Restriction spaces of A^∞

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Abstract

In the present paper it is shown that for certain totally disconnected Carleson sets E the restriction space $A_\infty(E) = \{f|_E : f \in A^\infty\}$ has a basis. Its isomorphism type is determined. The result disproves a claim of S. R. Patel in [12]. To prove our result we analyze restriction spaces $C_\infty(E) = \{f|_E : f \in C^\infty(\mathbb{R})\}$ and then, using a result of Alexander, Taylor and Williams, we show that $A_\infty(E) = C^\infty(E)$. Among our examples there is the classical Cantor set and sets of type $E = \{x_n : n \in \mathbb{N}\} \cup \{0\}$ where $(x_n)_{n\in\mathbb{N}}$ is a null sequence in $\mathbb{R}$ with certain properties.

In his paper [12] Patel claims the following result: let $E \subset [0, 2\pi[$ be a compact, totally disconnected Carleson set then the space of restrictions of A^∞ to E in its natural locally convex topology fails to have a Schauder basis. This result would have provided us with a wealth of quite natural counterexamples for the basis problem for nuclear Fréchet spaces. This problem has, of course, been solved in the negative long time ago by Mityagin and Zobin [7, 8, 9]. Further counterexamples have been given by Djakov and Mityagin [5], Djakov [4] and Moscatelli [10]. Quite recently the author of this note has given a very simple counterexample [19]. That the proof of Patel's result has a gap has been remarked widely. However it remained an interesting question whether the result is correct or not. Unfortunately it is not. We present examples of sets E fulfilling all the above mentioned assumption for which the restriction space $A_\infty(E)$ has a basis.

In this paper A^∞ will be considered as the space of all 2π periodic functions on $\mathbb{R}$ for which all negative Fourier coefficients vanish. E will always denote a compact subset of $\mathbb{R}$ and when it comes to considerations about A^∞ we will always automatically assume that $E \subset [0, 2\pi[$.

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We recall that the sets E which are zero sets of an A^∞-function have been characterized by Taylor and Williams [13] and Novinger [11] by the Carleson condition

$$\int_0^{2\pi} \log \frac{1}{d(x,E)}\, dx < \infty.$$

The sets E with the property that for any (periodic) C^∞-function f on $\mathbb{R}$ there is $g \in A^\infty$ such that f and g and all their derivatives coincide on E have been characterized by Alexander, Taylor and Williams [2] by the strong Carleson-condition (ATW-condition): there are constants C_1, C_2 such that

$$\frac{1}{b-a} \int_a^b \log \frac{1}{d(x,E)}\, dx \le C_1(b-a) \log \frac{1}{b-a} + C_2$$

for all $0 \le a < b \le 2\pi$.

For functional analytic terminology and results we refer to [6], for all notation concerning power series space, invariants like diametral dimension, (DN), (Ω) etc. and related results we refer also to the survey article [18].

1 Restriction spaces of $C^\infty(\mathbb{R})$

Let $E \subset \mathbb{R}$ be a closed set and 0 an accumulation point of E. We set

$$C_\infty(E) = \{f|_E \,:\, f \in C^\infty(\mathbb{R}) \text{ and } J(E) = \{f \in C^\infty(\mathbb{R}) \,:\, f|_E = 0\}.$$

Then we have in a natural way

$$C_\infty(E) \cong C^\infty(\mathbb{R})/J(E)$$

and this makes $C_\infty(E)$ a nuclear Fréchet space.

We want to characterize the functions $\varphi \in C_\infty(E)$. Of course such a characterization in terms of divided differences has been given by Whitney a long time ago, see [21] and there is a vast literature on this problem. We will give a description in this special case which fits to our purposes.

Lemma 1.1 *If $\varphi \in C_\infty(E)$, $\varphi = f|_E$ for $f \in C^\infty(\mathbb{R})$ then $f^{(p)}(0)$ is uniquely determined by φ for all $p \in \mathbb{N}_0$.*

Proof: We proceed by induction. $f^{(0)} = f(0) = \varphi(0)$. If $f^{(0)}, \ldots, f^{(p)}$ is determined, then we have for $x \in E$, $x \neq 0$

$$f^{(p+1)}(\xi) = \frac{(p+1)!}{x^{p+1}} \Big(f(x) - \sum_{j=0}^{p} \frac{f^{(j)}(0)}{j\,!}\, x^j \Big)$$

with suitable ξ between 0 and x.

For $x \to 0$ we have $f^{(p+1)}(\xi) \to f^{(p+1)}(0)$, hence we have

$$f^{(p+1)}(0) = \lim_{x \to 0, x \in E} \frac{(p+1)!}{x^{p+1}} \Big(\varphi(x) - \sum_{j=0}^{p} \frac{f^{(j)}(0)}{j\,!}\, x^j \Big).$$

In particular, this limit exists. □

Definition 1.2 *We set $\varphi^{(p)}(0) := f^{(p)}(0)$ for some $f \in C^\infty(\mathbb{R})$ with $f|_E = \varphi$.*

Corollary 1.3 *If $f \in J(E)$ then f is flat in 0, that is, $f^{(p)}(0) = 0$ for all p.*

Proof: This follows from Lemma 1.1 because f is an extension of 0 and $g \equiv 0$ is another one. □

Lemma 1.4 *$\delta_p : \varphi \mapsto \varphi^{(p)}(0)$ is a continuous linear form on $C^\infty(E)$.*

Proof: If δ_p^∞ is the same map considered on $C^\infty(\mathbb{R})$ and $\rho : C^\infty(\mathbb{R}) \to C_\infty(E)$ the restriction map then $\delta_p^\infty = \delta_p \circ \rho$, hence δ_p is continuous, due to the definition of the topology of $C_\infty(E)$. □

Lemma 1.5 *$\Delta(\varphi) := (\delta_p)_{p \in \mathbb{N}_0}$ defines a continuous, linear surjective map $C_\infty(E) \to \omega$.*

Proof: Continuity follows from Lemma 1.4, surjectivity from the E. Borel theorem. □

We set $J^\infty(0) := \{f \in C^\infty(\mathbb{R}) : f^{(p)}(0) = 0 \text{ for all } p\}$ and $J_\infty(0) := \{\varphi \in C_\infty(E) : \varphi^{(p)}(0) = 0 \text{ for all } p\} = \{f|E : f \in J^\infty(0)\}$.

If $f \in J^\infty(0)$ we have for all $x \in \mathbb{R}$ and $p \in \mathbb{N}$

$$f(x) = \frac{f^{(p)}(\xi)}{p\,!}\, x^p$$

where p is between 0 and x. Therefore for any $0 \le x \le R$ and $p \in \mathbb{N}_0$ we get, setting $\|f\|_M := \sup\{|f(t)| : t \in M\}$ for any function on a set M,

$$(1) \qquad |f(x)| \le \|f^{(p)}\|_{[0,R]} \frac{|x|^p}{p!}.$$

From now on we assume that $E = \{x_1, x_2, \dots\} \cup \{0\}$, where $x_n \searrow 0$. We set $\varepsilon_n = x_n - x_{n+1}$ and assume that $\varepsilon_n \ge \varepsilon_{n+1} > 0$ for all n.

Let $\chi \in \mathscr{D}([-\frac{1}{2}, +\frac{1}{2}])$, χ even and $\chi \equiv 0$ in a neighborhood of 0. We set $\chi_\varepsilon(x) := \chi(\frac{x}{\varepsilon})$. For any sequence $\xi \in \omega$ the function

$$f(x) = \sum_{n=1}^{\infty} \xi_n \, \chi_{\varepsilon_n}(x - x_n)$$

is in $C^\infty(\mathbb{R} \setminus \{0\})$ and $f(x_n) = \xi_n$ for all $n \in \mathbb{N}$.

Lemma 1.6 *Let f be as above. Then $f \in J^\infty(0)$ if, and only if, $\lim_{n\to\infty} \frac{|\xi_n|}{\varepsilon_n^p} = 0$ for all $p \in \mathbb{N}_0$.*

Proof: For all p we have

$$(2) \qquad \sup_{0<|x|\le x_N} |f^{(p)}(x)| = \sup_{n\ge N} |\xi_n| \, \|\chi_{\varepsilon_n}^{(p)}\|_{\mathbb{R}} = \|\chi^{(p)}\|_{\mathbb{R}} \sup_{n\ge N} \frac{|\xi_n|}{\varepsilon_n^p}.$$

This proves the result. □

We assume now that there is $q \in \mathbb{N}$ such that

$$(3) \qquad \sup_n \frac{x_n^q}{\varepsilon_n} < \infty.$$

Remark 1.7 *If condition (3) is fulfilled then for any scalar sequence ξ the following are equivalent*

1. $\lim_{n\to\infty} \frac{|\xi_n|}{\varepsilon_n^p} = 0$ *for all* $p \in \mathbb{N}_0$.
2. $\lim_{n\to\infty} \frac{|\xi_n|}{x_n^p} = 0$ *for all* $p \in \mathbb{N}_0$.

We set $\alpha_n := -\log x_n$. Because of $\sum_n x_n^q \le C \sum_n \varepsilon_n < \infty$ the space

$$\Lambda_\infty(\alpha) := \{\xi = (\xi_1, \xi_2 \dots) : |\xi|_p = \sup_n |\xi| e^{p\alpha_n} < \infty \text{ for all } p\}$$

is nuclear, due to the Grothendieck-Pietsch criterion (see [6, 28.15]). We obtain

Proposition 1.8 *If condition (3) is fulfilled then* $\Phi : \varphi \mapsto (\varphi(x_n))_{n\in\mathbb{N}}$ *maps* $J_\infty(0)$ *isomorphically onto* $\Lambda_\infty(\alpha)$.

Proof: If $\varphi \in J_\infty(0)$ and $f \in C^\infty(\mathbb{R})$ is any extension of φ then $f \in J^\infty(0)$ and, due to inequality (1) we have

$$|\varphi(x_n)| \le \frac{\|f^{(p)}\|_{[0,x_1]}}{p\,!}\, e^{-p\alpha_n}.$$

Since this holds for every extension f of φ we have

$$\sup_n |\varphi(x_n)|\, e^{p\alpha_n} \le s(f)$$

where s is a continuous seminorm on $J_\infty(0)$.

Obviously Φ is injective, surjectivity of Φ follows from Lemma 1.6. We have, using the notation of Lemma 1.6

$$\Phi^{-1}(\xi) = \sum_{n=1}^{\infty} \xi_n\, \chi_{\varepsilon_n}(x - x_n).$$

Continuity of Φ^{-1} follows from equation (2) with $N = 1$ or from the open mapping theorem. □

We will now investigate the structure of $C_\infty(E)$.

Theorem 1.9 *Let* $\varphi \in C(E)$. *Then* $\varphi \in C_\infty(E)$ *if and only if the following holds: There are numbers* A_p, $p \in \mathbb{N}_0$, *such that* $A_0 = f(0)$ *and for all* $p \in \mathbb{N}_0$ *we have*

$$(4) \qquad A_{p+1} = \lim_{n\to\infty} \frac{(p+1)!}{x_n^{p+1}} \Big(\varphi(x_n) - \sum_{j=0}^{p} \frac{A_j}{j\,!}\, x_n^j\Big).$$

In this case $A_p = \varphi^{(p)}(0)$ *for all* $p \in \mathbb{N}_0$.

Proof: Necessity follows from Lemma 1.1. From this Lemma also follows that necessarily $A_p = \varphi^{(p)}(0)$ for all $p \in \mathbb{N}_0$. We have to show that the condition is also sufficient.

Given the sequence A_p, $p \in \mathbb{N}_0$, there exists, due to the E. Borel theorem a function $g \in C^\infty(\mathbb{R})$ such that $g^{(p)}(0) = A_p$ for all $p \in \mathbb{N}_0$.

We consider the function $h = \varphi - g|_E$. We fix n_0 which will be determined later. For $n \geq n_0$ we have

$$\begin{aligned} h(x_n) &= \varphi(x_n) - g(x_n) \\ &= \sum_{j=0}^{p} \frac{A_j}{j\,!}\, x_n^j + \frac{\tilde{A}_{p+1}}{(p+1)!}\, x_n^{(p+1)} - g(x_n) \\ &= \frac{\tilde{A}_{p+1}}{(p+1)!}\, x_n^{(p+1)} - \frac{g^{(p+1)}(\xi)}{(p+1)!}\, x_n^{(p+1)} \\ &= (\tilde{A}_{p+1} - g^{(p+1)}(\xi))\frac{x_n^{(p+1)}}{(p+1)!} \end{aligned}$$

$\tilde{A}_{p+1}$ depends on n and converges to A_{p+1} for large n, due to (4), $\xi \in]0, x_n[$ comes from Taylor's formula with Langrange remainder. Hence we have

$$\lim_{n\to\infty} \frac{|h(x_n)|}{x_n^{p+1}} = \lim_{n\to\infty} \frac{1}{(p+1)!}\, |\tilde{A}_{p+1} - g^{(p+1)}(\xi)| = 0$$

for all $p \in \mathbb{N}_0$.

According to Lemma 1.6 and condition (2) there is a function $H \in J^\infty(0)$ such that $H(x_n) = h(x_n)$ for all $n \in \mathbb{N}$, that is $H|_E = h$. We set $f := g + H$. Then $f \in C^\infty(\mathbb{R})$ and $f|_E = \varphi$. $\square$

On $C^\infty(E)$ we consider for $p = 0, 1, \ldots$ the following seminorms

$$|\varphi|_p = \sup_n \left| \frac{p\,!}{x_n^p} \Big(\varphi(x_n) - \sum_{j=0}^{p-1} \frac{\varphi^{(j)}(0)}{j\,!}\, x_n^j \Big) \right|.$$

We fix p. For every n the function $\varphi \mapsto \Big| \ldots \Big|$ is a continuous seminorm, since the δ_p are continuous linear forms on $C_\infty(E)$. The supremum exists for all φ, hence, due to the Banach-Steinhaus theorem, the $|\cdot|_p$ are continuous seminorms on $C_\infty(E)$.

Theorem 1.10 *The norms $|\cdot|_p$, $p \in \mathbb{N}_0$, are a fundamental system of seminorms in $C_\infty(E)$.*

Proof: It suffices to show that $C_\infty(E)$ is complete in the topology generated by the $|\cdot|_p$. Let φ_k, $k \in \mathbb{N}$, be a Cauchy sequence with respect to the $|\cdot|_p$, $p \in \mathbb{N}_0$.

Since $|\varphi|_0 = \sup_{n\in\mathbb{N}} |\varphi(x_n)| = \sup\{|\varphi(x)| \,:\, x \in E\}$ the sequence φ_k converges uniformly on E to a function $\varphi \in C(E)$.

For every p the sequence

$$\frac{p\,!}{x_n^p}\Big(\varphi_k(x_n) - \sum_{j=0}^{p-1} \frac{\varphi_k^{(j)}(0)}{j\,!}\, x_n^j\Big), \quad k = 1, 2, \dots$$

converges uniformly in n. Therefore the right hand side of

$$\varphi_k^{(p+1)}(0) = \lim_{n\to\infty} \frac{(p+1)!}{x_n^{p+1}} \Big(\varphi_k(x_n) - \sum_{j=0}^{p} \frac{\varphi_k^{(j)}(0)}{j\,!}\, x_n^j\Big)$$

converges for all $p \in \mathbb{N}_0$. We set for $p \in \mathbb{N}_0$

$$A_{p+1} = \lim_{k\to\infty} \varphi_k^{(p+1)}$$

and arrive, by induction, at the condition (4) for $\varphi \in C(E)$. By Theorem 1.9 we get that $\varphi \in C^\infty(E)$.

The proof that $\lim_{k\to\infty} |\varphi_k - \varphi|_p = 0$ for all p is now standard. □

Remark 1.11 *The system of seminorms $|\cdot|_p$, $p \in \mathbb{N}_0$, is not increasing. To see this we choose $\varphi = P|_E$ where P is a polynomial of degree m. Then $|\varphi|_p = 0$ for $p > m$. Fundamental system of seminorms means here that every continuous seminorm s on $C_\infty(E)$ can be estimated in the form $s(\varphi) \le C \max_{p=0,\dots,P} |\varphi|_p$.*

2 $C_\infty(E)$ and $A_\infty(E)$

Lemma 2.1 *Condition (3) implies that $E = \{x_1, x_2, \dots\}$ is a Carleson set.*

Proof: We may assume that $0 < x_1 \le 1$ and obtain

$$\sum_{n=1}^{\infty} \varepsilon_n \log \frac{1}{\varepsilon_n} \le q \sum_{n=1}^{\infty} \varepsilon_n \log \frac{1}{x_n} \le q \int_0^1 \log \frac{1}{x}\, dx = q.$$

The second sum is a lower Riemann sum for the integral whence the second estimate. □

We will now carefully study the Carleson condition and also the strong Carleson condition of Alexander-Taylor-Williams (ATW-condition), see [2] . We start with a simple calculation:

For $0 \le a < b$ we obtain

$$\int_a^b \log \frac{1}{d(x,\{a,b\})} = (b-a)\log\frac{1}{b-a} + (1+\log 2)(b-a). \tag{5}$$

For $A < B$ and $a \in [\frac{A+B}{2}, B]$ we have

$$\int_a^B \log \frac{1}{d(x,\{A,B\})}\,dx = \int_a^B \log\frac{1}{B-a}\,dx = (B-a)\log\frac{1}{B-a} + (B-a).$$

For $a \in [A, \frac{A+B}{2}]$ we get

$$\begin{aligned} \int_a^B \log \frac{1}{d(x,\{A,B\})}\,dx &\le \int_A^B \log\frac{1}{d(x,\{A,B\})}\,dx \\ &= (B-A)\log\frac{1}{B-A} + (1+\log 2)(B-A) \\ &\le 2(B-a)\log\frac{1}{B-a} + 2(1+\log 2)(B-a) \end{aligned}$$

since $B - a \le B - A \le 2(B-a)$.

Therefore we have in both cases

$$\int_a^B \log \frac{1}{d(x,\{A,B\})}\,dx \le 2(B-a)\log\frac{1}{B-a} + 5(B-a). \tag{6}$$

In the same way we get for $b \in [A, B]$

$$\int_A^b \log \frac{1}{d(x,\{A,B\})}\,dx \le 2(b-A)\log\frac{1}{b-A} + 5(b-A). \tag{7}$$

We need another elementary inequality. For $0 < a \le b$ we have, using the mean value theorem, with $a < \xi < a+b$

$$\begin{aligned} (a+b)\log(a+b) - a\log a &= b(\log\xi + 1) \\ &\le b(\log(a+b)+1) \\ &\le b\log b + b\log 2 + b, \end{aligned} \tag{8}$$

and therefore

$$a\log\frac{1}{a} + b\log\frac{1}{b} \le (a+b)\log\frac{1}{a+b} + 2\,b. \tag{9}$$

Assume now that we have numbers $0 < a_1 \leq a_2 \leq \dots a_m$ with

$$\sum_{j=1}^{k} a_j \leq a_{k+1}.$$

for $k = 1, \dots, m-1$ We set $a = \sum_{j=1}^{m} a_j$ and we obtain by inductive use of estimate (9)

$$\sum_{j=1}^{m} a_j \log \frac{1}{a_j} \leq a \log \frac{1}{a} + 2\,a. \tag{10}$$

We return to our previous setting and we have shown:

Lemma 2.2 *If $x_{k+1} \leq \varepsilon_k$ for all $k \in \mathbb{N}$ then*

$$\frac{1}{b-a} \int_a^b \log \frac{1}{d(x,E)}\, dx \leq \log \frac{1}{b-a} + 16 \tag{11}$$

for $0 \leq a < b \leq x_1$.

Proof: First we apply for any j formulas (5), (6) or (7), respectively, to the interval $[\alpha_{j+1}, \alpha_j] = [x_{j+1}, x_j] \cap [a,b]$ and obtain in any case

$$\int_{\alpha_{j+1}}^{\alpha_j} \log \frac{1}{d(x,E)}\, dx \leq 2(\alpha_j - \alpha_{j+1}) \log \frac{1}{\alpha_j - \alpha_{j+1}} + 4\,(\alpha_j - \alpha_{j+1}). \tag{12}$$

If $b \in [\alpha_{m+1}, \alpha_m]$ then we obtain by use of formula (10)

$$\int_a^{\alpha_{m+1}} \log \frac{1}{d(x,E)}\, dx \leq 2(\alpha_{m+1} - a) \log \frac{1}{\alpha_{m+1} - a} + 8\,(\alpha_{m+1} - a). \tag{13}$$

Applying formula (9) to (12), with $j = m$, and (13), we arrive at

$$\int_a^b \log \frac{1}{d(x,E)}\, dx \leq 2(b-a) \log \frac{1}{b-a} + 16\,(b-a)$$

which is equivalent to (11). $\square$

We set now for $E \subset [0, 2\pi]$

$$A_\infty(E) = \{f|_E \,:\, f \in A^\infty\}.$$

From the result of Alexander, Taylor and Williams [2, Theorem 1.1.] we obtain

Theorem 2.3 *If $x_{n+1} \leq \varepsilon_n$ for all $n \in \mathbb{N}$ we have $C_\infty(E) = A_\infty(E)$.*

3 Structure of $\mathbb{C}^\infty(E)$

We will now investigate the linear topological structure of $C_\infty(E)$. Clearly it it nuclear and, being a quotient of $C^\infty(\mathbb{R})$, it has property (Ω). We will show that for suitable sequences $(x_n)_{n\in\mathbb{N}}$ it has also property (DN). The argument we will be using is due to Tidten. In fact the proof of the following theorem is an easy adaptation of the proof of Tidten [15, Satz 1] where we have Whitney jets and E is 1-perfect.

First we will define an increasing fundamental system of seminorms for $C_\infty(E)$. We set

$$R^p\varphi(x_n) = \varphi(x_n) - \sum_{j=0}^{p} \frac{\varphi^{(j)}(0)}{j\,!}\, x_n^j.$$

and define

$$\|\varphi\|_k := \max_{p=0,\ldots,k} \left\{ |\varphi^{(p)}(0)| + \sup_{n\in\mathbb{N}} \frac{|R^p\varphi(x_n)|}{x_n^p} \right\}.$$

Since $|\varphi^{(p)}(0)| \le |\varphi|_p$ and

$$\sup_{n\in\mathbb{N}} \frac{|R^p\varphi(x_n)|}{x_n^p} \le x_1\,|\varphi|_{p+1}$$

for all p the $\|\cdot\|_k$ are continuous seminorms on $C_\infty(E)$. Because

$$\frac{p\,!}{x_n^p}\, R^{p-1}\varphi(x_n) = \frac{p\,!}{x_n^p}\, R^p\varphi(x_n) + \varphi^{(p)}(0)$$

we have

$$|\varphi|_p \le p\,!\,\|\varphi\|_p$$

for all p. Therefore the $\|\cdot\|_k$ are a fundamental system of seminorms in $C_\infty(E)$.

Theorem 3.1 *If there is a constant C such that $x_n \le C\,x_{n+1}$ for all $n \in \mathbb{N}$, then $C_\infty(E)$ has property* (DN).

Proof: We follow the proof of Tidten [15, Satz 1]. We present it here, with the necessary changes (in fact, simplifications), for the convenience of the reader.

i) We want to show that there is a constant C_1, such that for $M > 1$, $k \in \mathbb{N}$ and $\varphi \in E_\infty(E)$ with $\|\varphi\|_{k-1} \le 1$ and $\|\varphi\|_{k+1} \le M$ we have:

$$\frac{|R^{k-1}\varphi(x_n)|}{x_n^k} \le C_1 M^{1/2}$$

for all $n \in \mathbb{N}$.

We set

$$Q := \frac{R^{k-1}\varphi(x_n)}{x_n^k}.$$

For $M \le x_1^{-2}$ we obtain i) with any $C_1 \ge x_1^{-2}$:

$$|Q| \le \frac{|R^k\varphi(x_n)|}{x_n^k} + \frac{1}{k\,!}\,|\varphi^{(k)}(0)| \le \|\varphi\|_k \le \|\varphi\|_{k+1} \le M \le x_1^{-2} \le C_1 \le C_1 M^{1/2}.$$

Let now $M > x_1^{-2}$. We consider wo cases.

In the case of $M^{1/2} \ge 1/x_n$ we obtain i) with any $C_1 \ge 1$.

$$|Q| = \frac{1}{x_n}\,\frac{|R^{k-1}\varphi(x_n)|}{x_n^{k-1}} \le \frac{1}{x_n}\,\|\varphi\|_{k-1} \le \frac{1}{x_n} \le M^{1/2} \le C_1 M^{1/2}.$$

It remains the case of $1/x_1 < M^{1/2} < 1/x_n$. Because of $x_1 > M^{-1/2}$ there is a maximal $m \in \mathbb{N}$ sucht that $x_m > M^{-1/2}$. For that m we have

$$x_{m+1} \le M^{-1/2} < x_m \le C\,x_{m+1}.$$

We set $\tilde{x} := x_{m+1}$ and we have

$$\tilde{x} \le M^{-1/2}, \quad \frac{1}{\tilde{x}} < C\,M^{1/2}, \quad x_n < M^{-1/2} < C\tilde{x}.$$

We obtain

$$\begin{aligned}
(14) \qquad \left|Q - \frac{1}{k\,!}\,\varphi^{(k)}(0)\right| &= \frac{|R^k\varphi(0)|}{x_n^k} \\
&= x_n\left|\frac{R^{k+1}\varphi(x_n)}{x_n^{k+1}} + \frac{1}{(k+1)\,!}\varphi^{(k+1)}(0)\right| \\
&\le x_n\,\|\varphi\|_{k+1} \le x_n\,M.
\end{aligned}$$

We set

$$\widetilde{Q} := \frac{R^{k-1}\varphi(\tilde{x})}{\tilde{x}^k}$$

and obtain, replacing in (14) x_n with $\tilde{x} = x_{m+1}$,

$$(15) \qquad \left|\widetilde{Q} - \frac{1}{k\,!}\,\varphi^{(k)}(0)\right| \le \tilde{x}\,M.$$

From (14) and (15) we obtain

$$|Q - \widetilde{Q}| \le (x_n + \tilde{x})M \le 2M^{1/2}. \tag{16}$$

Because of $\|\varphi\|_{k-1} \le 1$ we have

$$|\widetilde{Q}| = \frac{1}{\tilde{x}} \frac{|R^{k-1}\varphi(\tilde{x})|}{\tilde{x}^{k-1}} \le \frac{1}{\tilde{x}} \le C\,M^{1/2}. \tag{17}$$

From (16) and (17) we get:

$$|Q| \le |Q - \widetilde{Q}| + |\widetilde{Q}| < (C+2)\,M^{1/2}.$$

So, finally, w have shown the claim of i) with $C_1 = \max\{x_1^{-2}, 2C+1\}$.

ii) Let φ be like in i). From (4) we know that

$$\varphi^{(k)}(0) = \lim_{n\to\infty} k\,!\frac{R^{k-1}\varphi(x_n)}{x_n^k}.$$

Therefore i) implies $|\varphi^{(k)}(0)| \le k\,!\,C_1 M^{1/2}$.

We obtain

$$\frac{|R^k\varphi(x_n)|}{x_n^k} \le \frac{|R^{k-1}\varphi(x_n)|}{x_n^k} + \frac{1}{k\,!}\,|\varphi^{(k)}(0)| \le 2\,C_1 M^{1/2}.$$

and therefore

$$\begin{aligned} \|\varphi\|_k &= \max\left\{\|\varphi\|_{k-1}, |\varphi^{(k)}(0)| + \sup_{n\in\mathbb{N}} \frac{|R^k\varphi(x_n)|}{x_n^k}\right\} \\ &\le \max\{1, k\,!\,C_1 M^{1/2} + 2\,C_1 M^{1/2}\} \\ &\le C_2 M^{1/2} \end{aligned}$$

with $C_2 = (k\,! + 2)C_1$.

This implies easily that $\|\varphi\|_k \le C_2 \|\varphi\|_{k-1}^{1/2} \|\varphi\|_{k+1}^{1/2}$ for all $k \in \mathbb{N}$. □

4 Sets with one accumulation point

We made assumptions on the sequence $(x_n)_{n\in\mathbb{N}}$ in (3), in Lemma 2.2 and in Theorem 3.1. They all are fulfilled if we have with suitable $C > 0$

$$2x_{n+1} \le x_n \le C x_{n+1} \tag{18}$$

because this implies $x_{n+1} \le \varepsilon_n$ and therefore also $x_n = x_{n+1} + \varepsilon_n \le 2\varepsilon_n$.

Theorem 4.1 *If (18) is fulfilled then* $A_\infty(E) = C_\infty(E) \cong \Lambda_\infty(\alpha)$ *where* $\alpha_n = -\log x_n$.

Proof: By Theorem 2.3 we have $A_\infty(E) = C_\infty(E)$. Since (18) implies (3) we obtain from Proposition 1.8 that $J_\infty(0) \cong \Lambda_\infty(\alpha)$. Therefore have an exact sequence

$$0 \longrightarrow \Lambda_\infty(\alpha) \longrightarrow C_\infty(E) \longrightarrow \omega \longrightarrow 0$$

where ω denotes the space of all scalar sequences. Because of (18) the space $\Lambda_\infty(\alpha)$ is stable. For the diametral dimensions we get $\Delta(\Lambda_\infty(\alpha)) \cap \Delta(\omega) = \Delta(\Lambda_\infty(\alpha))$ and this is stable. So we obtain from [17, Proposition 4.2.] that $\Delta(C_\infty(E)) = \Delta(\Lambda_\infty(\alpha))$ and this is stable.

Clearly $C_\infty(E)$ has property (Ω) since it is a quotient of $C^\infty(\mathbb{R})$, by Theorem 3.1 it has also property (DN) and, of course it is nuclear. By Aytuna-Krone-Terzioğlu [1, Theorem 2.2] we get $\Lambda_\infty(E) \cong \Lambda_\infty(\alpha)$. □

Example 4.2 *Let* $x_n = 2^{-n}$. *Then (18) is fulfilled and* $C_\infty(E) = A_\infty(E) \cong H(\mathbb{C})$.

For the isomorphism we remark that, due to $\alpha_n = n \log 2$, the space $\Lambda_\infty(\alpha)$ is easily seen to be isomorphic to the space $H(\mathbb{C})$ of entire functions on $\mathbb{C}$.

5 The Cantor set

Let now E be the classical Cantor set. It is known since a long time that it is a Carleson set (see Beurling [3]). We will show that it fulfills also the ATW-condition.

For that we will use that $(3^k E) \cap [0,1] = E$ for all $k \in \mathbb{N}$. We will again need an elementary formula: For that let $M \subset [0,1]$ be a compact subset. We have for $a > 0$

$$\int_0^a \log \frac{1}{d(x, aM)}\, dx = a \log \frac{1}{a} + a \int_0^1 \log \frac{1}{d(t, M)}\, dt. \tag{19}$$

Let now $0 \le a < b < 1$ be given. We set $b - a := \gamma = 0, \gamma_1 \gamma_2 \ldots$ where the last term denotes the triadic expansion of γ, finite if possible. In a first step we restrict ourselves to the case of γ with a finite expansion, say $\gamma = 0, \gamma_1 \ldots \gamma_m$. We set $a_0 = a$ and $a_k = a + 0, \gamma_1 \ldots \gamma_k$, that means $a_{k+1} = a_k + \gamma_{k+1} 3^{-k-1}$. We obtain

$$\int_a^b \log \frac{1}{d(x,E)}\, dx = \sum_{k=0}^{m-1} \int_{a_k}^{a_{k+1}} \log \frac{1}{d(x,E)}\, dx.$$

Since γ_k takes only the values $0, 1, 2$ we have to estimate from above integrals over intervals of length 3^{-k-1} or $2 \cdot 3^{-k-1}$.

Now we consider the subdivision of $[0, 1]$ into 3^k intervals of length 3^{-k} and refer to the classical stepwise construction of the Cantor set. Some of the intervals, we call them windows, have already been excluded from the Cantor set, we call them white, some wait for treatment, we call them black.

We restrict now to the nontrivial case of $\gamma_{k+1} \neq 0$. Our interval of length 3^{-k-1} or $2 \cdot 3^{-k-1}$ hits at most two of the windows. If it is of length 3^{-k-1} and completely in a white window the worst case is (see equation(5))

$$\int_0^{3^{-k-1}} \log \frac{1}{x}\, dx = 3^{-k-1} \log \frac{1}{3^{-k-1}} + 3^{-k-1}.$$

If it is of length $2 \cdot 3^{-k-1}$ and completely in a white window we estimate roughly by 2 times the previous case and obtain for both cases

$$\int_{a_k}^{a_{k+1}} \log \frac{1}{d(x, E)}\, dx \leq 2 \int_0^{3^{-k-1}} \log \frac{1}{x}\, dx \leq 3^{-k} \log \frac{1}{3^{-k-1}} + 3^{-k}. \tag{20}$$

If it is completely in a black window we take into account that, by shifting the lower end of the window into zero and multiplying by 3^k we obtain E. The interval $[a_k, a_{k+1}]$, if nontrivial, extends to an interval of length $1/3$ or $2/3$. Therefore we have, estimating by the integral over the whole window and using (19),

$$\int_{a_k}^{a_{k+1}} \log \frac{1}{d(x, E)}\, dx \leq 3^{-k} \log \frac{1}{3^{-k-1}} + D_0\, 3^{-k}. \tag{21}$$

where

$$D_0 = \log 3 + \int_0^1 \log \frac{1}{d(x, E)}\, dx.$$

Therefore we have in all cases, estimating roughly by the sum of estimate (20) and estimate (21),

$$\begin{aligned}
\int_{a_k}^{a_{k+1}} \log \frac{1}{d(x, E)}\, dx &\leq 2 \cdot 3^{-k} \log \frac{1}{3^{-k-1}} + (D_0 + 1)\, 3^{-k} \\
&\leq 6\, \gamma_{k+1} 3^{-k-1} \log \frac{1}{3^{-k-1}} + 3(D_0 + 1)\, \gamma_{k+1} 3^{-k-1} \\
&\leq 6\, \gamma_{k+1} 3^{-k-1} \log \frac{1}{\gamma_{k+1} 3^{-k-1}} + D\, \gamma_{k+1} 3^{-k-1}
\end{aligned}$$

where $D = 6\log 2 + 3(D_0+1)$. Therefore

$$\int_a^b \log\frac{1}{d(x,E)}\,dx \le 6\sum_{k=0}^{m-1}\gamma_{k+1}3^{-k-1}\log\frac{1}{\gamma_{k+1}3^{-k-1}} + D\,(b-a). \tag{22}$$

To apply estimate (10), counting reversely, we need the following:

$$\sum_{k=n}^{m-1}\gamma_{k+1}3^{-k-1} \le 2\sum_{k=n}^{\infty}3^{-k-1} = 3^{-n} \le \gamma_\nu 3^{-\nu}.$$

where ν is the biggest number $\le n$ with $\gamma_\nu \ne 0$. If there is none we are done, we have to add no further summand.

From (22) and (10) we get now

$$\int_a^b \log\frac{1}{d(x,E)}\,dx \le 6\,(b-a)\log\frac{1}{b-a} + (D+15)\,(b-a)$$

for all triadic numbers in $[0,1[$. Since we know that E is Carleson, that is $\log\frac{1}{d(x,E)}$ is integrable over $[0,1]$, the left and the right hand side depend continuously on a and b. Therefore the estimate is true for all $0 \le a < b \le 1$.

Applying the result of Alexander, Taylor and Williams [2, Theorem 1.1.] we have shown:

Proposition 5.1 *If E is the classical Cantor set we have $A_\infty(E) = C_\infty(E)$.*

Remark 5.2 *Because of Corollary 1.3 the functions $f \in J(E)$ vanish on E including all their derivatives. That means $C_\infty(E) = \mathscr{E}(E)$, the space of Whitney jets on E.*

From Tidten [15, Folgerung, p.76] we know that $\mathscr{E}(E)$ is isomorphic to a complemented subspace of s. Clearly $C_\infty(E)$ is stable, because

$$C_\infty(E) \cong C_\infty[0,1/3] \oplus C_\infty[2/3,1] \cong C_\infty(E)^2.$$

Again, using Aytuna-Krone-Terziğlu [1, Theorem 2.2] (or Wagner [20, Theorem 1]), we obtain

Theorem 5.3 *If E is the classical Cantor set then $A_\infty(E) = C_\infty(E)$ and $A_\infty(E)$ has a basis. In fact, it is isomorphic to a power series space of infinite type.*

6 Final remarks

We return to the notation of Section 1 and assume that (3) holds. We define for $f \in J^\infty(0)$

$$Pf(x) = f(x) - \sum_{n=1}^{\infty} f(x_n)\chi_{\varepsilon_n}(x - x_n).$$

Then due to (1) and Lemma 1.6 P is a linear map from $J^\infty(0)$ to $J(E)$ which is continuous by estimates (1) and (2). We have shown:

Lemma 6.1 *If (3) is fulfilled, then P is a continuous projection in $J^\infty(0)$ onto $J(E)$.*

Corollary 6.2 *If (3) is fulfilled, then $J(E)$ has property (Ω).*

Proof: $J^\infty(0)$ has property (Ω) by Tidten [16, Satz 2.2] and (Ω) is inherited by complemented subspaces. $\square$

We obtain:

Theorem 6.3 *If there is $q \in \mathbb{N}$ and $C > 0$ such that $x_n^q \leq C\varepsilon_n$ and $x_n \leq Cx_{n+1}$ for all $n \in \mathbb{N}$, then there is a continuous linear extension operator from $C_\infty(E)$ to $C^\infty(\mathbb{R})$.*

Proof: We have the natural exact sequence

$$0 \longrightarrow J(E) \longrightarrow C^\infty(\mathbb{R}) \xrightarrow{\rho} C_\infty(E) \longrightarrow 0$$

where ρ is the restriction map. $J(E)$ has property (Ω) by Corollary 6.2, $C_\infty(E)$ has property (DN) by Theorem 3.1 and all spaces are nuclear. By the (DN)-(Ω)-splitting theorem (see [6, 30.1]) the sequence splits, hence ρ has a continuous linear right inverse, that is, there is a continuous linear extension operator. $\square$

Examples for this are not only exponentially decreasing sequences x_n but also, for example, $x_n = 1/n$, $n \in \mathbb{N}$.

Let us finally remark that for E being the classical Cantor set there is a continuous linear extension operator from $\mathbb{C}_\infty(E) = \mathscr{E}(E)$ to $C^\infty(\mathbb{R})$ by Tidten [15, Folgerung, p.76].

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