Bergische Universität Wuppertal

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Lehrstuhl für Optimierung und Approximation

Preprint BUW-AMNA-OPAP 10/10

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August 2010

http://www.math.uni-wuppertal.de

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Abstract

Modeling electric circuits that contain magnetoquasistatic (MQS) devices leads to a coupled system of differential-algebraic equations (DAEs). In our case, the MQS device is described by the eddy current problem already discretized in space (via edge-elements, e.g. the Finite Integration Technique). This yields a DAE with a properly stated leading term, which has to be solved in time domain. We are interested in structural properties of this system, which are important for numerical integration. Applying a standard projection technique, we are able to deduce topological conditions such that the tractability index of the coupled problem does not exceed two. Although index-2, we can conclude that the numerical difficulties for this problem are not severe due to linear dependence on index-2 variables.

Keywords: modified nodal analysis, differential-algebraic equations, tractability index, electromagnetic devices, Maxwell's Equations, Finite Integration Technique, consistent initialization

1. Introduction

Usually in a TCAD environment, electric circuits are simulated as networks of basic elements. In this context, devices such as complex semiconductors or even conductors and their interactions are described by corresponding

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subcircuits. That is, these devices are modeled via equivalent circuits containing only basic elements. Most often, the set-up of equations uses modified nodal analysis (MNA), which we also employ. Today, chip technology develops rapidly and the complexity of the above mentioned devices grows fast and plays a vital role in circuit design. This has two consequences. On the one hand, the corresponding equivalent circuits have become more and more complex and they contain already hundreds of parameters, most of them without a direct physical interpretation. On the other hand, the device simulation of spatially resolved (complex) models is influenced by secondary effects, such as the surrounding circuitry, which cannot any longer be neglected. This has motivated the idea of using distributed device models, represented by a system of partial differential equations (PDEs), to describe the behavior of the devices in the circuit. The resulting mathematical model couples DAEs describing the circuit and PDEs modeling the devices. Thus it gives a set of partial differential-algebraic equations (PDAE).

To numerically simulate electrical circuits described by such a model, we first discretize the PDEs in space (method of lines). This results in a coupled system of DAE to be solved in simulation.

A DAE is generally characterized by its index, which roughly measures the equation's sensitivity w.r.t. perturbations of the input and thus it reveals the expected numerical difficulties in simulation. Due to various facts and view point, there exist several index definitions, which all generalize the Kronecker index [9]. In this paper, we use as index framework the projector-based tractability index [8, 14]. This is due to the detailed view it reveals on the structure of the equations. We recall that for a large class of electric circuits described by equations from MNA, the tractability index is exclusively determined by the circuit's topology (e.g. [7]).

For electric circuits of basic elements refined by distributed elements, there exist already a couple of index results. For circuits containing semiconductor devices which are modeled by the drift-diffusion equation, it was shown in [2, 18, 20] that we can extend the topological index criteria of circuits containing just the basic elements.

We investigate electric circuits refined by spatially resolved MQS devices. The structural properties of the corresponding MQS field system, the so-called eddy current problem, were studied first in [21]. There a Kronecker-index analysis is given for the linear 2D problem in the magnetic vector potential formulation. In [17], the differential-index was used to obtain more general results for the linear 3D-case, where a gauging becomes necessary to obtain a uniquely solvable formulation. For electric circuits containing MQS devices a topological circuit-condition was shown to be sufficient to yield an overall problem of index-1.

Here, we extend the index analysis of the coupled field/circuit problem to a more general nonlinear setting and to the case of higher index. The topological conditions for index-1 and index-2 can be shown to be necessary.

This paper is organized as follows. In the next section, we first introduce the models for the electric circuits containing basic network elements and the distributed, but spatially discretized, MQS device models. Then we establish the coupling and state the coupled problem as a DAE. Section 3 contains the main part, which is devoted to the index analysis of the deduced DAE. At the begin of this section, we roughly recall the basics of the tractability index concept. Then we investigate the structural properties of the electrical circuit including MQS devices modeled by DAEs after the space discretization of the distributed devices. This section is concluded by a brief investigation of the consistent initialization for that system. Here we exploit the special structure. Eventually, we give a short numerical illustration on the simplest example and finish with conclusions.

2. Modeling

2.1. Electric Network Model

Let us consider an electric network consisting of capacitors, inductors, resistors, voltage and current sources with related incidence matrices (reduced): $A_\mathrm{C}, A_\mathrm{R}, A_\mathrm{L}, A_\mathrm{V}$ and A_I, which state the node-branch relation for each element type for the underlying digraph:

$$(A_\star)_{ij} = \begin{cases} 1, \text{ if branch } j \text{ leaves node } i \\ -1, \text{ if branch } j \text{ enters node } i \\ 0, \text{ if branch } j \text{ is not incident with node } i. \end{cases}$$

In fact, the rows of $A_\star$ refer to the network nodes. As usual, one node is identified as mass node. The corresponding row is skipped in $A_\star$.

The MNA leads to equations of the form (see e.g. [7])

$$A_\mathrm{C}\frac{\mathrm{d}}{\mathrm{d}t}q_\mathrm{C}(A_\mathrm{C}^\top e, t) + A_\mathrm{R}g_\mathrm{R}(A_\mathrm{R}^\top e, t) + A_\mathrm{L}j_\mathrm{L} + A_\mathrm{V}j_\mathrm{V} + A_\mathrm{I}i_\mathrm{s}(t) = 0, \tag{1a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\mathrm{L}}(j_{\mathrm{L}},t) - A_{\mathrm{L}}^{\top}e = 0, \tag{1b}$$

$$A_{\mathrm{V}}^{\top}e - v_{\mathrm{s}}(t) = 0 \tag{1c}$$

with time $t \in \mathcal{I},\ \mathcal{I} = [t_0,\, T]$. The given functions $q_{\mathrm{C}}(v,t)$, $g_{\mathrm{R}}(v,t)$, $\Phi_{\mathrm{L}}(j,t)$, $v_{\mathrm{s}}(t)$ and $i_{\mathrm{s}}(t)$ describe the constitutive relations for the circuit elements (for charge, resistance, flux, voltage source and current source, respectively). The unknowns are the node potentials $e : \mathcal{I} \to \mathbb{R}^n$, except of the mass node, as well as the currents $j_{\mathrm{L}} : \mathcal{I} \to \mathbb{R}^{n_L}$ through inductors and the currents $j_{\mathrm{V}} : \mathcal{I} \to \mathbb{R}^{n_V}$ through voltage sources (for n_L inductors and n_V voltage sources). The potential at the mass node is assigned to zero. Thus (1a) states the current balance at each network node, and (1b) and (1c) state the constitutive relations for inductances and voltage sources, respectively. Detail can be found in e.g. [20, 7].

For a mathematically consistent description, we need:

Assumption 2.1 (Soundness of circuits)**.** The circuit shall be connected and contains neither loops of voltage sources only nor cutsets of current sources only.

If Ass. 2.1 is violated, the circuit equations (plus initial conditions) would have either no solution of infinite many solutions due to Kirchhoff's laws.

Assumption 2.2 (Local Passivity)**.** The functions $q_{\mathrm{C}}(v,t), \Phi_{\mathrm{L}}(j,t)$ and $g_{\mathrm{R}}(v,t)$ are continuous differentiable with positive definite Jacobians:

$$C(v,t) := \frac{\partial q_{\mathrm{C}}(v,t)}{\partial v},\ L(j,t) := \frac{\partial \Phi(j,t)}{\partial j},\ G(v,t) := \frac{\partial g_{\mathrm{R}}(v,t)}{\partial v}.$$

Next, we add the electromagnetic field-elements to our system. That is, we enlarge our list of basic elements. This gives an extended circuit. In the MNA framework, we simply add the unknown current $j_{\mathrm{M}} \in \mathbb{R}^{n_M}$ through the field element (more precisely MQS device) to the current balance equation (1a) using the corresponding incidence matrix A_{M}. Then (1a) reads

$$A_{\mathrm{C}}\frac{\mathrm{d}}{\mathrm{d}t}q_{\mathrm{C}}(A_{\mathrm{C}}^{\top}e,t) + A_{\mathrm{R}}g_{\mathrm{R}}(A_{\mathrm{R}}^{\top}e,t) + A_{\mathrm{L}}j_{\mathrm{L}} + A_{\mathrm{V}}j_{\mathrm{V}} + A_{\mathrm{I}}i_{\mathrm{s}}(t) + A_{\mathrm{M}}j_{\mathrm{M}} = 0. \tag{2}$$

To obtain a uniquely solvable system we need further equations for the MQS device which describe the unknown currents j_{M} in terms of the other variables. This will involve the applied potentials $A_{\mathrm{M}}^{\top}e$. Before we discuss this, we now restate Ass. 2.1 in terms of graph theory for our extended circuit:

Assumption 2.3 (Soundness of extended circuit I)**.** The circuit shall be connected and the matrices

$$A_{\mathrm{V}} \quad \text{and} \quad [A_{\mathrm{C}}\, A_{\mathrm{R}}\, A_{\mathrm{L}}\, A_{\mathrm{V}}\, A_{\mathrm{M}}]^{\top}$$

have full column rank, i.e., it exists neither a loop containing only voltage sources nor a cutset containing only current sources.

Remark 2.4 (Incidence Matrices)**.** Recall, an incidence matrix (of a subgraph) has full column rank if there exists no loop; let $[A_1, A_2]$ denote the incidence matrix of a connected graph, $A_1^{\top}$ has full column rank iff there exists a spanning tree of elements from A_1; moreover, $A_2^{\top}$ has full column rank iff the graph contains no cutset of elements from A_1.

Later on, we need also:

Assumption 2.5 (Soundness of extended circuit II)**.** The circuit shall be connected and the matrices

$$A_{\mathrm{V}} \quad \text{and} \quad [A_{\mathrm{C}}\, A_{\mathrm{R}}\, A_{\mathrm{L}}\, A_{\mathrm{V}}]^{\top}$$

have full column rank, i.e., it exists neither a loop containing only voltage sources nor a cutset containing only current sources and MQS devices.

Furthermore, in stating the model as we do, we implicitly assume independent voltage and current sources only. Our results can be extended to broad class of controlled sources [7].

2.2. MQS Device Models

Next, we derive the MQS device model from Maxwell's space-discrete equations on a staggered grid. They can be obtained from any spatial discretization of Maxwell's Equations, here we use the notation of the Finite Integration Technique, [22]

$$\mathbf{C}\widehat{\mathbf{e}} = -\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\widehat{\mathbf{b}}}\,, \qquad \tilde{\mathbf{C}}\widehat{\mathbf{h}} = \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\widehat{\mathbf{d}}} + \widehat{\widehat{\mathbf{j}}}\,, \qquad \tilde{\mathbf{S}}\widehat{\widehat{\mathbf{d}}} = \mathbf{q}\,, \qquad \mathbf{S}\widehat{\widehat{\mathbf{b}}} = 0\,, \tag{3}$$

with discrete curl operators $\mathbf{C}$ and $\tilde{\mathbf{C}}$, divergence operators $\mathbf{S}$ and $\tilde{\mathbf{S}}$ (on the staggered grids). The variables are line-integrals of electric and magnetic field strength $\widehat{\mathbf{e}}$ and $\widehat{\mathbf{h}}$ (edges of the cells) and surface integrals of source current density, discrete magnetic flux density and displacement field $\widehat{\widehat{\mathbf{j}}}$, $\widehat{\widehat{\mathbf{b}}}$ and $\widehat{\widehat{\mathbf{d}}}$. The Maxwell's Equations are closed with the constitutive material relations:

$$\widehat{\widehat{\mathbf{b}}} = \mathbf{M}_\mu \widehat{\mathbf{h}} , \qquad \widehat{\widehat{\mathbf{d}}} = \mathbf{M}_\varepsilon \widehat{\mathbf{e}} , \qquad \widehat{\widehat{\mathbf{j}}} = \mathbf{M}_\sigma \widehat{\mathbf{e}} , \tag{4}$$

where matrices $\mathbf{M}_\mu$, $\mathbf{M}_\varepsilon$ and $\mathbf{M}_\sigma$ represent the permeabilities, permittivities and conductivities.

For low frequencies ("eddy current problem") the displacement current density can be negleced, when compared with the current density:

$$\max |\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\widehat{\mathbf{d}}}| \ll \max |\widehat{\widehat{\mathbf{j}}}|$$

Furthermore, we can reformulate the problem in terms of the magnetic vector potential $\widehat{\mathbf{a}} : \mathcal{I} \to \mathbb{R}^{n_M}$ (where n_M denotes the number of edges) with

$$\widehat{\mathbf{e}} = -\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathbf{a}} + \tilde{\mathbf{S}}^\top \mathbf{\Phi} \qquad \text{and} \qquad \widehat{\widehat{\mathbf{b}}} = \mathbf{C}\widehat{\mathbf{a}} \tag{5}$$

with electric scalar potential $\mathbf{\Phi}$. Starting from Ampère's law (second equation of (3)) neglecting the displacement current density, inserting material relations and using $\widehat{\mathbf{a}}$, this yields the following first-order DAE ("curl-curl equation"):

$$\mathbf{M}\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathbf{a}} + \mathbf{K}\widehat{\mathbf{a}} = \mathbf{M}\tilde{\mathbf{S}}^\top \mathbf{\Phi} , \tag{6}$$

where $\mathbf{M} := \mathbf{M}_\sigma$ and $\mathbf{K} := \tilde{\mathbf{C}}\mathbf{M}_\mu^{-1}\mathbf{C}$. The conductivity matrix is assumed constant in time but the matrix $\mathbf{M}_\mu^{-1}$ may depend nonlinearly on the flux $\widehat{\widehat{\mathbf{b}}}$ due to ferromagnetic saturation; this gives $\mathbf{K} = \mathbf{K}(\widehat{\mathbf{a}}) = \tilde{\mathbf{C}}\mathbf{M}_\mu^{-1}(\mathbf{C}\widehat{\mathbf{a}})\mathbf{C}$. The matrices $\mathbf{M}$ and $\mathbf{K}(\widehat{\mathbf{a}})$ are only (symmetric) positive semi-definite with a common kernel, where the singularity of $\mathbf{M}$ is due to non-conducting areas, while $\mathbf{K}$ is singular due to the nontrivial kernel of the curl operator. Using a gauging, the common kernel can be removed, [12]. As boundary conditions we assume that the tangential component of the vector potential $\widehat{\mathbf{a}}$ vanishes at the boundary of the domain.

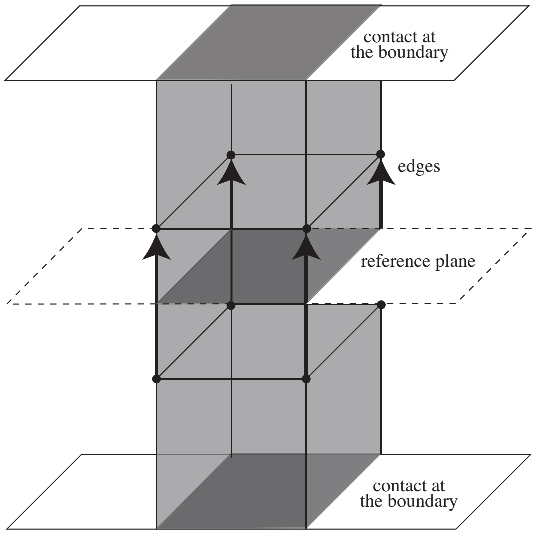


Figure 1: Coupling as given as in [5] (for a Cartesian grid).

MQS-device and Coupling

The coupling of the MQS device to the circuit is established by subdomains $\mathbf{\Omega}_{\mathrm{M}} \subset \mathbf{\Omega}$ which identify the areas, where an electric current is imposed by the coupled circuit. In this subdomain the electric scalar potential $\mathbf{\Phi}$ is related to the circuit voltage drop $v_{\mathrm{M}} = A_{\mathrm{M}}^{\top} e$ and the current density $\widehat{\widehat{\mathbf{j}}}$ is related to the branch current j_{M} of the electric circuit.

Let us consider a single solid conductor (see Fig. 1) with two perfect conducting contacts. The 0D-voltage drops must be distributed onto the 3D-grid; this defines an applied electric field on the edges. Since we are only interested in the line integrals of this field, let $\gamma \in \{-1, 0, 1\}^{n_{\mathrm{M}}}$ be a path from one contact to the other (within $\mathbf{\Omega}_{\mathrm{M}}$). Due to the linearity of Ohm's law (equation three in (4)), it is sufficient to consider an applied voltage $v_{\mathrm{M}} = 1\mathrm{V}$ and define a corresponding distribution matrix $\mathbf{X} \in \mathbb{R}^{n_{\mathrm{M}}}$, such that $\mathbf{X}^{\top}\gamma = 1$. A computational beneficial choice [5] is to impose the voltages only onto the edges crossing a reference plane (see Fig. 1) this yields a sparse coupling matrix (here given for the Cartesian case with aliged reference plane):

$$(\mathbf{X})_i = \begin{cases} \pm 1 & \text{if edge } i \text{ crosses the reference plane,} \\ 0 & \text{else,} \end{cases}$$

where the sign depends on the directions of the edges. By linearity, the distribution matrix $\mathbf{X}$ allows us to apply an arbitrary voltage excitation by multiplication

$$\tilde{\mathbf{S}}^\top \mathbf{\Phi} = \mathbf{X} v_\mathrm{M} \tag{7}$$

inserted into (6), which results in ($v_\mathrm{M} = A_\mathrm{M}^\top e$)

$$\mathbf{M} \frac{\mathrm{d}}{\mathrm{d}t} \widehat{\mathbf{a}} + \mathbf{K}(\widehat{\mathbf{a}})\widehat{\mathbf{a}} = \mathbf{M}\mathbf{X} A_\mathrm{M}^\top e. \tag{8}$$

The total current through the conductor is given by integrating over the reference cross section. We find by using Ohm's Law

$$j_\mathrm{M} = \mathbf{X}^\top \widehat{\widehat{\mathbf{j}}} = \mathbf{X}^\top \mathbf{M} \widehat{\mathbf{e}} = \mathbf{X}^\top \mathbf{M} \mathbf{X} v_\mathrm{M} - \mathbf{X}^\top \mathbf{M} \frac{\mathrm{d}}{\mathrm{d}t} \widehat{\mathbf{a}}$$

or equivalently (using the curl-curl equation (8))

$$j_\mathrm{M} = \mathbf{X}^\top \mathbf{K}(\widehat{\mathbf{a}})\widehat{\mathbf{a}} \ . \tag{9}$$

We denote the derivative of the curl-curl term w.r.t. $\widehat{\mathbf{a}}$ by

$$\frac{\mathrm{d}}{\mathrm{d}\widehat{\mathbf{a}}} \Big(\mathbf{K}(\widehat{\mathbf{a}})\widehat{\mathbf{a}} \Big) = \frac{\mathrm{d}}{\mathrm{d}\widehat{\mathbf{a}}} \mathbf{k}(\widehat{\mathbf{a}}) = \mathbf{k}_a'(\widehat{\mathbf{a}}),$$

(differential reluctivity matrix, [4]) with $\operatorname{Ker} \mathbf{k}_a'(\widehat{\mathbf{a}}) = \operatorname{Ker} \mathbf{K}(\widehat{\mathbf{a}})$.

Assumption 2.6 (Gauging/Structure)**.** For the model we have:

(a) The matrix pencil $[\mathbf{M}, \mathbf{k}_a']$ is positive definite, i.e., $\mathbf{c}^\top (\alpha \mathbf{M} + \mathbf{k}_a') \mathbf{c} > 0$ for all $\mathbf{c} \neq \mathbf{0}$ and $\alpha \neq 0$.

(b) Without loss of generality, we assume that

$$\mathbf{M} = \begin{bmatrix} \mathbf{D} & 0 \\ 0 & 0 \end{bmatrix},$$

where $\mathbf{D}$ is a positive definite, diagonal matrix.

(c) For the coupling term we assume: $\operatorname{Im} \mathbf{X} \subset \operatorname{Im} \mathbf{M}$.

Notice, that Ass. 2.6 (a) can be achieved by gauging, see [3]. Property (b) can be always achieved by a transformation, since $\mathbf{M}$ is symmetric positive semi-definite. Additionally, we note that by this assumption, we always have included non-conducting regions. Ass. 2.6 (c) states that coupling is only allowed at conductive parts, such that the coupling (7) makes sense.

Remark 2.7. Special choices of the conductivity matrix $\mathbf{M}$ yield certain conductor models (e.g stranded or foil conductor). Still, using algebraic manipulations, the coupling can be brought into the structure of (8) and (9), [17].

2.3. Coupled Problem

Assembling the equations of the MNA (2), (1b), (1c) for the extended circuit and the space discrete Maxwell equations (8), we can formulate the field/circuit coupled system

$$\begin{aligned} A_{\mathrm{C}}\frac{\mathrm{d}}{\mathrm{d}t}q_{\mathrm{C}}(A_{\mathrm{C}}^{\top}e,t) + A_{\mathrm{R}}g_{\mathrm{R}}(A_{\mathrm{R}}^{\top}e,t) + A_{\mathrm{L}}j_{\mathrm{L}} + A_{\mathrm{V}}j_{\mathrm{V}} + A_{\mathrm{I}}i_{\mathrm{s}}(t) & \\ +\mathbf{X}^{\top}\mathbf{K}(\widehat{\mathbf{a}})\widehat{\mathbf{a}} &= 0, \\ \frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\mathrm{L}}(j_{\mathrm{L}},t) - A_{\mathrm{L}}^{\top}e &= 0, \\ A_{\mathrm{V}}^{\top}e - v_{\mathrm{s}}(t) &= 0, \\ \mathbf{M}\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathbf{a}} + \mathbf{K}(\widehat{\mathbf{a}})\widehat{\mathbf{a}} - \mathbf{M}\mathbf{X}A_{\mathrm{M}}^{\top}e &= 0, \end{aligned} \tag{10}$$

where the MQS-current (9) is already inserted into the current balance (2). The unknows of (10) are e, j_{L}, j_{V}, $\widehat{\mathbf{a}}$ for which we will derive the structural analysis in the following.

3. Index Analysis

The tractability index is a projector-based approach. It provides an index characterization in terms of the original problem's unknowns, leads to a precise solution description and requires low smoothness of the involved functions [8, 14]. First, we summarize the key ingredients, then we apply the index concept to our coupled problem.

3.1. Tractability Index

We investigate the DAE

$$A\frac{\mathrm{d}}{\mathrm{d}t}d\,(x,t) + b\,(x,t) = 0 \tag{11}$$

with coefficient functions $A \in \mathbb{R}^{m\times n}$, $d(x,t) \in \mathbb{R}^n$ and $b(x,t) \in \mathbb{R}^m$ that are continuous in their arguments and are smooth. The unknown solution is, $x = x(t) \in \mathcal{D} \subset \mathbb{R}^m$, $t \in \mathcal{I} \subset \mathbb{R}$.

Now recall, a projector $Q : \mathbb{R}^m \to \mathbb{R}^m$ is an operator such that $Q^2 = Q$. For our later investigations we deal with a smaller class of DAEs with a so-called properly stated leading term.

Definition 3.1 ([14])**.** The DAE (11) has a properly stated leading term if

$$\operatorname{Ker} A \oplus \operatorname{Im} d'_x(x,t) = \mathbb{R}^n \quad \text{for all } \ x \in \mathcal{D},\ t \in \mathcal{I},$$

and if there is a representing projector $R \in C^1(\mathcal{I},\ \mathbb{R}^n)$, $\operatorname{Ker} A = \operatorname{Ker} R(t)$, $\operatorname{Im} d'_x(x,t) = \operatorname{Im} R(t)$ and $d(x,t) = R(t)\, d(x,t)$ for all $x \in \mathcal{D}$ and $t \in \mathcal{I}$.

For the index definition, we need

Definition 3.2 (Matrix Chain and Subspaces)**.** Given the DAE (11), we define recursively the following objects:

$$\begin{aligned}
G_0(x,t) &:= A d'_x(x,t)\,,\\
N_0(x,t) &:= \operatorname{Ker} G_0(x,t)\,,\\
P_0(x,t) &:= \mathbf{I} - Q_0(x,t)\,,\ Q_0(x,t)\,\text{projector onto } N_0(x,t)\,,\\
S_0(x,t) &:= \{z \in \mathbb{R}^m \,|\, b'_x(x,t)\, z \in \operatorname{Im} G_0(x,t)\}\,,\\
G_1(x,t) &:= G_0(x,t) + b'_x(x,t)\, Q_0(x,t)\,,\\
N_1(x,t) &:= \operatorname{Ker} G_1(x,t)\\
S_1(x,t) &:= \{z \in \mathbb{R}^m \,|\, b'_x(x,t)\, P_0(x,t)\, z \in \operatorname{Im} G_1(x,t)\}\,.
\end{aligned}$$

Definition 3.3 ([14])**.** The DAE (11) with a properly stated leading term is called DAE of (tractability) index-0 if

$$N_0(x,t) = \{0\} \text{ for all } \ x \in \mathcal{D},\ t \in \mathcal{I}$$

or otherwise it is of index-1 if

$$(N_0 \cap S_0)(x,t) = \{0\} \text{ for all } \ x \in \mathcal{D},\ t \in \mathcal{I}$$

or it is of index-2 if

$$(N_0 \cap S_0)(x,t) = \text{constant and } \ (N_1 \cap S_1)(x,t) = \{0\} \text{ for all } \ x \in \mathcal{D},\ t \in \mathcal{I}.$$

Solving a DAE with a properly stated leading term is advantageous especially in index-1 and index-2 cases:

Remark 3.4. Often DAEs are not given with a properly stated leading term, and not all DAEs can be formulated as such. If possible, it is worth to formulate the properly stated leading term, because

- the leading term $d(x,t)$ figures out precisely which derivatives are actually involved and
- for a large class of index-1 and index-2 DAEs it can be shown that BDF and Runge-Kutta methods are stability preserving [10, 11].

For electric circuits we refer to some standard results:

Remark 3.5. For the MNA (and nodal analysis) with merely basic elements the index does not exceed two, under the strictly passivity assumption [7]. More precisely, the MNA equations are of index-2 iff there are LI-cutsets or CV-loops.

Rewriting the coupled problem (10) in the abstract form of (11) with properly stated leading term gives

$$\begin{bmatrix} A_{\mathrm{C}} & 0 & 0 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{M} \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} A_{\mathrm{C}}^{+} A_{\mathrm{C}} q_{\mathrm{C}}(\cdot) \\ \Phi_{\mathrm{L}}(\cdot) \\ \mathbf{M}^{+}\mathbf{M}\widehat{\mathbf{a}} \end{bmatrix} + \begin{bmatrix} A_{\mathrm{R}} g_{\mathrm{R}}(\cdot) + A_{\mathrm{L}} j_L + A_{\mathrm{V}} j_V + A_{\mathrm{M}} \mathbf{X}^{\top} \mathbf{K}(\cdot)\widehat{\mathbf{a}} \\ -A_{\mathrm{L}}^{\top} e \\ A_{\mathrm{V}}^{\top} e \\ \mathbf{K}(\cdot)\widehat{\mathbf{a}} - \mathbf{M}\mathbf{X}A_{\mathrm{M}}^{\top} e \end{bmatrix} = f(\cdot) \tag{12}$$

with time dependent function f and unknown $x = [e, j_{\mathrm{L}}, j_{\mathrm{V}}, \widehat{\mathbf{a}}]$. Now, let Q_M be a constant projector onto $\operatorname{Ker}\mathbf{M}$, such that $Q_M = Q_M^{\top}$, which is always possible. Then for $P_M = \mathbf{I} - Q_M$ holds $P_M = \mathbf{M}^{+}\mathbf{M}$, where '+' indicates the (Moore-Penrose) pseudo-inverse. In fact, the DAE (12) has a properly stated leading term with representing projector

$$\mathbf{R} = \begin{bmatrix} A_{\mathrm{C}}^{+} A_{\mathrm{C}} & 0 & 0 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & \mathbf{M}^{+}\mathbf{M} \end{bmatrix}.$$

Lemma 3.6. *From Ass. 2.6 follows (Q_M constant projector onto* $\operatorname{Ker}\mathbf{M}$*)*

$$\operatorname{Ker}\left(\mathbf{M}+Q_M^\top \mathbf{k}_a'(\cdot)Q_M\right)=\{0\}.$$

Notice, although $Q_M^\top = Q_M$, we use still the transposed of Q_M to keep the underlying physical meaning visible.

3.2. Index Investigation

For an index-0 result we need to inspect

$$G_0(x,t)=Ad_x'(x,t)=\begin{bmatrix} A_{\mathrm{C}}C(\cdot)A_{\mathrm{C}}^\top & 0 & 0 & 0\\ 0 & L(\cdot) & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \mathbf{M}\end{bmatrix}. \tag{13}$$

Theorem 3.7 (Index-0)**.** *Let Ass. 2.2 and 2.6 be fulfilled. Additionally, either Ass. 2.3 or Ass. 2.5 shall be fulfilled. Then the DAE* (12) *has index-0 iff there exists no MQS device, no voltage sources and a tree containing capacitors only.*

Proof. We have to check that $G_0(x,t)$ is nonsingular. Since $C(\cdot)$ and $L(\cdot)$ are positive definite the matrix $G_0(x,t)$ is nonsingular iff the zero rows and columns disappear and $\operatorname{Ker}A_{\mathrm{C}}^\top=\{0\}$. The null space of $\operatorname{Ker}A_{\mathrm{C}}^\top$ is trivial iff the circuit has a tree containing capacitors only, see Remark 2.4. The block zero row and column disappears iff there exist no voltage sources. Lastly, $\operatorname{Ker}\mathbf{M}=\{0\}$ iff there exists no MQS devices. □

Remark 3.8. Certainly, Theorem 3.7 can be extended to include MQS devices that consist of conducting materials only, i.e, $\operatorname{Ker}\mathbf{M}=\{0\}$. This particular case will not be discussed further on.

For the next step we need

$$Q_0=\begin{bmatrix} Q_C & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \mathbf{I} & 0\\ 0 & 0 & 0 & Q_M\end{bmatrix},\quad b_x'(x,t)=\begin{bmatrix} A_{\mathrm{R}}G(\cdot)A_{\mathrm{R}}^\top & A_{\mathrm{L}} & A_{\mathrm{V}} & A_{\mathrm{M}}\mathbf{X}^\top\mathbf{k}_a'(\cdot)\\ -A_{\mathrm{L}}^\top & 0 & 0 & 0\\ A_{\mathrm{V}}^\top & 0 & 0 & 0\\ -\mathbf{M}\mathbf{X}A_{\mathrm{M}}^\top & 0 & 0 & \mathbf{k}_a'(\cdot)\end{bmatrix},$$

and

$$b'_x(x,t)\,Q_0 = \begin{bmatrix} A_{\mathrm{R}}G(\cdot)\,A_{\mathrm{R}}^\top Q_C & 0 & A_{\mathrm{V}} & A_{\mathrm{M}}\mathbf{X}^\top\mathbf{k}'_a(\cdot)Q_M \\ -A_{\mathrm{L}}^\top Q_C & 0 & 0 & 0 \\ A_{\mathrm{V}}^\top Q_C & 0 & 0 & 0 \\ -\mathbf{MX}A_{\mathrm{M}}^\top Q_C & 0 & 0 & \mathbf{k}'_a(\cdot)Q_M \end{bmatrix},$$

where Q_0 and Q_C are constant projectors onto $\mathrm{Ker}\,G_0(x,t)$ and $\mathrm{Ker}\,A_{\mathrm{C}}^\top$, respectively.

Theorem 3.9 (Index-1)**.** *Ass. 2.2, 2.6 as well as either Ass. 2.3 or 2.5 hold true and there exists at least one MQS device, voltage source or no tree containing capacitors only. The DAE* (12) *has index-1 iff there exist neither LIM-cutset nor a CV-loop with at least one voltage source.*

Proof. We need to compute $(N_0 \cap S_0)(x,t)$. By definition, we have

$$(N_0 \cap S_0)(x,t) = \mathrm{Ker}\,G_0(x,t) \cap \mathrm{Ker}\,(W_0 b'_x)(x,t)\,.$$

Let, as usual, $W_0(x,t)$ be a projector along $\mathrm{Im}\,G_0(x,t)$; hence $W_0^\top(x,t)$ is a projector onto $\mathrm{Ker}\,G_0^\top(x,t)$. It holds true that $\mathrm{Ker}\,G_0^\top(x,t) = \mathrm{Ker}\,G_0(x,t)$, i.e., we can choose $W_0^\top(x,t) = Q_0$. Hence

$$(N_0 \cap S_0)(x,t) = \mathrm{Im}\,Q_0 \cap \mathrm{Ker}\,W_0 b'_x(x,t)\,Q_0.$$

We compute:

$$W_0 b'_x(x,t)\,Q_0 = \begin{bmatrix} Q_C^\top A_{\mathrm{R}}G(\cdot)\,A_{\mathrm{R}}^\top Q_C & 0 & Q_C^\top A_{\mathrm{V}} & Q_C^\top A_{\mathrm{M}}\mathbf{X}^\top\mathbf{k}'_a(\cdot)Q_M \\ 0 & 0 & 0 & 0 \\ A_{\mathrm{V}}^\top Q_C & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_M^\top\mathbf{k}'_a(\cdot)Q_M \end{bmatrix}.$$

Let $z \in (N_0 \cap S_0)(x,t)$, then $Q_0 z = z$ and $W_0 b'_x(x,t)\,Q_0 z = 0$. From this, we obtain for $z = [z_1\ z_2\ z_3\ z_4]$

$$Q_C z_1 = z_1, \qquad (14)$$

$$Q_M z_4 = z_4, \qquad (15)$$

$$z_2 = 0, \qquad (16)$$

$$Q_C^\top A_{\mathrm{R}}G(\cdot)\,A_{\mathrm{R}}^\top Q_C z_1 + Q_C^\top A_{\mathrm{V}} z_3 + Q_C^\top A_{\mathrm{M}}\mathbf{X}^\top\mathbf{k}'_a(\cdot)Q_M z_4 = 0, \qquad (17)$$

$$A_{\mathrm{V}}^{\top} Q_C z_1 = 0, \tag{18}$$
$$Q_M^{\top} \mathbf{k}_a'(\cdot) Q_M z_4 = 0. \tag{19}$$

From $Q_M^{\top}\mathbf{M} = 0$ and (19) it follows immediately that

$$Q_M^{\top}\big(\mathbf{M} + \mathbf{k}_a'(\cdot)\big) Q_M z_4 = 0,$$

where $\mathbf{M} + \mathbf{k}_a'(\cdot)$ is positive definite by Ass. 2.6 and thus we archive $z_4 = 0$ using (15). Thus, from (14-19) we can conclude

$$Q_C^{\top} A_{\mathrm{R}} G(\cdot)\, A_{\mathrm{R}}^{\top} Q_C z_1 + Q_C^{\top} A_{\mathrm{V}} z_3 = 0, \tag{20}$$
$$A_{\mathrm{V}}^{\top} Q_C z_1 = 0, \tag{21}$$
$$Q_C z_1 = z_1 \tag{22}$$
$$z_2 = 0, \qquad z_4 = 0. \tag{23}$$

Left-multiplying (20) by $z_1^{\top}$ and using (21), we conclude that $z_1 \in \operatorname{Ker} A_{\mathrm{R}}^{\top} Q_C$ and thus $Q_C^{\top} A_{\mathrm{V}} z_3 = 0$. Now, using (22) we have both

$$z_1 \in \operatorname{Ker}\, [A_{\mathrm{C}}\, A_{\mathrm{R}}\, A_{\mathrm{V}}]^{\top} \text{ and } Q_C^{\top} A_{\mathrm{V}} z_3 = 0\,.$$

Then $(N_0 \cap S_0)(x,t) = \{0\}$ holds iff there exist neither a LIM-cutset nor a CV-loop with at least one voltage source. In a nutshell we have shown $z = 0$. □

We introduce the constant projector Q_{CRV} onto $\operatorname{Ker}[A_{\mathrm{C}}\ A_{\mathrm{R}}\ A_{\mathrm{V}}]^{\top}$ with $Q_{CRV} = Q_C Q_{C-VS}^{\top} Q_{R-CV}$ where Q_{C-V} and Q_{R-CV} a constant projector onto $\operatorname{Ker} Q_C^{\top} A_{\mathrm{V}}$ and $\operatorname{Ker} A_{\mathrm{R}}^{\top} Q_C Q_{C-V}^{\top}$, respectively, see [7]. With these new projectors we directly obtain:

Lemma 3.10. *The intersection $(N_0 \cap S_0)(x,t)$ can be described by*

$$(N_0 \cap S_0)(x,t) = \{z \in \mathbb{R}^n \,|\, z_1 \in \operatorname{Im} Q_{CRV},\ z_3 \in \operatorname{Im} Q_{C-V}, [z_2\ z_4] = 0\}\,. \tag{24}$$

Thus, the dimension of $(N_0 \cap S_0)(x,t)$ is constant.

Notice, the constant dimension is important for the index-2 case. Now, we can compute $G_1(x,t) = G_0(x,t) + b'_x(x,t)\,Q_0$ from the matrix chain:

$$G_1(x,t) = \begin{bmatrix} A_{\mathrm{C}}C(\cdot)\,A_{\mathrm{C}}^\top + A_{\mathrm{R}}G(\cdot)\,A_{\mathrm{R}}^\top Q_C & 0 & A_{\mathrm{V}} & A_{\mathrm{M}}\mathbf{X}^\top\mathbf{k}'_a(\cdot)Q_M \\ -A_{\mathrm{L}}^\top Q_C & L(\cdot) & 0 & 0 \\ A_{\mathrm{V}}^\top Q_C & 0 & 0 & 0 \\ -\mathbf{M}\mathbf{X}A_{\mathrm{M}}^\top Q_C & 0 & 0 & \mathbf{M}+\mathbf{k}'_a(\cdot)Q_M \end{bmatrix}.$$

Assumption 3.11 (Consistent Excitation)**.** The excitation is consistent, i.e., it holds $\operatorname{Ker}\mathbf{X} = \operatorname{Ker}\,(\mathbf{k}'_a\mathbf{X}) = \{0\}$, [17], and

$$\mathbf{X}^\top\mathbf{H}_{\mathbf{k}}(\cdot)\,\mathbf{X} \text{ is positive definite}$$

with $\mathbf{H}_{\mathbf{k}}(\cdot) = \mathbf{k}'_a(\cdot)\left(\mathbf{k}'_a(\cdot)^+ - \left(Q_M^\top\mathbf{k}'_a(\cdot)Q_M\right)^+\right)\mathbf{k}'_a(\cdot)$.

Remark 3.12. Ass. 3.11 is trivially fulfilled for 2D models, because the discrete curl operator is a regular matrix. It follows immediately the positive definiteness of the curl-curl matrix $\operatorname{Ker}\mathbf{k}'_a(\cdot) = \{0\}$ and thus $\mathbf{H}_{\mathbf{k}}(\cdot)$ is positive definite (the subtrahend is a generalized Schur Complement).

Theorem 3.13 (Index-2)**.** *The Assumptions 2.2, 2.6 and either*

i) Assumptions 2.3 and 3.11 or

ii) Assumption 2.5

hold true and there exists at least one MQS device, voltage source or no tree containing capacitors only. The DAE (12) *has index-2 iff there exist a LIM-cutset or a CV-loop with at least one voltage source.*

Proof. We show that $(N_1 \cap S_1)(x,t)$ is trivial. To this end, we inspect

$$\begin{aligned} \widetilde{S}_1(x,t) &= \{z \in \mathbb{R}^n \,|\, b'_x(x,t)\,P_0 z \in \operatorname{Im} G_1(x,t)\} \\ &= \{z \in \mathbb{R}^n \,|\, W_1(x,t)\,b'_x(x,t)\,P_0 z = 0\} \end{aligned}$$

where $W_1(x,t)$ is a projector with $(W_1G_1)(x,t) = 0$. We will show that $\left(N_1 \cap \widetilde{S}_1\right)(x,t)$ is trivial; since $S_1(x,t) \subset \widetilde{S}_1(x,t)$ holds, we obtain the desired result. We can choose $W_1(x,t)$ as

$$W_1(x,t) = \begin{bmatrix} Q_{CRV}^\top & 0 & 0 & -Q_{CRV}^\top A_{\mathrm{M}}\mathbf{X}^\top\mathbf{k}'_a(\cdot)\left(Q_M^\top\mathbf{k}'_a(\cdot)Q_M\right)^+ \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Q_{C-V}^\top & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\left(Q_M^\top \mathbf{k}_a'(\cdot)Q_M\right)^+ = Q_M\left(Q_M^\top \mathbf{k}_a'(\cdot)Q_M\right)^+ Q_M^\top$. Then we compute

$$W_1(x,t)\,b_x'(x,t)\,P_0 = \begin{bmatrix} 0 & Q_{CRV}^\top A_\mathrm{L} & 0 & Q_{CRV}^\top A_\mathrm{M}\mathbf{X}^\top \mathbf{H_k}(\cdot)\,P_M \\ 0 & 0 & 0 & 0 \\ Q_{C-V}^\top A_\mathrm{V}^\top P_C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where we employ $\mathbf{H_k}(\cdot) = \mathbf{k}_a'(\cdot)\left(\mathbf{k}_a'(\cdot)^+ - \left(Q_M^\top \mathbf{k}_a'(\cdot)Q_M\right)^+\right)\mathbf{k}_a'(\cdot)$. From $(W_1 b_x')(x,t)\,P_0 z = 0$ follows

$$Q_{CRV}^\top A_\mathrm{L} z_2 + Q_{CRV}^\top A_\mathrm{M}\mathbf{X}^\top T(\cdot)\,P_M z_4 = 0, \quad (25)$$

$$Q_{C-V}^\top A_\mathrm{V}^\top P_C z_1 = 0, \quad (26)$$

and $G_1(x,t)\,z = 0$ gives

$$\left(A_\mathrm{C} C(\cdot)\,A_\mathrm{C}^\top + A_\mathrm{R} G(\cdot)\,A_\mathrm{R}^\top Q_C\right) z_1 + A_\mathrm{V} z_3 + A_\mathrm{M}\mathbf{X}^\top \mathbf{k}_a'(\cdot)Q_M z_4 = 0, \quad (27)$$

$$z_2 - L^{-1}(\cdot)\,A_\mathrm{L}^\top Q_C z_1 = 0, \quad (28)$$

$$A_\mathrm{V}^\top Q_C z_1 = 0 \quad (29)$$

$$-\mathbf{M}\mathbf{X}A_\mathrm{M}^\top Q_C z_1 + \left(\mathbf{M} + \mathbf{k}_a'(\cdot)Q_M\right) z_4 = 0. \quad (30)$$

Left-multiplying (30) by $(Q_M z_4)^\top$ we obtain

$$Q_M z_4 = 0, \text{ that is, } z_4 = P_M z_4, \quad (31)$$

because $\mathbf{M} + \mathbf{k}_a'(\cdot)$ is positive definite by Ass. 2.6.
From (30) with $P_M = \mathbf{M}^+\mathbf{M}$ and $z_4 = P_M z_4$, we can conclude that

$$z_4 = \mathbf{X}A_\mathrm{M}^\top Q_C z_1 \quad (32)$$

since it holds $\mathbf{X} = P_M\mathbf{X}$ using Ass. 2.6 (c). Multiplying (27) from left by $(Q_C z_1)^\top$, using both (29) and (31), we obtain

$$Q_C z_1 \in \operatorname{Ker} A_\mathrm{R}^\top. \quad (33)$$

Putting (28),(33) and the definition of Q_C together, we have

$$Q_C z_1 \in \operatorname{Ker}\,[A_\mathrm{C}\, A_\mathrm{R}\, A_\mathrm{V}]^\top = \operatorname{Im} Q_{CRV}$$

that is, $Q_C z_1 = Q_{CRV} Q_C z_1$. Inserting (28) and (32) into (25) gives

$$Q_{CRV}^\top A_\mathrm{L} L^{-1}(\cdot)\, A_\mathrm{L}^\top Q_C z_1 + Q_{CRV}^\top A_\mathrm{M} \mathbf{X}^\top \mathbf{H_k}(\cdot)\, \mathbf{X} A_\mathrm{M}^\top Q_C z_1 = 0$$

where

i) $\mathbf{X}^\top \mathbf{H_k}(\cdot)\, \mathbf{X}$ is positive definite due to Ass. 3.11. Therefore it follows that $A_\mathrm{L}^\top Q_C z_1 = 0$ and $A_\mathrm{M}^\top Q_C z_1 = 0$ (using $Q_C z_1 = Q_{CRV} Q_C z_1$). Hence $Q_C z_1 \in \operatorname{Ker} [A_\mathrm{C}\, A_\mathrm{R}\, A_\mathrm{L}\, A_\mathrm{V}\, A_\mathrm{M}]^\top$, which is trivial due to Ass. 2.3.

ii) $\mathbf{H_k}(\cdot)$ is positive semidefinite since it consists of the Schur complement of the positive semidefinite $\mathbf{k}_a'(\cdot)$ and therefore it follows that $0 = A_\mathrm{L}^\top Q_C z_1$. Hence $Q_C z_1 \in \operatorname{Ker} [A_\mathrm{C}\ A_\mathrm{R}\ A_\mathrm{L}\ A_\mathrm{V}]^\top$, which is trivial due to Ass. 2.5.

Thus in both cases we find $Q_C z_1 = 0$, in other words, $P_C z_1 = z_1$. From (28) we deduce $z_2 = 0$. Using this, (27) can be written as

$$H_C(\cdot)\, P_C z_1 = -A_\mathrm{V} z_3,$$

where $H_C(\cdot) = A_\mathrm{C} C(\cdot)\, A_\mathrm{C}^\top + Q_C^\top Q_C$ is positive definite. Thus

$$z_1 = -H_C(\cdot)^{-1} A_\mathrm{V} z_3.$$

Multiplying (27) from left by $Q_C^\top$ leads to $z_3 \in \operatorname{Im} Q_{C-V}$. Together with (26) and $z_3 \in \operatorname{Im} Q_{C-V}$ this gives

$$Q_{C-V}^\top A_\mathrm{V}^\top H_C(\cdot)^{-1} A_\mathrm{V} Q_{C-V} z_3 = 0.$$

Hence we can conclude $A_\mathrm{V} z_3 = 0$ and from this we have $z_3 = 0$, since A_V has full column rank. Consequently from $H_C(\cdot)\, z_1 = 0$ follows $z_1 = 0$. Hence $\left(N_1 \cap \widetilde{S}_1\right)(x,t)$ and $(N_1 \cap S_1)(x,t)$ is trivial iff there exist a LIM-cutset or a CV-loop with at least one voltage source. $\square$

Remark 3.14. The index-2 variables are those components that depend on first derivatives of the input functions. They can be described by Tx, [6], where T is a constant projector onto $(N_0 \cap S_0)(x,t)$ with

$$T = \begin{bmatrix} Q_{CRV} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Q_{C-V} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then we can write $b(x,t) = b(Ux,t) + BTx$ and $d(x,t) = d(Ux,t)$ with $U = \mathbf{I} - T$. Thus index-2 variables enter our system linearly.
Using perturbation-index analysis, it has been shown for index-2 Hessenberg systems with linear index-2 variables, [1], and for index-2 circuits, [19], that the numerical difficulties in time-integration are moderate, because the differential (index-0) variables are not affected by derivatives of the numerical perturbations.

Remark 3.15. Notice that the introduction of j_{M} as an additional unknown of our coupled problem (cf. Section 2.3) would not change our index results.

4. Consistent Initialization

To solve a DAE numerically, it is important to start at least from an consistent initial value [9]:

Definition 4.1. a) A vector $x_0 \in \mathbb{R}^m$ is a consistent initial value of the DAE (11) if there exists a solution of (11) that fulfils $x_0 = x(t_0)$, $t_0 \in \mathcal{I}$.

b) A tuple $(x_0,\ y_0) \in \mathbb{R}^{m+n}$ is an operating point of the DAE (11) if $Ay_0 + b(x_0,t_0) = 0$ is fulfilled, $t_0 \in \mathcal{I}$.

c) A tuple $(x_0,\ y_0) \in \mathbb{R}^{m+n}$ is a consistent initialization of the DAE (11) if x_0 is a consistent value of (11) and $(x_0,\ y_0)$ is an operating point.

Firstly, we focus on the index-1 case, which can be handled very generally.

Theorem 4.2. *Let the DAE* (11) *have a properly stated leading term with representing projector R and let it have index-1. Then for any $t_0 \in \mathcal{I}$ and $x^0 \in \mathbb{R}^m$ (starting value), the following system*

$$\begin{aligned} Ay_0 + b(x_0,t_0) &= 0 \\ (I - R(t_0))\,y_0 + Rd(x_0,t_0) - Rd\left(x^0,t_0\right) &= 0 \end{aligned}$$

is locally uniquely solvable for $(x_0,\ y_0) \in \mathbb{R}^{m+n}$*, which provides a consistent initialization.*

For a proof see e.g. [15]. The starting value x^0 can be chosen arbitrarily. In the case of an index-2 DAE, the calculation of an consistent initialization becomes more complicated. At least for DAEs with linear index-2 components we can obtain a consistent value after one (implicit) Euler integration step starting from an operation point [6] (after the first step at the next time level). Recalling Remark 3.14, this hold in our case for (24).

5. Numerical Example

Here, we will prove the numerical importance of the index results, by discussing two simple examples that illustrate the different behavior of the index-1 and index-2 cases. The most simple problems are the following: Fig. 2,

(a) a circuit with no devices but a voltage source and a one-port MQS device ($A_\mathrm{V} = [1]$ and $A_\mathrm{M} = [-1]$) states an index-1 problem (see Fig. 2a),

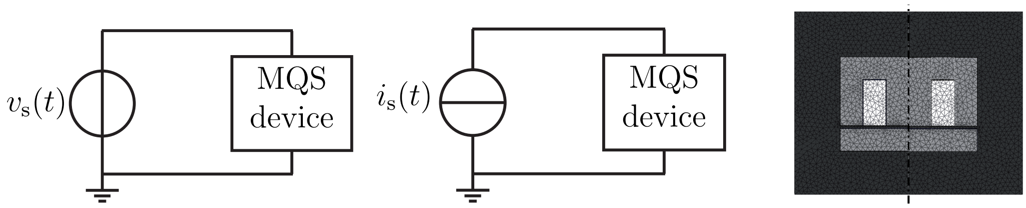
(b) a circuit with no devices but a current source and a one-port MQS device ($A_\mathrm{I} = [1]$ and $A_\mathrm{M} = [-1]$) states an index-2 problem (see Fig. 2b).

The tractability indeces of those particular problems agree with the Kronecker index and differentiation index results in [17, 21].

The simplest one-port MQS device is a (linear) inductor without eddy currents; we consider the axisymmetric PDE model Fig. 2c discretized by FEMM[1]. In this simple linear setup the conductor model is equivalently given by a series connection of a lumped resistance R and an inductance L, [13]. Hence, for this simple problem, we have the analytic solution of the voltages and currents of the coupled DAE problem at hand: in the index-2 setting, cf. Fig. 2b, with a sinusoidal current source

$$i_\mathrm{s}(t) = \sin(2\pi f t), \quad \text{with a frequency} \quad f = 1\mathrm{Hz}$$

[1] D. C. Meeker, Finite Element Method Magnetics, Version 4.2 (02Nov2009 Build), `http://www.femm.info`



(a) index-1 problem (b) index-2 problem (c) MQS device

Figure 2: Examples. a) voltage-driven MQS device is index-1, b) current-driven MQS device is index-2, c) axisymetric inductor model discretized by FEMM coupled to a circuit via its coil (white).

the voltage (at the RL-element) is analytically given by

$$e(t) = -2\pi L \cos(2\pi f t) - R i_s(t).$$

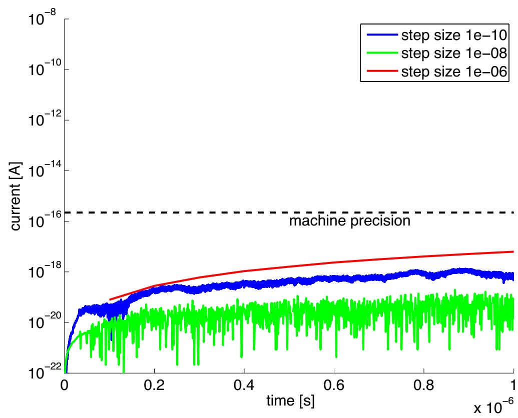
In the index-1 case, cf. Fig. 2a, a voltage source is connected to the MQS device and it is chosen accordingly to the previous case as $e(t) = v_s(t)$ and therefore the solution is given by $j_L(t) = i_s(t)$.

Fig. 3 shows the numerical error due to time-integration by the implicit Euler scheme using fixed time step sizes $h = 10^{-11}$s, 10^{-9}s and 10^{-7}s. Although both, the index-1 and index-2 circuits describe the same physical phenomenon, the error due to time-integration is much larger in the index-2 case, Fig. 3b compared to the index-1 case, Fig. 3a.

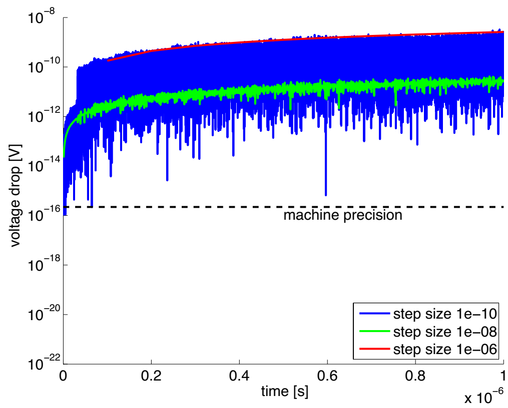
In the index-1 case all errors are below the machine precision (round-off errors) and hence they are all equally good. In contrast to these, the errors in the index-2 case are much larger and especially for small step sizes ($h = 10^{-11}$s) they become very large and oscillate. Nonetheless they are not propagated in time because they do not affect the differential components, see Remark 3.14. Only when using adaptive step size control, one has to take special care, i.e, exclude the index-2 voltages from the set of variables that are monitored, such that the oscillations do not require unreasonable step sizes.

6. Conclusion

In this paper we have modeled a network of lumped resistors, inductors, capacitors, independent current and voltage sources, and spatially distributed MQS devices by applying MNA. Starting from spatially discretized



(a) error in current (index-1 problem)



(b) error in voltage (index-2 problem)

Figure 3: Numerical Errors for a) the index-1 problem (voltage-driven) and b) the index-2 problem (current driven)

MQS devices, we have deduced a coupled system of DAEs with properly stated leading term. Then the structural properties of this system have been analyzed: We have proven that the index does not exceed two under certain conditions and we have generalized the topological index criteria of the electrical circuits. A simple numerical example illustrates the results.

Although the MQS devices are modelled as controlled current sources, our structural analysis shows that they behave topologically as inductances. This correspondes to the physcial effects covered by the eddy current problem.

Acknowledgement

The authors are indebted to the ICESTARS project (ICT214911), which is funded by the seventh framework programme of the European Union. And especially the third author is indebted to the "DAAD Doktoranden Programm" of the German Academic Exchange Service.

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