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RESEARCH ARTICLE

A high-order compact method for nonlinear Black-Scholes option pricing equations of American Options

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Due to transaction costs, illiquid markets, large investors or risks from an unprotected portfolio the assumptions in the classical Black–Scholes model become unrealistic and the model results in nonlinear, possibly degenerate, parabolic diffusion–convection equations.

Since in general, a closed–form solution to the nonlinear Black–Scholes equation for American options does not exist (even in the linear case), these problems have to be solved numerically. We present from the literature different compact finite difference schemes to solve nonlinear Black–Scholes equations for American options with a nonlinear volatility function. As compact schemes cannot be directly applied to American type options, we use a fixed domain transformation proposed by Ševčovič and show how the accuracy of the method can be increased to order four in space and time.

Keywords: nonlinear Black-Scholes equation; compact finite difference scheme; American options; high-order methods; fixed domain transformation; transaction costs

AMS Subject Classification: 65M06; 65M12; 65N06

1. Introduction

In the recent several years stock option was one of the most popular financial derivatives. There are many types of options on the market including European Call (Put) options, American Call (Put) options, Exotic options, etc.. But it was difficult to accurately price options until 1973 when Black and Scholes published their Black-Scholes model [5].

There exists two types of vanilla options. *A Call option* is a contract that gives the right to the holder to buy the underlying asset on a particular date at a specified value. *A Put option* is a contract that gives the right to the holder to sell the underlying asset on a particular date at a specified value. The price in the contract is called *exercise price* (or *strike price*). The date in the contract is called *expiration date* (or *exercise date, maturity date*). Options are also divided into two types according to the expiration date T. While *American options* can be exercised

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at any time before maturity date, *European options* can only be exercised at the maturity date. In this work we will focus on American options.

In an idealized financial market, the fair price of an American Call option can be obtained by solving the Black-Scholes partial differential equation (PDE)

$$V_t + \frac{\sigma_0^2}{2} S^2 V_{SS} + (r-q) S V_S - rV = 0, \quad 0 < S < S_f(t),\, t \in (0,T), \tag{1}$$

where t is the current date, σ_0 the (constant) volatility, S the price of the underlying asset, r the risk-free interest rate, q the dividend yield and V the option price. In the financial sense, the partial derivatives indicate the sensitivity of the option price V to the corresponding parameter and are called *Greeks*. The option delta is denoted by $\Delta = V_S$, the option gamma by $\Gamma = V_{SS}$ and the option theta by $\theta = V_t$, cf. [38].

Equation (1) is a backward-in-time parabolic PDE and is supplied with the terminal *pay-off condition*

$$V(S,T) = \max(S-E,0) =: (S-E)^+, \quad S \geq 0. \tag{2}$$

Since the value of an American Call option equals the value of a European Call option if no dividends are paid and the volatility is constant, we included the continuous dividend yield in (1).

Remark 1 discrete dividend payments

Let us note that also discrete dividend payments can be included here, cf. [38]. We assume that there is only one dividend payment of the dividend yield q during the lifetime of the option at the *dividend date* t_q. Neglecting other factors, such as taxes, the asset price S must decrease exactly by the amount of the dividend payment q at time t_q.

Thus we have the *jump condition*

$$S(t_q^+) = (1-q) S(t_q^-),$$

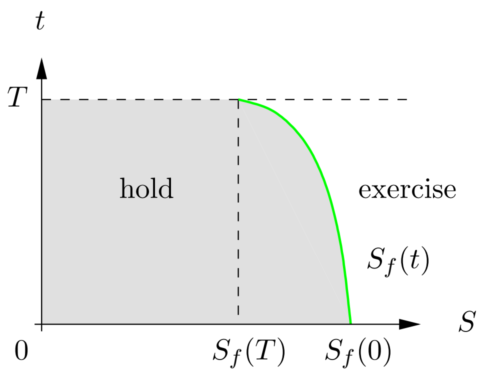
where t_q^-, t_q^+ denote the moments just before and after the dividend date t_q. This leads to the following effect on the option price:

$$V(S, t_q^-) = V((1-q)S, t_q^+), \tag{3}$$

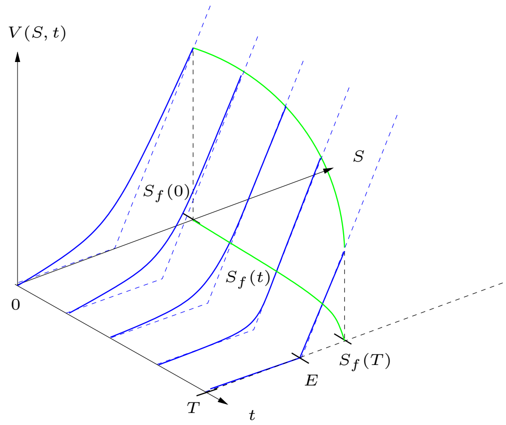
i.e. the value of the option at S and time t_q^- is the same as the value immediately after the dividend date t_q but at the asset value $(1-q)S$. In order to calculate the value of a Call option with one dividend payment we solve the Black–Scholes equation from expiry $t = T$ until $t = t_q^+$ and use the relation (3) to compute the values at $t = t_q^-$. Finally, we continue to solve the Black–Scholes equation backwards starting at $t = t_q^-$ using these values as the initial data. The boundary conditions, that are discussed next, do not need to be modified for this case.

Since American options can be exercised at any time before expiry, we need to find the optimal time t of exercise, known as the *optimal exercise time*. At this time, which mathematically is a *stopping time*, the asset price reaches the *optimal exercise price* or *optimal exercise boundary* $S_f(t)$.

This leads to the formulation of the problem for American options by dividing the domain $[0,\infty[\times[0,T]$ of (1) into two parts along the curve $S_f(t)$ and analyzing each of them (see Fig. 1(a)). Since $S_f(t)$ is not known in advance but has to be determined in the process of the solution, the problem is called *free boundary value problem*.



(a) Exercising and holding regions.



(b) Schematical value $V(S,t)$.

Figure 1. American Call.

For the American Call option the *spatial* domain is divided into two regions by the *free boundary* $S_f(t)$, the *stopping region* $S_f(t) < S < \infty$, $0 \le t \le T$, where the option is exercised or dead with $V(S,t) = S - E$ and the *continuation region* $0 \le S \le S_f(t)$, $0 \le t \le T$, where the option is held or stays alive and equation (1) is valid with the following boundary conditions at $S = 0$ and $S_f(t)$

$$\begin{aligned} V(0,t) &= 0 && \text{for } 0 \le t \le T, \\ V(S_f(t),t) &= S_f(t) - E && \text{for } 0 \le t \le T, \\ V_S(S_f(t),t) &= 1 && \text{for } 0 \le t \le T, \end{aligned} \tag{4}$$

Note that we need two conditions at the free boundary $S = S_f(t)$. One condition is necessary for the solution of (1) and the other one is needed for determining the position of the free boundary $S_f(t)$ itself. The second condition in (4) ('value matching' condition) is the continuity of the mapping $S \mapsto V(S,t)$ since $V(S,t) = (S-E)^+ = S - E$, in the exercise region $S \ge S_f(t)$. At $S = S_f(t)$ one requires additionally that $V(S,t)$ touches the payoff function tangentially ('high contact condition'), i.e. the function $S \mapsto \partial V(S,t)/\partial S$ should be continuous at $S = S_f(t)$. The conditions (4) are jointly referred as the *'smooth–pasting conditions'*. Note that this third condition can be derived from an arbitrage argument [38].

For the sake of simplicity we will assume $r > q$ in this work, and therefore we have $S_f(T) = rE/q$ for the American Call.

The structure of the value of an American Call can be seen in Fig. 1(b), where we notice that the free boundary $S_f(t)$ determines the position of the exercise.

This linear model is not very realistic [14], since the Black-Scholes model had been derived under very restrictive assumptions, such as frictionless, liquid and complete market. In a real financial market the traders actually work in a different environment: transaction cost arising [4, 9], the market is incomplete, illiquid, etc..

Although the Black-Scholes model has been used in practice, it has also caused some criticism, for example because the volatility is not observable. It is often deduced by calculating the *implied volatility* from sampled option prices by inverting the Black-Scholes formula. A widely observed unique property, the so-called volatility smile, is that these computed volatilities are not constant. This leads to a natural generalization of the Black-Scholes model replacing the constant volatility σ_0 in the model by a local volatility function $\sigma = \sigma(E,T)$, where E denotes the exercise price and T is the maturity.

In practice, transaction costs arise when trading securities. Although they are generally small for institutional investors, they lead to a notable increase in the option price. In the past years, different models have been proposed to relax unrealistic assumptions of the Black-Scholes model (1). These models result in fully *nonlinear Black-Scholes equations.*

Boyle and Vorst [9] derived an option price taking into account transaction costs that is equal to a Black-Scholes price but with a modified volatility of the form

$$\sigma = \sigma_0\sqrt{1+cA}, \quad A = \frac{\mu}{\sigma_0\sqrt{\Delta T}}, \quad c = 1. \tag{5}$$

Here, μ is the proportional transaction cost, ΔT denotes the transaction period, and σ_0 is the original volatility constant. Leland [26] computed $c = \sqrt{2/\pi}$.

A more complex model has been proposed by Barles and Soner [4]. In their model the nonlinear volatility reads

$$\sigma^2 = \sigma_0^2\bigl(1 + \Psi[\exp(r(T-t)a^2S^2V_{SS})]\bigr), \tag{6}$$

where r is the risk-free interest rate, T the maturity and $a = \mu\sqrt{\gamma N}$ where γ is the risk aversion factor and N is the number of options to be sold. The function Ψ is the solution of the nonlinear singular initial-value problem

$$\Psi'(A) = \frac{\Psi(A)+1}{2\sqrt{A\Psi(A)} - A}, \quad A \neq 0, \quad \Psi(0) = 0. \tag{7}$$

In the mathematical literature, only a few results can be found on the numerical solution of nonlinear Black-Scholes equations. The numerical discretization of the Black-Scholes equations with the nonlinear volatility (6) has been performed using explicit finite difference schemes (FDS) [4]. However, explicit schemes have the disadvantage that restrictive conditions on the discretization parameters (for instance, the ratio of the time and the space step) are needed to obtain stable and convergent schemes [36]. Moreover, the convergence order is only one in time and two in space. Düring et al. [14, 15] combined high-order compact difference schemes derived by Rigal [32] and techniques to construct numerical solutions with frozen values of the nonlinear coefficient of the nonlinear Black-Scholes equation

$$V_t + \frac{1}{2}\sigma(V_{SS})^2S^2V_{SS} + (r-q)SV_S - rV = 0, \quad 0 < S < S_f(t),\, t \in (0,T), \tag{8}$$

to linearize the formulation.

Since analytical solutions to nonlinear Black-Scholes equations can only be obtained in rather special cases [6–8], we will compute the option prices numerically. Here, we consider compact FDS for American options and focus on the transaction cost model of Barles and Soner (6). Instead of solving the singular differential equation (7) we propose to use some properties of $\Psi = \Psi(A)$ described recently in [10].

Theorem 1.1 [10] *The nonlinear volatility correction function Ψ, unique solution of (7) satisfies the following properties:*

(i) *Ψ is implicitly defined by*

$$A = \left(-\frac{\operatorname{arcsinh}\sqrt{(\Psi)}}{\sqrt{\Psi+1}} + \sqrt{\Psi} \right)^2, \quad \textit{if } \Psi > 0, \tag{9}$$

$$A = -\left(\frac{\arcsin\sqrt{(-\Psi)}}{\sqrt{\Psi+1}} - \sqrt{-\Psi} \right)^2, \quad \textit{if } 0 > \Psi > -1. \tag{10}$$

(ii) *Ψ is an increasing function mapping the real line onto the interval* $]-1,+\infty[$.

Many of the developed methods for solving the Black-Scholes equation can only be applied to European options, like the method of Liao and Khaliq [27] and methods derived by Rigal [32]. Hence, for American options another strategy is needed. Usually, the equation is transformed into the heat equation and the domain is modified into a semiunbounded one with a free boundary.

Exact analytical formulas for the free boundary $S_f(t)$ in (8) with conditions (4) are not known, but there exist several deductions of approximate formulas for American option estimation in the linear case. Recently, Ševčovič [33] proposed a new method to transform the free boundary problem for the early exercise boundary location into deduction of a time dependent nonlinear parabolic equation on a fixed domain, cf. Section 3.

Liao and Khaliq [27] offered an unconditionally stable compact FDS of fourth order both in space and time applied to European options.

This paper is organized as follows. First, in Section 2 we review from the literature a couple of compact high order finite difference schemes. As compact schemes cannot be directly applied to American type options, we will employ them using a novel fixed domain transformation proposed by Ševčovič, cf. [3, 33]. Finally, we use in Section 4 the approach in combination with the new method of Liao and Khaliq [27] for solving the nonlinear Black-Scholes equation (8) with transaction costs.

2. Compact schemes

Here we will review from [14, 15] a couple of standard difference schemes and compact schemes and also present the schemes developed by Rigal [32]. Before we start with presenting the schemes, we briefly mention a useful transformation of the nonlinear PDE (8) that will be the starting point for the methods. In order to transform the nonlinear Black-Scholes equation (8) with volatility (6) into a convection-diffusion problem, we use the following transformation [14]:

$$x(S) = \ln\frac{S}{E}, \quad \tau(t) = \frac{\sigma_0^2}{2}(T-t), \quad u = e^{-x}\frac{V}{E}.$$

Then (8) may be rewritten in the following form

$$u_\tau = \Big(1 + \Psi[e^{(K\tau+x)}a^2E(u_{xx}+u_x)]\Big)\,(u_{xx}+u_x) + Ku_x, \quad x \in \mathbb{R}, \tag{11}$$

where $0 \le \tau \le \sigma_0^2T/2$, $K = 2r/\sigma_0^2$. τ denotes the (scaled) time to maturity.

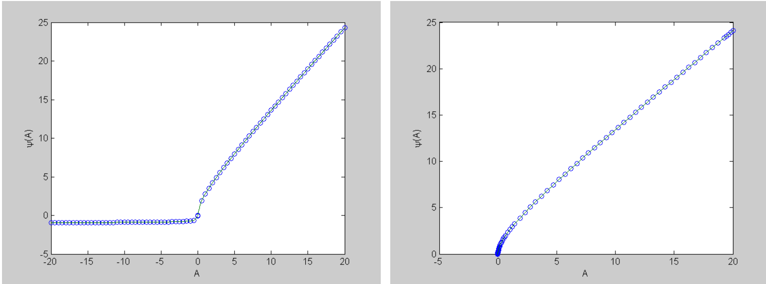


Figure 2. Solutions of (9), (10) (left figure) and ODE (7) (right figure, for $A > 0$), where $T = 1$, $K = 10$, $r = 0.1$, $q = 0.05$.

2.1 *Finite difference methods*

All considered difference schemes have two time levels. Let A^n and B^n be the system matrices of $A^nU^{n+1} = B^nU^n$

$$A^n = [a_{-1}, a_0, a_1], \qquad B^n = [b_{-2}, b_{-1}, b_0, b_1, b_2],$$

where a_i, b_i denote the main diagonals, superdiagonals and subdiagonals. The matrix A^n is tridiagonal and thus the obtained linear systems may be solved efficiently using the Thomas algorithm. We further assume the normalization conditions

$$\sum_{i=-1}^{1} a_i = \sum_{i=-2}^{2} b_i = 1$$

and following [15], we discretize the volatility correction in (6) as

$$s_i^n = \Psi\left[\exp\left(Knk + x_i\right)a^2 E\left(\frac{U_{i-2}^n - 2U_i^n + U_{i+2}^n}{4h^2} + \frac{U_{i+1}^n - U_{i-1}^n}{2h}\right)\right]. \qquad (12)$$

This formula gives an explicit discretization of the nonlinearity and uses a special stencil for the second derivative (spatial step $2h$ instead of h).

Another problem lies in the initial condition for $u(x, 0)$, as it is nondifferentiable in the point $x = 0$. Oosterlee et al. [29] solved this problem of reduced accuracy and proposed *a grid stretching technique*, which is based on an idea of placing more points in the neighborhood of the nondifferentiable payment condition.

Figure 2 shows the solutions of (9), (10) (on the left), and of the ODE (7) (on the right) and their spline interpolation. It is easily seen that the plots show indistinguishable results.

Let us also introduce the following notations. $\lambda = -(1 + K)$ denotes the linear part of the coefficient of the convection term in (11), i.e. (11) reads

$$u_\tau = \Psi[e^{(K\tau+x)}a^2E(u_{xx} + u_x)](u_{xx} + u_x) + u_{xx} - \lambda u_x, \quad x \in \mathbb{R}. \qquad (13)$$

Moreover, $h = \Delta x$, $k = \Delta\tau$ are the step sizes in space and time, $\alpha = \lambda h/2$ is the cell Reynolds number, $\zeta = k/h^2$ the parabolic mesh ratio and $\mu = k/h$ the hyperbolic mesh ratio.

2.2 *Classical finite difference schemes*

Here we summarize the most well-known schemes, cf. [15].

2.2.1 *The Forward-Time Central-Space explicit scheme (FTCS)*

This scheme is given by the coefficients

$$a_{\pm 1} = 0, \quad a_0 = 1, \quad b_{\pm 1} = \zeta \pm \frac{\mu}{2}(s_i^n - \lambda), \quad b_0 = 1 - 2\zeta - \frac{\zeta}{2}s_i^n, \quad b_{\pm 2} = \frac{\zeta}{4}s_i^n.$$

It is of order 1 in time, 2 in space, with a very strict stability condition $\zeta \leq 1/2$. The condition $|\alpha| \leq 1$ must be satisfied to avoid oscillations.

2.2.2 *The Backward-Time Central-Space semi-explicit scheme (BTCS)*

This scheme treats explicitly the nonlinearity and is given by

$$a_{\pm 1} = \mp\frac{\lambda}{2}\mu - \zeta, \quad a_0 = 1 + 2\zeta, \quad b_{\pm 2} = \frac{\zeta}{4}s_i^n, \quad b_{\pm 1} = \pm\frac{1}{2}\mu s_i^n, \quad b_0 = 1 - \frac{\zeta}{2}s_i^n.$$

This scheme is of order 1 in time and 2 in space. It is unconditionally stable and if $|\alpha| \leq 1$ is satisfied, then it is non-oscillatory.

2.2.3 *The Crank-Nicolson scheme (CN)*

This scheme, with an explicit treatment of the nonlinearity, is given by

$$a_{\pm 1} = \left(-\frac{\zeta}{2} \mp \frac{\mu}{4}\right)s_i^n - \frac{\zeta}{2} \pm \frac{\lambda}{4}\mu, \quad b_{\pm 1} = \left(\frac{\zeta}{2} \pm \frac{\mu}{4}\right)s_i^n + \frac{\zeta}{2} \mp \frac{\lambda}{4}\mu,$$
$$a_0 = 1 + \zeta(1 + s_i^n), \quad b_0 = 1 - \zeta(1 + s_i^n), \quad b_{\pm 2} = 0.$$

It is of the order 2 both in time and space and unconditionally stable.

2.3 *The compact schemes of higher order*

Rigal [32] introduced several FDS for linear convection-diffusion problems and Düring et al. [14, 15] applied them to the problem (11). These schemes are both compact two-level schemes of order 2 in time and 4 in space in the linear case. The nonlinearity is handled semi-implicitly as in the previous subsection.

2.3.1 *The R3A scheme*

In this scheme the coefficients are chosen as

$$a_{\pm 1} = \left(\frac{1}{12} - \frac{\zeta}{2}\right)(1 \mp \alpha) - \frac{\alpha^2\zeta}{6} + \frac{\alpha^2\zeta^2}{3}, \quad a_0 = \frac{5}{6} + \zeta + \frac{\alpha^2\zeta}{3} - \frac{2\alpha^2\zeta^2}{3},$$
$$b_{\pm 2} = \frac{\zeta}{4}s_i^n, \quad b_{\pm 1} = \left(\frac{1}{12} + \frac{\zeta}{2}\right)(1 \mp \alpha) \mp \frac{\alpha^2\zeta}{6} + \frac{\alpha^2\zeta^2}{6} \pm \frac{1}{2}\mu s_i^n,$$
$$b_0 = \frac{5}{6} - \zeta - \frac{\alpha^2\zeta}{3} - \frac{2\alpha^2\zeta^2}{3} - \frac{\zeta}{2}s_i^n.$$

It is stable in the linear case $s_i^n = 0$ if $\zeta \leq 1/(\sqrt{2}|\alpha|)$ and the scheme is non-oscillatory, for arbitrary α, cf. [32].

2.3.2 The R3B scheme

The coefficients for this scheme read

$$
\begin{aligned}
a_{\pm1} &= \left(\frac{1}{12} - \frac{\zeta}{2}\right)(1 \mp \alpha) - \frac{\alpha^2\zeta}{6} \mp \frac{\alpha^3\zeta^2}{3} - \frac{2\alpha^4\zeta^3}{3},\\
a_0 &= \frac{5}{6} + \zeta + \frac{\alpha^2\zeta}{3} + \frac{4\alpha^4\zeta^3}{3},\\
b_{\pm2} &= \frac{\zeta}{4}s_i^n, \qquad b_0 = \frac{5}{6} - \zeta - \frac{\alpha^2\zeta}{3} - \frac{4\alpha^4\zeta^3}{3} - 2\zeta s_i^n,\\
b_{\pm1} &= \left(\frac{1}{12} \mp \frac{\zeta}{2}\right)(1 + \alpha) + \frac{\alpha^2\zeta}{6} \mp \frac{\alpha^3\zeta^2}{3} + \frac{2\alpha^4\zeta^3}{3} - \left(\frac{\zeta}{4} \mp \frac{1}{2}\mu\right)s_i^n.
\end{aligned}
$$

It is unconditionally stable and non-oscillatory in the linear case $s_i^n = 0$, cf. [32].

3. The fixed domain transformation

Compact schemes that many authors applied to the Black-Scholes equation with transaction costs have one disadvantage: these schemes cannot be generalized to multi-dimensional problems, and are (directly) applicable to European type options only. However, with the fixed domain transformation introduced by Ševčovič [3, 33] we overcome this second shortcoming.

We consider the nonlinear Black-Scholes equation (8) for an American Call option, i.e. supplied with the terminal condition (2) and boundary conditions (4).

Equation (8) subject to (2), (4) is a backward-in-time parabolic free boundary problem. To solve this free boundary problem numerically, many different methods are developed, for instance the standard method consists in the reformulation to a linear complementary problem (LCP) and solution by a projected SOR method. Alternatively, penalty and front-fixing methods were developed (e.g. in [18], [28]). A disadvantage of these methods is the change of the underlying model. A different approach [21] is based on a recursive calculation of the early exercise boundary, estimating the boundary by Richardson interpolation. The explicit boundary tracking algorithms are for example a finite difference bisection scheme [24] or the front-tracking strategy of Han and Wu [20].

Here we consider the approach of Ševčovič [33] to simplify the numerical solution of (8), with (2), (4) for American call options and get rid of the (explicit) appearance of the free boundary. To do this, we need to transform the problem into a problem posed on a fixed, but unbounded domain additionally to the forward transformation in time. Then, the domain does not depend on the free boundary $S_f(t)$ anymore. All we need is to calculate an *algebraic constraint equation* for the position of the free boundary. To do so, we make the following substitution:

$$
\tau = T - t, \quad x = \ln\left(\frac{\varrho(\tau)}{S}\right) \Leftrightarrow S = e^{-x}\varrho(\tau), \quad \varrho(\tau) = S_f(T-\tau),
$$

such that $x \in \mathbb{R}^+$ and $\tau \in [0, T]$. The constructed *(synthetic) portfolio* will be

$$
\Pi(x,\tau) = V(S,\tau) - SV_S(S,\tau), \tag{14}
$$

made by buying $\Delta = V_S$ shares S and selling an option V.

Differentiating this artificial portfolio Π with respect to x and τ gives us

$$\Pi_x = V_S S_x - S_x V_S - S V_{SS} S_x = S^2 V_{SS} \tag{15}$$

$$\begin{aligned} \Pi_\tau &= V_S S_\tau + V_t t_\tau - S_\tau V_S - S(V_{SS} S_\tau + V_{St} t_\tau) \\ &= -V_t - \frac{\varrho'(\tau)}{\varrho(\tau)} S^2 V_{SS} + S V_{St} \\ &= -V_t - \frac{\varrho'(\tau)}{\varrho(\tau)} \Pi_x - S \partial_S(-V_t). \end{aligned} \tag{16}$$

Equation (15) will be used e.g. to reformulate the nonlinear volatility correction of Barles and Soner also in terms of function Π

Substituting (15), (16) into (8) we get

$$\Pi_\tau = \frac{1}{2} \partial_x (\tilde{\sigma}^2 \Pi_x) + \left(\frac{\tilde{\sigma}^2}{2} - b(\tau) \right) \Pi_x - r\Pi, \quad x \in \mathbb{R}^+, \, \tau \in (0, T), \tag{17}$$

where the time-dependent coefficient $b(\tau)$ reads

$$b(\tau) = \partial_\tau \log(\varrho(\tau)) + r - q. \tag{18}$$

The initial conditions (2) and boundary conditions (4) after substitution (14) transform to

$$\Pi(x, 0) = V(S, T) - S V_S(S, T) = \begin{cases} -E & \text{for } S > E \Leftrightarrow x < \ln \frac{r}{q} \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

$$\begin{aligned} &\Pi(x, \tau) = 0 \quad \text{as } x \to \infty, \; 0 \le \tau \le T, \\ &\Pi(0, \tau) = -E \quad \text{for } 0 \le \tau \le T. \end{aligned} \tag{20}$$

With the assumption $r \ge q$ we obtain the constraint equation

$$\varrho(\tau) = \frac{1}{2q} \tilde{\sigma}^2 \Pi_x(0, \tau) + \frac{rE}{q} \quad \text{with} \quad \varrho(0) = \frac{rE}{q}, \tag{21}$$

where $0 \le \tau \le T$ and the modified volatility function becomes

$$\tilde{\sigma}^2 = \sigma_0^2 \left(1 + \Psi[e^{r\tau} a^2 \Pi_x] \right). \tag{22}$$

Our transformed problem (17) subject to (19)–(21) with the volatility function (22) can be solved by the split-step FDS proposed by Ševčovič [33]. However, we will propose another scheme recently developed by Liao and Khaliq [27]. After solving the transformed problem with some suitable method, we calculate the value of the American call option $V(S, t)$ by transforming (14) back to the original variables. Since we know that

$$\frac{\Pi(x, \tau)}{S^2} = \frac{V(S, t)}{S^2} - \frac{V_S(S, t)}{S} = \partial_S \left(-\frac{V(S, t)}{S} \right),$$

we integrate the above equation from S to $S_f(t)$ with the boundary condition $V(S_f(t),t) = S_f(t) - E$ and we obtain

$$V(S, T-\tau) = \frac{S}{\varrho(\tau)} \left(\varrho(\tau) - E + \int_0^{\ln \frac{\varrho(\tau)}{S}} e^x \Pi(x,\tau)dx \right). \qquad (23)$$

yielding the price of an American call $V(S,t)$ in the presence of transaction costs.

4. The method of Liao and Khaliq

Liao and Khaliq [27] proposed a new efficient fourth-order compact scheme based on the Padé approximation. The method was applied to the nonlinear Black-Scholes equation for European options. In this section we will briefly review their work and show in the following Section 5 how it can be applied to American options.

Let us consider the following one-dimensional time dependent convection-diffusion equation

$$u_t = \beta u_{xx} + \lambda u_x - ru, \qquad (24)$$

where β, λ and r are constants. Instead of solving a single convection-diffusion equation (24), Liao and Khaliq transform it into a system of two equations. The following new unknown function $v(x,t) = u_x(x,t)$ is introduced, hence we consider

$$u_t = \beta u_{xx} + \lambda v - ru \qquad (25)$$

$$v_t = \beta v_{xx} + \lambda u_{xx} - rv. \qquad (26)$$

We state for $u(x,t)$ the initial conditions $u(x,0) = u_0(x)$ and boundary conditions

$$u(0,t) = b_0(t), \qquad u(1,t) = b_1(t).$$

For $v(x,t)$ the initial condition is simply the x-derivative of $u(x,t)$: $v(x,0) = u_0'(x)$.

In the sequel we will make frequent use of the standard central difference operator

$$\Delta_h^0 u_i := u_{i+1} - u_{i-1} = 2hD_h^0 u_i$$

and the standard second order difference operator

$$\Delta_h^2 u_i := u_{i+1} - 2u_i + u_{i-1} = h^2 D_h^2 u_i$$

Suppose now the spatial grid is uniform, i.e. N sub-intervals form the interval $[0,1]$ and $h = 1/N$. The second order approximation

$$v(h,t) = \frac{\partial u}{\partial x}(h,t) \approx D_h^0 u(h,t) = \frac{\Delta_h^0}{2h} u(h,t).$$

can be improved to fourth order if Δ_h^0 is replaced by $\Delta_h^0/(1+\frac{1}{6}\Delta_h^2)$

$$v(h,t) = \frac{\Delta_h^0}{2h(1+\frac{1}{6}\Delta_h^2)u(h,t)} + \mathrm{O}(h^4).$$

Doing so, we obtain a fourth order approximation

$$v(0,t) = \frac{3}{h}\left(u(2h,t) - u(0,t)\right) - 4v(h,t) - v(2h,t).$$

Hence, we can approximate the right boundary condition of v at $x=1$ as

$$v(1,t) = \frac{3}{h}\left(u(1-h,t) - u(1-2h,t)\right) - 4v(1-h,t) - v(1-2h,t).$$

Let us consider a more general system

$$u_t = \beta u_{xx} + f(u,v), \tag{27}$$

$$v_t = \lambda u_{xx} + \beta v_{xx} + g(u,v), \tag{28}$$

where the term λv is included in the general function $f(u,v)$ and there is only one diffusion term βu_{xx} in (27). The method starts from the Crank-Nicolson scheme

$$\frac{u_i^{n+1} - u_i^n}{k} = \frac{1}{2}\left(\frac{\beta}{h^2}\Delta_h^2 u_i^{n+1} + \frac{\beta}{h^2}\Delta_h^2 u_i^n + f_i^{n+1} + f_i^n\right), \tag{29}$$

$$\frac{v_i^{n+1} - v_i^n}{k} = \frac{1}{2}\left(\frac{\lambda}{h^2}\Delta_h^2[u_i^{n+1} + u_i^n] + \frac{\beta}{h^2}\Delta_h^2[v_i^{n+1} + v_i^n] + g_i^{n+1} + g_i^n\right), \tag{30}$$

where

$$f_i^{n+1} = f(u_i^{n+1}, v_i^{n+1}),\ f_i^n = f(u_i^n, v_i^n),\ g_i^{n+1} = g(u_i^{n+1}, v_i^{n+1}),\ g_i^n = g(u_i^n, v_i^n),$$

We improve this second order approximation to the fourth order by using instead of the previous approximation the Padé approximation

$$(u_{xx})_i \approx \frac{\Delta_h^2}{h^2(1+\frac{1}{12}\Delta_h^2)}. \tag{31}$$

We apply this Padé approximation in (29)–(30), multiply both sides by $1+\frac{1}{12}\Delta_h^2$:

$$\begin{aligned}\left(1+\frac{\Delta_h^2}{12} - \frac{\beta\zeta}{2}\Delta_h^2\right)u_i^{n+1} &= \left(1+\frac{\Delta_h^2}{12} - \frac{\beta\zeta}{2}\Delta_h^2\right)u_i^n \\ &\quad + \frac{k}{2}\left(1+\frac{\Delta_h^2}{12}\right)(f_i^{n+1} + f_i^n),\end{aligned} \tag{32}$$

$$\begin{aligned}\left(1+\frac{\Delta_h^2}{12}-\frac{\beta\zeta}{2}\Delta_h^2\right)v_i^{n+1}=\left(1+\frac{\Delta_h^2}{12}-\frac{\beta\zeta}{2}\Delta_h^2\right)v_i^n+\lambda\frac{\zeta}{2}\Delta_h^2(u_i^{n+1}+u_i^n)\\ +\frac{k}{2}\left(1+\frac{\Delta_h^2}{12}\right)(g_i^{n+1}+g_i^n),\end{aligned}\tag{33}$$

with the parabolic mesh ratio $\zeta = k/h^2$. The truncation error of (32)–(33) is $C_1k^2 + C_2k^4 + C_3h^4$ and since we solely have even powers w.r.t. the time step k, a Richardson extrapolation technique can improve the approximation to fourth order in time.

Suppose the solutions of (32) and (33) after l iterations are denoted by $u_i^{n+1^{(l)}}$ and $v_i^{n+1^{(l)}}$ respectively. To get $u_i^{n+1^{(l+1)}}$ and $v_i^{n+1^{(l+1)}}$, we first expand f_i^{n+1}:

$$f(u_i^{n+1},v_i^{n+1})=f(u_i^{n+1^{(l)}},v_i^{n+1^{(l)}})+\frac{\partial f}{\partial u}(u_i^{n+1^{(l)}},v_i^{n+1^{(l)}})(u_i^{n+1}-u_i^{n+1^{(l)}})\tag{34}$$

and insert it into (32), then solve the following equation for $u_i^{n+1^{(l+1)}}$

$$\begin{aligned}\left(1+\frac{1}{12}\Delta_h^2-\frac{\beta\zeta}{2}\Delta_h^2-\frac{k}{2}(1+\frac{1}{12}\Delta_h^2)\hat{\mathbf{J}}_i^{n+1^{(l)}}\right)u_i^{n+1^{(l+1)}}\\ =\left(1+\frac{1}{12}\Delta_h^2-\frac{\beta\zeta}{2}\Delta_h^2\right)u_i^n\\ +\frac{k}{2}(1+\frac{1}{12}\Delta_h^2)\left(f(u_i^{n+1^{(l)}},v_i^{n+1^{(l)}})-\hat{\mathbf{J}}_i^{n+1^{(l)}}u_i^{n+1^{(l)}}+f(u_i^n,v_i^n)\right),\end{aligned}\tag{35}$$

where $\hat{\mathbf{J}}_i^{n+1^{(l)}}=\frac{\partial f}{\partial u}(u_i^{n+1^{(l)}},v_i^{n+1^{(l)}})$. Once we have found $u_i^{n+1^{(l+1)}}$, we expand g_i^{n+1}

$$g(u_i^{n+1},v_i^{n+1})=g(u_i^{n+1^{(l+1)}},v_i^{n+1^{(l)}})+\frac{\partial g}{\partial u}(u_i^{n+1^{(l+1)}},v_i^{n+1^{(l)}})(v_i^{n+1}-v_i^{n+1^{(l)}}).\tag{36}$$

Substituting (36) into (34) we get:

$$\begin{aligned}\left(1+\frac{1}{12}\Delta_h^2-\frac{\beta\zeta}{2}\Delta_h^2-\frac{k}{2}(1+\frac{1}{12}\Delta_h^2)\tilde{\mathbf{J}}_i^{n+1^{(l)}}\right)v_i^{n+1^{(l+1)}}\\ =\left(1+\frac{1}{12}\Delta_h^2-\frac{\beta\zeta}{2}\Delta_h^2\right)u_i^n\\ +\frac{k}{2}(1+\frac{1}{12}\Delta_h^2)\left(f(u_i^{n+1^{(l)}},v_i^{n+1^{(l)}})-\tilde{\mathbf{J}}_i^{n+1^{(l)}}u_i^{n+1^{(l)}}+f(u_i^n,v_i^n)\right),\end{aligned}\tag{37}$$

where

$$\tilde{\mathbf{J}}_i^{n+1^{(l)}}=\frac{\partial g}{\partial v}(u_i^{n+1^{(l)}},v_i^{n+1^{(l)}}).$$

We then solve (37) for $v_i^{n+1^{(l+1)}}$. The two steps are repeated alternatively until convergence occurs.

Liao and Khaliq [27] made a careful von Neumann stability analysis for their scheme and proved that this high order compact scheme is unconditionally stable.

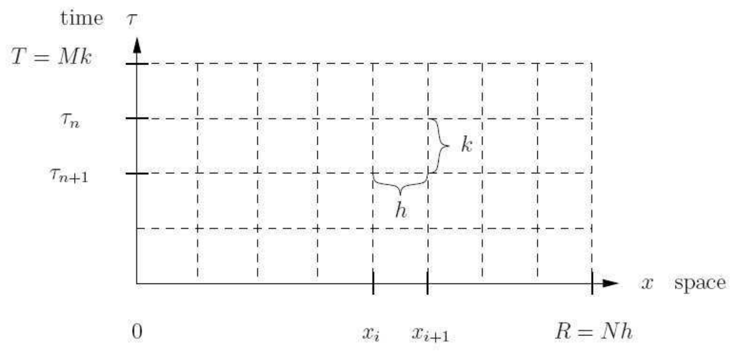


Figure 3. Uniform grid for American options.

5. The Numerical Solution of the American Option problem

In this section we will explain the steps, how to solve the nonlinear Black-Scholes equation (8) in case of American options.

We confine the unbounded domain $x \in \mathbb{R}^+$ and $\tau \in [0, T]$ to $x \in (0, R)$ with $R > 0$ large enough (see [33]). For the calculation Ševčovič chooses to take $R = 3$, since this is equivalent to $S \in (S_f(t)e^{-R}, S_f(t))$ and yields a good approximation for $S \in (0, S_f(t))$ (as the transformation was $S = S_f(t)e^{-x}$). We also take $h > 0$ as spatial step size, $k > 0$ as temporal step size, $x_i = ih$, $i \in [0, N]$, $R = Nh$, $\tau_n = nk$, $n \in [0, M]$, $T = Mk$, cf. [3] (see Fig. 3).

Our computations rely on the transformation technique of Section 3 and thus we compute numerically values of the synthetic portfolio $\Pi = \Pi(x, \tau)$ and use in the sequel the standard notation $\Pi_i^n \approx \Pi(x_i, \tau_n)$, $b_i^n \approx b(\tau_n)$ In all our computations the nonlinearity is discretized explicitly in the scheme, i.e. now s_i^n denotes the nonlinear volatility correction of Barles and Soner (6) written in terms of Π:

$$s_i^n = \Psi\big[e^{r\tau_n} a^2 D_h^+ \Pi_i^n\big] \tag{38}$$

We emphasize that another advantage of this synthetic portfolio formulation is that, compared to (12), we do not need to evaluate here a second derivative, i.e. we can use a smaller stencil. Now, the synthetic portfolio equation (17) can be written as a "semi-discretized" equation

$$\Pi_\tau = \frac{1}{2}\partial_x(\sigma_0^2(1 + s_i^n)\Pi_x) + \Big(\frac{\sigma_0^2}{2}(1 + s_i^n) - b^n\Big)\Pi_x - r\Pi, \tag{39}$$

i.e.

$$\Pi_\tau = \frac{\sigma_0^2}{2}(1 + s_i^n)\Pi_{xx} + \Big(\frac{\sigma_0^2}{2}(1 + s_i^n + \partial_x s_i^n) - b^n\Big)\Pi_x - r\Pi, \tag{40}$$

that has the form

$$\Pi_\tau = \beta\,\Pi_{xx} + \lambda\,\Pi_x - r\Pi, \tag{41}$$

and we will consider FDS for solving (41) with arbitrary parameters $\beta, r > 0$ and $\lambda \in \mathbb{R}$. Especially the scheme of Liao and Khaliq [27] fits to this equation and will be used.

We can sketch our procedure for the *time advancement* as follows:

$$\Pi^n \longrightarrow \varrho^n \longrightarrow b^n \longrightarrow \Pi^{n+1}.$$

Hence, assume Π^n is given, we describe in the next part how to determine ϱ^n and the coefficient b^n.

We consider the free boundary (21) in the point $x = 0$ and approximate the derivative in space by forward differences:

$$\varrho^n = \frac{1}{2q}\,\sigma_0^2\big(1 + s_0^n\big)D_h^+\Pi_0^n + \frac{rE}{q} \quad \text{with} \quad \varrho^0 = \frac{rE}{q}, \tag{42}$$

where $D_h^+\Pi_0^n = (\Pi_1^n - \Pi_0^n)/h$ is the forward difference quotient in point $x = 0$. Clearly, this finite difference approximation is only of first order, but this equation is just to determine the approximate location of the free boundary. We expect that our detailed numerical tests in [17] will show that the order of accuracy of the overall scheme is not reduced. Anyway, the approximation error can easily be increased by using one-sided finite difference approximations of higher order.

Next, the coefficent b^n is readily obtained by discreting (18)

$$b^n = \frac{\log \varrho^n - \log \varrho^{n-1}}{k} + r - q, \quad n = 2, 3, \ldots. \tag{43}$$

Only, in the first step, to compute b^1 we use with very high accuracy an asymptotic expansion for the free boundary.

Remark 1 For the American Call option (in contrast to the American Put option) it is possible to derive a series for the location of the optimal exercise boundary close to expiry using standard asymptotic analysis [38]. This local analysis of the free boundary $S_f(t)$ yields

$$S_f(t) \sim S_f(T)\left(1 + \xi_0\sqrt{\frac{1}{2}\sigma^2(T-t)} + \ldots\right), \qquad \text{as } t \to T, \tag{44}$$

where $\xi_0 = 0.9034\ldots$ is a *universal constant* of Call option pricing. Equation (44) can be rewritten as

$$\varrho(\tau) \sim \varrho(0)\left(1 + \xi_0\sqrt{\frac{1}{2}\sigma^2(\tau)} + \ldots\right), \qquad \text{as } \tau \to 0. \tag{45}$$

With only very few terms we get a fairly accurate result for the free boundary and thus equation (45) will serve us as a check for the case of a constant volatility $\widetilde{\sigma}^2 = \sigma^2$ (see Fig. 4). Note that this result is especially useful in the first time levels of a numerical calculation where rapid changes in $\varrho(\tau)$ influence the whole solution region.

Figure 4 shows the difference between the free boundary and the asymptotic free boundary taken with exercise date $T = 1$ year, exercise price $E = 10$ and dividend yield $q = 0.05$.

Finally we can compute the solution π^{n+1} on the next time level by solving the semi discretized equation (41) subject to the initial conditions (19) and boundary

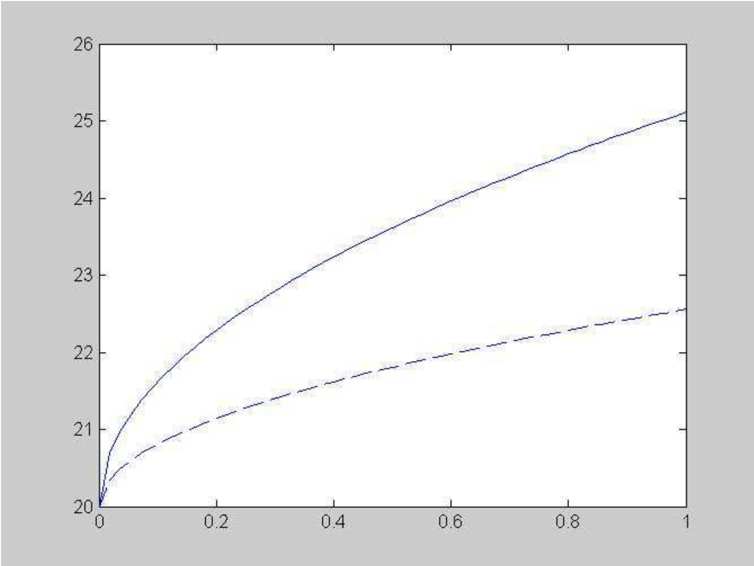


Figure 4. The free boundary (solid) and the asymptotic free boundary (dashed) for nonlinear Black-Scholes equation (8) with a nonlinear volatility (6) with $T = 1$, $E = 10$, $\sigma_0 = 0.2$, $r = 0.1$, $q = 0.05$, $a = 0.02$, $h = 0.1$, $k = 0.1$

conditions (20):

$$\begin{aligned} \Pi_i^0 &= \Pi(x_i, 0) = \begin{cases} -E & \text{for } x_i < \ln \frac{\varrho(0)}{E} = \ln \frac{r}{q} \\ 0 & \text{otherwise} \end{cases}, \\ \Pi_0^n &= -E, \\ \Pi_N^n &= 0. \end{aligned} \tag{46}$$

Sometimes high order methods need more (numerical) boundary conditions to close the system of equations. In these cases on uses high order extrapolation techniques, cf. [19] for the necessary order of accuracy.

Recall that at each time level n, we can calculate $V(S_i, t_n) = V(e^{-x_i}\varrho^n, T - \tau_n)$ with these values and proceed to the next time level $n+1$. From (23) we then know that:

$$V(S_i, t_n) = e^{-x_i}\big(\varrho^n - E + \mathcal{I}_i\big), \tag{47}$$

where

$$\begin{aligned} \mathcal{I}_i &= \sum_{j=0}^{i-1} \mathcal{I}_k + \int_{x_{i-1}}^{x_i} e^x \Pi(x, \tau) dx \\ &= \sum_{j=0}^{i-1} \mathcal{I}_k + \frac{x_i - x_{i-1}}{2} \Big(e^{x_{i-1}} \Pi_{i-1}^n + e^{x_i} \Pi_i^n \Big). \end{aligned}$$

Here, we use the *trapezoidal rule* in order to approximate the integral (23).

Figure 5 shows the computed value $V(S, 0)$ of an American Call option determined from the nonlinear Black-Scholes equation (8) using the discretization steps $h = 0.1$, $k = 0.1$ and the parameters: exercise date $T = 1$ year, exercise price $E = 10$ and dividend yield $q = 0.05$. with $T = 1$, $E = 10$, $\sigma_0 = 0.2$, $r = 0.1$,

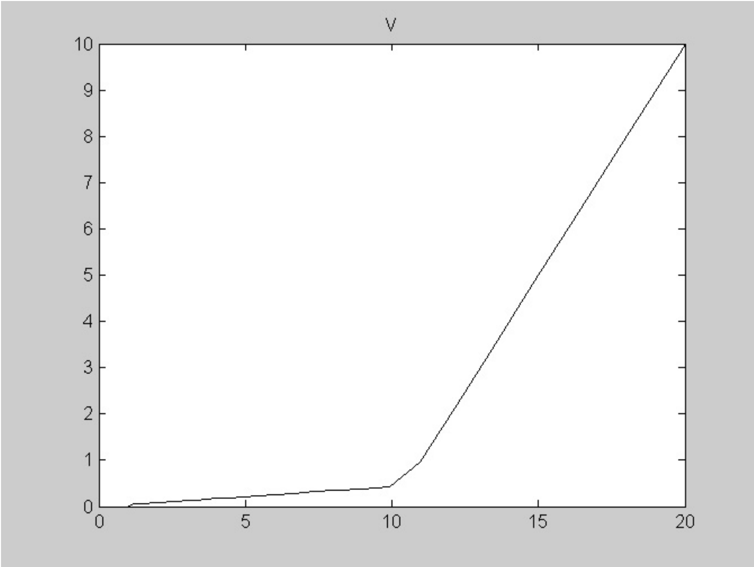


Figure 5. Price of the American Call option $V(S,0)$ in the presence of transaction costs (in case of nonlinear Black-Scholes equation).

$q = 0.05$, $a = 0.02$.

5.1 *The algorithm description*

The calculations of the price $V(S,t)$ for American options in the presence of transaction costs lead us to the following algorithm.

Algorithm 1 Computation of the price $V(S,t)$ for the American option.
Input parameters: σ, r, q, E, a, R, T, h, k, M, N, γ, α, β

(1) Use formula (22) for the Barles and Soner volatility model (6) and interpolate the solutions
(2) initialize Π^0, ϱ^0, $V(S,T)$
(3) calculate Π^{n+1} for each time level iteratively
  a) calculate from (38) the volatility correction s_i^n for the time step τ_n using results from the previous time step
  b) calculate the free boundary ϱ^n for the time step τ_n using (42)
  c) calculate the coefficient b^n from (43)
  d) calculate the solution Π^{n+1} using an FDS for solving (41), e.g. method of Liao and Khaliq [27]
(4) transform Π into V
(5) plot V for each time level and each stock price

Conclusions and Future Work

In this work we considered the numerical solution of nonlinear Black-Scholes equations in the presence of transaction costs in case of American options.

While we focused in this paper on standard options (known as *plain–vanilla options*) of American type, our future work will deal with extensions: forward and future contracts, options on futures, more general pay–off functions (e.g. 'cash–or–nothing call') with transaction costs and instalment options.

We presented several high-order compact schemes and discussed how to increase the order of accuracy to be fourth order both in time and space. It turned out in our preliminary tests that the method of Liao and Khaliq [27] performs better than other methods described in Section 2.2.

We showed that the compact methods can be used very efficiently to price American options using the fixed domain transformation technique of Ševčovič [34].

In a second follow-up paper [17] we will present concisely the numerical results of our comparison study including details about the obtained rates of convergence. Our future research will be directed towards the implementation of (discrete) artificial boundary conditions, cf. [16, 37], since (17) is posed on an unbounded domain.

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