Extension operators for real analytic functions on compact subvarieties of $\mathbb{R}^d$

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Abstract

Let X be a compact coherent real analytic subvariety of $\mathbb{R}^d$. It is shown that a continuous linear operator which extends real analytic functions on X to real analytic functions on $\mathbb{R}^d$ exists if and only if X is of type PL, which means that in every point of X the local complexification satisfies Hörmander's local Phragmén-Lindelöf condition. This is in particular true if X is a manifold.

In the present paper we study the question under which condition on a compact coherent subvariety $X \subset \mathbb{R}^d$ there is a continuous linear operator which extends the real analytic functions on X to the whole of $\mathbb{R}^d$ or, equivalently, under which condition on X the ideal of X in the algebra $\mathscr{A}(\mathbb{R}^d)$ is complemented. We solve the problem completely and it turns out that the characterizing condition is well known and appears in various other connections. An extension operator exists if and only if in every point of X the local complexification of X satisfies Hörmander's local Phragmén-Lindelöf condition. This condition is well studied, see for that Section 1 below. There is a significant difference to the corresponding problem for complex subvarieties of $\Omega \subset \mathbb{C}^d$. While our condition is purely local and, in fact, is a condition on the type of singularities, this aspect plays no role in the complex case. If $\Omega = \mathbb{C}^d$ there it is true for algebraic varieties (Zahariuta [27], Djakov-Mityagin [7]) and, more general, if and only if the variety is of strong Liouville type ([23], [24] with [29]). Moreover it is true for strictly pseudoconvex Ω with C^2-boundary and a complex manifold transversal at $\partial\Omega$ (Mityagin-Henkin [20, Theorem 4.2]). Here the difficulties one has to overcome at the boundary resemble those we have to overcome at the singularities. A proof of this result in the spirit of

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the present work has been given in [22]. A significant difference there is also to the differentiable case. Then coherence alone implies that the ideal of functions in $C^\infty(\mathbb{R}^d)$ vanishing on X is complemented (see Malgrange [14, Chap. VI] and Bierstone-Schwarz [1, Theorem 0.1.3]). The present work has been motivated by the study of vector-valued interpolation in Bonet, Domański and Vogt [3]. In Lemma 4.4 and Remark 4.6 there we give an explicit formula for an extension operator in the case where X is the unit circle in $\mathbb{R}^2$. As a particular case we obtain in the present paper that such an extension exists for any compact real analytic submanifold of $\mathbb{R}^d$ and, for purely 1-dimensional X, the only singularities which are admitted are self-intersections.

1 The local Phragmén-Lindelöf condition

Let X_a be the germ of a real analytic variety at a point $a \in \mathbb{R}^d$. This means, there is a neighborhood U_a of a in $\mathbb{C}^d$ and holomorphic functions $f_1, \ldots, f_m$ on U_a, so that

$$X_a = \{x \in \mathbb{R}^d \cap U_a \mid f_1(x) = \ldots = f_m(x) = 0\}.$$

Let J_a be the ideal of X_a in $\mathscr{O}_a$, i.e. all germs $(f, U) \in \mathscr{O}_a$ so that $f|_{U \cap X_a} = 0$. This ideal is finitely generated. We may assume that $f_1, \ldots, f_m$ are generators.

We put $V_a = \{z \in U_a \mid f_1(z) = \ldots = f_m(z) = 0\}$. V_a defines the germ of a complex variety at a, which is called the complexification of X_a. Each germ of a real analytic function on X_a can be uniquely extended to the germ of a holomorphic function on V_a. Notice that V_a is uniquely determined by X_a.

We recall the following definition which goes back to Hörmander (see [11, p. 176]) and plays an important role in [16, Theorem 4.7.], [18, Remark 3.12.]. In recent time it was carefully studied in the work of R. Braun, R. Meise and B. A. Taylor (see [6]).

In this definition and in the following remarks and lemmata let V be a complex subvariety of an open set $U \in \mathbb{C}^d$, $X = V \cap \mathbb{R}^d$ and $a \in X$.

Definition 1.1 *V satifies $PL_{loc}(a)$ (i. e. Hörmander's local Phragmén-Lindelöf condition at a) if there are $r_1 > r_2 > 0$ and $A > 0$ so that $B(a, r_1) \subset U$ and for any plurisubharmonic function u on $V_a \cap B(a, r_1)$ we have that (α) and (β) imply (γ) where*

(α) $u(z) \leq 1$; $z \in V \cap B(a, r_1)$

(β) $u(x) \leq 0$; $x \in X \cap B(a, r_1)$

(γ) $u(z) \leq A\,|\mathrm{Im}\, z|$; $z \in V \cap B(a, r_2)$

Here and throughout the paper $B(\xi, r) = \{z \in \mathbb{C}^d : |z - \xi| < r\}$, $|z| = \max_j |z_j|$.

From [6] we take the following lemma:

Lemma 1.2 *Let $\Omega \subset \Omega_0$ be open in $\mathbb{C}^d$, W a closed subvariety of Ω_0, $V = \Omega \cap W$, $\Omega_0 \subset \{z \mid |\mathrm{Im}\, z| \leq r\}$. We assume that for every plurisubharmonic function u on W with $u(z) \leq 1$ on W and $u(x) \leq 0$ on $W \cap \mathbb{R}^d$ we have $u(z) \leq A\,|\mathrm{Im}\, z|$ for $z \in V$.*
Then for every $\xi \in \mathbb{R}^d$ and $r_1 > r_2 > 0$ so that $B(\xi, r_1) \subset \Omega$, there is $A' > 0$ with the property that for every plurisubharmonic function u on $B(\xi, r_1) \cap V$, with $u(z) \leq 1$ on $B(\xi, r_1) \cap V$ and $u(z) \leq 0$ on $B(\xi, r_1) \cap V \cap \mathbb{R}^d$ we have $u(z) \leq A'\,|\mathrm{Im}\, z|$ on $V \cap \{z \mid |\mathrm{Re}\, z - \xi| < r_2\}$.

Applying this to $V \cap B(a, r_2) \subset V_a \cap B(a, r_1)$ in Definition 1.1 we obtain that the assertion will be true then, for $r_1 > r_2 > 0$ replaced by any $r_1 \geq \rho_1 > \rho_2 > 0$ with possibly changed A. We obtain the following consequences:

Lemma 1.3 *If V satisfies $PL_{loc}(a)$ then the germ V_a defined by V is the complexification of the germ X_a defined by X.*

Proof: By use of [10, Proposition III, 2.1, iii] this follows from the proofs of [17, Proposition 2.7] and [17, Lemma 2.8]. □

Lemma 1.4 *If V satisfies $PL_{loc}(a)$ then there is a neighborhood U of a so that V satisfies $PL_{loc}(b)$ for all $b \in U \cap X$*

In the previous definition and in the related results the primary object of study was a complex variety and how it is situated in relation to $\mathbb{R}^d$. In the present work the object of study is a real analytic variety. This leads to the following definition, where X_a is the germ of a real analytic variety at a.

Definition 1.5 *X_a is of type PL if its complexification V_a satisfies $PL_{loc}(a)$.*

Here a germ V_a is said to satisfy $PL_{loc}(a)$ if there is a V defining V_a which satisfies this condition. For the consistency of this definition cf. the remark after Lemma 1.2.

Lemma 1.6 *If X_a is of type PL, then also X_b for any b close to a.*

In terms of Definition 1.1 this holds for any $b \in X_a \cap B(a, r_1)$. An immediate consequence is the following proposition:

Proposition 1.7 *If X_a is of type PL then it is a coherent germ.*

Proof: By use of Lemma 1.3 and 1.4 this follows from [10, Proposition III, 2.8]. □

Let now X be a real analytic subvariety of some open $\omega \subset \mathbb{R}^d$. This means that X is a closed subset of ω so that for every $a \in X$ there are an open neighborhood U and real analytic functions $f_1, \ldots, f_m$ on U so that $X \cap U = \{x \in U \,:\, f_1(x) = \cdots = f_m(x) = 0\}$. This means, for any $a \in X$ we have the germ X_a as before.

Definition 1.8 *For $a \in X$ we say that X satisfies $PL_{loc}(a)$ if its germ in a is of type PL. We call X of type PL if it satisfies $PL_{loc}(a)$ for every $a \in X$.*

Theorem 1.9 *If X is of type PL then X is coherent.*

Since $PL_{loc}(\cdot)$ is obviously invariant under real analytic diffeomorphisms we obtain the following consequence of Lemma 1.6.

Proposition 1.10 *A homogenous real analytic subvariety of $\mathbb{R}^d$ is of type PL if and only if it satisfies $PL_{loc}(0)$.*

Lemma 1.11 *If X_a is the germ of a real analytic manifold, then it is of type PL.*

Proof: We may assume that there exist a real analytic chart $\varphi\colon X_a \longrightarrow B(0,r) \subset \mathbb{R}^d$, if n is the dimension of X_a. This can be extended to a holomorphic chart from a neighborhood of a in V_a onto a neighborhood of 0 in $\mathbb{R}^d$. Then the claim follows from the classical Phragmén-Lindelöf theorem. □

We denote by $S(X)$ the singular locus, i.e. the complement of the manifold points in X and by $\sigma(X)$ the PL-singular locus, i.e. the complement of the PL-points in X.

Theorem 1.12 *$\sigma(X)$ is a closed subset of X and $\sigma(X) \subset S(X)$.*

2 Main theorem

We are now in the position to formulate our main results. Before we do this we recall the linear topological background.

If $K \subset \mathbb{R}^d$ is compact we denote by $\mathscr{H}(K)$ the space of germs of holomorphic functions on K equipped with its natural topology of an inductive limit of Banach spaces given by

$$\mathscr{H}(K) = \lim\operatorname{ind}_{n\in\mathbb{N}} H^\infty(U_n)$$

where U_n runs through a basis of complex neighborhoods of K, which may be chosen as polynomial polyhedra (see [13, Lemma 5.4.1]). For the space $\mathscr{A}(\mathbb{R}^d)$ of real analytic functions on $\mathbb{R}^d$ we the set

$$\mathscr{A}(\mathbb{R}^d) = \lim\operatorname{proj}_{n\in\mathbb{N}}\mathscr{H}(K_n)$$

where $K_1 \subset K_2 \subset \cdots \nearrow \mathbb{R}^d$ is any compact exhaustion of $\mathbb{R}^d$.

There are other equivalent descriptions of this unique natural locally convex topology on $\mathbb{R}^d$ (see [15]). A sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathscr{A}(\mathbb{R}^d)$ converges to f if and only if f and all f_n extend to a common complex neighborhood U of $\mathbb{R}^d$ and converge to f in the space $H(U)$ of holomorphic functions on U equipped with the compact-open topology.

Let now X be a compact real analytic subvariety of $\mathbb{R}^d$. If it is coherent, then it admits a complexification V which we may assume of the form

$$V = \{z \in \mathbb{C}^d \,:\, |\operatorname{Im} z| < r,\ f_1(z) = \cdots = f_m(z) = 0\}$$

where $f_1(z), \ldots, f_m(z)$ are holomorphic on $\Omega = \{z \in \mathbb{C}^d \,:\, |\operatorname{Im} z| < r\}$.

Since for coherent X the restriction map $\mathscr{A}(\mathbb{R}^d) \longrightarrow \mathscr{A}(X)$ is surjective, the quotient topology of $\mathscr{A}(\mathbb{R}^d)$ coincides, by means of the open mapping theorem (see e.g. [19, 24.30]) with the topology

$$\mathscr{A}(X) = \lim\operatorname{ind}_{n\in\mathbb{N}} H^\infty(V_n)$$

where V_n runs through a basis of neighborhoods of X in V.

The continuity of a linear map $R : \mathscr{A}(X) \longrightarrow \mathscr{A}(\mathbb{R}^d)$ means that for any neighborhood U of X in V (or in $\mathbb{C}^d$) and any sequence $(f_n)_{n\in\mathbb{N}}$ in $H(U)$ converging uniformly to zero the sequence $(Rf_n)_{n\in\mathbb{N}}$ extends to a joint neighborhood $\omega \subset \mathbb{C}^d$ of $\mathbb{R}^d$ and converges to zero in $H(\omega)$.

In the following sections of the paper we will prove:

Proposition 2.1 *Let $X \subset D = \{x \in \mathbb{R}^d \,:\, |x| \le R\}$. Then the following are equivalent:*

1. *X is coherent and there exists a continuous linear extension operator $\mathscr{A}(X) \longrightarrow \mathscr{H}(D)$.*
2. *X is of type PL.*

From this we derive our main result:

Theorem 2.2 *X is coherent and there exists a continuous linear extension operator $\mathscr{A}(X) \to \mathscr{A}(\mathbb{R}^d)$ if and only if X is of type PL.*

Proof: If there is a continuous linear extension operator $\mathscr{A}(X) \to \mathscr{A}(\mathbb{R}^d)$ then, in particular, there is one $\mathscr{A}(X) \to \mathscr{H}(D)$, i.e. ρ and the result follows from Proposition 2.1.

To prove the converse direction we set e.g. $\Phi(z_1, \ldots, z_d) = (\arctan z_1, \ldots, \arctan z_d)$ defining a biholomorphic map from a complex neighborhood of $\mathbb{R}^d$ onto a complex neighborhood of $(-\frac{\pi}{2}, +\frac{\pi}{2})$ with $\Phi\mathbb{R}^d \subset \mathbb{R}^d$. Then ΦX (with ΦV) has the same properties as X and we get from Proposition 2.1 the existence of a continuous linear extension operator $E_0 : \mathscr{A}(\Phi X) \to \mathscr{H}(D)$ where $D = \{x \in \mathbb{R}^d \,:\, |x| \leq \pi\}$. We set for $f \in \mathscr{A}(X)$

$$E(f) := (E_0(f \circ \Phi^{-1})) \circ \Phi.$$

E is an extension operator, as claimed. □

We may formulate this also in the following way:

Theorem 2.3 *X is coherent and its ideal $\mathscr{J}_X \subset \mathscr{A}(\mathbb{R}^d)$ is complemented if and only if X is of type PL.*

Needless to say that in most cases we will know in advance that X is coherent, so in these cases an extension operator exists, resp. the ideal is complemented, if and only if X is of type PL.

3 Compact varieties of type PL

Let now X be a coherent compact real analytic subvariety of $\mathbb{R}^d$. Then it admits a complexification V which we may assume of the form

$$V = \{z \in \mathbb{C}^d \,:\, |\mathrm{Im}\, z| < r,\ f_1(z) = \cdots = f_m(z) = 0\}$$

where $f_1(z), \ldots, f_m(z)$ are holomorphic on $\Omega = \{z \in \mathbb{C}^d \,:\, |\mathrm{Im}\, z| < r\}$.

Lemma 3.1 *X is of type PL if and only if there is a constant A such that for every plurisubharmonic function u on V satisfying (α) and (β) we have (γ), where*

(α) $u(z) \leq 1,\ z \in V$

(β) $u(z) \leq 0,\ z \in X$

(γ) $u(z) \leq A\,|\mathrm{Im}\, z|,\; z \in V$.

Proof: Let V satisfy $PL_{loc}(\xi)$ for every $\xi \in X$. By an easy compactness argument we find $r > r_2 > 0$ and A_0 so that

$$u(z) \leq 1,\; z \in V$$
$$u(z) \leq 0,\; z \in X$$

implies

$$u(z) \leq A_0\,|\mathrm{Im}\, z|,\; z \in \{z \in V \,:\, |\mathrm{Im}\, z| < r_2\}.$$

For $z \in \{z \in V \,:\, r_2 \leq |\mathrm{Im}\, z| < r\}$ we have

$$u(z) \leq 1 \leq \frac{1}{r_2}|\mathrm{Im}\, z|.$$

Hence we obtain the result with $A = \max(A_0, \frac{1}{r_2})$.

The proof of the converse implication follows from Lemma 1.2. □

We set for $z \in V$

$$\begin{aligned} \omega_0(z) &= \sup\{u(z) \,:\, u \text{ plurisubharmonic on } V,\, u(z) \leq 0 \text{ on } X, \\ &\quad u(z) \leq 1 \text{ on } V\}, \\ \omega(z) &= \omega_0^*(x) = \limsup_{\zeta \to z} \omega_0(\zeta). \end{aligned}$$

Then Lemma 3.1 can also be expressed in the following way :

Lemma 3.2 *X is of type PL if and only if there is $A > 0$ so that*

$$\omega(z) \leq A\,|\mathrm{Im}\, z|,\; z \in V.$$

We set for $0 < \alpha < r$

$$V_\alpha = \{z \in V \,:\, \omega(z) < \alpha\}$$

and obtain, if V satisfies $PL_{loc}(\xi)$ for every $\xi \in X$, that

$$\{z \in V \,:\, |\mathrm{Im}\, z| < \frac{\alpha}{A}\} \subset V_\alpha. \tag{1}$$

For $f \in H^\infty(V_\alpha)$ we set

$$|f|_\alpha = \sup_{z \in V_\alpha} |f(z)|,\; |f|_0 = \sup_{z \in X} |f(z)|.$$

and obtain the following well-known result (see Zahariuta [28, 30]).

Lemma 3.3 *For $0 \le \alpha_1 < \alpha_2 < \alpha_3 \le r$ and $f \in H^\infty(V_{\alpha_3})$ we have*

$$|f|_{\alpha_2}^{\alpha_3-\alpha_1} \le |f|_{\alpha_1}^{\alpha_3-\alpha_2} |f|_{\alpha_3}^{\alpha_2-\alpha_1}.$$

Proof: We give a proof for the sake of completeness. Let f be holomorphic, bounded and non-constant on V_{α_3}. We put for $\alpha_3 - \alpha_2 > \varepsilon > 0$

$$u_\varepsilon(z) = \alpha_1 + (\alpha_3 - \alpha_1) \frac{\log |f(z)| - \log |f|_{\alpha_1}}{\log |f|_{\alpha_3} - \log |f|_{\alpha_1}} - \varepsilon.$$

Then u_ε is plurisubharmonic on V_{α_3}, $u_\varepsilon \le \alpha_1 - \varepsilon$ on V_{α_1} and $u_\varepsilon \le \alpha_3 - \varepsilon$ on V_{α_3}. Therefore the function

$$v_\varepsilon(z) = \begin{cases} \omega(z) & : \quad z \in V_{\alpha_1} \\ \max(u_\varepsilon(z), \omega(z)) & : \quad z \in V_{\alpha_3} \setminus V_{\alpha_1} \\ \omega(z) & : \quad z \in V \setminus V_{\alpha_3} \end{cases}$$

is plurisubharmonic on V and $v_\varepsilon \le 0$ on X, $v_\varepsilon \le 1$ on V. Therefore we have $v_\varepsilon \le \omega$ on V, hence $v_\varepsilon(z) \le \alpha_2$ for $z \in V_{\alpha_2}$, which implies

$$|f(z)|^{\alpha_3-\alpha_1} \le |f|_{\alpha_1}^{\alpha_3-\alpha_2-\varepsilon} |f|_{\alpha_3}^{\alpha_2+\varepsilon-\alpha_1}$$

for all $z \in V_{\alpha_2} \setminus V_{\alpha_1}$, hence for all $z \in V_{\alpha_2}$. With $\varepsilon \longrightarrow 0$ we obtain the result. □

4 Proof of the main theorem: sufficiency of PL

Let now R be chosen large enough so that $X \subset B(0,R) \cap \mathbb{R}^d$ and V chosen as in the previous section. We may assume that $r = 1$.

We set

$$D_\alpha = \{z \in \mathbb{C}^d \,:\, |\mathrm{Im}\, z| < \alpha,\ |\mathrm{Re}\, z| < R + \alpha\}.$$

and for $F \in H^\infty(D_\alpha)$

$$\|F\|_\alpha = \sup_{z \in D_\alpha} |F(z)|.$$

The sets D_α are a parametrized family of analytic polyhedra. From a well known result of Zahariuta [28, 30] (for a proof see also [8]) we obtain:

Lemma 4.1 *For $0 < \alpha_1 < \alpha_2 < \alpha_2' < \alpha_3$ and $\eta \in H^\infty(D_{\alpha_1})'$ we have*

$$\|\eta\|_{\alpha_2}^{*\,\alpha_3-\alpha_1} \le C\, \|\eta\|_{\alpha_1}^{*\,\alpha_3-\alpha_2'} \|\eta\|_{\alpha_3}^{*\,\alpha_2'-\alpha_1}.$$

Finally, we need the following Lemma which is an immediate consequence of the pseudoconvexity of D_α, the Cartan-Oka theory and the closed-graph theorem.

Lemma 4.2 *For every* $0 < \beta < \alpha \leq 1$ *there is a* $C > 0$ *such that for every* $f \in H^\infty(V \cap D_\alpha)$ *there is* $F \in H^\infty(D_\alpha)$ *with* $F|_{V\cap D_\alpha} = f$ *and* $\|F\|_\beta \leq C\,|f|_\alpha$.

We set $J^\infty(D_\alpha) := \{f \in H^\infty(D_\alpha) \,:\, f|_V = 0\}$ and obtain by the same arguments as in Lemma 4.2:

Lemma 4.3 *For every* $0 < \beta < \alpha$ *there is a* $C > 0$ *such that for every* $f \in J^\infty(D_\alpha)$ *there are* $g_j \in H^\infty(D_\beta)$, $j = 1, \ldots, m$, *with* $f = \sum_{j=1}^m g_j\, f_j$ *on* D_β *and* $\sup_{j=1,\ldots,m} \|g_j\|_\beta \leq C\,|f|_\alpha$.

We put $D = \{x \in \mathbb{R}^d \,:\, |x| \leq R\}$ and set up the following exact sequence

$$\mathscr{H}(D)^m \xrightarrow{\ \sigma\ } \mathscr{H}(D) \xrightarrow{\ \rho\ } \mathscr{A}(X) \longrightarrow 0 \tag{2}$$

where ρ is the restriction map and $\sigma(g_1, \ldots, g_m) = \sum_{j=1}^m g_j\, f_j$. We consider the (LB)-space $\mathscr{H}(D)$ to be graded by $\mathscr{H}(D) = \bigcup_\alpha H^\infty(D_\alpha)$ and $\mathscr{A}(X)$ by $\mathscr{A}(X) = \bigcup_\alpha H^\infty(V_\alpha)$.

If we set B_α the unit ball in $H^\infty(D_\alpha)$ or in $H^\infty(V_\alpha)$, respectively, then the exact sequence (2) is tame in the following sense:

Lemma 4.4 *For every* $0 < \beta < \alpha \leq A$ *there is* $C > 0$ *such that*

$$\frac{1}{C}\,\sigma(B_\alpha^m) \subset B_\alpha \subset C\,\sigma(B_\beta^m) \text{ and } \rho(B_\alpha) \subset B_\alpha \subset C\,\rho(B_{\frac{\beta}{A}}).$$

Proof: The first inclusion is obvious, the second follows from Lemma 4.3. For the last two inclusions we remark that

$$V \cap D_{\frac{\alpha}{A}} \subset V_\alpha \subset V \cap D_\alpha.$$

This follows from (1) and the fact that, by definition, $\omega(z) \geq |\mathrm{Im}\, z|$.

Then the third inclusion is again obvious and the fourth follows from Lemma 4.2. □

We are now in the position to prove the first part of our main result.

Proposition 4.5 *The map* ρ *in the exact sequence (2) has a continuous linear right inverse.*

Proof: We set $K = \mathscr{H}(D)^m / \ker\sigma \cong \ker\rho$ and, by dualization, we obtain an exact sequence of Fréchet spaces

$$(3) \qquad 0 \longrightarrow \mathscr{A}(X)' \xrightarrow{\rho'} \mathscr{H}(D)' \xrightarrow{\sigma'} K' \longrightarrow 0.$$

On $\mathscr{A}(X)'$ we use the fundamental system of seminorms $\|\eta\|_k := |\eta|^*_{\frac{1}{k}}$ and obtain from Lemma 3.3

$$\|\ \|^*_k \leq \|\ \|^{*1-\vartheta_k}_{k-1} \|\ \|^{*\vartheta_k}_{k+1}$$

with $\vartheta_k = \frac{k+1}{2k}$.

On K' we use the fundamental system of seminorms $\|\eta\|_k := |||\sigma'(\eta)|||^*_{\frac{r}{k}}$ where

$$|||(g_1, \dots, g_m)|||_\alpha = \max_{j=1,\dots,m} \|g_j\|_\alpha.$$

From Lemma 4.1 we get

$$\|\ \|_k \leq \|\ \|^{1-\tau_k}_{k-1} \|\ \|^{\tau_k}_{k+1}$$

for any $\tau_k > \frac{k+1}{2k}$. Choosing, for $k > 1$, $\frac{k}{2(k-1)} > \tau_k > \frac{k+1}{2k}$ we have $\vartheta_k > \tau_{k+1}$ for all $k \in \mathbb{N}$.

By [22, Theorem 6.1.] the exact sequence (3) splits and therefore ρ' has a left inverse. By dualization and use of reflexivity we obtain that ρ has a right inverse. □

5 Proof of the main theorem: necessity of PL

We are now going to prove the converse of Proposition 4.5. We are assuming that X is coherent so we can make the same assumptions on the complexification of X as in the beginning of Section 3 and use the notation of the previous section. We assume that there is a continuous linear extension operator $\mathscr{A}(X) \to \mathscr{H}(D)$. In particular the restriction map $\mathscr{H}(D) \to \mathscr{A}(X)$ is surjective.

We set $W_\alpha = V \cap D_\alpha$ and for $f \in H^\infty(W_\alpha)$ we set $\|f\|_\alpha = \sup_{z \in W_\alpha} |f(z)|$ and $\|f\|_0 = \sup_{z \in X} |f(z)|$. In complete analogy to Lemma 4.1 and Lemma 3.3 we obtain:

Lemma 5.1 *For $0 < \alpha_1 < \alpha_2 < \alpha_2' < \alpha_3$ and $\eta \in H^\infty(W_{\alpha_1})'$ we have*

$$\|\eta\|^{*\alpha_3-\alpha_1}_{\alpha_2} \leq C\, \|\eta\|^{*\alpha_3-\alpha_2'}_{\alpha_1} \|\eta\|^{*\alpha_2'-\alpha_1}_{\alpha_3}.$$

Lemma 5.2 *For $0 < \alpha_1 < \alpha_2 < \alpha_3$ and $f \in H^\infty(\tilde{D}_{\alpha_3})$ we have*

$$\|f\|^{\alpha_3-\alpha_1}_{\alpha_2} \leq \|f\|^{\alpha_3-\alpha_2}_{\alpha_1} \|f\|^{\alpha_2-\alpha_1}_{\alpha_3}.$$

We have set $\tilde{D}_\alpha = \{z : V_D(z) < \alpha\}$, where V_D is the pluricomplex Green function of D (see [13, p. 207]), with corresponding sup-norms. In the following Lemma we denote by B_α and $\tilde{B}_\alpha$ the unit balls of $H^\infty(W_\alpha)$ resp. $H^\infty(\tilde{D}_\alpha)$ and obtain:

Lemma 5.3 *If $\varphi \in L(\mathscr{A}(X), \mathscr{H}(D))$ then there is an $0 < \varepsilon \leq 1$ and for every $0 < \alpha \leq r$ a $C_\alpha > 0$ so that $\varphi(B_\alpha) \subset C_\alpha \tilde{B}_{\varepsilon\alpha}$.*

Proof: We set for $0 < \alpha \leq r$

$$\tau(\alpha) = \sup\{\beta \leq r : \ \exists\, C > 0 \text{ with } \varphi(B_\alpha) \subset C\, \tilde{B}_\beta\}.$$

From Lemmas 5.1 and 5.2 and from [19, 29.17] we conclude that τ is an increasing and concave function $(0, r] \to (0, r]$. Therefore we have an $\varepsilon > 0$ with $\tau(\alpha) > \varepsilon\alpha$ for all $0 < \alpha \leq r$ which proves the result. □

Lemma 5.4 *If ρ has a continuous linear right inverse then there is $\varepsilon > 0$ so that for any $0 < \alpha_1 < \alpha_2 < \alpha_3 \leq r$ and $f \in H^\infty(W_{\alpha_3})$ we have*

$$\|f\|_{\varepsilon\alpha_2} \leq \|f\|_{\alpha_1}^{\frac{\alpha_3-\alpha_2}{\alpha_3-\alpha_1}} \|f\|_{\alpha_3}^{\frac{\alpha_2-\alpha_1}{\alpha_3-\alpha_1}}.$$

Proof: Let φ be the right inverse. Then, by use of Lemma 5.3, we get $\varepsilon > 0$ and constants $C_\alpha > 0$ so that $\|\varphi f\|_{\varepsilon\alpha} \leq C_\alpha \|f\|_\alpha$ for all $0 < \alpha \leq r$ and $f \in H^\infty(W_\alpha)$. Now we apply Lemma 5.2 to get the following chain of inequalities for $0 < \alpha_1 < \alpha_2 < \alpha_3 \leq r$ and $f \in H^\infty(W_{\alpha_3})$. We remark that for some $\varepsilon' > 0$ we have $W_{\varepsilon'\alpha} \subset V \cap \tilde{D}_\alpha \subset \tilde{D}_\alpha$.

$$\begin{aligned} \|f\|_{\varepsilon'\varepsilon\alpha_2} &\leq \|\varphi f\|_{\varepsilon\alpha_2} \\ &\leq \|\varphi f\|_{\varepsilon\alpha_1}^{\frac{\alpha_3-\alpha_2}{\alpha_3-\alpha_1}} \|\varphi f\|_{\varepsilon\alpha_3}^{\frac{\alpha_2-\alpha_1}{\alpha_3-\alpha_1}} \\ &\leq C_{\alpha_1}^{\frac{\alpha_3-\alpha_2}{\alpha_3-\alpha_1}} C_{\alpha_3}^{\frac{\alpha_2-\alpha_1}{\alpha_3-\alpha_1}} \|f\|_{\alpha_1}^{\frac{\alpha_3-\alpha_2}{\alpha_3-\alpha_1}} \|f\|_{\alpha_3}^{\frac{\alpha_2-\alpha_1}{\alpha_3-\alpha_1}}. \end{aligned}$$

By applying this to f^n, taking n-th roots and letting n go to infinity we obtain the result. □

We obtain:

Proposition 5.5 *If ρ has a continuous linear right inverse then there is $A > 0$ so that $\omega(z) \leq A|\mathrm{Im}\, z|$ for $z \in V$.*

Proof: For given $0 < \alpha < r$ we apply Lemma 5.4 to $0 < \beta < \alpha < \gamma < r$ and obtain for any bounded holomorphic function f on W_γ

$$\|f\|_{\varepsilon\alpha} \leq \|f\|_\beta^{\frac{\gamma-\alpha}{\gamma-\beta}} \|f\|_\gamma^{\frac{\alpha-\beta}{\gamma-\beta}}.$$

Now we let β tend to 0 and get

$$\|f\|_{\varepsilon\alpha} \leq \|f\|_0^{\frac{\gamma-\alpha}{\gamma}} \|f\|_\gamma^{\frac{\alpha}{\gamma}}. \tag{4}$$

Here we used that $\lim_{\beta\to 0+} \|f\|_\beta = \|f\|_0$. $\lim_{\beta\to 0+} \|f\|_\beta \geq \|f\|_0$ is evident. To prove the converse inequality choose some sequence $\beta_n \downarrow 0$. For every $n \in \mathbb{N}$ there is $x_n \in \overline{D}_{\beta_n}$ such that $\|f\|_{\beta_n} = |f(x_n)|$. The sequence $(x_n)_n$ has a convergent subsequence $x_{n_m} \to x \in X$. Since $\|f\|_\beta$ is decreasing for $\beta \downarrow 0$ we have $\lim_{\beta\to 0+} \|f\|_\beta = |f(x)| \leq \|f\|_0$.

From (4) we get for any function $u(z) = c \log |f(z)|$ where f is holomorphic on V and $c > 0$, so that $u(z) \leq 0$ on X and $u(z) \leq 1$ on W_γ

$$u(z) \leq \frac{\alpha}{\gamma},\ z \in W_{\varepsilon\alpha}.$$

The same remains true for u of the form $u(z) = \max_{j=1,\ldots,m} c_j \log |f_j(z)|$.

Let now u be a plurisubharmonic function on V, $u(z) \leq 0$ on X and $u(z) \leq 1$ on V. By [9, Theorem 5.3.1] there is an extension of u to a plurisubharmonic function λ on an open Stein neighborhood $D \subset \mathbb{C}^d$ of V.

We fix some Stein open set $G \subset \mathbb{C}^d$ so that $W_\gamma \subset\subset G \subset\subset D$. By [12, Theorem 2.6.3.] and Dini's theorem there is, for given $\delta > 0$ a continuous plurisubharmonic function $\mu \geq \lambda$ on G so that $\mu(z) \leq \delta$ on X and $\mu(z) \leq 1 + \delta$ on $W_\gamma \subset\subset G$. For μ there are holomorphic functions $f_1, \ldots, f_m$ on G and positive constants $c_1, \ldots, c_m$ so that

$$\mu(z) - \delta \leq v(z) := \max_{j=1,\ldots,m} c_j \log |f_j(z)| \leq \mu(z),\ z \in \overline{W}_\gamma.$$

Therefore we have $v(z) \leq \delta$ on X and $u(z) \leq 1 + \delta$ on W_γ, hence

$$v(z) \leq \frac{\alpha}{\gamma} + \delta,\ z \in W_{\varepsilon\alpha}.$$

Finally we get

$$u(z) \leq \frac{\alpha}{\gamma} + 2\delta,\ z \in W_{\varepsilon\alpha}$$

for every $\alpha < \gamma < r$ and $\delta > 0$ which implies

$$u(z) \leq \frac{\alpha}{r},\ z \in W_{\varepsilon\alpha}.$$

Therefore

$$\sup\{\omega_0(z) \,:\, z \in V,\, |\mathrm{Im}\, z| \le \varepsilon\alpha\} \le \frac{\alpha}{r}.$$

This yields the result with $A = \frac{1}{\varepsilon r}$ for $|\mathrm{Im}\, z| \le \varepsilon r$. For $\varepsilon r \le |\mathrm{Im}\, z| \le r$ we proceed as in the proof of Lemma 3.1. □

6 Examples

Examples of varieties which satisfy or do not satisfy the local Phragmén-Lindelöf condition and criteria for that can be found in [11, 16, 18, 4, 5] and, in a recent very comprehensive study, in Braun, Meise and Taylor [6]. We will present some of these examples and special cases for the sake of completeness. The proofs, given in the spirit of the present paper, might be of independent interest.

We resume the notation of Section 1. Let X_a be the germ of a real analytic variety, V_a its complexification. Throughout this section we assume V_a to be relatively compact. In particular there is $R > 0$ so that $V_a \subset \Omega = \{z \,:\, |\mathrm{Im}\, z| < r\}$. We define for $z \in V_a$

$$\omega_a(z) = \limsup_{\zeta \to z} \sup\{u(\zeta) \,:\, \text{ u plurisubharmonic on } V_a,\, u \le 1, u \le 0 \text{ on } X_a\}.$$

Then ω_a defines the germ of an extremal plurisubharmonic function at a. From Lemma 1.2 we conclude:

Lemma 6.1 *Let $\tilde{X}_a$ be a representation of the germ X_a with $\tilde{X}_a \subset X_a$ and a complexification $\tilde{V}_a \subset V_a$. Let $\tilde{\omega}_a$ be the extremal function of $\tilde{X}_a$ in $\tilde{V}_a$ then there is a neighborhood U of a in $V_a \cap \tilde{V}_a$ and a constant $C > 0$ so that*

$$\frac{1}{C}\,\omega_a \le \tilde{\omega}_a \le C\,\omega_a$$

on U.

For two germs ω_a and $\tilde{\omega}_a$ we set $\omega_a \sim \tilde{\omega}_a$ if there is a constant $C > 0$ so that

$$\frac{1}{C}\,\omega_a \le \tilde{\omega}_a \le C\,\omega_a.$$

in a neighborhood of a in $V_a \cap \tilde{V}_a$ and call such germs equivalent. Then we obtain from Lemma 6.1

Lemma 6.2 *Up to equivalence the germ of ω_a depends only on the germ of X_a.*

We can then rephrase Definition 1.8 in the following way:

Proposition 6.3 *The germ X_a is of type PL if and only if $\omega_a(z) \sim |\mathrm{Im}\, z|$.*

We wish now to give an interpretation of ω_a in certain cases. We may assume that $a = 0$ and omit the suffix. Let $\tilde{V}$ be an analytic variety, $\varphi : \tilde{V} \longrightarrow V$ a holomorphic map onto V so that $\varphi : \tilde{V} \setminus \varphi^{-1}(D) \longrightarrow V \setminus D$ is a p-sheeted locally biholomorphic covering map, where D is a thin set which contains $V \setminus V_{reg}$ and thin means that it is contained in the 0-set of a nonconstant holomorphic function.

We set $\tilde{X} = \varphi^{-1}(X)$ and

$$\tilde{\omega}(z) = \limsup_{\zeta \to z} \sup\{u(\zeta) \,:\, u \text{ plurisubharmonic and } u \leq 1 \text{ on } \tilde{V}, u \leq 0 \text{ on } \tilde{X}\}.$$

Then $\tilde{\omega}$ is plurisubharmonic on $\tilde{V}$ and we obtain:

Lemma 6.4 *$\tilde{\omega} = \omega \circ \varphi$.*

Proof: As $\omega \circ \varphi \leq \tilde{\omega}$ is obvious we have to prove the reverse inequality.

On $V \setminus D$ we set

$$u(z) = \max_{z = \varphi(\tilde{z})} \tilde{\omega}(\tilde{z}).$$

Due to the assumption this is a well defined plurisubharmonic function on $V \setminus D$. We extend it by $u(z) = \limsup_{\zeta \to z} u(\zeta)$ to a plurisubharmonic function on V. Obviously $u \leq 1$ and $u \leq 0$ on $X \cap V \setminus D$.

We choose a non-zero holomorphic function f on V which vanishes on D. We may assume that $|f(z)| \leq 1$ for all $z \in V$. Let $\varepsilon > 0$. For $z \in V$ we set

$$u_\varepsilon(z) = u(z) + \varepsilon \log |f(z)|.$$

Then $u_\varepsilon(z) \leq u(z) \leq 1$ for all $z \in V$ and $u_\varepsilon(x) \leq u(x) \leq 0$ on $X \cap V \setminus D$. Since $u_\varepsilon(z) = -\infty$ on D we have $u_\varepsilon(x) \leq 0$ for all $x \in X$. This implies $u_\varepsilon(z) \leq \omega(z)$ for all $z \in V_a$. Letting $\varepsilon \to 0+$ we get $u(z) \leq \omega(z)$ for all $z \in V$ with $f(z) \neq 0$ and therefore for all $z \in V$. This implies $\tilde{\omega}(\zeta) \leq \omega(\varphi(\zeta))$for all $\zeta \in \tilde{V} \setminus \varphi^{-1}(D)$ and therefore the result. □

We apply this in a twofold way. First we assume that V is the germ at 0 of a pure ν-dimensional complex variety in $\mathbb{C}^d$ and $X = V \cap \mathbb{R}^d$, with the following properties:

1. after a real change of variables the projection $z \mapsto z' = (z_1, \ldots, z_\nu)$ defines a p-sheeted branched covering map $\varphi : V \to U$ where $U \subset \mathbb{C}^\nu$ is an open neighborhood of 0,

2. $\varphi^{-1}(U \cap \mathbb{R}^\nu) = X$.

In this case we call X *a real analytic cover* and Lemma 6.4 says that $\omega(z) \sim |\operatorname{Im} z'|$. So we have proved

Proposition 6.5 *If X_a is a real analytic cover then it is of type PL.*

The second way we wish to apply Lemma 6.4 is the following: We assume $\tilde{V}$ to be a complex manifold and p=1. This is fulfilled if $\varphi : \tilde{V} \longrightarrow V$ is a desingularization of V.

Let X_a be purely p-dimensional. Let φ be a desingularization of the following form: $\tilde{V}$ a p-dimensional complex submanifold of a bounded neighborhood U of 0 in $\mathbb{C}^q$, $\varphi : \tilde{V} \to V_a$ a proper holomorphic map onto V_a so that $\varphi(0) = a$, $\varphi(\tilde{V} \cap \mathbb{R}^q) = X_a$ and $\varphi : \varphi^{-1}(W) \to W$ bijective where $W \subset V_{a,reg}$ and $V_a \setminus W$ is contained in the zero set of a nonzero holomorphic function on V_a. Here $V_{a,reg}$ denotes the set of regular points in V_a.

Proposition 6.6 *Under these assumptions for any $\tilde{V}_0 \subset\subset \tilde{V}$ there is $C > 0$ with*

$$\frac{1}{C} |\operatorname{Im} z| \leq \omega_a \circ \varphi(z) \leq C |\operatorname{Im} z|$$

for all $z \in \tilde{V}_0$.

Proof: This is a consequence of Lemma 6.4 and, since $\tilde{V}$ is a manifold, of Lemma 1.11 together with a compactness argument as in the proof of Lemma 3.1. □

As an immediate consequence we have:

Theorem 6.7 *Under the above assumptions the germ X_a is of type PL if and only if there is a constant $C > 0$ so that $|\operatorname{Im} w| \leq C |\operatorname{Im} \varphi(w)|$ for all w on some neighborhood of 0.*

As an example we consider an irreducible purely 1-dimensional germ X_0 of type PL. Then there is a desingularization $\varphi(w) = (\varphi_1(w), \ldots, \varphi_d(w))$, $w \in \tilde{V}$ where $\tilde{V}$ is a neighborhood of 0 in $\mathbb{C}$ and we conclude from Theorem 6.7 that at least one of the φ_j has a zero of order 1 in 0. However, then it is locally invertible and we may assume that $\varphi_j(w) = w$ which means that X_0 is regular. Hence we have proved the following result of Braun, Meise and Taylor [6]:

Corollary 6.8 ([6], Proposition 3.16.) *A purely 1-dimensional irreducible germ is of type PL if and only if it is regular.*

To have concrete examples we consider the polynomials $P_1(x,y) = y^2 - x^2 + x^4$, $P_2(x,y) = y^2 - x^3 + x^5$ and $P_3(x,y) = y^2 - x^4 + x^6$ in two variables. All have compact zero varieties X_j, $j = 1,2,3$. The varieties X_1 (lemniscate) and X_3 are of type PL, X_2 not. To see this we need only to consider the germ in 0. For X_1 and X_2 it consists of two regular branches. X_2 has a pinch point at 0 and we can refer to Corollary 6.8 or use $u(x,y) = \max(-\operatorname{Re} x, 0)$ and the points $(x,y) = (-t, it^{3/2}\sqrt{1-t^2})$ with $t > 0$ near 0. In consequence, by Theorem 2.3, the principal ideals (P_1) and (P_3) generated by the respective polynomials are complemented in $\mathscr{A}(\mathbb{R}^d)$, the principal ideal (P_2) not.

7 Further results and applications

Let E be a complete locally convex space. An E-valued function on $\mathbb{R}^d$ is called real analytic if $y \circ f$ is real analytic for every $y \in E'$. This means that f can be developed locally into a power series which converges under every seminorm p, however the domain of convergence may depend on P. By $\mathscr{A}(\cdot, E)$ we denote the respective spaces of E-valued real analytic functions.

It was not known, even for Fréchet spaces, whether for coherent X every E-valued real analytic function can be extended to a real analytic function on $\mathbb{R}^d$. For Fréchet spaces very restrictive sufficient conditions are contained in [2] and [25]. We give, for compact coherent X, a complete solution. We obtain as an immediate consequence of Theorem 2.2:

Corollary 7.1 *The following are equivalent:*

1. *For every E the restriction map $\mathscr{A}(\mathbb{R}^d, E) \longrightarrow \mathscr{A}(X, E)$ is surjective.*
2. *X is of type PL.*

The Corollary is also true if E is restricted to the class of Fréchet spaces. In this case an extension to non-compact X will be contained in a forthcoming paper. We use Theorem 2.2 and Corollary 7.1 to discuss the case of a homogeneous variety. Let us first fix the notation.

Let $X \subset \mathbb{R}^d$ be a coherent homogeneous subvariety and $X_1 = X \cap S^{d-1}$. We assume $X \neq \{0\}$ and we set $\mathbb{R}^d_* = \mathbb{R}^d \setminus \{0\}$, $X_* = \mathbb{R}^d_* \cap X$. Since X_* is, by $x \longrightarrow (\log|x|, x/|x|)$, analytically homeomorphic to $\mathbb{R} \times X_1$ we see easily that X_1 is of type PL if and only if X_* is of type PL. Moreover $\mathscr{A}(X_*)$ can be identified with $\mathscr{A}(X_1, \mathscr{A}(\mathbb{R}))$ and $\mathscr{A}(\mathbb{R}^d_*)$ with $\mathscr{A}(S^{d-1}, \mathscr{A}(\mathbb{R}))$. As a consequence of our previous results we obtain:

Theorem 7.2 *Under these assumptions the following are equivalent:*

1. *X_* is of type PL (resp. X_1 is of type PL).*
2. *There is a continuous linear extension operator $\mathscr{A}(X_1) \longrightarrow \mathscr{A}(S^{d-1})$.*
3. *There is a continuous linear extension operator $\mathscr{A}(X_*) \longrightarrow \mathscr{A}(\mathbb{R}^d_*)$.*

To prove (1) $\Rightarrow$ (3) we are using Corollary 7.1 with X_1 and $E = \mathscr{A}(\mathbb{R})$. In fact, in some special cases one would also use the same argument with different E, e. g. functions with certain asymptotic behavior, to obtain extension of real analytic functions preserving such behavior. A more thorough discussion has to be done elsewhere.

Let us specify some cases of interest. Let P be an irreducible homogeneous polynomial in d variables so that $\dim_{\mathbb{R}} X_a = d-1$ in every $a \in X$ (strong dimension condition). We set $X = \{x \in \mathbb{R}^d : P(z) = 0\}$. Hörmander showed in [11] that $P(D) : \mathscr{A}(\mathbb{R}^d) \longrightarrow \mathscr{A}(\mathbb{R}^d)$ is surjective if and only if $X_* := X \setminus \{0\}$ is of type PL. (Note that without the dimension condition one would here also have the elliptic ones.) The same condition describes the P as above for which $P(D)$ has a continuous linear right inverse in $C^\infty(\mathbb{R}^d)$ (see Meise-Taylor-Vogt [16] and, for corresponding results for systems of equations, Palamodov [21]). In both cases the PL-condition appears, by means of a technical proof, as equivalent to a global Phragmén-Lindelöf condition on the complex variety. Up to now it was not known, what kind of significance it might have for the real variety. The following result will give an interesting explanation.

By our assumptions X is coherent, $\mathscr{J}_X = P \cdot \mathscr{A}(\mathbb{R}^d)$ and (1) in Theorem 7.2 is equivalent to X being of type PL (see [18, Theorem 3.13]), which is, by Proposition 1.10, equivalent to X_0 (the germ of X in 0) being of type PL. On the basis of Theorem 7.2 and the arguments used in Section 4 above one can show the equivalence of (1) and (2) in the following theorem (see [26]). We obtain from this, [18, Theorem 3.13], [11] and [16].

Theorem 7.3 *For P as above the following are equivalent:*

1. *X is of type PL.*
2. *The principal ideal of P is complemented in the algebra $\mathscr{A}(\mathbb{R}^d)$.*
3. *$P(D) : \mathscr{A}(\mathbb{R}^d) \longrightarrow \mathscr{A}(\mathbb{R}^d)$ is surjective.*
4. *$P(D) : C^\infty(\mathbb{R}^d) \longrightarrow C^\infty(\mathbb{R}^d)$ has a continuous linear right inverse.*

Complemented, of course, here means the existence of a continuous linear projection. The following considerations will provide an interpretation of (2) in Theorem 7.3.

Finally we will consider the problem of continuous linear division. Let $P \in \mathbb{R}[x_1, \ldots, x_d]$ be an irreducible polynomial. We assume that $X := \{x \in \mathbb{R}^d : P(x) = 0\}$ be compact, or P homogeneous, and $\dim_{\mathbb{R}} X_a = d-1$ for every $a \in X$ (strong dimension condition). Then $\mathscr{J}_X = P \cdot \mathscr{A}(\mathbb{R}^d)$. Moreover $f \mapsto P \cdot f$ is an isomorphism $\mathscr{A}(\mathbb{R}^d) \to \mathscr{J}_X$.

It follows that $\mathscr{J}_X$ is complemented in $\mathscr{A}(\mathbb{R}^d)$ if and only if multiplication with P has a continuous linear left inverse in $\mathscr{A}(\mathbb{R}^d)$ or, equivalently, multiplication with P has a continuous linear right inverse in the space of analytic functionals $\mathscr{A}(\mathbb{R}^d)'$ (continuous linear division operator).

Theorem 7.4 *Under our assumptions on P, the existence of a continuous linear division operator in $\mathscr{A}(\mathbb{R}^d)'$ is equivalent to X being of type PL.*

This is in sharp contrast to the case of distributions, because the principal ideal $P \cdot C^\infty(\mathbb{R}^d)$ is always complemented in $C^\infty(\mathbb{R}^d)$ (see [1]).

Coming back to the examples at the end of Section 6 we see that P_1 and P_3 admit a continuous linear division operator in $\mathscr{A}(\mathbb{R}^2)'$, P_2 not, while all three admit such an operator in the space of distributions, resp. a left inverse L_j for P_j, $j = 1, 2, 3$ in $C^\infty(\mathbb{R}^2)$. This means, given L_2, there will always be functions $f \in \mathscr{A}(\mathbb{R}^2)$ so that $L_2 f \in C^\infty(\mathbb{R}^2) \setminus \mathscr{A}(\mathbb{R}^2)$, the critical points of course being in X.

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