Right inverses for $P(D)$ in spaces of real analytic functions

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Dedicated to the memory of Leon Ehrenpreis

Abstract In the present paper we study the question, when a linear partial differential operator $P(D)$ with constant coefficients admits a continuous linear right inverse in the space $A(\mathbb{R}^n)$ of real analytic functions on $\mathbb{R}^n$ (or, more general, in $A(\Omega)$ where Ω is a open subset of $\mathbb{R}^n$). To obtain a necessary condition we investigate when $P(D)$ admits solvability 'with real analytic parameter' in $A(\Omega)$ and solve it completely for convex Ω, using a different approach than the one used in DOMAŃSKI [7]. To obtain a sufficient condition we show that the global real analytic Cauchy problem is solvable if and only if the principal part of $P(D)$ is hyperbolic. In this way we get a complete solution of our main problem for $A(\mathbb{R}^2)$ and, in the homogeneous case, for $A(\Omega)$ where Ω is the open unit ball in $\mathbb{R}^n$.

Introduction

Let $\Omega \subset \mathbb{R}^n$ be open and $P \in \mathbb{C}[z_1, \dots, z_n]$. We study the linear partial differential operator with constant coefficients $P(D_1, \dots, D_n)$, with $D_j = -i\frac{\partial}{\partial x_j}$, acting on the space $A(\Omega)$ of real analytic functions on Ω. We want to know when $P(D)$ admits a continuous linear right inverse in $A(\Omega)$.

We should recall that $P(D)$ needs not to be surjective in $A(\Omega)$ even for $\Omega = \mathbb{R}^n$. It had been conjectured by De Giorgi and Cattabriga [5] and shown by Piccinini [18, 19] that not every linear differential operator $P(D)$ with constant coefficients is surjective in $A(\mathbb{R}^n)$. Their examples were operators whose principal part has a

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mute variable. Such operators we study in Section 1 of the present paper and we characterize completely when they are surjective in $A(\Omega \times \mathbb{R})$ for convex Ω (the last variable being assumed to be mute), so extending results of [22]. The same characterization has been obtained in Domański [7]. However methods and proofs there are completely different, so that it appears useful to present our approach. Surjectivity of $P(D)$ in $A(\Omega)$ for convex Ω has been characterized by Hörmander [8] in terms of a Phragmén-Lindelöf condition for plurisubharmonic functions on the zero variety of the principal part $P_m(D)$ of $P(D)$. In [13] Meise, Taylor and Vogt characterized, for convex Ω, the polynomials P such that $P(D)$ admits a continuous linear right inverse in $C^\infty(\Omega)$, also in terms of a Phragmén-Lindelöf condition on the zero variety of P. While surjectivity in $A(\Omega)$ depends only on the principal part, the existence of right inverses in $C^\infty(\Omega)$ does not and the perturbation conditions for lower order parts are unknown up to now. The condition for surjectivity with a mute variable, or surjectivity with parameter dependence connects both. It depends only on the principal part, which has to admit a continuous linear right inverse in $C^\infty(\Omega)$.

Surjectivity with parameter dependence or surjectivity in $\Omega \times \mathbb{R}$ is, of course, a necessary condition for the existence of a continuous linear right inverse of $P(D)$ in $A(\Omega)$ which yields a necessary condition which depends only on the principal part. We do not know whether the existence of a right inverse depends in fact only on the principal part. In some cases, however, the necessary condition turns out to be also sufficient. This is the case for dimension 2 and $\Omega = \mathbb{R}^2$. Here the necessary condition means that the principal part has to be hyperbolic.

We then show, for arbitrary dimension, that hyperbolicity of the principal part is equivalent to unique solvability of the global Cauchy problem in the real analytic functions, for real analytic data. This result is interesting in its own and yields a complete characterization of the $P(D)$ for which there exists a continuous linear solution operator in $A(\mathbb{R}^2)$. Another case where the necessary condition is also sufficient and we get such a characterization is the case of homogeneous operators and Ω a bounded set with C^1-boundary, for instance, the unit ball.

1 Preliminaries

Throughout the paper we denote by $A(\Omega)$ the linear space of real analytic functions on the open set $\Omega \subset \mathbb{R}^n$ equipped with its natural locally convex topology, which is as well of (PDF)-type and ultrabornological (see [12]). This implies by Grothendieck's (or de Wilde's) open mapping theorem that any continuous linear surjective map from $A(\Omega)$ to $A(\Omega)$ is open.

We will use the following condition $\mathrm{HPL}(\Omega, \mathrm{loc})$ introduced in Hörmander [8]. $\mathscr{K}(\Omega)$ denotes the convex, compact subsets of Ω and $\mathrm{PSH}(W)$ the plurisubharmonic functions on a complex variety W. For any compact convex set $K \subset \mathbb{R}^p$ we denote by $h_K(x) := \sup\{\langle x, \xi \rangle : \xi \in K\}$, $x \in \mathbb{R}^p$, the support function of K.

Let V be the germ of a complex variety at $\xi \in \mathbb{R}^n$. V satisfies HPL(Ω, loc) if there are open sets $U_1 \subset\subset U_2 \subset\subset U_3 \subset\subset \mathbb{C}^n$ with $\xi \in U_1$ such that for each $K \in \mathscr{K}(\Omega)$ there exists $K' \in \mathscr{K}(\Omega)$ and $\delta > 0$ such that each $u \in \text{PSH}(U_3 \cap V)$ satisfying (α) and (β), also satisfies (γ), where

(α) $u(z) \le h_K(\operatorname{Im} z) + \delta$, $z \in U_3 \cap V$
(β) $u(z) \le 0$, $z \in U_2 \cap \mathbb{R}^n \cap V$
(γ) $u(z) \le h_{K'}(\operatorname{Im} z)$, $z \in U_1 \cap V$.

For detailed information on this and other related Phragmén-Lindelöf conditions we refer to [8] and [16]. For unexplained notation and results on partial differential equations we refer to [9].

2 Solvability with real analytic parameter

We will use the following notation: For $P \in \mathbb{C}[z_1, \ldots, z_n]$ we set $P^+ = P$, considered as a polynomial in $\mathbb{C}[z_1, \ldots, z_{n+1}]$, and for open $\Omega \subset \mathbb{R}^n$ we consider $P^+(D)$ as acting in $A(\Omega^+)$ where $\Omega^+ = \Omega \times \mathbb{R}$.

We say that $P(D)$ is *solvable in $A(\Omega)$ with real analytic parameter* if $P^+(D) : A(\Omega^+) \longrightarrow A(\Omega^+)$ is surjective. Solvability in $A(\Omega)$ with a real analytic parameter has been investigated in a different context in [22] and a characterization for $\Omega = \mathbb{R}^n$ was given there. A complete characterization has been given in Domański [7]. Many results of this section, in particular, Theorem 1 can be found also there. However our approach and the methods of proof are entirely different.

Various kinds of parameter dependence in different spaces have also been studied recently in Bonet-Domański [1],[2]. Real analytic parameter-dependence in $\mathscr{D}'(\Omega)$ has been studied and characterized in Domański [6].

The following Lemma improves the necessary condition in [22, Proposition 3.1].

Lemma 1. *If $\Omega \subset \mathbb{R}^n$ is convex and $P^+(D)$ surjective in $A(\Omega^+)$ then $P_m(D)$ has a right inverse in $C^\infty(\Omega)$.*

Proof: Let $V = \{z \in \mathbb{R}^n : P_m(z) = 0\}$ be the zero variety of P_m and $V^+ = V \times \mathbb{R}$ the same for P_m^+. By [16, Theorem 3.3], it suffices to show that V satisfies HPL(Ω, loc) at zero and, by [8, Lemma 4.1], we have at our disposal HPL(Ω^+, loc) at any point $\xi^0 \in V^+ \cap \mathbb{R}^{n+1}$ with $|\xi^0| = 1$. We apply it to $\xi^0 = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. We find $0 < r_1 < r_2 < r_3$ and for every $K \in \mathscr{K}(\Omega)$ a $K' \in \mathscr{K}(\Omega^+)$ and $\delta > 0$ such that each $u \in \text{PSH}(U_3^+ \cap V^+)$ satisfying a. and b. also satisfies c., where

a. $u(z) \le h_{K\times\{0\}}(\operatorname{Im} z) + \delta, \qquad z \in U_3^+ \cap V^+$.
b. $u(x) \le 0, \qquad x \in U_2^+ \cap \mathbb{R}^{n+1} \cap V^+$.
c. $u(z) \le h_{K'}(\operatorname{Im} z), \qquad z \in U_1^+ \cap V^+$.

We have set $U_j^+ = \{z \in \mathbb{C}^{n+1} : |z - \xi^0| < r_j\}$ and put $U_j = \{z \in \mathbb{C}^n : |\zeta| < r_j\}$. We remark that $h_{K\times\{0\}}(x) = h_K(x_1, \ldots, x_n)$ for $x \in \mathbb{R}^{n+1}$.

Let now u be a plurisubharmonic function on $U_3 \cap V$ and u^+ be the same function acting on $U_3^+ \cap V^+$. Notice that for $(z_1,\dots,z_{n+1}) \in U_3^+$ we have $(z_1,\dots,z_n) \in U_3$. We assume that

α. $u(z) \leq h_K(\operatorname{Im} z) + \delta, \qquad z \in U_3 \cap V.$
β. $u(x) \leq 0, \qquad x \in U_2 \cap \mathbb{R}^n \cap V.$

Then u^+ satisfies a. and b., hence c. For $z \in U_1$ we set $\tilde{z} = (z_1,\dots,z_n,1)$. Then $\tilde{z} \in U_1^+$ and we have

γ. $u(z) = u^+(\tilde{z}) \leq h_{K'}(\operatorname{Im}\tilde{z}) = h_{K''}(\operatorname{Im} z)$

where $K'' = \pi K'$ and $\pi : (x_1,\dots,x_{n+1}) \mapsto (x_1,\dots,x_n)$. Clearly $K'' \subset\subset \Omega$. □

For convex Ω we obtain a complete characterization of differential polynomials $P(D)$ which admit solvability with a real analytic parameter. The same characterization has been given by a different method in DOMAŃSKI [7, Theorem 6.1].

Theorem 1. *For convex Ω the following are equivalent:*
1. $P^+(D)$ is surjective in $A(\Omega^+)$.
2. $P_m(D) : C^\infty(\Omega) \longrightarrow C^\infty(\Omega)$ admits a continuous linear right inverse.

PROOF: One implication is Lemma 1, the other [22, Proposition 3.2]. □

If we take into account [16, Corollary 3.14] then we get as a special case for $\Omega = \mathbb{R}^n$ [22, Theorem 3.4].

Theorem 2. *For $n > 1$ the following are equivalent:*
1. $P^+(D)$ is surjective in $A(\mathbb{R}^{n+1})$.
2. $P_m(D)$ is surjective in $A(\mathbb{R}^n)$ and P_m has no elliptic factor.

As an immediate consequence of Theorem 1 we obtain:

Corollary 1. *If $P_m(z) = P_m(z_1,\dots,z_p)$ with $1 \leq p < n$ then the following are equivalent:*
1. $P(D)$ is surjective in $A(\mathbb{R}^n)$.
2. $P_m(D_1,\dots,D_p) : C^\infty(\mathbb{R}^p) \longrightarrow C^\infty(\mathbb{R}^p)$ admits a continuous linear right inverse.

Since by a theorem of Grothendieck elliptic $P_m(D_1,\dots,D_p) : C^\infty(\mathbb{R}^p) \longrightarrow C^\infty(\mathbb{R}^p)$ for $p \geq 2$ never admits a continuous linear right inverse (see Trèves [20, Theorem C.1]) this explains the examples of DI GIORGI, CATTABRIGA and PICCININI. A somewhat more general formulation is:

Theorem 3. *If there is $N \neq 0$ in $\mathbb{R}^n$ such that $P_m(z + \lambda N)$ does not depend on λ for all $z \in \mathbb{R}^n$ then the following are equivalent:*
1. $P(D)$ is surjective in $A(\mathbb{R}^n)$.
2. $P_m(D) : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ admits a continuous linear right inverse.

PROOF: By a linear transformation, we may assume that $N = e_n$ and the result follows from Corollary 1. Assertions 2. in both results are then seen to be equivalent, because $P_m(D) = P_m(D_1,\dots,D_{n-1}) \otimes \mathrm{id}_{C^\infty(\mathbb{R})}$ acting on $C^\infty(\mathbb{R}^n) = C^\infty(\mathbb{R}^{n-1}) \widehat{\otimes} C^\infty(\mathbb{R})$. □

By use of a theorem of LANGENBRUCH [11] we obtain for general open $\Omega \subset \mathbb{R}^n$:

Corollary 2. *If P satisfies the assumptions of Theorem 3, $\Omega \subset \mathbb{R}^n$ is open and $P(D)$ is surjective in $A(\Omega)$, then $P_m(D) : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ admits a continuous linear right inverse.*

PROOF: By [11] $P(D)$ is surjective in $A(\mathbb{R}^n)$ and we can apply Theorem 3. □

We will need it in the following more special version:

Lemma 2. *If $n > 1$, $\Omega \subset \mathbb{R}^n$ open and $P(D)$ is surjective in $A(\Omega^+)$, then $P_m(D)$ has no elliptic factor.*

PROOF: By [11] $P^+(D)$ is surjective in $A(\mathbb{R}^{n+1})$ and we can apply Theorem 2. □

3 Right inverses in $A(\Omega)$ necessary condition

We begin with a simple observation:

Lemma 3. *If $P(D)$ has a right inverse in $A(\Omega)$ then $P^+(D)$ is surjective in $A(\Omega \times \mathbb{R})$.*

PROOF: If we identify $A(\Omega \times \mathbb{R}) \cong A(\Omega)\widehat{\otimes}A(\mathbb{R})$ then $P^+(D)$ corresponds to $P(D) \otimes \mathrm{id}_{A(\mathbb{R})}$ which has $R \otimes \mathrm{id}_{A(\mathbb{R})}$ as a right inverse, where R is a continuous linear right inverse for $P(D)$. In particular $P^+(D)$ is surjective. □

Lemma 3 and Lemma 2 together imply:

Proposition 1. *Let $n > 1$ and $\Omega \subset \mathbb{R}^n$ open. If $P(D) : A(\Omega) \longrightarrow A(\Omega)$ admits a continuous linear right inverse, then P_m has no elliptic factor.*

For convex Ω we can use Theorem 1 to sharpen the necessary criterion.

Proposition 2. *Let $n > 1$ and $\Omega \subset \mathbb{R}^n$ open and convex. If $P(D) : A(\Omega) \longrightarrow A(\Omega)$ admits a continuous linear right inverse, then so does $P_m(D) : C^\infty(\Omega) \longrightarrow C^\infty(\Omega)$.*

Homogeneous polynomials which admit a continuous linear right inverse are carefully studied in [13]. In the following case we get a sharp criterion and even a complete characterization:

Proposition 3. *Let $n > 1$ and $\Omega \subset \mathbb{R}^n$ open, convex and bounded with C^1-boundary. If $P(D) : A(\Omega) \longrightarrow A(\Omega)$ admits a continuous linear right inverse, then P_m is, up to a constant factor, a product of real linear forms.*

PROOF: This follows from Proposition 2 and [13], Theorem 3.8. □

Example 1. If Ω is the unit ball in $\mathbb{R}^d$ and P is homogeneous, then the following are equivalent:
1. $P(D)$ admits a continuous linear right inverse in $A(\Omega)$.
2. $P^+(D)$ is surjective in $A(\Omega^+)$.
3. $P(D)$ admits a continuous linear right inverse in $C^\infty(\Omega)$
4. P is, up to a constant factor, a product of real linear forms.

The only thing to prove is 4. $\Rightarrow$ 1. But this is done just by integration. We must construct a right inverse only for P being a real linear form L, which we may assume to be $L(x) = x_1$. Then $f \mapsto \int_0^x f(\xi, x_2, \ldots, x_d)d\xi$ is a right inverse.

4 Operators with hyperbolic principal part

The existence of a right inverse of $P(D)$ in $A(\Omega)$ depends in all cases treated up to now only on the principal part $P_m(D)$. Therefore it might also be of interest to mention that also “hyperbolicity” in the sense of global existence and uniqueness for the Cauchy problem follows from and is even equivalent to the hyperbolicity of the principal part, and in this case the right inverse can be given in a very explicit way.

We will consider $P(D)$ as acting not only in $A(\Omega)$ but also in $C^\infty(\Omega)$ and in the Gevrey classes $\gamma^{(s)}(\Omega)$ for $s > 1$ defined as follows:

$$\gamma^{(s)}(\Omega) = \{f \in C^\infty(\Omega) : \forall K \subset\subset \Omega, \varepsilon > 0\, \exists C\, \forall \alpha, x \in K : |f^{(\alpha)}(x)| \leq C\varepsilon^{|\alpha|}(|\alpha|!)^s\}.$$

Here $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0$ and $|\alpha| = \sum \alpha_j$. $\gamma^{(s)}(\Omega)$ is a Fréchet space when equipped with the seminorms

$$\|f\|_n = \sup_{\substack{x\in K_n \\ \alpha}} |f^{(\alpha)}(x)| \frac{n^{|\alpha|}}{(|\alpha|!)^s},$$

where K_n runs through a compact increasing exhaustion of Ω. For matter of convenience we set $\gamma^{(+\infty)}(\Omega) = C^\infty(\Omega)$.

Let us remark that $\gamma^{(s)}(\Omega) = \mathscr{E}_{(\omega)}(\Omega)$ with $\omega(t) = t^{1/s}$ in the sense of BRAUN-MEISE-TAYLOR [3].

We set $\gamma_0^{(s)}(\mathbb{R}^n) = \gamma^{(s)}(\mathbb{R}^n) \cap \mathscr{D}(\mathbb{R}^n)$ equipped with its natural (LF)-topology.

P is called $\gamma^{(m/m-1)}$-hyperbolic with respect to N if there are fundamental solutions $E_\pm \in \gamma_0^{(m/m-1)'}(\mathbb{R}^n)$ of $P(D)$ with support in the cones

$$H_\pm = \{x \mid \langle x, \pm N\rangle > 0\} \cup \{0\}.$$

From [15, Proposition 2.12] and [9, Theorem 12.7.5] we obtain:

Proposition 4. *If P_m is hyperbolic with respect to N, then*

1. *P is $\gamma^{(m/m-1)}$-hyperbolic with respect to N.*
2. *The Cauchy problem in Proposition 5 is is uniquely solvable in $\gamma^{(m/m-1)}(\mathbb{R}^n)$ for all data $g \in \gamma^{(m/m-1)}(\mathbb{R}^n)$ and $f_0, \ldots, f_{m-1} \in \gamma^{(m/m-1)}(\mathbb{R}^{n-1})$.*

We use it to show the unique solvability of the Cauchy problem in $A(\mathbb{R}^n)$.

Proposition 5. *Let $N = e_1$, and we set $x = (x_1, x')$. If P_m is hyperbolic with respect to N, then the Cauchy problem*

$$P(D)f = g, \quad \frac{\partial^k}{\partial x_1^k} f(0, x') = f_k(x'), \quad k = 0, \ldots, m-1$$

is uniquely solvable for all $g \in A(\mathbb{R}^n)$, $f_0, \ldots, f_{m-1} \in A(\mathbb{R}^{n-1})$ with $f \in A(\mathbb{R}^n)$.

PROOF: Since the assumption implies, by [8, Theorem 6.5], that $P(D) : A(\mathbb{R}^n) \to A(\mathbb{R}^n)$ is surjective, it is easily seen that we may assume $g = 0$. Since $f_0, \dots, f_{m-1} \in \gamma^{(m/m-1)}(\mathbb{R}^{n-1})$ there is, by assumption and Proposition 4, a unique solution $f \in \gamma^{(m/m-1)}(\mathbb{R}^n)$ of the Cauchy problem.

Due to the Cauchy-Kowalewski theorem this solution is real analytic in a neighborhood U of $\{0\} \times \mathbb{R}^{n-1}$.

Let $x \in \mathbb{R}^n$, $x_1 > 0$. We choose $\phi \in \gamma^{(m/m-1)}(\mathbb{R}^n)$, such that $\phi \equiv 1$ in a neighborhood of 0 and $\operatorname{supp}(\phi^2 - \phi) \subset x - V$, where $V = U \cap \{\xi \mid |\xi_1| < x_1\}$. Using the fundamental solution $E_+ \in \gamma_0^{(m/m-1)'}(\mathbb{R}^n)$, which exists by Proposition 4, we set

$$T = \phi E_+ \text{ and } P(D)T = \delta - S.$$

Then $\operatorname{supp} S \subset x - V$ and $\operatorname{supp} T \subset H_+ \cap \operatorname{supp} \phi$. We obtain

$$0 = (P(D)f) * T = f * (P(D)T) = f - S * f,$$

i.e. $f(\xi) = S_y(f(\xi - y))$ for all ξ.

For $y \in \operatorname{supp} S$ we have $x - y \in V$, and the same holds for all ξ in a neighborhood of x. Therefore f is real analytic in a neighborhood of x. An analogous argument applies for $x_1 < 0$. $\square$

Theorem 4. *If P_m is hyperbolic, then $P(D)$ admits a continuous linear right inverse in $A(\mathbb{R}^n)$.*

PROOF: Let P_m be hyperbolic with respect to N. We may assume $N = e_1$. We set $R(g) := f$ where g is the unique solution of the Cauchy problem in Proposition 5 with $f_0 = \cdots = f_{m-1} = 0$. R is clearly a linear right inverse for $P(D)$, it is continuous because the inverse of the 'Cauchy map' χ is continuous (see the proof of Proposition 6). $\square$

Proposition 6. *If the Cauchy problem*

$$P(D)f = 0, \quad \frac{\partial^k}{\partial x_1^k} f(0, x') = f_k(x'), \quad k = 0, \dots, m-1$$

is uniquely solvable for all $f_0, \dots, f_{m-1} \in A(\mathbb{R}^{n-1})$ with $f \in A(\mathbb{R}^n)$, then $P_m(D)$ is hyperbolic with respect to e_1.

PROOF: For $x \in \mathbb{R}^n$ we set again $x = (x_1, x')$. Let χ be the 'Cauchy map' $A(\mathbb{R}^n) \longrightarrow A(\mathbb{R}^{n-1})^m$, i.e.

$$\chi(\varphi) = (\varphi(0, x'), \varphi'(0, x'), \dots, \varphi^{(m-1)}(0, x'))$$

where all derivatives are taken with respect to the first variable. By assumption χ is surjective, hence bijective and therefore, due to the de Wilde-Grothendieck theorem, a topological isomorphism.

We consider the functions $\varphi_\zeta(x) := e^{ix\zeta}$, $P(\zeta) = 0$. Then

$$\begin{aligned}\chi(\varphi_\zeta) &= (\varphi_\zeta(0,x'), i\zeta_1\varphi_\zeta(0,x'),\ldots,(i\zeta_1)^{m-1}\varphi_\zeta(0,x'))\\ &= \varphi_\zeta(0,x')(1,i\zeta_1,\ldots,(i\zeta_1)^{m-1}).\end{aligned}$$

Since χ^{-1} is continuous we have the following estimates for φ_ζ, $P(\zeta)=0$:

$$\forall\, r\ \exists\, R\ \forall\, \varepsilon>0\ \exists\, \delta>0, C: \|\varphi_\zeta\|_{r,\delta} \leq C\|\chi(\varphi_\zeta)\|_{R,\varepsilon}$$

where

$$\begin{aligned}\|\varphi_\zeta(x)\|_{r,\delta} &= \sup_{\substack{|x|\leq r\\ |y|\leq\delta}} e^{-y\xi-x\eta} = e^{r|\eta|+\delta|\xi|}\\ \|\varphi_\zeta(0,x')\|_{R,\varepsilon} &= \sup_{\substack{|x'|\leq r\\ |y'|\leq\varepsilon}} e^{-y'\xi'-x'\eta'} = e^{R|\eta'|+\varepsilon|\xi'|}.\end{aligned}$$

Therefore, taking in $A(\mathbb{R}^{n-1})^m$ the maximum of the 'norms', we have for $P(\zeta)=0$

$$\|\chi(\varphi_\zeta)\|_{R,\varepsilon} = (1+|\zeta|)^{m-1}e^{R|\eta'|+\varepsilon|\xi'|}.$$

With the quantifiers as above and $c=\log C$ we obtain

$$r|\eta|+\delta|\xi| \leq c+(m-1)\log(1+|\zeta_1|)+R|\eta'|+\varepsilon|\xi'|.$$

Looking for the solutions of $P(\zeta)=0$ for real ζ', i.e. $\eta'=0$, and choosing $r=1$ we obtain for every $\varepsilon>1$ a C_ε so that

$$|\eta_1| \leq C_\varepsilon+(m-1)\log(1+|\zeta_1|)+\varepsilon|\xi'|$$

which implies

$$|\eta_1|-(m-1)\log(1+|\eta_1|) \leq C_\varepsilon+(m-1)\log(1+|\xi_1|)+\varepsilon|\xi'|$$

and, for large $|\eta_1|$,

$$\frac{1}{2}|\eta_1| \leq C_\varepsilon+(m-1)\log(1+|\xi_1|)+\varepsilon|\xi'|. \tag{1}$$

Assume that there is $\xi\in\mathbb{R}^n$, $\eta\in\mathbb{R}$ so that $P_m(\xi+i\eta e_1)=0$. We set $g(z)=P_m(\xi+i(\eta+z)e_1)$ for $z\in\mathbb{C}$. Since $g(0)=0$ and $g\not\equiv 0$ there is k, so that $g(z)=z^k g_0(z)$ and $|g_0(z)|\geq A>0$ in a neighborhood $U_r(0)$.

Now $t^{-m}P(t(\xi+i(\eta+z)e_1))=g(z)+h(z)$ where $|h(z)|\leq\frac{M}{t}$ for z in $U_r(0)$ and all $t>0$. We apply the theorem of Rouché for large t to the disc $U_\rho(0)$ with $\rho=Ct^{-1/k}$ where C is chosen such that $AC^k>M$ and obtain z_t with $|z_t|\leq Ct^{-1/k}$ such that for $\zeta_t=t(\xi+i(\eta+z_t)e_1)$ we have $P(\zeta_t)=0$.

We assume now $\eta\neq 0$ and apply inequality (1) with $\varepsilon>0$, such that $\varepsilon|\xi'|<\frac{1}{2}|\eta|$, to ζ_t for large t. Then instead of η_1 we have $\mathrm{Im}(it(\eta+z_t)$ and we can estimate

$$|\mathrm{Im}(it(\eta+z_t))| = t|\eta+\mathrm{Re}z_t| \geq t(|\eta|-|z_t|) \geq t(|\eta|-Ct^{-1/k}).$$

Instead of ξ_1 we have $t(\xi_1-\mathrm{Im}z_t)$ and the estimate

$$|t(\xi_1-\mathrm{Im}z_t)| \leq t(|\xi_1|+Ct^{-1/k}).$$

So (1) takes for all large enough t the form

$$\frac{t}{2}(|\eta|-Ct^{-1/k}) \leq C_\varepsilon+(m-1)\log(t(|\xi_1|+Ct^{-1/k}))+\varepsilon(|\xi'|t).$$

Dividing by t and letting $t\to+\infty$ we get a contradiction to the choice of ε. Hence η has to be zero. $\square$

So finally we obtain the following characterization:

Theorem 5. *The following are equivalent:*

1. *The Cauchy problem in Proposition 5 ('inhomogeneous Cauchy problem') is solvable.*
2. *The Cauchy problem in Proposition 6 ('homogeneous Cauchy problem') is solvable.*
3. *P_m is hyperbolic with respect to e_1.*

5 Case of n=2

We may use this to prove a complete characterization for n=2. We assume Ω to be open in $\mathbb{R}^2$.

Lemma 4. *If $n=2$ and $P^+(D)$ is surjective in $A(\Omega^+)$ then P_m is, up to a constant factor, the product of real linear forms.*

PROOF: P_m decomposes into irreducible factors, as follows:

$$P_m(z_1,z_2) = Az_2^{m_1}\prod_{\mu=1}^{m_2}(z_1+a_\mu z_2)$$

where $A\in\mathbb{C}$, $a_1,\ldots,a_{m_2}\in\mathbb{C}$ and $m_1+m_2=m$. If $a_\mu\in\mathbb{C}\setminus\mathbb{R}$ then $D_1+a_\mu D_2$ is elliptic which, by Lemma 2, cannot occur. $\square$

We arrive at the theorem:

Theorem 6. *For $n=2$ the following are equivalent:*

1. *$P(D)$ admits a continuous linear right inverse in $A(\Omega)$ for some open convex set $\Omega\subset\mathbb{R}^2$.*
2. *$P(D)$ admits a continuous linear right inverse in $A(\mathbb{R}^2)$.*
3. *$P^+(D)$ is surjective in $A(\mathbb{R}^3)$.*
4. *P_m is, up to a constant factor, the product of real linear forms.*

PROOF: 2. $\Rightarrow$ 1. is obvious, 1. $\Rightarrow$ 4. follows from Lemma 3 and Lemma 4, 2. $\Rightarrow$ 3. is Lemma 3, 3. $\Rightarrow$ 4. is Lemma 4. It remains to prove 4. $\Rightarrow$ 2. We notice that $P_m(D)$ is hyperbolic, even with respect to every non-characteristic direction. Theorem 4 then gives the result. $\square$

Example 2. Consider the polynomial $P(x,t) = x^2 + it \in \mathbb{C}[x,t]$. Then $P(D_x, D_t) = \partial/\partial t - \partial^2/\partial x^2$ is the heat operator in one space dimension. By Theorem 6 it admits a continuous linear right inverse in $A(\mathbb{R}^2)$, while it does not admit such an inverse in $C^\infty(\mathbb{R}^2)$, since it is hypoelliptic (see [21, Theorem 3.3] or [13, Corollary 2.11]).

6 Case of convex Ω with boundary

In this section we return to the case handled in Proposition 2 and Example 1. We collect the information we have up to now in the following theorem:

Theorem 7. *If $\Omega \subset \mathbb{R}^n$, $n > 1$, is a bounded, open, convex set with C^1-boundary, then the following are equivalent:*

1. *$P^+(D)$ is surjective in $A(\Omega^+)$.*
2. *$P_m(D)$ admits a continuous linear right inverse in $C^\infty(\Omega)$.*
3. *P_m is proportional to a product of real linear forms.*
4. *$P(D)$ is $\gamma^{(m/m-1)}$-hyperbolic in every noncharacteristic direction.*
5. *$P(D)$ admits a continuous linear right inverse in $\gamma^{(m/m-1)}(\Omega)$.*
6. *$P^+(D)$ admits a continuous linear right inverse in $\gamma^{(m/m-1)}(\Omega^+)$.*
   If P is homogeneous then there is also equivalent:
7. *$P(D)$ admits a continuous linear right inverse in $A(\Omega)$.*

PROOF: 1. $\Rightarrow$ 2. This is Lemma 1
2. $\Rightarrow$ 3. Follows from [13, Theorem 3.8].
3. $\Rightarrow$ 4. Since P_m is hyperbolic in every non-characteristic direction, this follows from Lemma 4.
4. $\Rightarrow$ 5. Follows from [14, Theorem 4.6].
5. $\Rightarrow$ 6. Obvious tensor argument, or also from [14, Theorem 4.6], since 5. implies 3. and 3. implies 3. for P_m^+.
6. $\Rightarrow$ 1. Follows from [14, Corollary 5.11].
7. $\Rightarrow$ 1. is always true, as follows from Lemma 3.
Let now P be homogeneous.
3. $\Rightarrow$ 7. We have only to show that a real differential operator of order 1 has a continuous linear right inverse in $A(\Omega)$. We may assume that $P(D) = \partial/\partial x_1$. We act in a similar way as in the proof of Example 1. Set $\omega = \{x' \in \mathbb{R}^{n-1} :$ exists x_1 with $(x_1, x') \in \Omega\}$ and for $x' \in \omega$ let $]\gamma_1(x'), \gamma_2(x')[= \{x_1 : (x_1, x') \in \Omega\}$. We put $\tilde{\gamma} = \frac{1}{2}(\gamma_1 + \gamma_2)$. Then $\tilde{\gamma} \in C(\omega)$. By Whitney's approximation theorem (see [17, Theorem 1.6.5]) we find $\gamma \in A(\omega)$ such that $|\gamma(x') - \tilde{\gamma}(x')| < \frac{1}{2}(\gamma_1(x') - \gamma_2(x'))$ for all $x' \in \omega$. Then $x' \mapsto (\gamma(x'), x')$ is a real analytic section of ω to Ω and $f \mapsto \int_{\gamma(x')}^{x_1} f(\xi, x') d\xi$ is a continuous linear right inverse. $\square$

The author wants to thank P. Domański for suggesting the use of Whitney's approximation theorem in 3. $\Rightarrow$ 7, which lead to a considerable improvement of the author's original statement, where real analyticity of the boundary had been assumed. We have even:

Remark 1. 3. $\Rightarrow$ 7. in Theorem 7 holds for any convex, open, bounded Ω.

The last part we can formulate also in the following way:

Theorem 8. *If Ω is bounded, convex with C^1-boundary, then the only non-constant irreducible homogenous differential operators $P(D)$ which admit a continuous linear right inverse are, up to a factor, directional derivatives of order one.*

We end the paper by two more special cases. First we assume that $\Omega \subset \mathbb{R}^n$ and $\omega \subset \mathbb{R}^{n-1}$ are convex and open and $\{0\} \times \omega \subset \Omega \subset \mathbb{R} \times \omega$. So Ω might, for instance, be the open unit ball.

We obtain the following analogue to Proposition 5. For some of the tools we will be using we refer to the proof of Proposition 5.

Proposition 7. *Assume that $P_m(x) = x_1^m$. Then the Cauchy problem*

$$P(D)u = 0, \quad \frac{\partial^k}{\partial x_1^k} f(0,x') = f_k(x'), \quad k = 0,\dots,m-1$$

is uniquely solvable for all $f_0,\dots,f_{m-1} \in A(\omega)$ with $u \in A(\mathbb{R} \times \omega)$.

PROOF: Due to the Cauchy-Kowalewski theorem we find an open neighborhood $W \supset \{0\} \times \omega$ in $\mathbb{R}^n$ and $u_0 \in A(W)$ which solves the Cauchy problem. We choose $\varphi \in \gamma^{(m/m-1)}(\Omega)$ with $\operatorname{supp} \varphi \subset W$ such that $\{x : \varphi(x) = 1\}$ contains a neighborhood of $\{0\} \times \omega$.

Then $w := P(D)(\varphi u_0) \in \gamma^{(m/m-1)}(\mathbb{R} \times \omega)$, $\operatorname{supp} w \subset W$ and $w \equiv 0$ in a neighborhood of $\mathbb{R} \times \omega$. By the assumption and [15, Corollary 2.11] there are fundamental solutions E_+ and E_- with support in $[0,+\infty) \times \{0\}$ and $(-\infty,0] \times \{0\}$, respectively. By decomposition of w in an 'upper' and a 'lower' part, convolution with E_+ or E_- and putting the results again together we obtain $v \in \gamma^{(m/m-1)}(\mathbb{R} \times \omega)$ with $P(D)v = w$ and $v \equiv 0$ in a neighborhood of $\{0\} \times \omega$.

We set now $u := u_0 - v$ and obtain a solution in $\gamma^{(m/m-1)}(\mathbb{R} \times \omega)$ of the Cauchy problem such that u is real analytic in a neighborhood of $\{0\} \times \omega$. Now we proceed like in the proof of Proposition 5, using E_+ and E_-. $\square$

In the following we set $N(X) = \{f \in A(X) : P(D)f = 0\}$ for any open subset $X \subset \mathbb{R}^n$.

Theorem 9. *Under the assumptions of Proposition 7 we obtain:*

1. *The restriction map $N(\mathbb{R} \times \omega) \to N(\Omega)$ is surjective.*
2. *$N(\Omega)$ is complemented in $A(\Omega)$.*
3. *$P(D)$ has a continuous linear right inverse in $A(\Omega)$.*

PROOF: $f \to u(f)$ where $u(f)$ is the unique solution of the Cauchy problem of Proposition 7 with $f_k(x') = \frac{\partial^k}{\partial x_1^k} f(0,x')$ is a continuous linear extension operator $N(\Omega) \to N(\mathbb{R} \times \omega)$. This proves 1. Composition with the restriction $A(\mathbb{R}) \to A(\Omega)$ gives the required projection to prove 2. Now $P(D)$ is surjective in $A(\Omega)$, which follows from an easy evaluation of the Phragmén-Lindelöf condition in [8], or from our Remark 1 together with the fact that surjectivity depends only on the principal part (see [8]). Then it is also open (see the Preliminaries). Together with 2., this shows 3.. $\square$

Example 3. If $\Omega = \{(t,x) \in \mathbb{R}^2 : t^2 + x^2 < 1\}$ is the open unit ball in $\mathbb{R}^2$, then the heat operator $\partial/\partial t - \partial^2/\partial x^2$ has a continuous linear right inverse in $A(\Omega)$.

Finally we consider the case of a non-characteristic half-space.

Theorem 10. *Let $\Omega = \{x : \langle x, N\rangle < \gamma\}$ where $P_m(N) \neq 0$. Then the following are equivalent:*

1. *$P(D)$ admits a continuous linear right inverse in $A(\Omega)$.*
2. *P_m is hyperbolic with respect to N.*

PROOF: If 1. is given then, by Proposition 2, $P_m(D)$ admits a continuous linear right inverse in $C^\infty(\Omega)$ and therefore, by [13, Proposition 3.2], $P_m(D)$ is hyperbolic with respect to N.

To prove the converse we may assume that $N = e_1$ and $\Omega = \{x : x_1 < 1\}$. The map which assigns to every $f \in A(\Omega)$ the restriction to Ω of the unique solution $u \in A(\mathbb{R}^n)$ of the Cauchy problem

$$P(D)u = 0, \quad \frac{\partial^k}{\partial x_1^k} u(0,x') = \frac{\partial^k}{\partial x_1^k} f(0,x'), \quad k = 0, \ldots, m-1$$

is a continuous projection in $A(\Omega)$ onto $N(\Omega)$. Since, by [13, Proposition 3.2] and [13, Proposition 4.12], $P(D)$ is surjective in $A(\Omega)$ we obtain, like in the proof of Theorem 9, assertion 1. $\square$

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