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Interval Calculus in Maple

The Extension `intpakX` to the Package `intpak` of the Share-Library

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The software is freely available from
`http://www.math.uni-wuppertal.de/wrswt/software.html`

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Abstract

The package `intpak` of the Share-Library of Maple places a first experimental interval–arithmetic package at the user's disposal. It contains among others the new data type `interval` (long number intervals), the corresponding basic arithmetic operations, lattice–operations as intersection and union, some basic (long number) interval functions as well as a command which transforms a given expression automatically into an interval expression.

The extension `intpakX` supplements the package `intpak` in essential points. E. g. the so-called extended interval division is provided, it contains the realization of an interval Newton–procedure, a complex disc–arithmetic, an extension of the exponential function to disc intervals as well as the realization of various algorithms for confining the range of a complex polynomial (with centred multiplication, with area–optimal multiplication). Moreover, by providing suitable procedures the extension aims at graphically vizualizing the verification algorithms (e. g. search for zeros, linear, quadratic range confinements, disc–interval calculus). The monitor output of the graphic routines is (in contrast to the figures in this report) coloured.

The source code (about 2000 lines Maple–code) of the extension `intpakX` is freely available.

Key Words: MapleV, Interval Analysis, Validated Computations, Disc Arithmetic, Visualization of Self Validating Numerical Algorithms

MSC: 65G05, 65G10, 68M15

1 About the Program Package `intpak`

1.1 Extent of Function

The interval package `intpak` included in the share-library of Maple contains

– the new data type `interval` (long number intervals),

– the corresponding basic arithmetic operations,

– basic interval functions and

– some auxiliary functions.

The Data Type `interval`

A variable x is of type `interval` if x is either an empty list or if $x = [x_1, x_2]$ is a list with two elements. The interval ends x_i, $i = 1, 2$ must fulfill one of the following requirements:

– x_i is a real number of type `float` or x_i is equal to 0.

– $x_i \in \{-\texttt{infinity}, \texttt{infinity}\}$.

– x_i is a constant predefined in Maple (to these belong `Pi, gamma, Catalan, FAIL, false, true and infinity`).

Remark: Numbers of type `integer` or of type `fraction` are not allowed as interval ends!

Basic Arithmetic Operations

The basic operations implemented are &+, &−, &∗, &/ and `inv`. Their input parameters must be either of type `interval` or of type `num_or_FAIL`. The output is an intervall.

A variable is of type `num_or_FAIL` if its value is a number (i. e. the variable is of type `numeric`), −`infinity` or a constant predefined in Maple (s. above).

Remark: The priorities for the so–called 'inert' operators &+, &−, &∗, &/ are unfortunately set in Maple such that &+ and &− have a higher priority than &∗ and &/. Because of this, one has to use lots of brackets in terms!

Basic Interval Functions

The interval functions implemented are &`sqr`, &`sqrt`, &`ln`, &`exp`, &∗∗, &`intpower`, &`sin`, &`cos`, &`tan`, &`arcsin`, &`arccos`, &`arctan`, &`sinh`, &`cosh` und &`tanh`. It is assumed that the corresponding mathematical functions in Maple deviate maximally by 0.6 ulp from the exact results. This assumption is, however, confirmed nowhere in the Maple manual.

Most names are self–explaining. The operation &`intpower` corresponds to the function x^n, n a natural number. The operator &∗∗ corresponds to the function x^α, where α can mean an interval, an integer or a number of type `float`.

Auxiliary Functions

The function `construct` generates from a number or a pair of numbers an element of type `interval`. As optional parameter may be entered the string 'rounded'. In this case the interval ends are rounded by one ulp from above or from below. This is realized with the functions `ru` (`Interval_Round_Up`) and `rd` (`Interval_Round_Down`). Notice that the input parameters of `ru` and `rd` must by of type `float`.

Also contained are the functions `midpoint`, `width`, &`intersect`, &`union` and `is_in`. The function `width` calculates the diameter and `midpoint` a confinement of the centre of an interval.

For converting a term into an interval term resp. into an interval function, there are the commands `‘convert/interval‘` and `inapply`.

1.2 First Calculation Examples

The interval package `intpak` can be called up the following way:

```
> with(share):

See ?share and ?share,contents for information about the share library
> with(intpak);
```

$$[init]$$

Notice that in Maple the distinction between upper and lower case is significant.

Example 1: The data type `interval`

Interval ends of type `integer` are *not* admitted:

```
> x:=[1,2];
```

$$x := [1, 2]$$

```
> type(x,interval);
```

$$false$$

```
> x:=construct(1,2); type(x,interval);
```

$$x := [1., 2.]$$
$$true$$

Generate confining interval:

```
> construct(1,rounded);
```

$$[.999999999, 1.000000001]$$

```
> y:=construct(1,infinity,rounded);
```

$$y := [.999999999, \infty]$$

```
> type(y,interval);
```

$$true$$

Diameter and confinement of interval centre:

```
> width(x); width(y);
```

$$1.$$
$$\infty$$

```
> midpoint(x);
```

$$[1.499999999, 1.500000001]$$

Example 2: Set–theoretic operations

```
> x:=[1.,3.]; y:=[2.,infinity]; z:=[4.,5.];
```

$$x := [1., 3.]$$
$$y := [2., \infty]$$
$$z := [4., 5.]$$

```
> x &union y;
```

$$[1., \infty]$$

```
>  x &union z;
```

$$[1., 3.], [4., 5.]$$

The resulting set consists of two intervals. Thus, &`union` does **not** calculate the interval hull. For this the procedure &`Convex_Hull` is available in `intpakX`.

```
>  x &intersect y;
```

$$[2., 3.]$$

```
>  x[1];
```

$$1.$$

```
>  is_in(x[1],y);
```

$$\mathit{false}$$

```
>  is_in(z,y);
```

$$\mathit{true}$$

Example 3: Arithmetic Operations and Interval Functions

```
>  [1.,2.]  &+ 3 &* 0; # Wrong priorities of operators
```

$$[0, 0]$$

```
>  [1.,2.]  &+ (3 &* 0);
```

$$[.999999999, 2.000000001]$$

```
>  [-1.,2.]  &intpower 3;
```

$$[-1.000000001, 8.000000001]$$

```
>  &sqr(&cosh(1)) &- &sqr(&sinh(1));
```

$$[.9999999919, 1.000000008]$$

Example 4: Value domain confinements
Confinement of the value domain of $f(x) := x^3 - x^2 - x + 1$ on the interval $[0, 0.5]$ by evaluation with intervals, using the mean value form and taking into consideration the monotony properties of f.

```
>  x:='x'; # release variable
```

$$x := x$$

```
>  f:=x^3-x^2-x+1;
```

$$f := x^3 - x^2 - x + 1$$

```
>  F:=inapply(f,x); # Transformation of f into an interval function
```

$$F := x \to (x \,\text{`\&intpower`}\, 3) \,\text{`\&+`}\, (((-1) \,\text{`\&*`}\, (x \,\text{`\&intpower`}\, 2)) \,\text{`\&+`}\, (((-1) \,\text{`\&*`}\, x) \,\text{`\&+`}\, 1))$$

Transformation of the derivative f' of f into an interval function

```
>  dF:=inapply(diff(f,x),x);
```

$$dF := x \to (3 \,\text{`\&*`}\, (x \,\text{`\&intpower`}\, 2)) \,\text{`\&+`}\, (((-2) \,\text{`\&*`}\, x) \,\text{`\&+`}\, (-1))$$

Confinement of the value domain of f on the interval [0,0.5]

```
> X:=[0,0.5];
```

$$X := [0,\ .5]$$

```
> mid_X:=midpoint(X);
```

$$mid\_X := [.2499999999,\ .2500000002]$$

```
> r_i:=F(X); # evaluation with intervals
```

$$r\_i := [.2499999994,\ 1.125000003]$$

```
> r_m:=F(mid_X) &- ( dF(X) &* (X &- mid_X) ); # mean value form
```

$$r\_m := [.2031249977,\ 1.203125003]$$

The evaluation with intervals of f' on the interval $X = [0, 0.5]$ shows that f is monotonically decreasing in X.

```
> dF([0,0.5]);
```

$$[-2.000000003,\ -.2499999985]$$

The exact value domain of f on X is the interval $[0.375, 1]$. A very sharp confinement of the value domain can be calculated in the following way:

```
> r_e:=construct(F(X[2])[1],F(X[1])[2]);
```

$$r\_e := [.3749999993,\ 1.000000003]$$

`F(X[2])` calculates the evaluation with intervals of f at the point `X[2]`$=$ 0.5. `F(X[2])[1]` gives the lower interval end of `F(X[2])`, thus a **safe** lower bound for the minimum of f on the interval $[0, 0.5]$.

In order to verify if the 'exact' confinement of the value domain `r_e` is contained in the intersection of the confinements `r_i` and `r_m` calculated above, e. g. the procedure `is_in` may be used.

```
> is_in(r_e, r_i &intersect r_m);
```

$$true$$

1.3 Necessary Corrections and Changes

In this section some errors in the implementation of the `intpak`–procedure `` `convert/interval` ``, removed by the extension `intpakX`, are discussed. Moreover, we will indicate some changes in the definition of the type `interval_comp` and the implementation of the procedures `is_in`, `Interval_power` and `construct` made with respect to `intpakX`.

The procedures `` `convert/interval` `` and `inapply`

The command `inapply` should(!) transform a term into an interval function, and is therefore very useful for generating procedures built up out of the package `intpak` (see also example 4, section 1.2). the transformation takes place in two steps. First the entered term is transformed, using the command `` `convert/interval` ``, into an interval term. Then, from the interval term is generated with the Maple command `inapply` an interval function.

Unfortunately, there have crept into the implementation of the command `‘convert/interval‘` some errors which become noticeable when calling up `inapply`:
1. The call–up

```
>  inapply(0.5*t,t);
```

```
Error, (in type/interval\_comp) too many levels of recursion
```

leads to an error message. For the procedure `‘convert/interval‘` enters an endless loop if a product with a factor of type `float` occurs in a term.

2. Transforming $f(t) := \sqrt{t}$ into an interval function:

```
>  f:=inapply(sqrt(t),t);
```

$$f := t \to t\,‘\&\hat{}‘\,\frac{1}{2}$$

Actually, one would have expected the output f :=&*sqrt*. But Maple transforms the term $\sqrt{t}$ into $t^{1/2}$ before the procedure `inapply` is started. If one e. g. wants to calculate $f([4., 9.])$, then the term is given back unevaluated, as the operator &ˆ is not defined in `intpak`:

```
>  f([4.,9.]);
```

$$[4.,\,9.]\,\&\hat{}\,\frac{1}{2}$$

The correct operator notation for the procedure `Interval_power` defined in `intpak` would be &∗∗. However, the call–up

```
>  [4.,9.]  &** (1/2);
```

$$[4.,\,9.]\,\&**\frac{1}{2}$$

also does not deliver the desired result, as the procedure `Interval_power` does not permit rational numbers in the second argument.

```
>  [4.,9.]  &** 0.5;
```

$$[1.999999999,\,3.000000001]$$

finally gives the desired result.

3. In the transfer of the interval package from release 4 to release 5 has crept in an additional error. The global variable `Interval_fnlist` responsible for transforming the standard functions into the corresponding interval functions has gone lost.

```
>  inapply(sin(x),x);
```

```
Error, (in convert/interval) wrong number (or type) of parameters in
function subs
```

In order to be also able to handle terms containing standard functions one therefore has to use the extension `intpakX`. The variable `Interval_fnlist` is initialized automatically by loading `intpakX`.

With the improved version of the procedure `` `convert/interval` `` contained in the package `intpakX` one obtains e. g.:

```
> f:=inapply(0.5*t,t);
```

$$f := t \to .5\,\text{`\&*`}\,t$$

```
> f(2);
```

$$[.999999999,\ 1.000000001]$$

```
> f:=inapply(sqrt(t),t);
```

$$f := \&sqrt$$

```
> f([4.,9.]);
```

$$[1.999999999,\ 3.000000001]$$

```
> f:=inapply(sin(x)+x,x);
```

$$f := x \to \text{`\&sin`}(x)\,\text{`\&+`}\,x$$

```
> f(0);
```

$$[0,\ 0]$$

The Procedure `Interval_power`

In order to enable the evaluation of e. g. the second derivative of $f(x) := \sqrt{x}$

$$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$$

with intervals, the procedure `Interval_power` (alias &**) was completed such that also rational numbers are permitted as second parameter.

Calculating $f''(4)$ with a floating point arithmetic

```
> evalf(-1/4*4^(-3/2));
```

$$-.03125000000$$

Evaluation of $f''(4)$ with intervals

```
> df2:=inapply(diff(sqrt(x),x$2),x);
```

$$df2 := x \to (\frac{-1}{4})\,\text{`\&*`}\,(x\,\text{`\&**`}\,(\frac{-3}{2}))$$

```
> df2(4);
```

$$[-.03125000004,\ -.03124999997]$$

The data types `interval_comp` and `interval`

According to the definition of the type `interval_comp` the in Maple predefined constants `Pi, gamma, Catalan, false` and `true` are permitted as interval ends in `intpak`. However, this is not considered in the implementation of the type `interval` as well as in the implementation of the basic operations and the basic interval functions, and therefore leads to unexpected results or error messages. Examples:

```
> type([1.,Pi],interval);

Error, (in intpak/max) cannot evaluate boolean

> &sinh([-infinity,Pi]);

Error, (in Interval\_ulp) improper op or subscript selector

> 1.  &+ [Pi,infinity];
```

$$[1. + \pi - \mathrm{Float}(1, -8 + \pi), \infty]$$

In the last case there is no error message, but the result is not of type `interval`. A similar behaviour occurs with the other constants mentioned above. Therefore, the interval package `intpakX` contains a changed version of the data type `interval_comp` which does no longer permit the Maple constants `Pi, gamma, Catalan, false` and `true` as interval components (i. e. as interval ends).

This does not mean an essential restriction, as in the execution of interval operations the constants `Pi, gamma` and `Catalan` are transformed into numbers of type `float`, anyway. Example:

```
> &sin(Pi);
```

$$[-.979323846265\,10^{-11}, .102067615375\,10^{-10}]$$

The Procedure `is_in`

The `intpak`–procedure `is_in` has two input parameters, and verifies if the first parameter is contained in the second (in a set–theoretical sense). As input parameters are allowed variables of type `interval`, numbers of type `numeric` and the values `FAIL`, `infinity` and `-infinity`. In particular, numbers of type `rational` which until now have lead to **wrong** results are also allowed. Example

```
> Digits; # accuracy
```

$$10$$

```
> is_in(1/3,[0.3333333332,0.3333333333]);
```

true

The result is obviously **wrong**, as

$$1/3 \notin [0.3333333332, 0.3333333333].$$

Of course, erroneous results occur also, if numbers of type `float` are used whose length exceeds the current value of the variable `Digits`.

```
>  is_in(1.9999999999,[2.,2.]);
```

$$true$$

Also the Maple command `evalb` gives out **wrong** results, as floating point numbers occur in the logical term.

```
>  evalb((0.3333333332 <= 1/3) and (1/3 <= 0.3333333333));
```

$$true$$

```
>  evalb(2.  <= 1.9999999999);
```

$$true$$

If all input parameters are, however, rational numbers, then the result under `evalb` is correct, whereas `is_in` leads in this case to a **wrong** result.

```
>  is_in(1/3,3333333333/10^10);
```

$$true$$

```
>  evalb(1/3 = 3333333333/10^10);
```

$$false$$

Thus, in order to improve the procedure `is_in` in such a way that, when using rational or 'too long' floating point numbers, a correct result is given out, the exact rational long number arithmetic of Maple has to be used.

In order to achieve this, a number of type `float` must be able to be converted into a number of type `rational` without 'conversion errors'. The Maple command `convert/rational` can unfortunately not be used for this.

```
>  convert(0.3333333333,rational);
```

$$\frac{1}{3}$$

Using the Maple command `op` it seems, however, to be possible to perform error–free conversions.

```
>  x:=0.3333333333;
```

$$x := .3333333333$$

Converting `x` into a rational number:

```
>  x_rational:=op(1,x) * 10^op(2,x);
```

$$x\_rational := \frac{3333333333}{10000000000}$$

Is `x = x_rational` ?

```
>  evalb((x <= x_rational) and (x_rational <= x));
                                  true
```

Is `x = 1/3` ?

```
>  evalb((x <= (1/3)) and ((1/3) <= x));
                                  true
```

Is `x_rational = 1/3` ?

```
>  evalb((x_rational <= (1/3)) and ((1/3) <= x_rational));
                                  false
```

The extension `intpakX` contains the command `‘intpakX/greater‘` which checks if the first input parameter is greater than the second using the transformation shown above and the long number arithmetic of Maple. With this command the procedure `is_in` was changed in such a way that it gives out a correct result even if one of the numbers is rational or if the length of the entered numbers exceeds the value of the variable `Digits`. Example:

```
>  Digits;
                                   10
>  is_in(1.9999999999999999,[2.,2.]);
                                  false
>  is_in(1/3,[0.3333333332,0.33333333333333333]);
                                  false
```

The Procedure `construct`

As parameters of an `intpak` interval operation are also permitted numbers of type `numeric`, as well as Maple–constants. Before performing the operation these are converted into intervals using the procedure `construct`. However, these intervals are not always a confinement of the actual value of the entered number which leads to erroneous results. Examples:

```
>  (1/3) &- 0.3333333333;
                                 [0, 0]
>  1.0000000001 &- 1.;
                                 [0, 0]
```

In both cases $[0, 0]$ is obviously **not** a confinement of the exact result. The cause of this error is in the procedure `construct`. On entering a rational or ’too long’ number without the optional parameter `rounded`, it **does not** generate a confining interval.

```
>  construct(1/3);
                        [.3333333333, .3333333333]
```

```
> construct(1.0000000001);
```

$$[1.000000000, 1.000000000]$$

Similarly to dealing with the procedure `is_in`, the procedure can be corrected in this case using the exact rational arithmetic of Maple. With the improved `intpakX`-version of `construct` one then obtains

```
> construct(1/3);
```

$$[.3333333333, .3333333334]$$

```
> (1/3) &- 0.3333333333;
```

$$[0, .1000000001\, 10^{-9}]$$

```
> construct(1.0000000001);
```

$$[1.000000000, 1.000000001]$$

```
> 1.0000000001 &- 1;
```

$$[0, .1000000001\, 10^{-8}]$$

2 The Extension `intpakX`

2.1 The Installation of `intpakX`

The commands of the extension `intpakX` are stored in the file `intpakX.m`. It can be called off via ftp from the server `iamk4515.mathematik.uni-karlsruhe.de` in the directory `/pub/iwrmm/maple/software`. The file was generated with Maple V Release 5.

The file can be read with `with` if the path to the directory in which it is filed is stored in the system variable `libname`. The variable `libname` generally contains only the path to the Maple-library. It is initialized automatically when starting Maple.

```
> restart;
> libname;
```

"C:\\PROGRAMME\ \MAPLE V RELEASE 5/lib"

If the `share`-library of Maple is loaded with `with`, then the path to the directory containing the `share`-packages is stored in `libname`.

```
> with(share);

See ?share and ?share,contents for information about the share library
> libname;
```

"C:\\PROGRAMME\ \MAPLE V RELEASE 5/lib",
"C:\\PROGRAMME\ \MAPLE V RELEASE 5/share"

If the file `intpakX.m` is filed in this `share`-directory, then it can be read with `with` like all other `share`-packages. As `intpakX` is based on the interval package `intpak` and contains some changed commands, the package `intpak` has to be loaded before. (If the reading of `intpakX` does not work the first time, then the system should be restarted with `restart`.)

```
> with(intpak):

Share Library:  intpak

Authors: Connell, Amanda E. and Corless, Robert.

Description:  Interval Arithmetic Package
> with(intpakX);

Authors: Geulig, Ilse and Kraemer, Walter (supervisor),
University of Karlsruhe, IWRMM

Description: Extension of the Interval Arithmetic Package intpak
```

[&*cadd*, &*cdiv*, &*cdiv_opt*, &*cmult*, &*cmult_opt*, &*csub*, *centred_form_eval*, *cexp*, *complex_disc_plot*, *compute_all_zeros*, *compute_all_zeros_with_plot*, *compute_combined_range*, *compute_mean_value_range*, *compute_monotonic_range*, *compute_naive_interval_range*, *compute_range*, *compute_range3d*, *compute_taylor_form_range*, *ext_int_div*, *horner_eval_cent*, *horner_eval_opt*, *init*, *interval_list_plot*, *interval_list_plot3d*, *is_in*, *mid*, *rel_diam*, *subdivide_adaptive*, *subdivide_equidistant*]

Unfortunately this way of proceeding generally works only under Windows, as under Linux the different users may not put their own files into the `share`-directory. Let now be `/users/maple/software` the directory containing the file `intpakX.m`. If one extends (during a Maple session) the system variable `libname` by the path to this directory, then the package `intpakX` can be read again with `with`.

Example for loading the interval package `intpak` and the extension `intpakX`:

```
> restart;
> with(share):

See ?share and ?share,contents for information about the share library
> with(intpak);
```

[*init*]

Extension of the system variable `libname` by the path to the directory in wich `intpakX.m` is stored.

```
> libname:=libname, "/users/maple/software"; # under Linux
```

libname := "/usr/local/maple/lib", "/usr/local/maple/share", "/users/maple/software"

```
> libname:=libname,"C:\\users\\maple\\software":  # under Windows
> with(intpakX);
```

With the procedures with graphical output (see e. g. sections 3 and 4.3) are used routines frome the `plots`-package as well as the `geometry`-package. Therefore these packages are usually included (without the user's further doing) when `intpakX` is included.

2.2 Function Range of the Extension `intpakX`

The package `intpakX` places at the user's disposal the following extensions to the interval package `intpak`:

1. Verified calculation of zeros
   - the extended interval division `ext_int_div`,
   - the realization of an extended interval Newton-procedure, (`compute_all_zeros, compute_all_zeros_with_plot`)
2. Complex disc arithmetics
   - the new data type `complex_disc` (complex disc interval),
   - the procedure `mid` for determining the centre of an interval (resp. an approximation of the centre lying certainly in the entered interval),
   - basic arithmetic operations for disc intervals (usual 'centred' defintions) (&`cadd`, &`csub`, &`cmult`, &`cdiv`),
   - area-optimal multiplication and division of two disc intervals (&`cmult_opt`, &`cdiv_opt`),
   - the exponential function for disc intervals, (`cexp`),
   - various algorithms for safe confinement of the range of a complex polynomial, (`horner_eval_cent, horner_eval_opt, centred_form_eval`)
3. Range confinements with graphical output
   - confinement of the range of a real-valued function in one real variable by successive partition of the start interval into subintervals (`compute_range`)
   - confinement of the range of a real-valued function in two real variables by successive partition of the start interval into subintervals (`compute_range3d`).
4. A series of procedures for graphical illustration of the procedures mentioned above.

3 Ranges with Graphical Output

3.1 Functions in One Variable

Two simple possibilities to confine the range of a function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ over an interval $[x] \subseteq D$ were presented already in example 4, section 1.2. Namely the interval-evaluation of f (if it exists) and the mean value form (if the interval-evaluation of f' over $[x]$ exists.

An improved confinement is obtained if the interval $[x]$ is partitioned and over each subinterval is calculated a range confinement. The interval hull of these subrange confinements is then a confinement of the range of f over $[x]$. If the partition is continued successively, then the initial range confinement can be improved successively.

This is realized by the procedure `compute_range`. The procedure demands three input parameters

– a function `f`,

– the start interval `xstart` (may be entered either as interval or as `range`)

– the number of `iterationsteps` to be performed. It is used as an interrupt criterion.

The order of the three parameters mentioned above is compulsory. In addition, the entering of four optional parameters (in any order) is possible

– Remitting a parameter `Nx = n`, `n` an integer greater than or equal to 1, effects the partitioning of the start interval into `n` intervals before all.

– the optional parameter `linear` (`quadratic`) effects the linear (quadratic) convergence of the procedure. On entering `linear`, the 'naive' interval evaluation is used for determining the subrange confinements. If `quadratic` is remitted as parameter, the procedure `compute_combined_range` is used for determining the subrange confinements, combining interval-evaluation, mean-value form and monotony-test. If none of these two parameters is entered, then in the first three iteration steps interval-evaluation is used for determining the range confinements, and from step 4 on is used the procedure `compute_combined_range`.

– the optional parameter `adaptive` effects the adaptive partitoning of the current interval list, and therefore generally leads to a calculation-time reduction.

– the optional parameter `colorlist = [color1,color2,...]` determines the colours used the graphical illustration of each iteration step. `color1, color2`, etc. must be colours predefined in Maple, e. g. `blue, red, green, magenta, coral, brown` etc. This does, however, not influence the illustration of the last iteration step which is always illustrated in yellow.

For reasons of clearness, only the graphical illustration of the last three iteration steps and the function f are given out. The graphical illustration of all iteration steps is, however, stored in the global variable `q`. The variable `q` is a table. If 3 iteration steps were performed, then the entries `q[1]`, `q[2]` and `q[3]` contain the illustrations of each iteration step, however, not the graph of the function. This is stored in the table entry `q[4]`.

Also the calculated range confinements are stored — in the global variable `r`. It is also a table and `r[i]` contains the range confinement calculated in the `i`-th step.

The current partition of the start interval is stored in the global variable `list_of_intervals`. The corresponding subrange confinements are stored in the variable `list_of_ranges`.

Examples:
Confinement of the range of the function

```
> f:=x->exp(-x^2)*sin(Pi*x^3);
```

$$f := x \to e^{(-x^2)} \sin(\pi\, x^3)$$

over the interval $X := [0.5, 2.]$ using the procedure `compute_range`

```
> X:=[0.5,2.];
```

$$X := [.5,\ 2.]$$

```
> compute_range(f,X,4);

initial range confinement =     [-.7788007834, .7788007834]

range confinement after iteration step 4 =
[-.3233867682,.6103317518]
```

The initial range confinement is $\approx [-0.78, 0.78]$. The range confinement after 4 iteration steps, that is after partitioning into $2^4 = 16$ subintervals, is $\approx [-0.32, 0.61]$. The graphical output can be found in Figure 1.

The same range confinement is obtained already after 1 iteration step if the start interval is partitioned before all into 2^3 intervals and the optional parameter `quadratic` is given:

```
> compute_range(f,X,1,Nx = 2^3,quadratic);

initial range confinement =     [-.7788007834, .7788007834]

range confinement after partitioning into 8 subintervals =

[-.3233867682,.6639743998]

range confinement after iteration step 1 =
[-.3233867682,.6103317518]
```

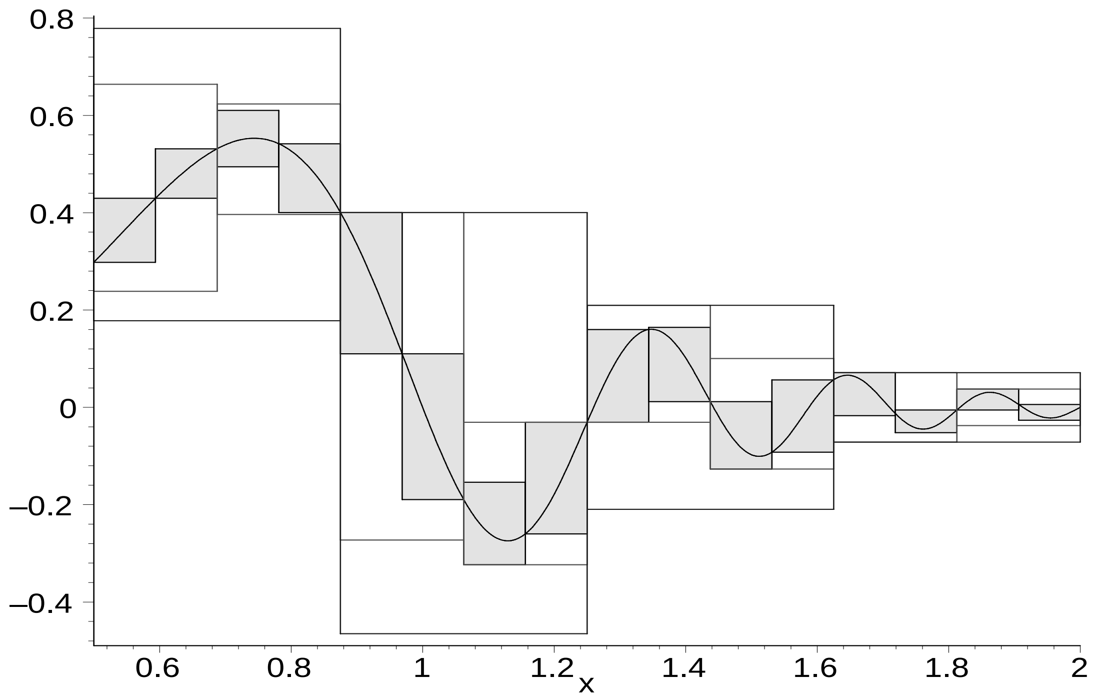


Figure 1: refining a range confinement by partitioning the argument domain into subintervals

Using the parameter `adaptive` can generally reduce the number of subintervals considerably:

```
>  compute_range(f,[0.5,2.],6,adaptive);

initial range confinement =     [-.7788007834, .7788007834]

range confinement after iteration step 6 =
[-.2834388814,.5563221618]
```

The current partitioning of the start interval is stored in the variable `list_of_intervals`. Therefore, the total number of subintervals can be determined simply at any time. In the example above it is determined in the following way (after 6 iteration steps):

```
>  nops(list_of_intervals);
                              24
```

Without the parameter `adaptive` the number of subintervals after 6 iteration steps would be $2^6 = 64$.

In order to illustrate only the last iteration step and the function f, the `plots`-command `display` can be used. The result of the following call-up is found in Figure 2.

```
>  display([q[7],q[6]],title=‘Adaptive partitioning after 6
iterationsteps‘,titlefont=[TIMES,BOLD,12]);
```

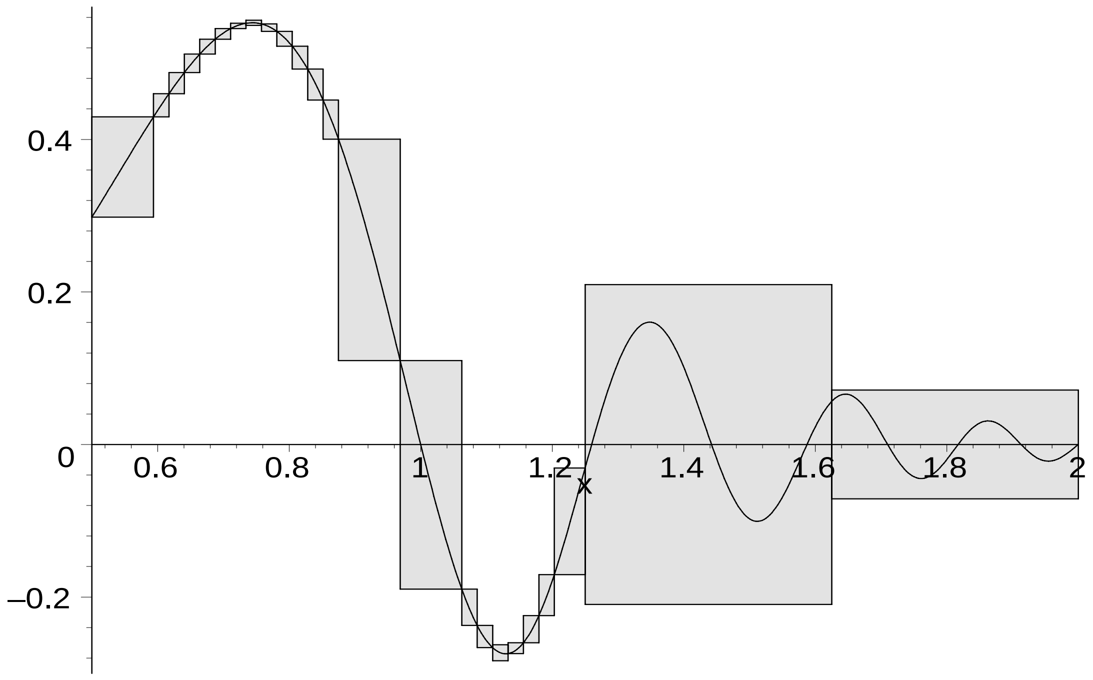


Figure 2: adaptive partitioning

3.2 Functions in Two Variables

The Procedure `compute_range3d` calculates range confinements for real-valued functions in two real variables over a two–dimensional interval $X \times Y$.

Its input parameters are (analogous to `compute_range`, however, **without** the optional parameters `linear/quadratic` and `adaptive`)

- the function `f`,
- a real interval, or a domain `X`,
- a real interval, or a domain `Y`,
- the number of iteration steps to be performed.

The order of these parameters is compulsory. With this procedure, too, can be set a series of optional parameters

– the parameters `Nx = n` and `Ny = m` effect a corresponding partitioning of the start interval (axe-parallel right-angle) in the `x`-`y`-direction,

– the parameter `colorlist` has the same meaning as in `compute_range`,

– the parameter `cutout = r` determines the strength of the lines in the illustration of the calculated confinements. Here, `r` should be 0, 1 or a fraction with $0 < r < 1$.

In addition, any options of the `plot3d`–command can be used.

For determining the subrange confinements in `compute_range3d` the 'naive' interval-evaluation is used. In each iteration step the subintervals are partitioned just in one direction. If e. g. two iteration steps are performed, then in the first step is partitioned in `x`-direction and in the second in `y`-direction.

Example: Determining a range confinement for the function

```
>  f:=(x,y)->exp(-x*y)*sin(Pi*x^2*y^2);
```

$$f := (x,\, y) \to e^{(-x\,y)}\sin(\pi\, x^2\, y^2)$$

over th interval $X \times Y = [\pi/8, \pi/2] \times [\pi/8, \pi/2]$.

```
>  X:=[evalf(Pi)/8,evalf(Pi)/2]; Y:=X;
```

$$X := [.3926990818,\ 1.570796327]$$

$$Y := [.3926990818,\ 1.570796327]$$

```
>  compute_range3d(f,X,Y,4);
```

```
initial range confinement =    [-.8570898115, .8570898115]

range confinement after iteration step 1 =
[-.8570898115, .8570898115]

range confinement after iteration step 2 =
[-.6800891261, .8570898115]

range confinement after iteration step 3 =
[-.6800891261, .8486122905]

range confinement after iteration step 4 =
[-.5093193828, .7559256232]
```

After each iteration step the calculated range confinement is given out. Only the illustration of the last iteration step and the graphical illustration of the function f are given out. Also in this case, the graphical illustrations of the other iteration steps are stored in the global variable `q`.

The graphical output of the procedure can, as in any other 3d-graphics in Maple, be edited thereafter with the commands from the graphics-menue. The desired graphic-options can, however, also be called up directly as parameters. E. g. the command

```
>  compute_range3d(f,X,Y,3,cutout=9/10,color=yellow,
>          lightmodel=light2,axes=framed,titlefont=[TIMES,BOLD,12],
            title=‘range confinement by partitioning into
            subintervals‘,);
```

generates the graphics in Figure 3.

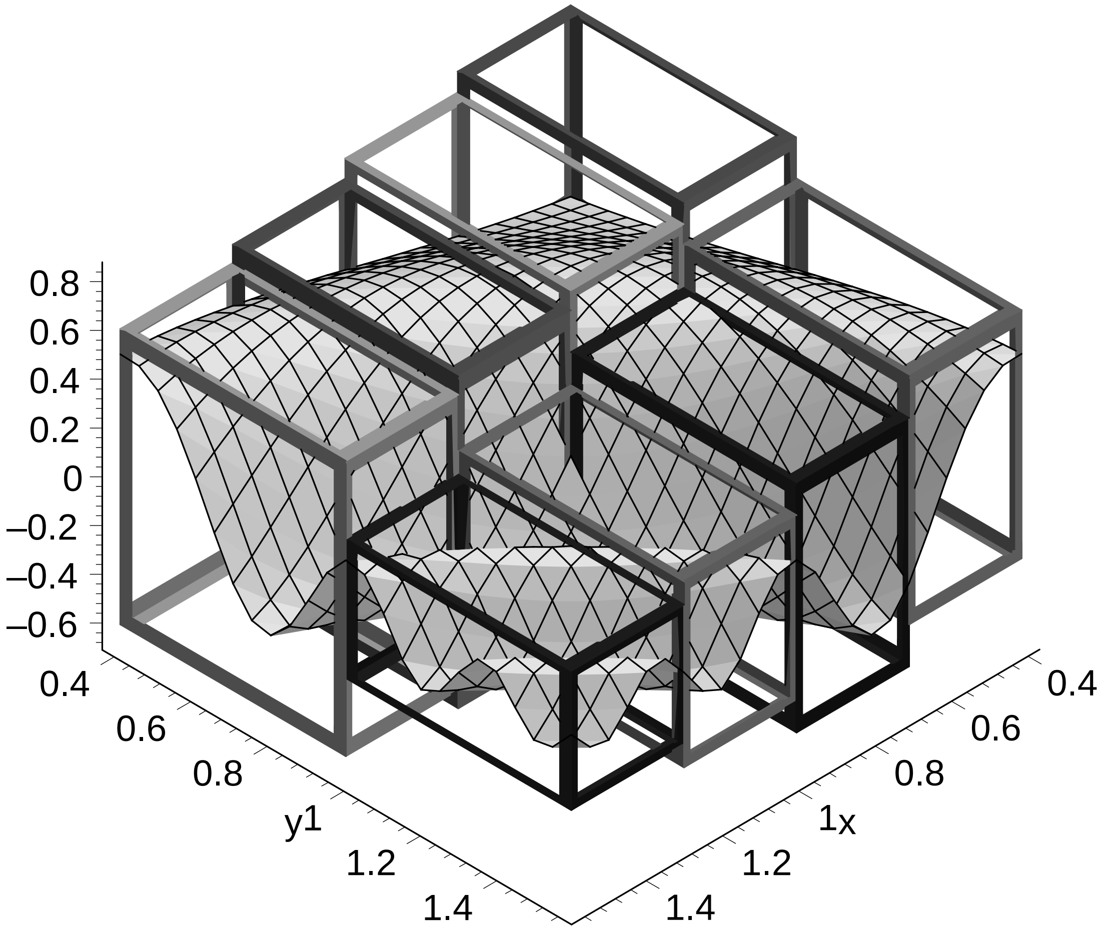


Figure 3: range confinement of a function in two variables

4 Verified Calculation of Zeros

The extension `intpakX` contains a realization of the extended interval-Newton-iteration

$$\begin{cases} [x]^0 & , \quad \text{real start interval} \\ [x]^{k+1} := N([x]^k) \cap [x]^k, & k = 0, 1, 2, \ldots, \end{cases}$$

where

$$N([x]) := m([x]) - \frac{f(m([x]))}{f'([x])}$$

denotes the interval-Newton-operator and $m([x])$ usually the centre of the interval $[x]$.

If $f : D \subset \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function on D and $[x]^0 \subset D$ a real interval for which the interval-evaluation $f'([x]^0)$ exists, then the interval-Newton-iteration calculates confinements of all zeros of f contained in $[x]^0$. In addition, using this procedure the existence and uniqueness of simple zeros of f in the given start interval can be proved. A more detailed description of this procedure can be found in [8].

The case $0 \in f'([x]^0)$ is permitted! Therefore, for performing this procedure one needs the extended interval division and the subtraction of an extended interval of a real number (extended interval subtraction, see page 24).

4.1 Extended Interval Division and Extended Interval Subtraction

Let $\mathbb{IR}$ be the set of real intervals and

$$\mathbb{IR}^* := \mathbb{IR} \cup \{[-\infty, r] \,|\, r \in \mathbb{R}\} \cup \{[l, +\infty] \,|\, l \in \mathbb{R}\} \cup \{[-\infty, +\infty]\}$$

the set of extended intervals.

The definitions of extended interval division and extended interval subtraction used in `intpak` resp. `intpakX` correspond to the definitions used in [15]. The interval operations defined in this way are inclusion-isotonic.

For two real intervals $[x] = [\underline{x}, \overline{x}]$ and $[y] = [\underline{y}, \overline{y}]$, the extended interval division is defined as follows

$$[x]/[y] := \begin{cases} [x] \cdot [1/\overline{y}, 1/\underline{y}], & \text{if } 0 \notin [y] \\ [-\infty, +\infty], & \text{if } 0 \in [x] \text{ and } 0 \in [y] \\ [\overline{x}/\underline{y}, +\infty], & \text{if } \overline{x} < 0 \text{ and } \underline{y} < \overline{y} = 0 \\ [-\infty, \overline{x}/\overline{y}] \cup [\overline{x}/\underline{y}, +\infty], & \text{if } \overline{x} < 0 \text{ and } \underline{y} < 0 < \overline{y} \\ [-\infty, \overline{x}/\overline{y}], & \text{if } \overline{x} < 0 \text{ and } 0 = \underline{y} < \overline{y} \\ [-\infty, \underline{x}/\underline{y}], & \text{if } 0 < \underline{x} \text{ and } \underline{y} < \overline{y} = 0 \\ [-\infty, \underline{x}/\underline{y}] \cup [\underline{x}/\overline{y}, +\infty], & \text{if } 0 < \underline{x} \text{ and } \underline{y} < 0 < \overline{y} \\ [\underline{x}/\overline{y}, +\infty], & \text{if } 0 < \underline{x} \text{ and } 0 = \underline{y} < \overline{y} \\ [\,], & \text{if } 0 \notin [x] \text{ and } 0 = [y]. \end{cases}$$

As the data type `interval` admits the points −`infinity` and `infinity` as interval ends, the extended interval division can be included without difficulty in the interval package. The corresponding command in `intpakX` is called `ext_int_div`. Examples:

```
> ext_int_div([1.,2.],[-1.,1.]);
```

$$[-\infty, -.999999999], [.999999999, \infty]$$

```
> ext_int_div([-2.,-1.],[0,2.]);
```

$$[-\infty, -.4999999999]$$

For $r \in \mathbb{R}$ and an interval $[y] \in I\mathbb{R}^* \cup \{[\,]\}$ the extended interval subtraction is defined by

$$r - [y] := \begin{cases} [r-\overline{y}, r-\underline{y}], & \text{if} \quad [y] = [\underline{y}, \overline{y}] \in I\mathbb{R} \\ [-\infty, +\infty], & \text{if} \quad [y] = [-\infty, +\infty] \\ [r-\overline{y}, +\infty], & \text{if} \quad [y] = [-\infty, \overline{y}] \\ [-\infty, r-\underline{y}], & \text{if} \quad [y] = [\underline{y}, +\infty] \\ [\,], & \text{if} \quad [y] = [\,]. \end{cases}$$

This is already realized by the subtraction operator &− contained in `intpak`. Examples:

```
> 1 &- [1.,infinity];
```

$$[-\infty, 0]$$

```
> 1 &- [-infinity,1.];
```

$$[0, \infty]$$

```
> 1 &- [];
```

$$[]$$

```
> 1 &- [-infinity,infinity];
```

$$[-\infty, \infty]$$

4.2 Extended Interval-Newton-Iteration

The procedure `compute_all_zeros` computes confinements of all zeros of a continuously differentiable function in a given entered start interval using the interval-Newton-iteration.

The input parameters of the procedure `compute_all_zeros` are

– the function `f` whose zeros are supposed to be calculated,

– the start interval `xstart` of the iteration and

– the desired relative diameter `eps` of the zero-confinements to be calculated.

The relative diameter of a real interval is defined as follows

$$d_{rel}([x]) := \begin{cases} \dfrac{d([x])}{\langle [x] \rangle}, & \text{if } 0 \notin [x] \\ d([x]), & \text{otherwise.} \end{cases}$$

Hereby, $d([x])$ denotes the diameter and $\langle [x] \rangle$ the minimum absolute value of the interval $[x]$.

The order of the parameters mentioned above is compulsory. As optional fourth parameter the desired accuracy, i. e. the value of the system variable `Digits` within the procedure can be entered. The fourth parameter should therefore be a positive integer greater than or equal to 10. If this fourth parameter is missing, then the accuracy is adapted to the relative accuracy needed and the length of the input-parameters, but in any case is greater than or equal to the current value of the variable `Digits`.

Given out is the accuracy used, the computed confinements of the zeros, and for each confinement the information if existence and uniqueness of a zero in the given interval was proved.

If the calculated interval was only a *potential* confinement of a zero, then it may contain either one, various or no zero of f at all.

The computed zero-confinements are stored in a global variable `zeros` and can therefore be used further in any way. `zeros` is a table and the access to an entry of the table is made in the usual way, e. g. using `zeros[2]`.

Further global variables initialized in the procedure are the table `infos` containing the additional informations, the number of calculated zero-confinements `N` and the step counter `iter_counter`.

Example 1: Calculating all zeros of

```
>  f:=x->2*exp(tan(cos(x))) - sin(x) + cos(2*x);
```

$$f := x \to 2\, e^{\tan(\cos(x))} - \sin(x) + \cos(2\, x)$$

in the interval $[0, 8]$.

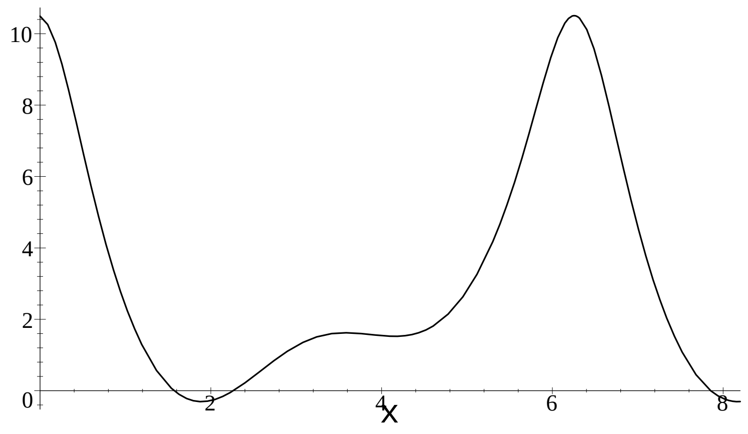


Figure 4: illustration of the function $f(x) := 2\mathrm{e}^{\tan(\cos(x))} - \sin(x) + \cos(2x)$

The function has three simple zeros in the interval $[0, 8]$ (see Figure 4). Two of them can be given exactly, namely $\frac{\pi}{2}$ und $\frac{5\pi}{2}$. An approximation of the third zero can be determined with the command `fsolve`.

Testing if $\pi/2$ and $5\pi/2$ are zeros of f:

```
>  f(Pi/2);
```

$$0$$

```
>  f(5*Pi/2);
```

$$0$$

Calculating the third zero with `fsolve`, `Digits=30`:

```
>  Digits:=30:  zero3:=fsolve(f(x),x=2..2.5); Digits:=10:
```

$$\mathit{zero3} := 2.26480074200004996505814286126$$

Calculating the zero-confinements, `Digits=20` (fourth input parameter):

```
>  compute_all_zeros(f,[0,8.],10^(-10),20);
```

```
Digits =    20
```

$$[7.8539816339705772082, 7.8539816339775304044]$$
$$\textit{contains exactly one zero}$$
$$[2.2648007419999768034, 2.2648007420001249007]$$
$$\textit{contains exactly one zero}$$
$$[1.5707963267948966180, 1.5707963267948966204]$$
$$\textit{contains exactly one zero}$$

```
number of zero confinements:    3

number of iteration steps:    21
```

Verifying if the zeros of f are contained in the calculated zero-confinements:

```
>  is_in(evalf(Pi/2,30),zeros[3]);
```

$$\mathit{true}$$

```
>  is_in(evalf(5*Pi/2,30),zeros[1]);
```

$$\mathit{true}$$

```
>  is_in(zero3,zeros[2]);
```

$$\mathit{true}$$

Example 2: Confining the zero of

```
>  f:=x->(x-1)^3;
```

$$f := x \to (x-1)^3$$

with start interval $[-3., 4,]$. The relative diameter of the confinement should be $\leq 10^{-50}$. (The accuracy used is adapted automatically.)

```
>  compute_all_zeros(f,[-3.,4.],10^(-50));
```

```
Digits =    55
```

$$\begin{array}{l}[.999999999999999999999999999999999999999999999999999997208,\\ \quad 1.000000000000000000000000000000000000000000000000000005551]\end{array}$$

$$\textit{potential zero confinement}$$

```
number of zero confinements:    1

number of iteration steps:    163
```

Diameter and relative diameter of the calculated zero-confinement:

```
>  Digits:=55:
>  diam:=width(zeros[1]);
```

$$\mathit{diam} := .58302\,10^{-50}$$

```
>  `relative_diam`:=rel_diam(zeros[1]);
```

$$\mathit{relative\_diam} :=$$
$$.583020000000000000000000000000000000000000000000000001628\,10^{-50}$$

```
>  Digits:=10:
```

4.3 Graphical Illustration

For graphical illustration of the interval-Newton-iteration, the procedure `compute_all_zeros_with_plot` is at the user's disposal. It calculates analogously to the procedure `compute_all_zeros` zero-confinements with the interval-Newton-iteration. Additionally, however, each iteration step is illustrated graphically.

Here, too, can be entered as optional fourth parameter the value of the variable `Digits`. Further, the input of a fifth optional parameter is possible, stating how many iteration steps may be performed maximally. If this fifth parameter is missing, then the maximum number of iteration steps must be entered interactively. The input must end with a colon or a semicolon.

Example: Calculating a zero-confinement of the function

```
>  f:=x->exp(sin(x-1))-1;
```

$$f := x \to e^{\sin(x-1)} - 1$$

in the interval $[0, 3.]$:

```
>  compute_all_zeros_with_plot(f,[0.,3.],10^(-3));
>  10;
```

The output of the procedure call-up is found on the pages 29 and 30. The slopes of the dotted lines are given by the smallest resp. greatest slope of all tangents to the graph of the function in the current argument domain `xalt`. The lines intersect in the point (expansion point, f(expansion point)). The intersection points of these lines with the `x`-axis are important auxiliary quantities for determining the next iteration of the extended interval-Newton-iteration (see page 22).

5 Disc Arithmetics

Based on the real interval operations, also a complex interval arithmetic can be defined. In the extension `intpakX`, such an arithmetic for disc intervals is realized.

A disc interval with centre $z_0 \in \mathbb{C}$ and radius $r > 0$

$$Z = \langle z_0,\, r\rangle := \{z \in \mathbb{C} \,|\, |z - z_0| \le r\}$$

is stored in `intpakX` as a list with three entries: real part of the centre z_0 of Z, imaginary part of z_0 and radius r.

The name of this new data type is `complex_disc`. As components of this new type are permitted numbers of type `numeric`, i. e. especially numbers of type `integer` and of type `fraction`, too.

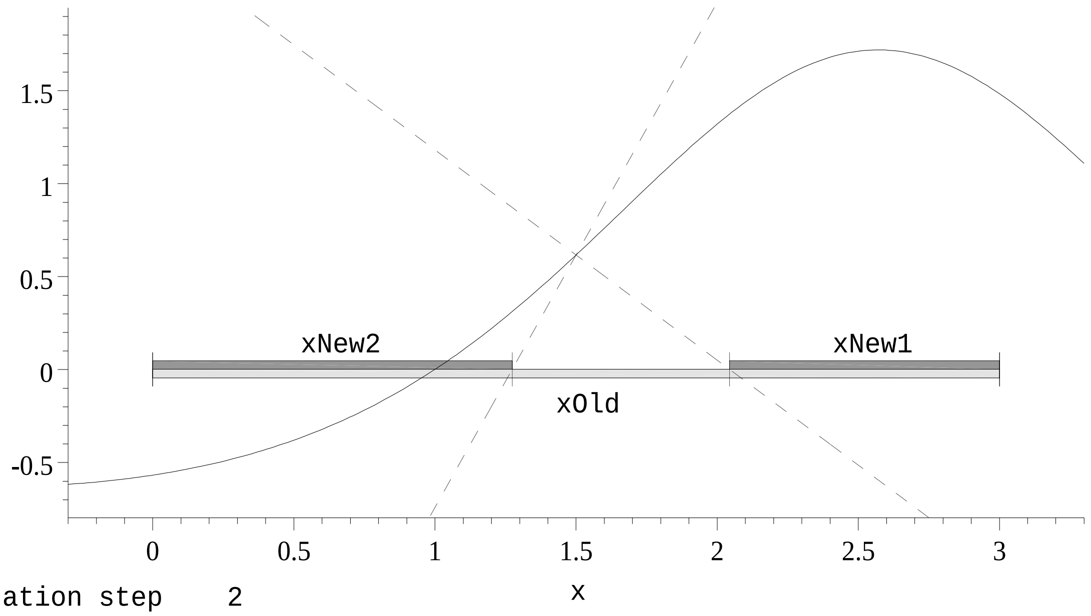
For graphical illustration of a disc interval can be used the procedure `complex_disc_plot`. It has as input parameter a variable of type `complex_disc`. As further optional parameters can be entered the usual illustration options of the Maple-command `plot`.

5.1 Arithmetic (Disc-)Operations

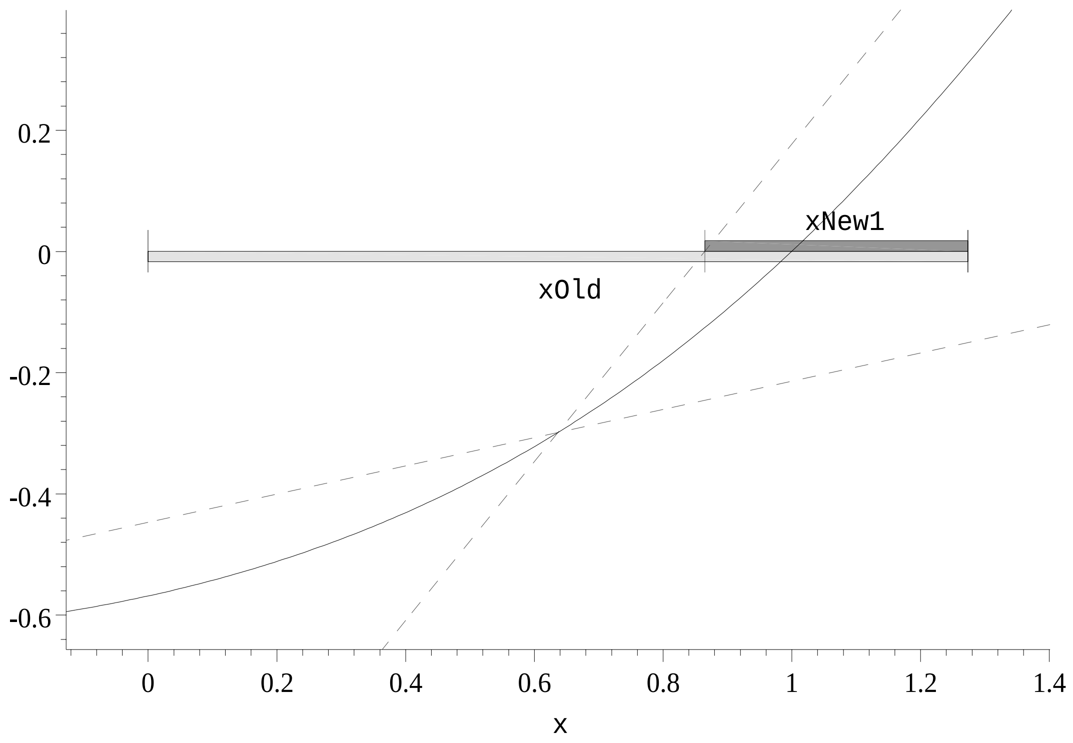
The basic arithmetic operations for disc intervals are usually defined (see e. g. [1]) as follows.

Let $A = \langle a,\, r_a\rangle$ and $B = \langle b,\, r_b\rangle$ be two disc intervals. Then

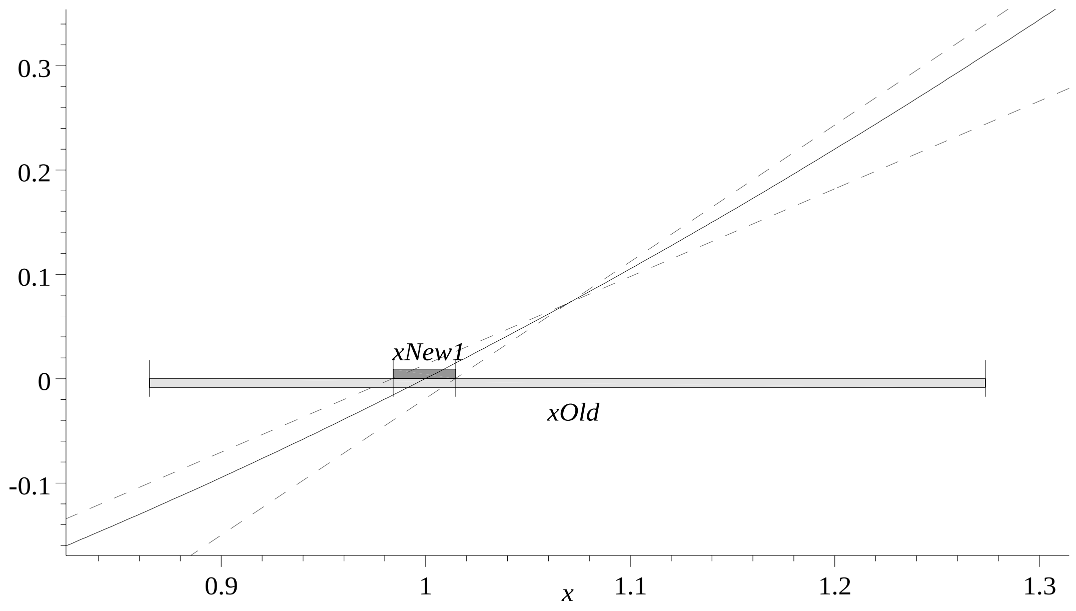
```
Digits =    10

Iteration step    1
xOld=   [0, 3.]
xNew1=   [2.043797652, 3.]
xNew2=   [0, 1.273700327]
```



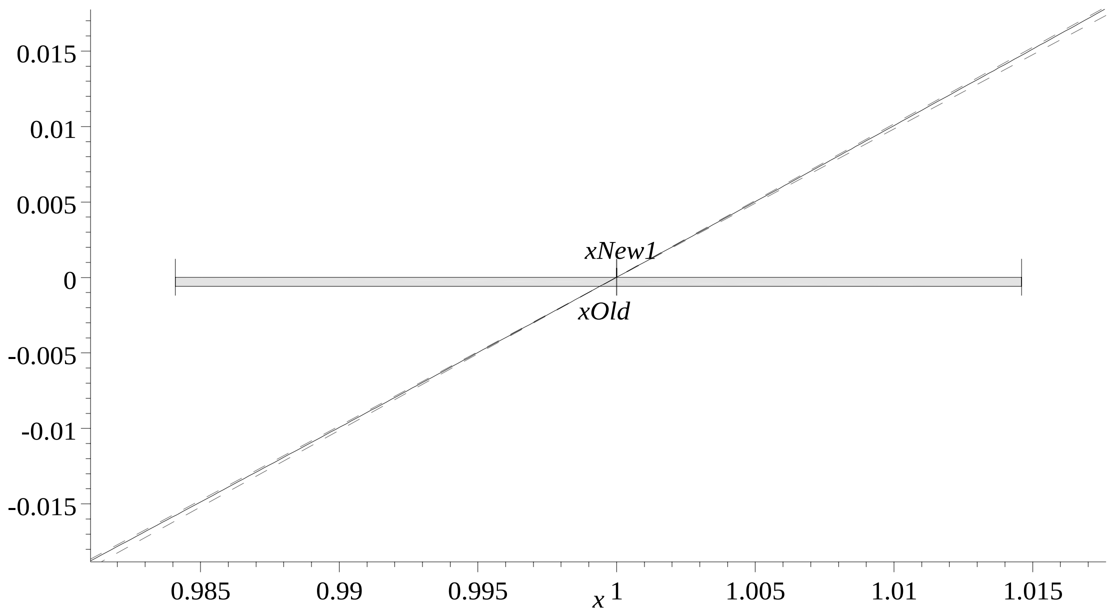
```
Iteration step    2
xOld=   [0, 1.273700327]
xNew1=   [.8650186701, 1.273700327]
```



```
Iteration step    3
xOld=   [.8650186701, 1.273700327]
xNew1=   [.9840864652, 1.014594170]
```



```
Iteration step    4
xOld=   [.9840864652, 1.014594170]
xNew1=   [.9999902275, 1.000010447]
```



[.9999902275, 1.000010447]

contains exactly one zero

```
number of zero confinements:    1
number of iteration steps:    4
```

$$\begin{aligned}
A + B &:= \langle a + b,\ r_a + r_b \rangle \\
A - B &:= \langle a - b,\ r_a + r_b \rangle \\
A \cdot B &:= \langle a \cdot b,\ |a| r_b + |b| r_a + r_a r_b \rangle \\
1 \,/\, B &:= \left\langle \frac{\overline{b}}{b\overline{b} - r_b^2},\ \frac{r_b}{b\overline{b} - r_b^2} \right\rangle, \quad 0 \notin B \\
A \,/\, B &:= A \cdot (1 \,/\, B), \quad 0 \notin B
\end{aligned}$$

where $|a| = \sqrt{a_1^2 + a_2^2}$ denotes the absolute value of the complex number $a = a_1 + i\, a_2$ and $\overline{b} = b_1 - i\, b_2$ the complexe conjugate of $b = b_1 + i\, b_2$.

For the operations defined in this way, we have with the notations as above

$$\begin{array}{lcl}
A \pm B & = & \{a \pm b \,|\, a \in A,\ b \in B\} \\
A \cdot B & \overset{\text{i. a.} \neq}{\supseteq} & \{a \cdot b \,|\, a \in A,\ b \in B\} \\
1 \,/\, B & = & \{1/b \,|\, b \in B\} \\
A \,/\, B & \overset{\text{i. a.} \neq}{\supseteq} & \{a/b \,|\, a \in A,\ b \in B\}
\end{array}$$

The operations for disc intervals defined above are realized in the extension `intpakX` by the operators &`cadd`, &`csub`, &`cmult` and &`cdiv`. For the inversion, there is no extra operator.

Area-Optimal Multiplication and Division

The (so-called **centred**) multiplication of two disc intervals $A = \langle a,\ r_a \rangle$ and $B = \langle b,\ r_b \rangle$ defined above delivers, for given centre $a \cdot b$, an optimal confinement of the resulting point complex $\{\alpha \cdot \beta \,|\, \alpha \in A,\ \beta \in B\}$. This confinement, however, is not area-optimal.

Determining an area-optimal confinement of the point–result set under multiplication of two disc intervals is more tedious and leads to solving an equation of third degree (see [11]).

For $A = \langle a,\ r_a \rangle$ and $B = \langle b,\ r_b \rangle$, the **area-optimal** multiplication is defined as

$$\begin{aligned}
A \cdot_{opt} B \ := \ \langle ab\,(1 + x_0),\ \ & (3\,|ab|^2 {x_0}^2 \\
& + 2\,(|ab|^2 + |r_a b|^2 + |r_b a|^2)\, x_0 \\
& + |r_a b|^2 + |r_b a|^2 + (r_a r_b)^2\,)^{\frac{1}{2}} \rangle
\end{aligned}$$

where x_0 is the non-negative zero of the polynomial

$$P(x) \ = \ 2\,|ab|^2 x^3 + (|ab|^2 + |r_a b|^2 + |r_b a|^2) x^2 - r_a^2 r_b^2$$

if `grad`$(P) \geq 2$ (otherwise, we set $x_0 = 0$).

The area-optimal division of two disc intervals is then defined as

$$A/_{opt}B := A \cdot_{opt} (1/\,B).$$

The package `intpakX` contains a realization of the area-optimal multiplication (&`cmult_opt`) and the area-optimal division (&`cdiv_opt`) of two disc intervals.

General Procedure when Implementing the Basic Operations

Let A and B be two disc intervals, which can be displayed on the calculator exactly and let $* \in \{+, -, \cdot, /\}$. In order to obtain a safe confinement C on the machine of the exact result complex $A * B$, during the implementation was proceeded as follows:

1. Calculate a real machine interval cx confining the real part of the centre of the result interval, and a real machine interval cy confining the imaginary part.
2. Calculate the radius r of the resulting circle as:

   $$\begin{aligned} r1 &:= \text{sup(formula for the radius evaluated with intervals)} \\ r2 &:= \triangle(r1 + d(cx)) \\ r &:= \triangle(r2 + d(cy)) \end{aligned}$$

   where $\triangle$ denotes the rounding from above and $d(cx)$, $d(cy)$ the diameter of cx resp. cy.
3. Set $C = \langle m(cx) + i \cdot m(cy), r\rangle$. $m(cx)$ and $m(cy)$ denote the centre of cx resp. cy.

In order to determine the centre of an interval, the procedure `mid` is used. In contrast to the `intpak`-procedure `midpoint`, it does not calculate a confinement of the centre of an interval, but a number (approximation of the centre of the interval) lying certainly within the entered interval.

A Numerical Example

For $A = \langle 1, 1\rangle,\ B = \langle -1 + i, 1\rangle$ we have

$$A + B = \langle i, 2\rangle$$
$$A - B = \langle 2 - i, 2\rangle$$
$$1\,/\,B = \langle -1 - i, 1\rangle$$

Calculation with Maple

```
>  A:=[1,0,1]:  B:=[-1,1,1]:
>  A &cadd B;
                              [0, 1.000000000, 2.000000005]
```

```
>  A &csub B;
```
$$[2.000000000, -1.000000000, 2.000000007]$$
```
>  1 &cdiv B;
```
$$[-1.000000001, -1.000000001, 1.000000051]$$

Using centred multiplication (see p. 31) one obtains

$$A \cdot B = \langle -1+i, 2+\sqrt{2}\rangle \approx \langle -1+i, 3.414213562\rangle$$
$$A \,/\, B = \langle -1-i, 2+\sqrt{2}\rangle$$

```
>  A &cmult B;
```
$$[-1.000000000, 1.000000000, 3.414213579]$$
```
>  A &cdiv B;
```
$$[-1.000000001, -1.000000001, 3.414213685]$$

and using area-optimal multiplication leads to

```
>  A &cmult_opt B;
```
$$[-1.390388204, 1.390388204, 2.969562256]$$
```
>  A &cdiv_opt B;
```
$$[-1.390388219, -1.390388219, 2.969562345]$$

In Figure 5 on p. 33 are displayed simultaneously the resulting point set of the product of A and B, the centred confinement and the area-optimal confinement of the set.

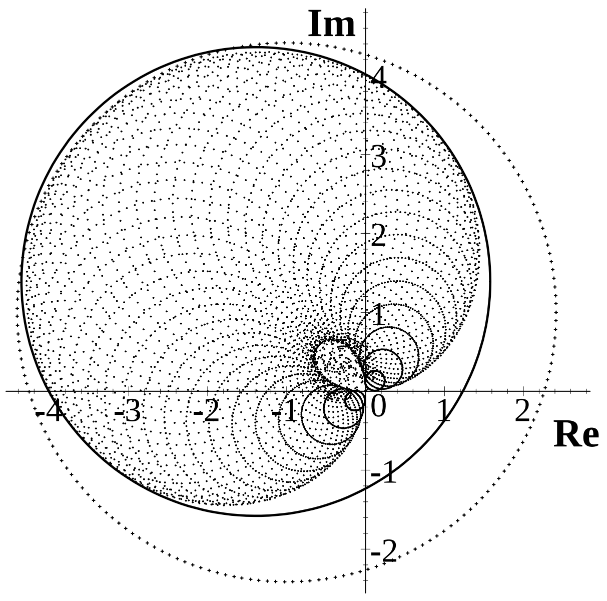


Figure 5: centred and area-optimal confinement of $\langle 1,1\rangle \cdot \langle -1+i, 1\rangle$

5.2 Range of Complex Polynomials

A first possible application of the in `intpakX` defined disc arithmetic is the determination of safe confinements for the range of a polynomial with complex coefficients over a disc interval.

For this, there are three procedures

1. `horner_eval_cent` (Horner-scheme using centred multiplication &`cmult`),
2. `horner_eval_opt` (Horner-scheme using area-optimal multiplication &`cmult_opt`),
3. `centred_form_eval` (centred form for complex polynomials).

The procedures `horner_eval_opt` and `centred_form_eval` generally give certainly better confinements than the procedure `horner_eval_cent`. They have, however, a substantially higher demand for time and memory.

The first input parameter of each procedure is a (complex) polynomial in the variable z. The denomination of this variable is compulsory! As second parameter must be entered a number or variable of type `complex_disc`.

Example 1: Confining the range of

$$\begin{aligned} p(z) \; := \; & (0.15 - 0.1i) + (0.15 - 0.12i)z + (-0.2 - 0.2i)z^2 \\ & +(0.1 + 0.3i)z^3 + (0.1 - 0.2i)z^4 + (0.1 - 0.2i)z^5 \\ & +(0.2 - 0.2i)z^6 + (0.1 - 0.2i)z^7 + (0.2 - 0.1i)z^8 \\ & +(0.1 - 0.1i)z^9 \end{aligned}$$

over the interval $Z = \langle -0.1 + 0.2i,\, 0.9\rangle$.

```
> p_H:=horner_eval_cent(p,Z);
```
$$p\_H := [.1590115281, -.04050517670, 3.058832329]$$
```
> p_Hopt:=horner_eval_opt(p,Z);
```
$$p\_Hopt := [.2219721872, .2917174855, 2.243412729]$$
```
> p_C:=centred_form_eval(p,Z);
```
$$p\_C := [.1590115281, -.04050517670, 1.717944237]$$

In this example the procedure `centred_form_eval` gives the best confinement. The graphical illustration of the range of p over Z and the calculated confinements can be found in Figure 6.

Generating the graphics

For illustrating the range of p over Z the command `complexplot` from the `plots`-package can be used. Each command should be ended by a colon (otherwise

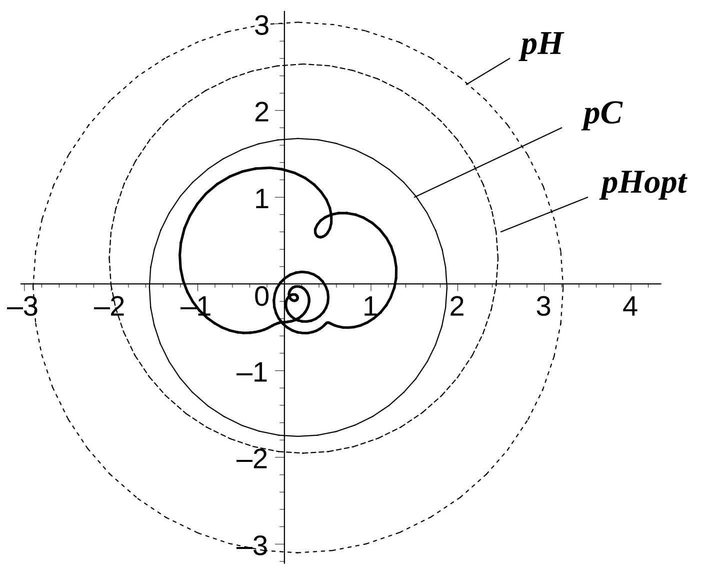


Figure 6: confinement of the range of a complex polynomial

very much, generally unnecessary, information is given out).

```
>  c1:=complexplot(subs(z=Z[1]+I*Z[2]+Z[3]*(cos(t)+I*sin(t)),p),
                       t=0..2*Pi,color=black,thickness=3,numpoints=200):
```

For graphical illustration of the computed confinements the command `complex_disc_plot` was used.

```
>  c2:=complex_disc_plot(p_H,color=black,thickness=2,linestyle=4):
>  c3:=complex_disc_plot(p_Hopt,color=black,thickness=2,linestyle=3):
>  c4:=complex_disc_plot(p_C,color=black,thickness=2):
```

The labelling of the graphic was generated with a series of `plot`- and `textplot`-commands. `textplot` is a command from the `plots`-Package.

```
>  s1:=plot([[2.1,2.3],[2.6,2.6]],color=black,thickness=2):
>  t1:=textplot([3,2.8,pH],font=[TIMES,BOLDITALIC,12]):
>  s2:=plot([[1.5,1],[3.2,1.8]],color=black,thickness=2):
>  t2:=textplot([3.7,2,pC],font=[TIMES,BOLDITALIC,12]):
>  s3:=plot([[2.5,0.6],[3.5,1]],color=black,thickness=2):
>  t3:=textplot([4.2,1.2,pHopt],font=[TIMES,BOLDITALIC,12]):
```

For simultaneous display of the generated graphics the command `display` from the `plots`-package can be used.

```
>  display([c1,c2,c3,c4,s1,s2,s3,t1,t2,t3],scaling=constrained);
```

Example 2: Confinement of the value of

$$p(z) := (z-i)^4(z-1-i)^5$$

at the place $z = 1$. The exact value is $p(1) = 4i$.

```
>   subs(z=1,p);
```

$$4\,I$$

Calculation of confinements using the disc arithmetic:

```
>   Digits:=60:
>   horner_eval_cent(p,1);
```

$[0,\ 4.00000000000000000000000000000000000000000000000000000000000,$
$\quad .281310000000000000000000000000000000000000000000000000000045\ 10^{-55}]$

```
>   horner_eval_opt(p,1);
```

$[0,\ 4.00000000000000000000000000000000000000000000000000000000000,$
$\quad .281310000000000000000000000000000000000000000000000000000045\ 10^{-55}]$

```
>   centred_form_eval(p,1);
```

$[0,\ 4.00000000000000000000000000000000000000000000000000000000000,$
$\quad .112542000000000000000000000000000000000000000000000000000010\ 10^{-54}]$

All three procedures give out as confinement of $p(1)$ a disc interval with centre $4i$ and the radii of the computed confinements are around 10^{-55}, if one calculates with 60 digits (`Digits = 60`).

5.3 The Exponential Function for Disc Intervals

The image of a disc interval $Z = \langle c,\, r\rangle$ under the exponential function is in general not a disc. If as centre of the resulting interval is prescribed the point $\exp(c)$, then

$$\exp(Z) := \langle \mathsf{e}^c,\ |\mathsf{e}^c|(\mathsf{e}^r - 1)\rangle$$

defines an optimal confinement (under given centre $\exp(c)$ of the resulting point complex $\{\exp(z) \mid z \in Z\}$).

A detailed discussion of images of disc intervals under the exponential function is found in [4].

The realization of the exponential function for disc intervals `cexp` from the extension `intpakX` has as input parameter a variable of type `complex_disc` or a complex number and gives a safe confinement of the resulting point complex.

Example: Confinement of the image of $Z = \langle 0, \pi + 1\rangle$ under the exponential function

```
> Cexp:=cexp([0,0,evalf(Pi+1)]);
```
$$Cexp := [1.000000000, 0, 61.90292461]$$

Graphical illustration of the computed disc interval confinement:

```
> c1:=complex_disc_plot(Cexp,color=black,thickness=3,numpoints=400):
```

Image of the boundary of $\langle 0, \pi + 1\rangle$ under the exponential function (Attention: `plots` must be loaded beforehand with `with`!):

```
> c2:=complexplot(exp(polar(Pi+1,phi)),phi=0..2*Pi,
                        color=black,thickness=3,numpoints=400):
```

For illustrating inner points of the range, the radius r is varied from 0 to $\pi + 1$ for fixed angle. Example:

```
> c2:=complexplot(exp(polar(r,0.5)),r=0..Pi+1,
                        color=black,thickness=3,numpoints=400):
```

The different graphic commands are bundled afterwards with `display`. On display should be entered as additional option `scaling=constrained`. In order to display a cut out piece, the `plot`-option `view` was used.

The graphical illustration in Figure 7 shows that the centred confinement of the image of $\langle 0, \pi + 1\rangle$ under the exponential function is the optimal confinement for prescribed result-centre $\exp(0) = 1$. It is far from area-optimal. Figure 8 shows a cut out piece around the origin.

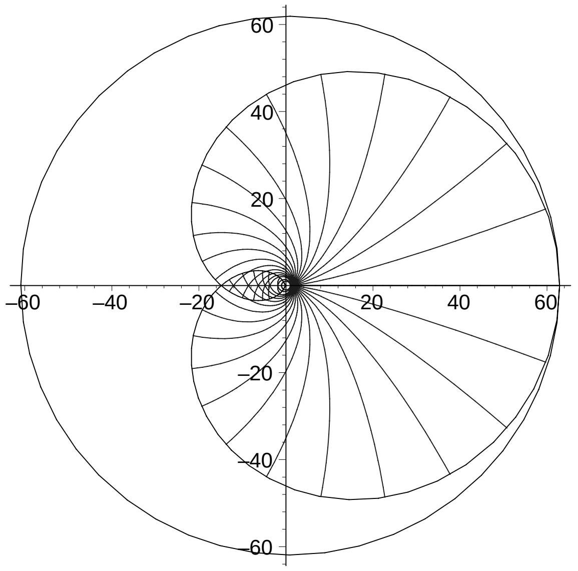


Figure 7: image of the disc interval $\langle 0, \pi + 1\rangle$ under the exponential function and centred disc interval confinement of $\exp(\langle 0, \pi + 1\rangle)$

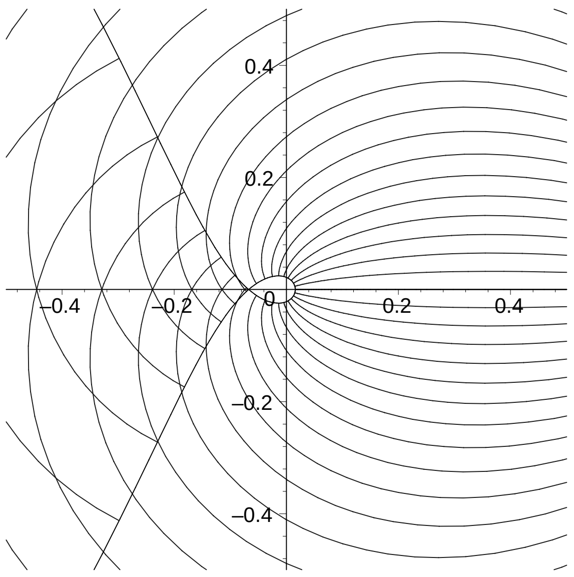


Figure 8: illustration of a cut out piece around the origin

The exponential function is $2\pi\mathtt{i}$ periodic, and as the diameter of the interval $Z = \langle 0,\ \pi + 1\rangle$ is greater than 2π, there is an area in the image Z of the exponential function, in which every point has exactly two preimages. This area can be recognized by its double hatching. Besides, in the illustration of the cut out piece can be observed the drop-like image-free area around the origin.

6 Evaluation and Outlook

Using verification algorithms it is possible, if occasion arises, to prove with the calculator automatically the existence and uniqueness of a solution for a given problem, and also to compute a (narrow) confinement of the exact solution. The results obtained in this way have the same mathematical quality as results obtained e. g. by using computer algebra systems, i. e. by automatic formula manipulations. Thereby it turns out to be a great advantage that verification algorithms can handle safely numerical input-data with errors within tolerance. In such cases infinitely many problems are solved simultaneously. For an entire family of problems it is e. g. proved that each has a unique solution.

Whenever computer algebra packages make use of numerical routines (e. g. when computing determined integrals), a verification algorithm should be used if possible. The results obtained are then mathematically safe (a property usually expected when working with a CA-system). Pretended solutions resp. far off approximations are then excluded.

Also by using interval methods, the graphic abilities of computer algebra systems can be improved resp. made secure. That this is necessary, is impressively shown by

Example 3 in the appendix.

Computer algebra and verification numerics complete each other ideally. Thus the computer becomes for the mathematician, but also for the engineer, a safe mathematical tool. Especially in view of processors which are becoming faster and faster and more powerful, the symbiosis of symbolic calculus and safe numerical routines should be pushed forward massively.

7 Appendix: Questionable Maple Results

The following examples show that also results obtained with computer algebra systems should be examined carefully. In order to improve the reliability and to disclose more areas of application, verification algorithms should be integrated additionally into such systems. All examples have been computed using Maple V.

Example 1: Wrong computation of minimum and maximum

```
> f:=x->x^2+sin(x)+cos(2*x);
```

$$f := x \to x^2 + \sin(x) + \cos(2\,x)$$

Attempt of calculating the range of f over the interval [-2.,0] using the Maple-functions `minimize` and `maximize`:

```
> minimize(f(x),x,-2..0);
```

$$1$$

```
> r1:=evalf(%);
```

$$\mathit{r1} := 1.$$

```
> maximize(f(x),x,-2..0);
```

$$4 - \sin(2) + \cos(4)$$

```
> r2:=evalf(%);
```

$$\mathit{r2} := 2.437058952$$

```
> range_f:=[r1,r2];
```

$$\mathit{range\_f} := [1., 2.437058952]$$

Figure 9 shows that `range_f` is obviously **not** a confinement of the range of f over the interval $[-2., 0]$!

Range confinement using `compute_range`:

```
> compute_range(f,[-2.,0],3,adaptive,Nx=4,quadratic):
> range_f:=r[3];
```

$$\mathit{range\_f} := [-.2786237965, 2.437058957]$$

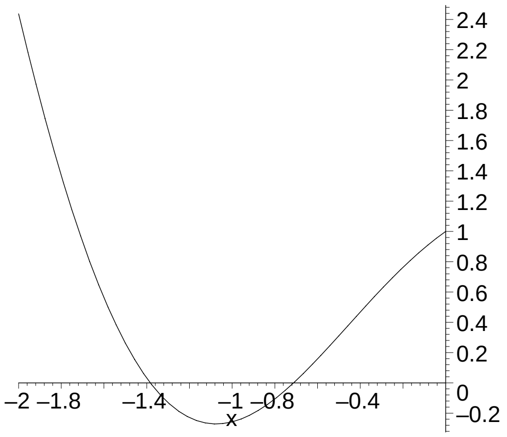


Figure 9: illustrating the funktion $f(x) := x^2 + \sin(x) + \cos(2x)$

Coarse estimation of the maximal error in determining the minimum:

```
>  max(seq(width(list_of_ranges[i]),i=8..11));
```

$$.0499309122$$

Graphical illustration of the last iteration step:

```
>  display([q[3],q[4]],view=[-1.8..-0.51,-0.28..1.5]);
```

This command generates figure 10.

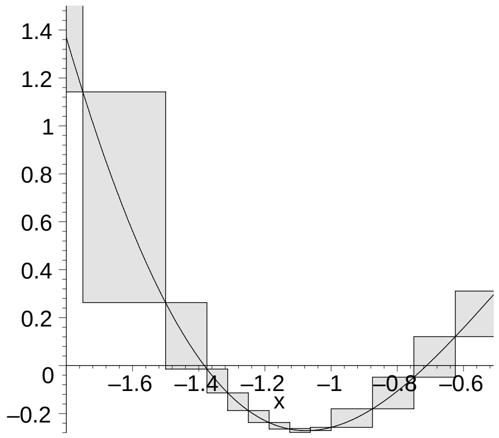


Figure 10: verfied range confinement

Example 2: Questionable integration result and limit calculation

```
>  int(x/cosh(x), x=1..2);
```

$$-2\,I\ln(1+I\,e^2)-\frac{1}{2}\ln(1+I\,e^2)\,\pi-I\,\mathrm{dilog}(1+I\,e^2)+\frac{1}{2}\ln(e-I\,e^{(-1)})\,\pi-I\,\mathrm{dilog}(I\,e^2)$$
$$+\frac{1}{2}\ln(e+I\,e^{(-1)})\,\pi+\frac{1}{2}\pi+I\ln(1+I\,e)+\frac{1}{2}\ln(1+I\,e)\,\pi+I\,\mathrm{dilog}(1+I\,e)$$
$$-\frac{1}{2}\ln(e^{(1/2)}-I\,e^{(-1/2)})\,\pi+I\,\mathrm{dilog}(I\,e)-\frac{1}{2}\ln(e^{(1/2)}+I\,e^{(-1/2)})\,\pi$$

Is this formula correct?

```
>  for k from 2 to 5 do evalf(Int(x/cosh(x), x=0..10^k)); od;
```

$$1.831931188$$
$$1.831931188$$
$$.2300979673\,10^{-28}$$
$$0$$

Does

$$\int_0^t \frac{x}{\cosh(x)}dx$$

really converge with increasing t towards 0?

Example 3: Unreliable graphics

The following Maple command should generate a circle. In fact, however, the result is Figure 11:

```
>  implicitplot(x^2+y^2 = 1, x=-1..1, y=-15..50, numpoints=10000,
scaling=constrained);
```

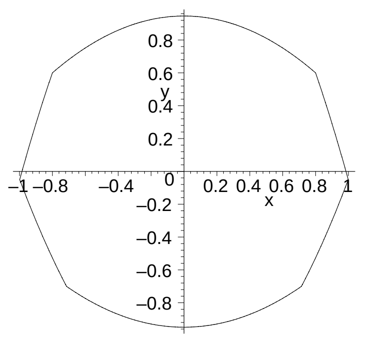


Figure 11: A circle?

Now it is attempted to generate a daisy with 90 blossom leaves (the result is given in Figure 12):

```
> r:= 1/2*sin(90*t):  plot( [(1+r)*cos(t), (1+r)*sin(t), t=0..2*Pi]);
```

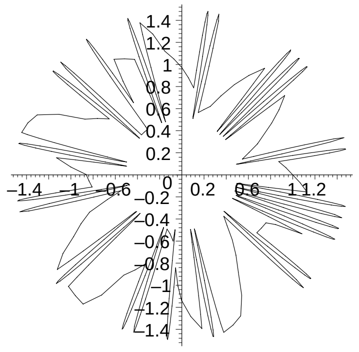


Figure 12: A daisy with 90 blossom leaves?

If the graphic routines are supported by suitable verification steps [3], then the results in the figures are, as expected, a circle resp. a stilized blossom.

References

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