Felix Hausdorff and the *Hausdorff edition**

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Hausdorff and Mongré

Felix Hausdorff (b. November 8, 1868, at Breslau, d. January 26, 1942, Bonn) studied mathematics at Leipzig, Freiburg and Berlin between 1887 and 1891 (dissertation) and started research in applied mathematics related to work of his teacher, the astronomer H. Bruns.[1] After his habilitation (1895) he taught at Leipzig university and a local commercial school. He moved in a milieu of Leipzig intellectuals and artists, strongly influenced by the early work of F. Nietzsche, striving for a cultural modernization of late 19th century Germany.

Between 1897 and 1904 (with some additional later contributions) Hausdorff published two philosophical books, a poem collection, and a satirical theater play under the pseudonym *Paul Mongré*. The play ridiculized the honor codex of late 19th century German *Bildungsbürger* adapting to the values of the Wilhelminean officer corps. At the time it was quite successful. It had about 300 performances between 1904 and 1930 at about 40 towns, among them Berlin, Budapest, Prag, Strassburg, Wien, and Zürich.[2] Moreover, Mongré regularly contributed cultural critical essays to the *Neue Deutsche Rundschau*, a leading intellectual journal.[3] In his second book, *The Chaos in Cosmic Selection* (Hausdorff 1898), he critically decomposed metaphysical remnants in contemporay concepts of space and time. He combined Niezschean and Kantian views and enriched them by mathematical arguments in terms of Cantorian set theory and stepwise generalized transformations, comparable to F. Klein's approach in the *Erlanger Programm*. In Hausdorff's perspective, the generalization of transformations from Euclidean via differentiable and continuous to any point transformation would lead to a general transfinite set as symbol for some structureless fictitious "absolute". The progression of "absolute" or "transcendent" time might then be perceived as *any* order structure on the set of the "transcendent" world content, without any perceivable relation to the order of the empirical or phenomenological time ordering. That served Mongré as an argument that "the absolute" has to be considered as essentially void of objective meaning. He thus proudly proclaimed the "end of metaphysics".[4]

During this period, Hausdorff reoriented his mathematical work towards the new field of transfinite set theory. He gave one of the first lecture courses on the topic in summer 1901 and contributed important results to it, among others the *Hausdorff recursion* for aleph exponentiation and deep methods for the classification of order structures (confinality, gap types, general ordered products, and eta-alpha sets).[5] Hausdorff considered the contemporary attempts to secure axiomatic foundations for set theory as premature. Working on the basis of a "naive" concept of set (expressedly understood as a semiotic tool of thought), he nevertheless achieved an exceptionally high precision of argumentation. Although his set theoretical studies prior to 1910 concentrated on order structures (remember that his earliest interest in transfinite set theory was triggered by the tremendous amount of possible different modes of progression of a fictitious "transcendent time point" in a transfinite set), he contributed crucial insights in foundational questions, most importantly his *maximal chain principle* (related to Zorn's lemma, but different from it), a characterization of *weakly inaccessible cardinals* (in present terminology) and the *universality property* for order structures of what he called " eta-alpha sets". The latter became one of the roots of "saturated structures" in model theory of the 1960s. Moreover, Hausdorff hit upon the importance of the *generalized continuum hypothesis* in his studies of eta-alpha sets.[6]

The "*Grundzüge*"

In summer 1910 Hausdorff started teaching at Bonn university as "extraordinarius" (associate professor) and broadened his perspective on set theory as a general symbolical basis for

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[1]See (Hausdorff Werke, V (2005), forthcoming).

[2]Information due to a communication of U. Roth.

[3]Cf. (Vollhardt/Roth 2002); more in (Hausdorff Werke, vol. VIII (forthcoming)).

[4]Cf. (Stegmaier 2002) and (Hausdorff Werke, VII (2004), 49–61); for relations to mathematics (Hausdorff Werke, vol. VI (forthcoming)).

[5](Hausdorff Werke, vol. I (forthcoming)); detailed references to Hausdorff's publications in all volumes.

[6]See (Hausdorff Werke, II (2002), 600ff.), (Moore 1982, 116 etc.) and (Felgner 2002).

mathematics. In early 1912 he found a beautiful axiomatic characterization of topological spaces by neighbourhood systems and started to compose a monograph on “basic features of set theory” (*Grundzüge der Mengenlehre*). It was finished two years later, after he had moved to Greifswald university in 1913 on a call to an “ordinary” (full) professorship, and became his *opus magnum* (Hausdorff 1914*b*)[7].

In this book, Hausdorff showed how set theory could be used as a working frame for mathematics more broadly. It contained three parts, (I) *general set theory* and order structures, (II) *topological spaces* and their basic properties, (III) *measure theory* and integration. While set theory was introduced in a non-axiomatic style, although with extraordinary precision, topological spaces and measure theory were given an axiomatic presentation. In part (II), Hausdorff published his *neighbourhood axioms* for general spaces, found two years earlier, introduced separation and countability axioms, studied connectivity properties and other concepts. This part of the book contained the first comprehensive treatment of the theory of metric spaces, initiated by M. Fréchet in 1906, and laid the basis for an important part of the traditon of general topology of the coming century.[8]

In part (III) he gave a lucid introduction to measure theory, building upon the work of E. Borel and H. Lebesgue. In a paper published shortly before the book, and added in content as an appendix to the latter, Hausdorff gave a negative answer to Lebesgue’s question (for $n \geq 3$), whether a (finitely) additive content function invariant under congruences can be defined on *all* subsets of Euclidean $\mathbb{R}^n$ (Hausdorff 1914*a*). Using the axiom of choice, he “constructed” a partition of the 2-sphere (up to a countable residual set), in which each part is congruent to the union of two of them. This was the starting point for the later paradoxical constructions of measure theory by Banach and Tarski.[9]

An intense reception of the *Grundzüge* started only after World War I, and most strongly in the rising schools of modern mathematics in Poland, around the journal *Fundamenta Mathematicae*, and the Soviet Union mainly among N. Lusin’s students around P. Alexandroff. Between the latter and Hausdorff there arose a close scientific exchange and intellectual friendship, interrupted only after 1933. All in all, the *Grundzüge* became one of the founding documents of *mathematical modernism* in the sense of the 1920/30s. In a lecture course in 1923 Hausdorff introduced an axiomatic basis for probability theory, which anticipated Kolmogorov’s axiomatization of 1933.[10]

Real analysis and descriptive set theory

In his own research, Hausdorff took up questions in real analysis, now informed by the new “basic features” of general set theory. His introduction of what are now called *Hausdorff measure* and *Hausdorff dimension* (Hausdorff 1919) became of long-lasting importance in the theory of dynamical systems, geometrical measure theory and the study of “fractals”, which arose broad and even popular interest in the last third of the 20th century.[11]

Other important technical contributions dealt with summation methods of infinite divergent series and a generalization of the Riesz-Fischer theorem, which established the now well known relation between L^p function spaces and l^q series of Fourier coefficients, for $\frac{1}{p} + \frac{1}{q} = 1$, and opened the path for later developments in harmonic analysis on topogical groups (Hausdorff 1923).[12]

Like in the case of his ealier studies of order structures such investigations led Hausdorff back to foundational questions of set theory. Already in the *Grundzüge* he had been able to show that certain Borel sets were either countable or of the cardinality of the continuum. In 1916 Hausdorff, and independently P. Alexandroff, could show that any Borel set in a separable metrical space is of cardinality $\aleph_0$ or of the continuum. That was an important step forward for a strategy proposed by G. Cantor to clarify the continuum hypothesis. Although this goal could not be achieved along this road, it led to the development of an extended field of investigation on the border region between set theory and analysis, now dealt with in *descriptive set theory*.[13]

When Hausdorff revised his *opus magnum* for a second edition in the later 1920s, he rewrote the parts on descriptive set theory and topological spaces completely, extending the first considerably and concentrating the second on metrical spaces. As other books on general set and general topology had appeared in the meantime, he omitted these parts; thus the socalled “second editon” was a completely new book on specialized topics of set theory (Hausdorff 1927).

Last years at Bonn

In 1921 Hausdorff had returned to Bonn university, now as a full professor and colleague of E. Study and (a little later) O. Toeplitz. After the rise to power of the Nazi regime, life and work conditions deterioriated steadily and more and more drastically for Hausdorff and other people of Jewish origin (F. Hausdorff had distached himself from religion during the 1890s, his wife had converted to protestantism). While he was still regularly emeritated in early 1935, his colleague O. Toeplitz was dismissed and left Nazi-Germany for Palestine shortly before the outbreak of the second World War. Hausdorff’s attempts for emigration came too late to be successful and his contacts to local mathematicians reduced essentially to one sensible and upright colleague, E. Bessel-Hagen.[14]

When Felix Hausdorff, his wife Charlotte and a sister of her were ordered to leave their house for a local internment regime in January 1942, they opted for suicide rather than suffering further persecution. At that time their (adult) daughter was living at Jena.

[7] (Purkert 2002)
[8] See (Epple/Herrlich e.a. 2002) and further commentaries in (Hausdorff Werke, II (2002), 745–772).
[9] (Chatterji 2002)
[10] (Hausdorff Werke, V (2005)), cf. (Hochkirchen 1999).
[11] Commentaries in (Hausdorff Werke, IV (2001), 44– 54,) and in (Brieskorn 1996).
[12] Cf. S. Chatterji’s commentaries in (Hausdorff Werke, IV (2001), 163– 171, 182–190).
[13] (Koepke/Kanovei 2002)
[14] (Neuenschwander 1996)

She could escape from deportation and managed to hide in the Harz region until the end of the war and the downfall of the Nazi regime.

The *Nachlass* and the *Hausdorff edition*

Hausdorff's voluminous *Nachlass* was handed over to a local friend of his. It survived the end of the war with only minor damages (Hausdorff n.d.). It now lies at the University library at Bonn and has been made accessible for research by a detailed catalogue [http://www.aic.uni-wuppertal.de/fb7/hausdorff/findbuch.asp].

An interdisciplinary group of 21 persons from four countries, under the direction of E. Brieskorn and W. Purkert (Bonn), is preparing a collected edition of the scientifc, literary, and philosophical works of Felix Hausdorff in nine volumes (Hausdorff Werke). This edition has been generously supported by the *Deutsche Forschungs Gemeinschaft* and the *Akademie der Wissenschaften Nordrhein-Westfalen*. It is now run as a long-term project of the latter. Four volumes have already been published or are in preparation for immediate publication; the remaining five are scheduled to follow during the next few years [http://www.aic.uni-wuppertal.de/fb7/hausdorff/baende.htm].

Call for support in retrieving Hausdorff correspondence

Volume IX of the edition will contain the available scientific, literary, and personal correspondence of F. Hausdorff. Very informative parts of the exchange between P. Alexandroff and F. Hausdorff have been preserved, and also letters of Mongré/Hausdorff to the Nietzsche archive and to literary intellectuals of the turn to the 20th century; but much of the correspondence seems to be lost. Most of what is known, is due to patient and laborious collecting activities of E. Brieskorn, the driving force behind the editorial project.

Readers of this note who know about letters from or to Felix Hausdorff are heartily invited to get into contact with the editorial office and to communicate their findings to its scientific coordinator:

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