means of pieces of parametric hypersurfaces, the latter represented by their coordinate functions depending on corresponding sets of $k \leq n-1$ parameters. The numerical examples given throughout the paper demonstrate the discussed visualization approaches and illustrate some properties of the parametric solution sets.

2 Preliminaries

Denote by $\mathbb{R}^n, \mathbb{R}^{n\times m}$ the set of real vectors with n components and the set of real $n\times m$ matrices, respectively. A real compact interval is $[a] = [a^-, a^+] := \{a \in \mathbb{R} \mid a^- \leq a \leq a^+\}$. By $\mathbb{IR}^n, \mathbb{IR}^{n\times m}$ we denote the sets of interval n-vectors and interval $n \times m$ matrices, respectively. The end-point functionals $(\cdot)^-$, $(\cdot)^+$ are applied to interval vectors and matrices componentwise.

Theorem 2.1. *If $A(p)$ is nonsingular for all $p \in [p]$ then Σ^p is compact and connected.*

Proof. Since $A(p)$ is non-singular for every $p \in [p]$, $A^{-1}(p)$ exists for $p \in [p]$ and $x(p) := A^{-1}(p) \cdot b(p)$ is a function of k variables $p \in \mathbb{R}^k$ which is continuous. Since $[p_\nu]$, $\nu = 1, \ldots, k$ are connected and compact, the same holds for the image Σ^p of $x(p)$. □

An obvious set-theoretical description of the parametric solution set is given by the following

Theorem 2.2.
$$\Sigma^p\left(A(p), b(p), [p]\right) := \bigcup_{\tilde{p}\in[p]} \left\{x \in \mathbb{R}^n \mid A(\tilde{p}) \cdot x = b(\tilde{p})\right\}.$$

In particular, if $A(p)$ is square nonsingular for all $p \in [p]$ then

$$\Sigma^p\left(A(p), b(p), [p]\right) := \bigcup_{\tilde{p}\in[p]} \{x(\tilde{p}) = A^{-1}(\tilde{p}) \cdot b(\tilde{p})\}.$$

Denote by $\partial\Sigma^p$ the boundary of the parametric solution set Σ^p. In what follows we will characterize $\partial\Sigma^p$ by pieces of parametric hypersurfaces. Hypersurface is an $(n-1)$-dimensional surface embedded in n-dimensional space. A hypersurface is therefore the set of solutions to a single equation

$$f(x_1, \ldots, x_n) = 0.$$

Definition 1. *A hypersurface in n-dimensional space is called* parametric *if it is defined by n coordinate functions*

$$x_i = x_i(p_1, \ldots, p_k), \qquad i = 1, \ldots, n$$

depending on $k \leq n-1$ parameters.

Parametric hypersurface (PHS) is then a hypersurface parameterized by its coordinates. The parametric hypersurfaces will be denoted by n-dimensional vectors $x(p)$ where p are k-dimensional parameter vectors and $k \leq n-1$. A parametric hypersurface is called degenerated if it is defined by $k < n-1$ parameters. In a 3D space degenerated parametric hypersurfaces are curves and points.

Particular pieces (parts) of a parametric hypersurface will be obtained for specified ranges of the parameters $p_j^- \leq p_j \leq p_j^+$, $j = 1, \ldots, k$. Therefore, particular parts of a parametric hypersurface will be denoted by $x(p)\big|_{p\in[p]}$ and will be also called restricted PHS-s.

3 Special Cases

In this section we demonstrate how to utilize the plotting functions in *Mathematica* and Maple for visualizing the solution set of some particular cases of 2D and 3D parametric linear systems.

Consider the parametric linear system (1–4). If $A(p)$ is nonsingular for all $p \in [p]$ and the number of parameters is $k \leq n-1$, then $x(p) = A^{-1}(p) \cdot b(p)$ defines a PHS (curve for $k = 1$) in $\mathbb{R}^n$ which fully characterizes the parametric solution set (5), respectively its boundary.

For $n = 2$, $k = n-1$, e.g., the parametric solution set is a parametric curve which x, y coordinates are functions of one uncertain parameter $p \in [p]$

$$\begin{aligned} x &= x_1(p) = \{A^{-1}(p) \cdot b(p)\}_1 \\ y &= x_2(p) = \{A^{-1}(p) \cdot b(p)\}_2. \end{aligned}$$

Many computer algebra systems, e.g. *Mathematica* [15] and Maple [8] have built-in functions for drawing 2D and 3D parametric plots. In 2D, *Mathematica* and Maple effectively generate a sequence of points by varying the parameter p, then form a curve by joining these points.

Example 3.1. *($n = 2$, $k = 1$) Let*

$$A(p) = \begin{pmatrix} 3p & 1 \\ -2 & 3p-1 \end{pmatrix}, \quad b(p) = \begin{pmatrix} 2p \\ p \end{pmatrix}, \quad p \in [0, 1].$$

For these data, the parametric solution set is that part of the parametric curve

$$x(p) = A^{-1}(p) \cdot b(p) = \begin{pmatrix} -3p + 6p^2 \\ 4p + 3p^2 \end{pmatrix} / (2 - 3p + 9p^2)$$

which is obtained for $0 \leq p \leq 1$.

The following Maple code visualizes on Fig. 1 the parametric solution set as a part of the corresponding parametric curve.

```
> A:=[[3*p, 1], [-2, 3*p-1]]:
> b:=[2*p, p]:
> x:=linalg[linsolve](A, b):
> plot([x[1], x[2], p=0..1]);
```

The same figure can be generated in *Mathematica* by using its kernel function `ParametricPlot`.

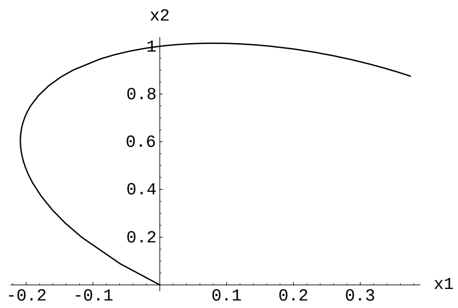

Figure 1: The parametric solution set of the system from Example 3.1.

Mathematica and Maple can be also used for plotting either one- or two-parameter sets of points in a 3D space. The built-in functions `plot3D` in Maple or `ParametricPlot3D` in *Mathematica* create three-dimensional space curves and surfaces, parameterized by one or two coordinates respectively. In *Mathematica* the surface is formed from a collection of quadrilaterals. The corners of the quadrilaterals have coordinates corresponding to the values of $z = z(u, v)$ when the parameters u and v take on values in a regular grid. The option `PlotPoints` of the *Mathematica* function `ParametricPlot3D` allows a user to specify the number of sample points used. The *Mathematica* package `Graphics‘ParametricPlot3D‘` extends the kernel function `ParametricPlot3D` by providing an alternative to the `PlotPoints` option where the sampling may be specified by giving a step size in each coordinate. The package also introduces `PointParametricPlot3D` function for plotting either one- or two-parameter sets of points in space.

Example 3.2. *($n = 3$, $k = 2$) Let*

$$A(p) = \begin{pmatrix} 1 & p & q \\ p & 2 & p \\ q & p & 3 \end{pmatrix}, \quad b(p) = \begin{pmatrix} 1 \\ p^2 \\ q^2 \end{pmatrix}, \quad \begin{matrix} p \in [0, 1] \\ q \in [0, 0.9]. \end{matrix}$$

The following *Mathematica* code generates the snail represented on Fig. 2A.

```
In[1]:= m = {{1,p,q}, {p,2,p}, {q,p,3}};
        b = {1, p^2, q^2};
        s = LinearSolve[m, b];
        ParametricPlot3D[s, {p,0,1}, {q,0,0.9}];
```

Example 3.3. *($n = 3$, $k = 1$) For*

$$A(p) = \begin{pmatrix} 1 & p & p \\ p & 2 & p \\ p & p & 3 \end{pmatrix}, \quad b(p) = \begin{pmatrix} 1 \\ p^2 \\ p^2 \end{pmatrix}, \quad p \in [-0.2, 0.1],$$

the one-parameter solution set is a 3D curve represented on Fig. 2B.

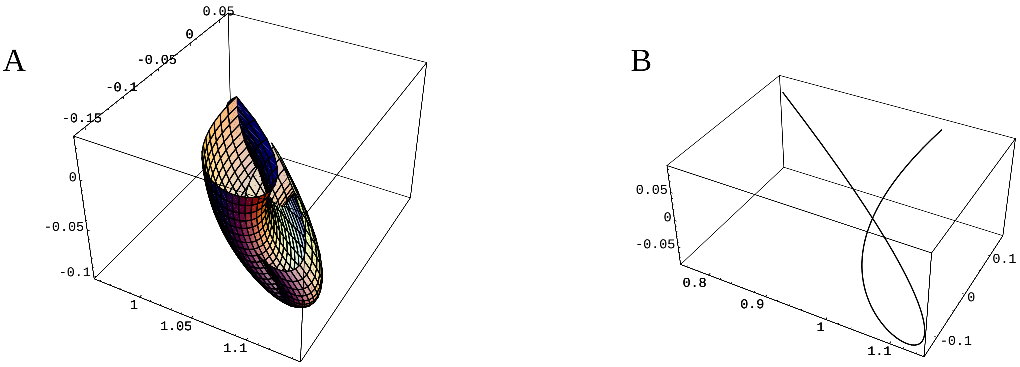

Figure 2: One- (B) and two-parameter (A) solution sets corresponding to the 3D linear systems from Examples 3.3 and 3.2, respectively.

For 2D systems involving more than one parameter and having nonsingular matrices for all values of the parameters, we can get a good impression of the parametric solution set by drawing a set of one-parameter curves obtained for the parameters after the first one taking on values in a grid of points within the parameter intervals. This approach is implemented in a *Mathematica* function `ParametricSSet` which is part of the package `IntervalComputations‘SolutionSets‘`. The following Example illustrates the usage of this function.

Example 3.4. *($n = 2$, $k = 2$) Let*

$$A(p) = \begin{pmatrix} p & q-1 \\ q & p \end{pmatrix}, \quad b(p) = \begin{pmatrix} -q+1/3 \\ q \end{pmatrix}, \quad p \in [-2, -1], \quad q \in [3, 5].$$

The following *Mathematica* code first loads the package, then defines the arguments and calls the visualization function.

```
In[5]:= << IntervalComputations‘SolutionSets‘
In[6]:= m = {{p, q-1}, {q, p}}; b={-q+1/3, q};
        tr = {p->Interval[{-2, -1}],  q->Interval[{3, 5}]};
        ParametricSSet[m, b, tr];
```

The function `ParametricSSet[mat, vec, tr]` has three obligatory arguments: `mat` is the parametric matrix of the system, `vec` is the right-hand side vector, and `tr` is a list of *Mathematica* rules specifying the parameters and their interval values. By default, `ParametricSSet` draws a set of one-parameter curves taken on an uniform mesh in the parameter intervals after the first one. The default value for the mesh step size is 1% of the interval width. Fig. 3A represents the generated graphics image.

By an optional argument `StepSize`, one can specify particular values for the step size of all parameter values after the first one. The graphics on Fig. 3B is drawn by specifying `StepSize->{0.1}` regarding the second parameter `q`. Although the set of one-parameter curves on Fig. 3B is more shaggy than that on Fig. 3A, the shape of the parametric solution set is still well visible.

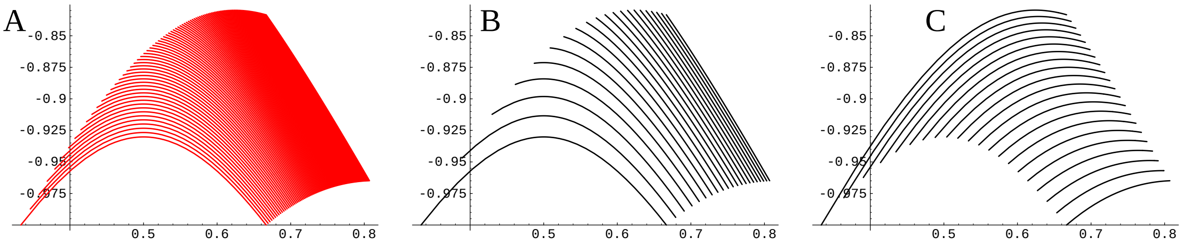

Figure 3: The parametric solution set of Example 3.4 built by 1-parameter curves drawn on an uniform mesh for the parameters after the first one. A: the default value for the `StepSize` which is 1% of the interval width, B: `StepSize`$= 0.1$ for the parameter q, C: `StepSize`$=$ 0.05 for the parameter p.

```
In[9]:= tr = {q->Interval[{3, 5}],  p->Interval[{-2, -1}]};
        ParametricSSet[m, b, tr, StepSize->{0.05}];
```

One can change the order in which the parameters are enlisted in the third argument of the function `ParametricSSet`, and this way to represent the parametric solution set by another set of one-parameter curves. The result of the execution of the above code can be seen on Fig. 3C.

Based on the representation given by Theorem 2.2, a 2D parametric solution set can be visualized by a set of one-parameter curves drawn in a mesh of values for the parameters after the first one. This is the only approach for visualization the solution set of a 2D linear system involving nonlinear dependencies. For example, Fig. 4 shows the graphics produced by the function `ParametricSSet` for the system

$$A(p) = \begin{pmatrix} a*b & 1 \\ 3 & b^2 \end{pmatrix}, \quad b(p) = \begin{pmatrix} a/b \\ \cos(a) \end{pmatrix}, \quad a \in [1, 9], \quad b \in [2, 3].$$

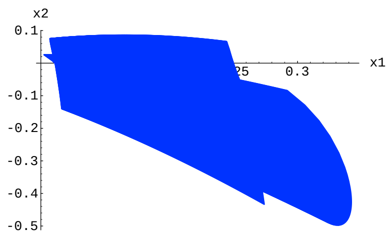

Figure 4: The parametric solution set of a 2D linear system involving nonlinear dependencies.

Unfortunately, the above approach is not applicable for drawing disconnected 2D solution sets and in the case of 3D systems. An important disadvantage is the big size of the graphics image which increases with decreasing the mesh step size. That is why, in the next section we derive another approach for visualization only the boundary of the parametric solution set. To this end, the parametric solution set is characterized by parametric hypersurfaces.

4 Main Results

In this section we consider parametric linear systems involving affine-linear dependencies. The following definitions and notations will be used. Let $\mathcal{K} = \{1, \dots, k\}$. For $n \leq k$, define $Q(n-1, k)$ as the set of all possible subsets of $\mathcal{K}$ containing $n-1$ elements

$$Q(n-1,k) := \{q = \{i_1, \dots, i_{n-1}\} \mid q \subset \mathcal{K}, \mathrm{Card}(q) = n-1\}\,.$$

For $k < n$, $Q(n-1,k) := \{q \mid q = \mathcal{K}\}$, that is $Q(n-1,k)$ consists of only one set which is the set $\mathcal{K}$ itself. For $n \leq k$ the dimension of $Q(n-1,k)$ is $\mathrm{Card}(Q(n-1,k)) = \binom{k}{n-1} = \frac{k!}{(k-n+1)!(n-1)!}$. For a set of indexes $q = \{i_1, \dots, i_{n-1}\} \in Q(n-1,k)$, the vector $(p_{i_1}, \dots, p_{i_{n-1}})$ will be denoted by p_q.

For a vector $p = (p_1, \dots, p_k) \in \mathbb{R}^k$ and $q \in Q(n-1,k)$, define $\tilde{q} = \mathcal{K} \setminus q$ and two vectors $p_q \in \mathbb{R}^{n-1}$, $p_{\tilde{q}} \in \mathbb{R}^{k-n+1}$ by

$$\begin{aligned} p_q &:= (p_{i_1}, \dots, p_{i_{n-1}}), \\ p_{\tilde{q}} &:= (p_{i_n}, \dots, p_{i_k}). \end{aligned}$$

For $n \leq k$, the vectors p_q and $p_{\tilde{q}}$ split the original vector p into two nonintersecting subvectors defined by the set of indexes q.

Denote by $U(k-n+1) := \left\{u \in \mathbb{R}^{k-n+1} \mid |u| = (1, \dots, 1)^\top\right\}$ the set of all $(k-n+1)$-dimensional sign vectors. For $[a] = [a^-, a^+] \in \mathbb{IR}^{k-n+1}$ and $u \in U(k-n+1)$,

$$\{a^u\}_i := \begin{cases} a_i^- & \text{if } u_i = -1 \\ a_i^+ & \text{if } u_i = 1 \end{cases}, \quad i = 1, \dots, k-n+1.$$

Thus for an interval vector $[a]$, a^u denotes a real vector whose elements are corresponding interval end-points. The dimension of the set $U(k-n+1)$ is $\mathrm{Card}(U(k-n+1)) = 2^{k-n+1}$.

Our first theorem characterizes exactly the boundary $\partial\Sigma^p$ by parts of parametric hypersurfaces in the special case when the number of the parameters is less than or equal to the dimension of the system.

Theorem 4.1. *If $A(p)$ is nonsingular for all $p \in [p]$ and $k \leq n$, then*

$$\partial\Sigma^p = \bigcup_{q \in Q(n-1,k)} \left\{ x(p_q, p_{\tilde{q}}^-)\Big|_{p_q \in [p_q]},\ \ x(p_q, p_{\tilde{q}}^+)\Big|_{p_q \in [p_q]} \right\}.$$

Proof. By definition, every tuple of $n-1$ parameters determines one PHS in the nD-space. So, if $k \leq n-1$, $\partial\Sigma^p$ consists of a piece of exactly one PHS, possibly degenerated if $k < n-1$, which is the parametric solution set itself

$$\partial\Sigma^p = x(p)\Big|_{p \in [p]} = \Sigma^p,$$

where $x(p) = A^{-1}(p) \cdot b(p)$ and $[p] \in \mathbb{IR}^k$, $k \leq n-1$.

Let $k = n$. For a fixed $q \in Q(n-1,k)$, $p_{\tilde{q}}$ is a one-component vector. Define the set of restricted PHS-s

$$\Sigma^{p_q, p_{\tilde{q}}} := \bigcup_{t \in [p_{\tilde{q}}]} x(p_q, t)\Big|_{p_q \in [p_q]}.$$

It is evident that $\Sigma^{p_q, p_{\tilde{q}}} \equiv \Sigma^p$. By Tarski's theorem [13], Σ^p, respectively $\Sigma^{p_q, p_{\tilde{q}}}$, is a semialgebraic set, that is a subset of $\mathbb{R}^n$ which is a finite Boolean combination of sets of the form $\{x \in \mathbb{R}^n : f(x) > 0\}$ and $\{x \in \mathbb{R}^n : g(x) = 0\}$, where f and g are polynomials in $x_1, \ldots, x_n$ over the reals [14]. From the theorem, defining a Fourier-Motzkin-like elimination process for parameter-dependent linear systems [6], it follows that eliminating the parameter $p_{\tilde{q}}$ we will obtain an equivalent representation of Σ^p, resp. $\Sigma^{p_q, p_{\tilde{q}}}$, as a semialgebraic set where the involved polynomials contain the end-points $p_{\tilde{q}}^-$ and $p_{\tilde{q}}^+$ instead of the parameter $p_{\tilde{q}}$. Since some of these polynomials define also the boundary of the set, the PHS-s $x(p_q, p_{\tilde{q}}^-)$ and $x(p_q, p_{\tilde{q}}^+)$ are boundaries of the set $\Sigma^{p_q, p_{\tilde{q}}}$. Due to $\Sigma^{p_q, p_{\tilde{q}}} \equiv \Sigma^p$, these end-point PHS-s are boundaries of the parametric solution set Σ^p, too. For $q_1, q_2 \in Q(n-1,k)$, $q_1 \neq q_2$ and fixed $t_1 \in [p_{\tilde{q}_1}]$, $t_2 \in [p_{\tilde{q}_2}]$, in general $x(p_{q_1}, t_1) \neq x(p_{q_2}, t_2)$, but $\Sigma^{p_{q_1}, p_{\tilde{q}_1}} \equiv \Sigma^{p_{q_2}, p_{\tilde{q}_2}}$. Thus, when q traces the set $Q(n-1,k)$, $p_{\tilde{q}}$ traces all the parameters, respectively all the boundary parametric hypersurfaces, which proves the theorem. $\square$

Figures 3B and 3C illustrate the above Theorem.

Remark 4.1. If $k = n$ and for some $q_1 \in Q(n-1,k)$, some $\lambda \in \{-,+\}$, $x(p_{q_1}, p_{\tilde{q}_1}^\lambda)$ is degenerated, than there exist $q_2 \in Q(n-1,k)$, $\mu \in \{-,+\}$ such that $x(p_{q_2}, p_{\tilde{q}_2}^\mu)$ is non-degenerated and $x(p_{q_1}, p_{\tilde{q}_1}^\lambda) \in x(p_{q_2}, p_{\tilde{q}_2}^\mu)$.

Example 4.1. *Consider the parametric linear system*

$$\begin{pmatrix} p_1 & p_2 - 1 \\ p_2 & p_1 \end{pmatrix} \cdot x = \begin{pmatrix} -p_2 + 1/3 \\ p_2 \end{pmatrix}, \qquad p_1 \in [-2,-1], \quad p_2 \in [3,5].$$

For $p_1 \in [-2,-1]$, $p_2 \in [3,5]$, $A(p)$ *is nonsingular and*

$$A^{-1}(p) = \begin{pmatrix} p_1 & 1 - p_2 \\ -p_2 & p_1 \end{pmatrix} / (p_1^2 + p_2 - p_2^2).$$

We have $Q(n-1,k) = Q(1,2) = \{\{1\},\{2\}\}$. *For* $q = \{1\}$,

$$x(p_1, p_2^-) = \begin{pmatrix} -2(9 + 4p_1)/3 \\ 8 + 3p_1 \end{pmatrix} / (-6 + p_1^2),$$

$$x(p_1, p_2^+) = \begin{pmatrix} -2(30 + 7p_1) \\ 5(14 + 3p_1) \end{pmatrix} / (-60 + 3p_1^2).$$

For $q = \{2\}$,

$$x(p_2, p_1^-) = \begin{pmatrix} 2 - 9p_2 + 3p_2^2 \\ -p_2(8 - 7 + 3p_2) \end{pmatrix} / (-12 - 3p_2 + 3p_2^2),$$

$$x(p_2, p_1^+) = \begin{pmatrix} 1 - 6p_2 + 3p_2^2 \\ -p_2(-4 + 3p_2) \end{pmatrix} / (-3 - 3p_2 + 3p_2^2).$$

The corresponding parts of the above boundary curves are drawn on Figure 5. Compare Fig. 5 and Fig. 3.

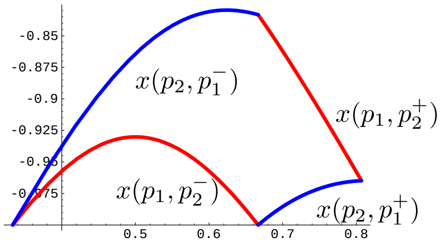

Figure 5: Boundary curves for the parametric solution set from Example 4.1.

Let $k > n$ and $q \in Q(n-1, k)$ be fixed. Since $k > n$, the corresponding "free" parameters $p_{\tilde{q}} \in \mathbb{R}^{k-n+1}$ are more than one. Let $k - n + 1 = 2$. For fixed $t_1 \in \{[p_{\tilde{q}}]\}_1$, define a set of restricted PHS-s

$$\Sigma^{p_q, t_1} \;:=\; \{\; x(p_q, t_1, t_2)\Big|_{p_q \in [p_q]} \mid t_2 \in \{[p_{\tilde{q}}]\}_2 \}.$$

Then the whole parametric solution set Σ^p can be considered as a family Σ^{p_q} of sets defined by one free parameter

$$\Sigma^{p_q} \;:=\; \bigcup_{t_1 \in \{[p_{\tilde{q}}]\}_1} \Sigma^{p_q, t_1}. \tag{6}$$

By Theorem 4.1, $\partial\Sigma^{p_q, t_1} = \{\; x(p_q, t_1, t_2^-)\Big|_{p_q \in [p_q]}, \;\; x(p_q, t_1, t_2^+)\Big|_{p_q \in [p_q]}\}$, where $t_2 = \{[p_{\tilde{q}}]\}_2$. Thus, for $k - n + 1 = 2$ we obtain

$$\partial\Sigma^{p_q} \;\subseteq\; \partial\Sigma^{p_q, t_1^\lambda} \;=\; \bigcup_{u \in U(k-n+1)} x(p_q,\; p_{\tilde{q}}^u)\Big|_{p_q \in [p_q]},$$

where $t_1 \in \{[p_{\tilde{q}}]\}_1$ and $\lambda \in \{+, -\}$. By induction on the number of free parameters and by varying $q \in Q(n-1, k)$, we prove the following theorem which generalizes Theorem 4.1 for $k > n$.

Theorem 4.2. *If $A(p)$ is nonsingular for all $p \in [p]$, then*

$$\partial\Sigma^p \;\subseteq\; \bigcup_{q \in Q(n-1,k)} \; \bigcup_{u \in U(k-n+1)} x(p_q,\; p_{\tilde{q}}^u)\Big|_{p_q \in [p_q]} \;\subseteq\; \Sigma^p. \tag{7}$$

The set in the middle of the relation (7) will be called set of end-point parametric hypersurfaces.

Example 4.2. *Consider the parametric linear system*

$$\begin{pmatrix} 1 & p_1 \\ p_1 & p_2 \end{pmatrix} \cdot x = \begin{pmatrix} p_3 \\ p_3 \end{pmatrix}, \qquad \begin{array}{l} p_1 \in [0, 1] \\ p_2 \in [-4, -1] \\ p_3 \in [0, 2]. \end{array}$$

For the parameters varying within their intervals, $A(p)$ is nonsingular and

$$A^{-1}(p) = \begin{pmatrix} p_2 & -p_1 \\ -p_1 & 1 \end{pmatrix} / (-p_1^2 + p_2).$$

$Q(n-1,k) = \{\{1\},\{2\},\{3\}\}$, then we obtain the following set of PHS-s. For $q = \{1\}$ the corresponding end-point PHS-s are

$$\begin{aligned}
&x(p_1, p_2^-, p_3^-) = x(p_1, p_2^+, p_3^-) = (0,0)^\top,\\
&x(p_1, p_2^-, p_3^+) = \begin{pmatrix} 8+2p_1 \\ -2+2p_1 \end{pmatrix} / (4+p_1^2), \qquad x(p_1, p_2^+, p_3^+) = \begin{pmatrix} 2+2p_1 \\ -2+2p_1 \end{pmatrix} / (1+p_1^2).
\end{aligned}$$

Parts of these PHS-s corresponding to $p_1 \in [0,1]$ are presented on Figure 6A. For $q = \{2\}$ the corresponding end-point PHS-s are

$$\begin{aligned}
&x(p_2, p_1^-, p_3^-) = x(p_2, p_1^+, p_3^-) = (0,0)^\top,\\
&x(p_2, p_1^-, p_3^+) = (2, 2/p_2)^\top, \quad x(p_2, p_1^+, p_3^+) = (2,0)^\top.
\end{aligned}$$

Those parts of the above PHS-s corresponding to $p_2 \in [-4,-1]$ are presented on Figure 6B.

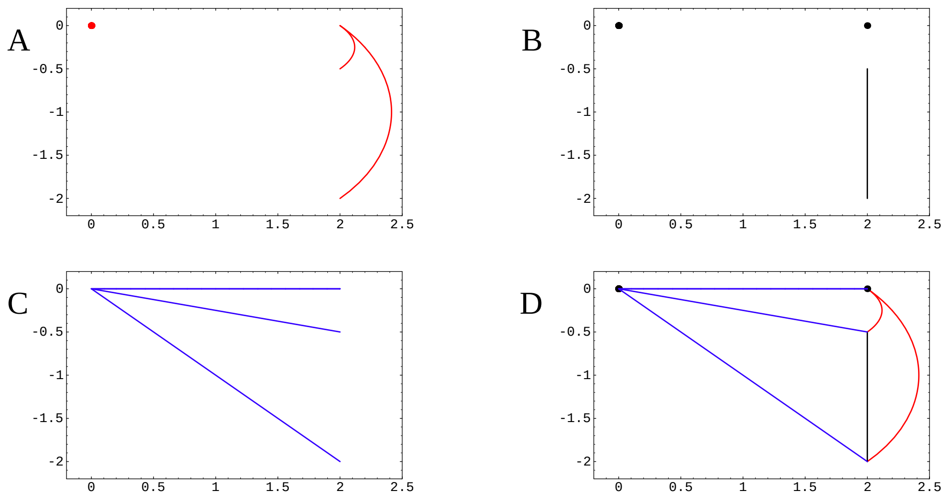

Figure 6: Parts of the end-point parametric hypersurfaces for the system from Example 4.2. A: $x(p_1, p_1^u)\big|_{p_1\in[0,1]}$, B: $x(p_2, p_2^u)\big|_{p_2\in[-4,-1]}$, C: $x(p_3, p_3^u)\big|_{p_3\in[0,2]}$, where $u \in U(2)$. D: The parametric solution set represented by the set of all end-point parametric hypersurfaces for the system from Example 4.2.

For $q = \{3\}$ the corresponding end-point PHS-s are

$$\begin{aligned}
&x(p_3, p_1^-, p_2^-) = (p_3, -p_3/4)^\top, \quad x(p_3, p_1^+, p_2^-) = x(p_3, p_1^-, p_2^+) = (p_3, 0)^\top,\\
&x(p_3, p_1^-, p_2^+) = (p_3, -p_3)^\top.
\end{aligned}$$

Those parts of the above PHS-s corresponding to $p_3 \in [0,2]$ are presented on Figure 6C. The set of all end-point PHS-s restricted to the ranges of the corresponding parameters is presented on Figure 6D.

The set of all end-point PHS-s contains exceed parametric hypersurfaces. The dimension of this set is also quite big, $\mathrm{Card}\Big(\bigcup_{q\in Q}\ \bigcup_{u\in U}\ x(p_q,\ p_{\tilde{q}}^{u})\Big)\ =\ \binom{k}{n-1}2^{k-n+1}$, growing with the dimension of the system and the number of free parameters. In the 3D case drawing exceed parametric surfaces will make the plotting function take much longer to render the surface. Therefore, we need a mechanism for filtering only the boundary PHS-s from the set of all end-point parametric hypersurfaces. Since Remark 4.1 remains valid also for $k > n$, all degenerated end-point PHS-s may be eliminated from the set of end-point PHS-s. One criterion for eliminating particular end-point PHS-s is given by the following.

Theorem 4.3. *Let $A(p)$ be nonsingular for all $p \in [p]$ and $k > n$. For fixed $q \in Q(n-1,k)$ and $\lambda \in U(k-n+1)$, the corresponding restricted PHS $x\left(p_q,\ p_{\tilde{q}}^{\lambda}\right)\big|_{p_q\in[p_q]} \notin \partial\Sigma^p$ if there exist another piece of PHS, defined by $r \in Q(n-1,k)$ and $\mu \in U(k-n+1)$, such that*

$$\square\ x(p_q, p_{\tilde{q}}^{\lambda})\Big|_{p_q\in[p_q]}\ \subset\ \square\ x(p_r, p_{\tilde{r}}^{\mu})\Big|_{p_r\in[p_r]}.$$

Example 4.3. *Consider the parametric linear system*

$$\begin{pmatrix}2p_1 & -p_2\\ p_2 & 2p_1\end{pmatrix}\cdot x = \begin{pmatrix}p_3\\ p_3\end{pmatrix}, \qquad \begin{array}{l} p_1 \in [1,2]\\ p_2 \in [-1,2]\\ p_3 \in [-2,2].\end{array}$$

The matrix is nonsingular for all values of the parameters within their intervals and

$$A^{-1}(p) = \begin{pmatrix}2p_1 & p_2\\ -p_2 & 2p_1\end{pmatrix}/(4p_1^2+p_2^2).$$

$Q(n-1,k) = \{\{1\},\{2\},\{3\}\}$, then we obtain the following set of end-point PHS-s. For $q = \{1\}$

$$\begin{aligned} x(p_1,p_2^-,p_3^-) &= -x(p_1,p_2^-,p_3^+) = \begin{pmatrix}2-4p_1\\ -2-4p_1\end{pmatrix}/(1+4p_1^2),\\ x(p_1,p_2^+,p_3^-) &= -x(p_1,p_2^+,p_3^+) = \begin{pmatrix}-1-p_1\\ 1-p_1\end{pmatrix}/(1+p_1^2).\end{aligned}$$

The corresponding pieces of these PHS-s are presented on Figure 7A. For their hulls we have

$$\square\ x(p_1,p_2^-,p_3^-)\Big|_{p_1\in[p_1]} = \begin{pmatrix}[\frac{-\sqrt{2}}{2+\sqrt{2}},\frac{-6}{17}]\\ [\frac{-6}{5},\frac{-10}{17}]\end{pmatrix}, \square\ x(p_1,p_2^+,p_3^-)\Big|_{p_1\in[p_1]} = \begin{pmatrix}[-1,-3/5]\\ [-1/5,0]\end{pmatrix}.$$

For $q = \{1\}$ no one of the restricted PHS-s satisfies Theorem 4.3.
For $q = \{2\}$,

$$\begin{aligned} x(p_1^-,p_2,p_3^-) &= -x(p_1^-,p_2,p_3^+) = \begin{pmatrix}-4-2p_2\\ -4+2p_2\end{pmatrix}/(4+p_2^2),\\ x(p_1^+,p_2,p_3^-) &= -x(p_1^+,p_2,p_3^+) = \begin{pmatrix}-8-2p_2\\ -8+2p_2\end{pmatrix}/(16+p_2^2).\end{aligned}$$

Figure 7: Restricted end-point PHS-s for the system from Example 4.3 and $q = \{1\}$ (A), $q = \{2\}$ (B).

The corresponding pieces of these PHS-s are presented on Figure 7B. For their hulls we have

$$\begin{aligned}
\square\; x(p_1^-, p_2, p_3^-)\Big|_{p_2\in[p_2]} &= \begin{pmatrix} [\frac{1}{\sqrt{2}(-2+\sqrt{2})},\ \frac{-2}{5}] \\ [\frac{1}{\sqrt{2}(-2+\sqrt{2})},\ 0] \end{pmatrix}, \\
\square\; x(p_1^+, p_2, p_3^-)\Big|_{p_2\in[p_2]} &= \begin{pmatrix} [\frac{1}{4-4\sqrt{2}},\ \frac{-6}{17}] \\ [\frac{-10}{17},\ \frac{-1}{5}] \end{pmatrix}.
\end{aligned}$$

Since $\square\; x(p_1^+, p_2, p_3^-)\big|_{p_2\in[p_2]} \subset \square\; x(p_1^-, p_2, p_3^-)\big|_{p_2\in[p_2]}$, by Theorem 4.3, $x(p_1^+, p_2, p_3^-)\big|_{p_2\in[p_2]}$ is not a boundary curve. Analogously $x(p_1^-, p_2, p_3^+)\big|_{p_2\in[p_2]}$ is not a boundary curve. Both curves are represented on Figure 7B as dashed curves. For $q = \{3\}$,

$$\begin{aligned}
x(p_1^-, p_2^-, p_3) &= \begin{pmatrix} p_3/5 \\ 3p_3/5 \end{pmatrix}, & x(p_1^+, p_2^-, p_3) &= \begin{pmatrix} 3p_3/17 \\ 5p_3/17 \end{pmatrix} \\
x(p_1^-, p_2^+, p_3) &= \begin{pmatrix} p_3/2 \\ 0 \end{pmatrix}, & x(p_1^+, p_2^+, p_3) &= \begin{pmatrix} 3p_3/10 \\ p_3/10 \end{pmatrix}.
\end{aligned}$$

The corresponding pieces of these PHS-s are presented on Figure 8A and their hulls are

$$\begin{aligned}
\square\; x(p_1^-, p_2^-, p_3)\Big|_{p_3\in[p_3]} &= \begin{pmatrix} [\frac{-2}{5},\ \frac{2}{5}] \\ [\frac{-6}{5},\ \frac{6}{5}] \end{pmatrix}, & \square\; x(p_1^+, p_2^-, p_3)\Big|_{p_3\in[p_3]} &= \begin{pmatrix} [\frac{-6}{17},\ \frac{6}{17}] \\ [\frac{-10}{17},\ \frac{10}{17}] \end{pmatrix} \\
\square\; x(p_1^-, p_2^+, p_3)\Big|_{p_3\in[p_3]} &= \begin{pmatrix} [-1,\ 1] \\ [0,\ 0] \end{pmatrix}, & \square\; x(p_1^+, p_2^+, p_3)\Big|_{p_3\in[p_3]} &= \begin{pmatrix} [\frac{-3}{5},\ \frac{3}{5}] \\ [\frac{-1}{5},\ \frac{1}{5}] \end{pmatrix}.
\end{aligned}$$

Since $\square\; x(p_1^+, p_2^-, p_3)\big|_{p_3\in[p_3]} \subset \square\; x(p_1^-, p_2^-, p_3)\big|_{p_3\in[p_3]}$, by Theorem 4.3, $\square\; x(p_1^+, p_2^-, p_3)\big|_{p_3\in[p_3]}$ is not a boundary curve. Since $\square\; x(p_1, p_2^+, p_3^-)\big|_{p_1\in[p_1]} \subset \square\; x(p_1^-, p_2, p_3^-)\big|_{p_2\in[p_2]}$, $\square\; x(p_1, p_2^+, p_3^+)\big|_{p_1\in[p_1]} \subset \square\; x(p_1^-, p_2, p_3^+)\big|_{p_2\in[p_2]}$, then by Theorem 4.3 $\square\; x(p_1, p_2^+, p_3^-)\big|_{p_1\in[p_1]}$ and $\square\; x(p_1, p_2^+, p_3^+)\big|_{p_1\in[p_1]}$ are not boundary curves, too. The set of restricted end-point PHS-s that remains after the application of Theorem 4.3 is drawn on Figure 8B. Obviously, there are end-point PHS-s that are not boundaries but cannot be eliminated by the above criterion. For producing a best looking graphics the exceed end-point PHS-s can be eliminated manually by enumerating the elements of the set of end-point parametric hypersurfaces .

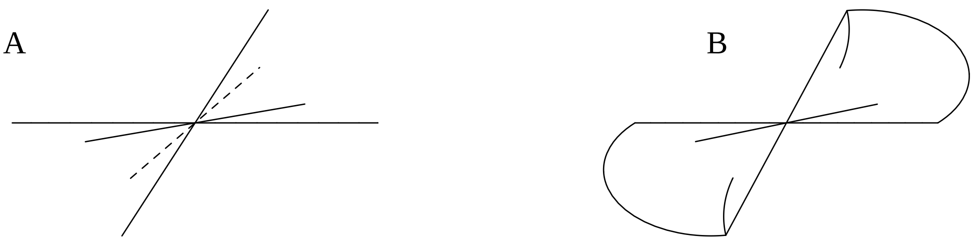

Figure 8: For the system from Example 4.3, A: end-point PHS-s for $q = \{3\}$; B: the set of end-point PHS-s for $q = \{1\}, \{2\}, \{3\}$ after the application of Theorem 4.3.

5 Conclusion

We characterized the boundary of a parametric solution set by parts of parametric hypersurfaces. In view that some environments like *Mathematica* and Maple support tools for drawing parametric hypersurfaces, the presented approach is much more straightforward for visualization of parametric solution sets than the approach based on a combination of Fourier-Motzkin like elimination and CAD for visualization of inequalities. Furthermore, the inequalities plotting functions cannot visualize the parametric solution set in the case $k \leq n - 1$.

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