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Modelling and Simulation of Forced Oscillators with Random Periods

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Abstract In nanoelectronics, the miniaturisation of circuits causes uncertainties in the components. An uncertainty quantification is achieved by the introduction of random parameters in corresponding mathematical models. We consider forced oscillators described by time-dependent differential algebraic equations, where a random period appears. A corresponding uncertainty quantification results from a modelling based on a transformation to a unit time interval. We apply the technique of the generalised polynomial chaos to resolve the stochastic model. Thereby, a Galerkin approach yields a larger coupled system of differential algebraic equations satisfied by an approximation of the random process. We present numerical simulations of an illustrative example.

1 Introduction

Uncertainty quantification becomes important in nanoelectronics, since the downscaling of circuits produces undesired but inevitable variations in the components. In the mathematical models, corresponding physical parameters are substituted by random variables to describe the uncertainties. We consider the traditional modelling of electric circuits by differential algebraic equations (DAEs), where the time-dependent solution becomes a random process now.

On the one hand, forced oscillators with random parameters, where the period of the input signals is constant and deterministic, have been investigated in [5–8]. On the other hand, autonomous oscillators with random parameters have been considered in [10]. Thereby, the period depends on the random parameters, since no input signals appear. Now we analyse the case of forced oscillators, where the period of the input signals is assumed to be a random variable modelling an own uncertainty. This setting represents a mixture of the two previous cases.

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Since the period is given by a random variable, the domain of dependence differs for a single oscillation. We achieve a model for uncertainty quantification by a transformation to a unit time interval as for autonomous oscillators. The stochastic model can be resolved by a quasi Monte-Carlo simulation, for example. We use the technique of the generalised polynomial chaos (gPC), see [1,2,12], in the numerical simulation to investigate a more sophisticated approach. A Galerkin method results in a larger coupled system of DAEs, which yields an approximation of the periodic random process. To illustrate the modelling and the simulation, we apply a transistor amplifier supplied by an input with random period as test example.

2 Modelling of Uncertainties in Period

The mathematical modelling of electric circuits is based on network approaches, which yield systems of DAEs, see [3]. We consider general systems of the form

$$A(\mathbf{p})\mathbf{x}'(t,\mathbf{p}) = \mathbf{f}(t,\mathbf{x}(t,\mathbf{p}),\mathbf{p}), \tag{1}$$

where $\mathbf{x} : [t_0,t_1] \to \mathbb{R}^n$ represents unknown node voltages, branch currents and possibly other quantities. The singular matrix $A \in \mathbb{R}^{n\times n}$ and the right-hand side $\mathbf{f}$ include physical parameters $\mathbf{p} = (p_1,\ldots,p_q)^\top$ from some relevant set $Q \subseteq \mathbb{R}^q$. Hence the solution $\mathbf{x}$ of (1) depends on time as well as the parameters. If the matrix A is regular, then the system (1) consists of implicit ordinary differential equations (ODEs).

We assume that the chosen parameters exhibit some uncertainties. Consequently, we replace the parameters by independent random variables $\mathbf{p} : \Omega \to Q$ according to some probability space $(\Omega,\mathscr{A},\mu)$. We use a classical random distribution for each parameter like Gaussian, uniform, beta, etc. Given a function $f \in L^1(\Omega)$ depending on the random parameters, we denote the expected value by

$$\langle f(\mathbf{p})\rangle := \int_\Omega f(\mathbf{p}(\omega))\,\mathrm{d}\mu(\omega) = \int_Q f(\mathbf{p})\rho(\mathbf{p})\,\mathrm{d}\mathbf{p} \tag{2}$$

with the probability density function $\rho : Q \to \mathbb{R}$. The expected value (2) yields an inner product $\langle f(\mathbf{p})g(\mathbf{p})\rangle$ for two functions $f,g \in L^2(\Omega)$ depending on the random parameters. We also apply the expected value (2) to vector-valued or matrix-valued functions by components.

We investigate forced oscillators, i.e., the right-hand side of (1) includes periodic input signals with time rate T. Now let the period T also be a random variable. Two cases imply the same model:

i) Although the period corresponds to the input signals, it is chosen in dependence on the selected parameters, i.e., $T = T(\mathbf{p})$. In contrast to the case of autonomous oscillators, see [10], we assume that the period can be evaluated directly for a given tuple of parameters. Hence the period inherits some uncertainties from the random parameters.

ii) The period is considered as an additional independent random parameter due to an own uncertainty. Thus we introduce the period to the set of parameters. Without loss of generality, we can write $T = T(\mathbf{p})$ as in the scenario (i), where the special case $T(\mathbf{p}) = p_1$ is given, for example.

Consequently, let $\mathbf{f}(t+T(\mathbf{p}),\cdot,\mathbf{p}) = \mathbf{f}(t,\cdot,\mathbf{p})$ for all $t \in \mathbb{R}$ and each $\mathbf{p} \in Q$ in (1). We assume that the solution of the dynamical system (1) inherits the periodicity, i.e,

$$\mathbf{x}(t+T(\mathbf{p}),\mathbf{p}) = \mathbf{x}(t,\mathbf{p}) \qquad \text{for all } \ t \ \text{ and each } \ \mathbf{p} \in Q. \tag{3}$$

We restrict our attention to a single cycle of each periodic solution. Since a single cycle is given in a time interval $[0,T(\mathbf{p})]$, the domain of dependence differs in case of random parameters. We want to compare the periodic solutions, which represent realisations for different random parameters. In particular, an expected value and a corresponding variance describe a kind of comparison of the realisations with respect to the underlying random distribution. However, a direct definition of an expected value or a variance corresponding to the single cycles of the random process is not feasible, because the domains of dependence differ.

As for autonomous oscillators, see [10], we transform the time intervals $[0,T(\mathbf{p})]$ into the unit interval $[0,1]$. The transformed solution reads

$$\tilde{\mathbf{x}}(\tau,\mathbf{p}) := \mathbf{x}(\tau T(\mathbf{p}),\mathbf{p}) \qquad \text{for each } \ \mathbf{p} \in Q \tag{4}$$

with the independent variable τ. It follows the periodicity

$$\tilde{\mathbf{x}}(\tau+1,\mathbf{p}) = \tilde{\mathbf{x}}(\tau,\mathbf{p}) \qquad \text{for all } \ \tau \ \text{ and each } \ \mathbf{p} \in Q. \tag{5}$$

The same relations are given for the input signals in the right-hand side of (1).

The transformation (4) changes the DAEs (1) into the equivalent system

$$A(\mathbf{p})\tilde{\mathbf{x}}'(\tau,\mathbf{p}) = T(\mathbf{p})\,\mathbf{f}(\tau T(\mathbf{p}),\tilde{\mathbf{x}}(\tau,\mathbf{p}),\mathbf{p}). \tag{6}$$

Due to (5), the corresponding periodic boundary conditions read

$$\tilde{\mathbf{x}}(0,\mathbf{p}) = \tilde{\mathbf{x}}(1,\mathbf{p}) \qquad \text{for each } \ \mathbf{p} \in Q. \tag{7}$$

We apply the stochastic model (6),(7) in case of random periods, where the solution is the periodic random process $\tilde{\mathbf{x}}$. The original random process $\mathbf{x}$ satisfying (1) is obtained via the transformation (4). The expected value of $\tilde{\mathbf{x}}$ can be seen as a reference shape of the oscillations in the standardised time interval $[0,1]$, where the locations are relative to the input signals. The variance of $\tilde{\mathbf{x}}$ characterises the discrepancies with respect to the reference shape.

3 Numerical Simulation

The stochastic model (6),(7) can be resolved by a quasi Monte-Carlo simulation, for example. Common numerical techniques yield the solutions of the resulting periodic boundary value problems like multiple shooting methods, finite difference methods or harmonic balance. Typically, a large number of samples is required to achieve sufficiently accurate approximations.

Alternatively, we derive a technique based on the gPC, see [1,2,12]. The gPC has already been applied to forced oscillators with constant periods in [5–8]. Assuming finite second moments, the random process satisfying (6) can be represented via

$$\tilde{\mathbf{x}}(\tau, \mathbf{p}(\omega)) = \sum_{i=0}^{\infty} \mathbf{v}_i(\tau) \Phi_i(\mathbf{p}(\omega)). \tag{8}$$

A complete set of basis polynomials $\Phi_i : Q \to \mathbb{R}$ is involved, where we consider an orthonormal system, i.e., $\langle \Phi_i \Phi_j \rangle = \delta_{ij}$ with the Kronecker delta. Each random distribution implies a corresponding polynomial basis. The multivariate polynomials are just products of the orthogonal univariate polynomials. Hence the basis polynomials are known explicitly. The time-dependent coefficient functions satisfy the equation

$$\mathbf{v}_i(\tau) = \langle \tilde{\mathbf{x}}(\tau, \mathbf{p}) \Phi_i(\mathbf{p}) \rangle. \tag{9}$$

The series (8) converges point-wise for each τ in $L^2(\Omega)$. The coefficient functions (9) inherit the smoothness of the random process under certain assumptions.

The unknown coefficient functions can be determined by either a stochastic collocation or the stochastic Galerkin approach, see [11,12]. In a stochastic collocation method, approximations of the probabilistic integrals (9) are computed. We apply the stochastic Galerkin method in the following. A truncation of the series (8) at the mth term yields an approximation of the random process. Inserting this finite approximation in the DAEs (6) causes the residual

$$\mathbf{r}(\tau, \mathbf{p}) := A(\mathbf{p}) \left(\sum_{i=0}^{m} \mathbf{v}_i'(\tau) \Phi_i(\mathbf{p}) \right) - T(\mathbf{p})\, \mathbf{f}\left(\tau T(\mathbf{p}), \sum_{j=0}^{m} \mathbf{v}_j(\tau) \Phi_j(\mathbf{p}), \mathbf{p} \right).$$

The Galerkin approach demands that the residual is orthogonal with respect to the space spanned by the applied basis functions, i.e.,

$$\langle \mathbf{r}(\tau, \mathbf{p}) \Phi_l(\mathbf{p}) \rangle = \mathbf{0} \qquad \text{for each } \tau \text{ and } l = 0, 1, \ldots, m.$$

It follows the larger coupled system of DAEs

$$\sum_{i=0}^{m} \langle \Phi_l(\mathbf{p}) \Phi_i(\mathbf{p}) A(\mathbf{p}) \rangle \mathbf{v}_i'(\tau) = \left\langle \Phi_l(\mathbf{p})\, T(\mathbf{p})\, \mathbf{f}\left(\tau T(\mathbf{p}), \sum_{j=0}^{m} \mathbf{v}_j(\tau) \Phi_j(\mathbf{p}), \mathbf{p} \right) \right\rangle \tag{10}$$

for $l = 0, 1, \ldots, m$. A constant matrix appears in the left-hand side of the complete system. The coefficient functions inherit the periodicity of the random process due to (9). Hence we arrange the boundary conditions

$$\mathbf{v}_i(0) = \mathbf{v}_i(1) \qquad \text{for } i = 0, 1, \ldots, m. \tag{11}$$

The periodic boundary value problem (10),(11) can be solved by common numerical techniques again. Often the probabilistic integral in the right-hand side of (10) cannot be calculated explicitly. Gaussian quadrature yields an approximation of the right-hand side evaluations.

A special case appears in case of a constant matrix, i.e., $A(\mathbf{p}) = A_0$. Due to the orthogonality of the basis polynomials, the coupled system (10) simplifies to

$$A_0 \mathbf{v}_l'(\tau) = \left\langle \Phi_l(\mathbf{p})\, T(\mathbf{p})\, \mathbf{f}\left(\tau T(\mathbf{p}), \sum_{j=0}^{m} \mathbf{v}_j(\tau)\Phi_j(\mathbf{p}), \mathbf{p} \right) \right\rangle \tag{12}$$

for $l = 0, 1, \ldots, m$. Hence the constant matrix corresponding to the left-hand side of the complete system becomes block-diagonal.

In contrast to a Monte-Carlo simulation, the gPC problem (10),(11) has to be solved just once. A simulation based on the larger coupled system from the stochastic Galerkin method is often more efficient than a quasi Monte-Carlo simulation in case of linear systems of differential equations (ODEs, DAEs or PDEs). The above approach is also feasible for linear systems (1) with time-dependent inputs. However, autonomous oscillators are described by nonlinear systems of ODEs or DAEs in most instances. In the nonlinear case, the efficiency of the stochastic Galerkin approach requires further investigations.

The solution of boundary value problems of dynamical systems with random parameters via the gPC using either a stochastic collocation or the stochastic Galerkin approach is analysed more detailed in [9].

4 Illustrative Example

We apply a transistor amplifier shown in Figure 1 (left). A mathematical modelling yields a nonlinear system of DAEs for the unknown five node voltages, see [4]. The differential index of the DAEs is one. We arrange the input signal

$$U_{\text{in}}(t) = 0.4 \sin\left(\tfrac{2\pi}{T} t\right)$$

with period T. A corresponding periodic solution for $T = 0.01$ is depicted in Figure 1 (right). The voltages U_1, U_2, U_3 exhibit the form of sine waves. In contrast, the voltage U_4 and the output voltage U_5 behave nonlinearly due to the transistor.

Now we consider a random period

$$T(p) = 0.01(1 + 0.1p),$$

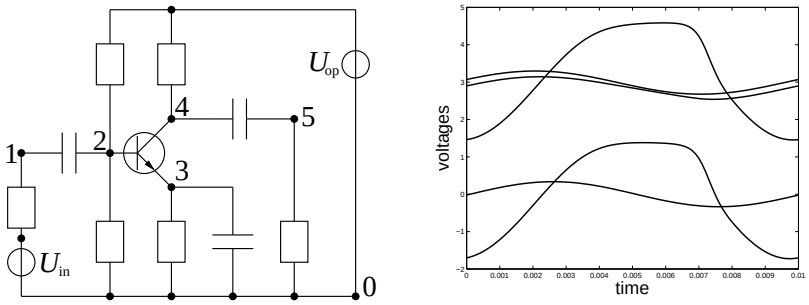

Fig. 1 Circuit of transistor amplifier (left) and deterministic periodic solution for $T = 0.01$ (right).

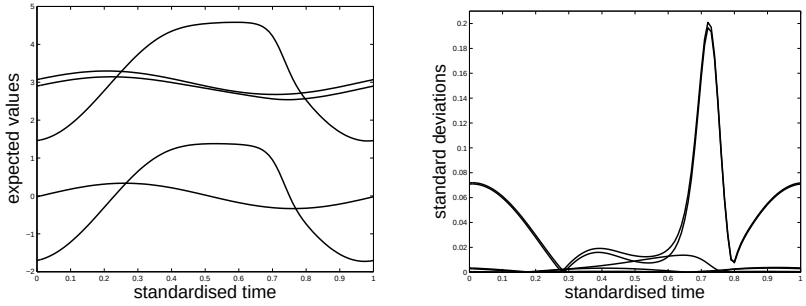

Fig. 2 Expected values (left) and standard deviations (right) for node voltages with random period.

where p represents a standardised random variable with a beta distribution according to the probability density function

$$\rho(p) = C(\alpha,\beta)(1-p)^{\alpha}(1+p)^{\beta} \qquad \text{for } -1 \le p \le 1$$

with a constant $C(\alpha,\beta)$. Hence the random period itself is distributed of beta type. We choose $\alpha = \beta = 2$. It follows the expected value $\langle T(p)\rangle = 0.01$, the standard deviation $\sigma(T(p)) \doteq 3.8 \cdot 10^{-4}$ and the range of the random period includes variations up to 10%. The other physical parameters are chosen deterministic.

The gPC expansion (8) includes the Jacobi polynomials. We apply polynomials up to degree $m = 3$. The larger coupled system exhibits the structure (12). The periodic boundary value problem (12),(11) is solved by a finite difference method using asymmetric formulas of second order (BDF2). Figure 2 illustrates the resulting approximations of the expected values (degree 0) and the standard deviations corresponding to the five node voltages. The expected values are similar to the deterministic solution shown in Figure 1 (right). The standard deviation of the voltages U_1, U_2, U_3 is relatively low. In contrast, the voltages U_4 and U_5 feature a subdomain in time (near $\tau = 0.7$), where a relatively high standard deviation appears. The standard deviation of U_4 and U_5 is nearly the same, since the shape of the oscillations

agrees. Furthermore, Figure 3 illustrates the coefficient functions (9) of the output voltage U_5. The magnitude of the coefficient functions decreases rapidly for increasing degree, which reflects the convergence of the gPC representation (8).

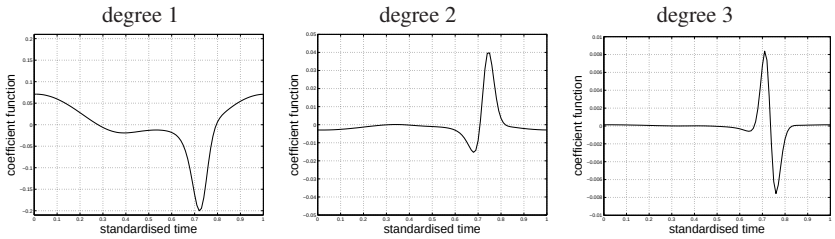

Fig. 3 Coefficient functions of output voltage in gPC.

For comparison, we compute a reference solution via a quasi Monte-Carlo simulation using $K = 1000$ samples. The periodic boundary value problems (6),(7) are resolved by a finite difference method of second order again. Alternatively, we solve the periodic boundary value problems (12),(11) for different orders m now. The maximum absolute differences between the approximations of the expected values and the variances corresponding to the output voltage are shown in Table 1. The differences corresponding to the other node voltages have the same or a smaller magnitude. As hoped for, the accuracy of the gPC approximations improves for increasing order m. In particular, a linear approximation ($m = 1$) is not sufficiently accurate, whereas nonlinear polynomials of a low order yield an adequate numerical solution. The differences do not decrease for $m \geq 4$ any more. To achieve a better agreement for large orders m, the number K of samples has to be increased in the Monte-Carlo simulation and a higher accuracy has to be demanded in all involved finite difference methods.

Table 1 Maximum differences between approximations from gPC for different order m and approximations from quasi Monte-Carlo simulation using $K = 1000$ samples corresponding to output voltage U_5.

	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
expected value	$2.6 \cdot 10^{-3}$	$8.4 \cdot 10^{-5}$	$5.1 \cdot 10^{-6}$	$4.3 \cdot 10^{-6}$	$4.5 \cdot 10^{-6}$
variance	$4.5 \cdot 10^{-3}$	$2.2 \cdot 10^{-4}$	$5.3 \cdot 10^{-5}$	$3.6 \cdot 10^{-5}$	$3.7 \cdot 10^{-5}$

5 Conclusions

A modelling of forced oscillators with random periods has been introduced, which defines a corresponding random process. We have constructed a numerical technique based on the generalised polynomial chaos for solving the stochastic model. A Galerkin approach changes the underlying system of differential algebraic equations into a larger coupled system of differential algebraic equations. We presented numerical simulations of a test example, which confirm that the stochastic Galerkin approach is feasible in this application. Further investigations are required for statements on the efficiency of the technique in comparison to stochastic collocation methods or quasi Monte-Carlo simulations. In particular, more test examples have to be considered for a discussion of the efficiency.

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