Analysis on Singular Complex Spaces

Habilitationsschrift

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1 Introduction

The Cauchy-Riemann equations – usually expressed by use of the Cauchy-Riemann operator $\overline{\partial}$ – play a central role in Complex Analysis. One can even say that they are at the heart of the theory as they can be used to define holomorphic functions. Many of the oldest, classical problems in Complex Analysis are closely related to the Cauchy-Riemann equations, e.g. the fundamental Cousin problems are equivalent to the solution of some $\overline{\partial}$-equations.

Let M be a Hermitian (complex) manifold and $G \subset\subset M$ a relatively compact domain in M with boundary bG. Then there exist deep and multifaceted relations between the solvability and regularity of the Cauchy-Riemann equations

$$\overline{\partial} u = f \tag{1}$$

on G on the one hand and the geometry of the domain G and its boundary bG on the other hand. Here, various notions of convexity play a crucial role. We will mention some important examples for this phenomenon later on.

Closely related to the $\overline{\partial}$-operator are the complex Laplacian and the complex Green operator which are essential for Complex Analysis, as well. The properties of these operators are also tightly connected to geometric properties of the underlying spaces. We denote by

$$\overline{\partial}_w : L^2_*(G) \to L^2_*(G)$$

the $\overline{\partial}$-operator in the sense of distributions on L^2-forms.[1] It is a densely defined closed operator, and so there is the Hilbert space adjoint operator $\overline{\partial}_w^*$ which is again closed and densely defined. Now, a complex Laplace operator can be defined as

$$\Box = \overline{\partial}_w \overline{\partial}_w^* + \overline{\partial}_w^* \overline{\partial}_w.$$

The question whether this operator is invertible in an appropriate sense is known as the $\overline{\partial}$-Neumann problem. The resulting complex Green operator $N = \Box^{-1}$ is usually called the $\overline{\partial}$-Neumann operator.

To illustrate the deep connection between analysis and geometry on complex manifolds, we will now discuss briefly three fundamental results of Complex Analysis involving the operators $\overline{\partial}$, $\Box$ and N, namely the Theorem of Hodge, Kohn's solution of the $\overline{\partial}$-Neumann problem [K2, K3] and Hörmander's L^2-estimates [H6]. We will later consider the question what happens to these milestones of the theory if singularities come into play, i.e. if we study the situation on singular complex spaces instead of complex manifolds.

Let us first consider the Theorem of Hodge. If M is a compact complex manifold, then the complex Laplace operator $\Box$ is a self-adjoint elliptic operator with closed range, and this yields the orthogonal decomposition

$$L^2_*(M) = \ker \Box \oplus \operatorname{range} \Box. \tag{2}$$

[1] We use the notation $\overline{\partial}_w$ for this operator to indicate that this is the $\overline{\partial}$-operator in a weak sense.

We can thus define the $\overline{\partial}$-Neumann operator

$$N : L^2_*(M) \to \operatorname{Dom}\Box \subset L^2_*(M) \tag{3}$$

as follows: let $Nu = 0$ for $u \in \ker\Box$, and let Nu be the unique preimage of u orthogonal to $\ker\Box$ for $u \in \operatorname{range}\Box$.[2] Ellipticity of $\Box$ implies that the groups

$$\mathcal{H}^{p,q}(M) := \ker\Box \cap L^2_{p,q}(M) \cong H^{p,q}(M) \cong H^q(M, \Omega^p) \tag{4}$$

are of finite dimension. Here, we denote by $\mathcal{H}^{p,q}(M)$ the $\Box$-harmonic (p,q)-forms, by $H^{p,q}(M)$ the Dolbeault cohomology, and by $H^q(M,\Omega^p)$ the q-th cohomology group of the sheaf of germs of holomorphic p-forms on M. The last equivalence relies on the Theorem of Dolbeault. The $\overline{\partial}$-Neumann operator N gains two derivatives due to ellipticity of the Laplacian $\Box$, and so it is a compact operator by the Rellich embedding theorem. The spectral theorem implies that $L^2_{p,q}(M)$ has an orthonormal basis consisting of eigenforms of $\Box$, the eigenvalues are non-negative, occur with finite multiplicity and do not have an accumulation point.

The question to what extend this nice situation can be discovered also on open manifolds is called the $\overline{\partial}$-Neumann problem. If we consider the problem on a relatively compact domain G with boundary bG in an arbitrary Hermitian manifold M, then we have to realize that the situation is much more complicated for we obtain a non-coercive boundary value problem. We need a geometric condition on the boundary to ensure that the $\Box$-operator has closed range. Here, pseudoconvexity turned out to be a suitable concept.

Assume first that bG is smooth and strongly pseudoconvex, i.e. that it is given as the zero set of a smooth strictly plurisubharmonic function such that the gradient is not vanishing in a neighborhood of bG.[3] Kohn showed that the range of the complex Laplacian $\Box$ is closed under this assumption, $\Box$ is self-adjoint, (2) and (4) are valid as before. Kohn obtained subelliptic estimates of order 1 for the complex Laplacian at the boundary, so that the $\overline{\partial}$-Neumann operator gains exactly one derivative. Again, the space of $\Box$-harmonic (p,q)-forms is of finite dimension and N is compact on (p,q)-forms for $q \geq 1$.

Another crucial development of Complex Analysis in the 1960s is Hörmander's L^2-theory for the $\overline{\partial}$-operator which (amongst other things) provides another approach to the $\overline{\partial}$-Neumann problem that does not involve subelliptic estimates. Hörmander avoided the regularity problem at the boundary by an artful use of suitable weight functions in his L^2-estimates. So, let us drop all conditions on the boundary of G and assume instead that G itself is pseudoconvex, i.e. that there exists a continuous strictly plurisubharmonic exhaustion function for G. Under this assumption, Hörmander showed that the $\overline{\partial}$-equation is solvable in the L^2_{loc}-sense and in the C^∞-sense on G in all degrees, i.e. that $H^q(G,\Omega^p) = 0$ for all $q \geq 1$ and all p.

[2] Generally, the $\overline{\partial}$-Neumann operator N can be defined in the same manner even if the range of $\Box$ is not closed by setting $Nu = 0$ for $u \in (\operatorname{range}\Box)^\perp$. But then N is not a bounded operator. N defined in that way is bounded exactly if $\Box$ has closed range.

[3] A C^2-function is plurisubharmonic (or strictly plurisubharmonic) if the complex Hessian has only non-negative (or only positive) eigenvalues. From this one can deduce that strongly pseudoconvex domains are locally biholomorphic to strongly convex domains.

If we assume that G is contained in a Stein manifold[4], for example if G is an open set in $\mathbb{C}^n$, then the converse is also true. In that case, vanishing of the cohomology groups $H^q(G, \mathcal{O}) = H^q(G, \Omega^0)$ for all $q \geq 1$ implies that G itself is Stein (see [L1]). But it is not difficult to see that Stein manifolds possess a continuous strictly plurisubharmonic exhaustion function. Hence, a domain $G \subset\subset M$ in a Stein manifold M is pseudoconvex exactly if the $\overline{\partial}$-equation is solvable in the L^2_{loc}-category on G, and this is the case exactly if G itself is Stein.[5]

After having illustrated the interplay between geometry and analysis on complex manifolds by these important examples, we will discuss in this thesis the question to what extend their relations are understood on singular complex spaces. Singular complex spaces occur naturally as the zero set of holomorphic functions and are thus an important object in many areas of mathematics. But, whereas geometric and algebraic methods – particularly the theory of coherent analytic sheaves – are very well developed on singular spaces, the most analytic tools are still missing. For instance, a singular complex space is Stein exactly if it is pseudoconvex, i.e. if there exists a continuous strictly plurisubharmonic exhaustion function (see [N]). But, in contrast to the situation on manifolds which we discussed above, no analytic characterization involving some $\overline{\partial}$-equation is known.

Starting from the milestones of Complex Analysis on manifolds mentioned above, we will discuss in this thesis some of the recent developments in an area which we may call "Analysis on Singular Complex Spaces" with a special focus on the contribution of the author. That comprises the joint papers with E. Zeron [RZ1, RZ2] and M. Colţoiu [CR] as well as the articles [R3, R4, R5, R6, R7, R8]. Much material also originates from the two preprints [R9, R10]. Some results stem from the joint preprint with N. Øvrelid [OR].

We tried to arrange this exposition self-contained in the sense that the reader should be able to understand all the principles and methods without consulting the references, but we skip most of the details. A special focus is placed on the L^2-theory for the $\overline{\partial}$-operator which was developed recently in the two preprints [R9, R10]. Here, we give more details.

With all my heart I thank my wife Julia for her great interest in my work, her loving support and ongoing encouragement in hard times.

[4]We call a complex space X, particularly a complex manifold X, Stein if Cartan's Theorem B holds, i.e. if $H^q(X, \mathcal{F}) = 0$ for any coherent analytic sheaf $\mathcal{F}$ on X and all $q \geq 1$. A complex space X is Stein exactly if X is holomorphically convex and holomorphically spreadable. Here, we say that X is holomorphically convex if for each sequence $\{x_k\}_{k\in\mathbb{N}} \subset X$ without accumulation point in X there exists a holomorphic function $f \in \mathcal{O}(X)$ such that f is unbounded on $\{x_k\}$. X is said to be holomorphically spreadable if for each point $x_0 \in X$ there exist finitely many holomorphic functions $f_1, ..., f_l \in \mathcal{O}(X)$ such that x_0 is an isolated point in the common zero set $\{x \in X : f_1(x) = ... = f_l(x) = 0\}$. For more details, we refer to the works of Cartan, Serre, Grauert and Remmert (see [GR1]).

[5]In general, the implication that pseudoconvex spaces are holomorphically convex is called Levi problem.

2 L^2-theory for the $\overline{\partial}$-operator

The L^2-theory for the $\overline{\partial}$-operator is exceptionally important, on the one hand because the Hilbert space machinery is a very mighty tool which enables a far-reaching theory, on the other hand because L^2-methods have led to fundamental advanes in many areas. For instance, Hörmander's L^2-theory does not only yield a solution of the Levi problem, but also the Ohsawa-Takegoshi L^2-extension theorem.[6] The Ohsawa-Takegoshi L^2-extension and its variations in turn play a central role in many further results, like e.g. Siu's proof of the analyticity of the level sets of Lelong numbers [S7] or the invariance of plurigenera [S8]. These are good examples of how analytic methods on manifolds have led to fundamental advances in geometry, and so there is a hope that a suitable L^2-theory for the $\overline{\partial}$-operator on singular complex spaces will also lead to some new developments.

In the present exposition, we will describe an L^2-theory for the $\overline{\partial}$-operator for $(0,q)$ and (n,q)-forms on complex Hermitian spaces[7] of pure dimension n with isolated singularities.[8] This theory was developed in [R9, R10]. The general philosophy is to use a resolution of singularities to obtain a regular model of the L^2-cohomology.

In general, it seems that studying the $\overline{\partial}$-equation for (n,q)-forms is the most interesting case on singular complex spaces so that we can live with the fact that not too much is known about (p,q)-forms for $0 < p < n$. The reason is that many of the interesting concepts and problems in algebraic or complex geometry are linked to canonical sheaves. Let us just mention L^2-extension and vanishing theorems, invariance of plurigenera, the Kodaira dimension, the abundance conjecture, finite generation of the canonical ring.

2.1 The $\overline{\partial}$-operator on singular complex spaces

2.1.1 L^2-cohomology of singular complex spaces

When we consider the $\overline{\partial}$-operator on singular complex spaces, the first problem is to define an appropriate Dolbeault complex in the presence of singularities. It turns out that it is very fruitful to investigate the $\overline{\partial}$-operator in the L^2-category (simply) on the complex manifold consisting of the regular points of a complex space. One reason lies in Goresky and MacPherson's notion of intersection (co-)homology (see [GM1, GM2]) and the conjecture of Cheeger, Goresky and MacPherson, which states that the L^2-deRham cohomology on the regular part of a projective variety Y (with respect to the restriction of the Fubini-Study metric and the exterior derivate in the sense of distributions) is naturally isomorphic to the intersection cohomology of

[6] Here, various further developments of Hörmander's L^2-estimates come into play, like the twisted L^2-estimates of Ohsawa–Takegoshi [OT], Donelly–Fefferman [DF] or McNeal [McN].

[7] We call a reduced and paracompact complex space X Hermitian if the regular part $X - \operatorname{Sing} X$ carries a Hermitian metric which is locally the restriction of a Hermitian metric in some complex number space where X is represented locally. In other words, the metric h on $X - \operatorname{Sing} X$ extends smoothly to the singular set. We write (X, h) for a Hermitian space with metric h. For example, a projective variety is Hermitian with the the restriction of the Fubini-Study metric.

[8] Some of the results are also for arbitrary singularities, but the picture is almost complete for spaces with only isolated singularities.

middle perversity $IH^*(Y)$ of Y:

Conjecture 2.1. (Cheeger–Goresky–MacPherson [CGM])
Let $Y \subset \mathbb{CP}^N$ be a projective variety. Then there is a natural isomorphism

$$H^k_{(2)}(Y - \operatorname{Sing} Y) \cong IH^k(Y). \tag{5}$$

The early interest in this conjecture was motivated in large parts by the hope that one could then use the natural isomorphism (5) and a classical Hodge decomposition $H^k = \oplus H^{p,q}$ for $H^k_{(2)}(Y - \operatorname{Sing} Y)$ to put a pure Hodge structure on the intersection cohomology of Y. Ohsawa proved the conjecture of Cheeger, Goresky and MacPherson under the assumption that Y has only isolated singularities (see [O3]), while it is still open in general. Ohsawa's proof depends on an earlier work of Saper who constructed a complete Kähler metric on $Y - \operatorname{Sing} Y$ such that the L^2-deRham cohomology in this complete metric equals the intersection cohomology of Y (see [S2]). Ohsawa used a family of such complete metrics which degenerate to the incomplete restriction of the Fubini-Study metric on $Y - \operatorname{Sing} Y$, and showed that the L^2-cohomology is stable under the limit process by the use of strong L^2-estimates going back to Donnelly and Fefferman (see [DF]). The principle of this method is a very important tool for analysis on singular complex spaces.

Eventually, using methods not directly related to L^2-cohomology (D-modules), Saito established a pure Hodge decomposition in the sense of Deligne for the intersection cohomology of singular varieties ([S1]). However, it still seems interesting to get a classical Hodge decomposition on $H^k_{(2)}(Y - \operatorname{Sing} Y)$ and to investigate the relation to Saito's construction via an isomorphism (5). This was done in special cases by Zucker [Z2] and Ohsawa [O4]. In the case of isolated singularities, a decomposition of $H^k_{(2)}(Y - \operatorname{Sing} Y)$ in terms of Dolbeault cohomology groups was in fact established by Pardon and Stern (in [PS2]). Combined with Ohsawa's solution of the Cheeger-Goresky-MacPherson conjecture, this gives a Hodge decomposition for the intersection cohomology on such varieties. See also the work of Nagase [N] and Fox and Haskell [FH] for related decomposition results on projective normal complex surfaces.

The conjecture of Cheeger, Goresky and MacPherson is closely related to the (more famous) Zucker conjecture which states that the L^2-cohomology of a Hermitian locally symmetric space is isomorphic to the intersection cohomology of middle perversity of its Baily-Borel compactification [Z1]. Zucker's conjecture was proved independently by Looijenga [L3] and by Saper and Stern [SS].

Another reason why it is interesting to study the L^2-cohomology of the $\overline{\partial}$-operator on the regular part of the variety is the arithmetic genus of complex varieties. If M is a compact complex manifold of dimension n, the arithmetic genus

$$\chi(M) := \sum (-1)^q \dim H^{n,q}(M) \tag{6}$$

is a birational invariant of M. The conjectured extension of the classical Hodge decomposition to projective varieties led MacPherson also to ask wether the arithmetic genus $\chi(M)$ extends to a birational invariant of all projective varieties (see [M]). We may formulate MacPherson's question slightly more general, extending it to Hermitian complex spaces:

Conjecture 2.2. *If X is a compact Hermitian space of pure dimension n, then*

$$\chi_{(2)}(X - \operatorname{Sing} X) := \sum_{q=0}^{n} (-1)^q \dim H^{n,q}_{(2)}(X - \operatorname{Sing} X) = \chi(M),$$

where $\pi : M \to X$ is any resolution of singularities.

Due to the incompleteness of the metric on $X - \operatorname{Sing} X$, one has to be very careful with the definition of L^2-Dolbeault cohomology groups for they depend on the choice of some kind of boundary condition for the $\overline{\partial}$-operator. We will explain that now more precisely and return to the arithmetic genus later.

2.1.2 Closed L^2-extensions of the $\overline{\partial}$-operator and Serre duality

If N is any Hermitian complex manifold of dimension n, let

$$\overline{\partial}_{cpt} : A^{p,q}_{cpt}(N) \to A^{p,q+1}_{cpt}(N) \tag{7}$$

be the $\overline{\partial}$-operator on smooth forms with compact support in N. We may consider $\overline{\partial}_{cpt}$ as a densely defined operator on L^2-forms

$$\overline{\partial}_{cpt} : L^2_{p,q}(N) \to L^2_{p,q+1}(N).$$

This operator then is graph-closable and has various closed extensions. The two most important are the maximal closed extension

$$\overline{\partial}_{max} : L^2_{p,q}(N) \to L^2_{p,q+1}(N),$$

that is the $\overline{\partial}$-operator in the sense of distributions, and the minimal closed extension

$$\overline{\partial}_{min} : L^2_{p,q}(N) \to L^2_{p,q+1}(N)$$

given by the closure of the graph of $\overline{\partial}_{cpt}$ in $L^2_{p,q}(N) \times L^2_{p,q+1}(N)$.[9]

We denote by $H^{p,q}_{max}(N)$ the L^2-Dolbeault cohomology on N with respect to the maximal closed extension $\overline{\partial}_{max}$, and by $H^{p,q}_{min}(N)$ the L^2-Dolbeault cohomology with respect to the minimal closed extension $\overline{\partial}_{min}$.

To get an impression of the difference between the two kinds of Dolbeault cohomology, consider the following example: If D is a relatively compact domain with smooth strongly pseudoconvex boundary in a complex manifold, then $H^{p,q}_{max}(D)$ is canonically isomorphic to the C^∞-Dolbeault cohomology $H^q(D, \Omega^p)$ if $q \geq 1$. This

[9] We should mention that there are other common notations for $\overline{\partial}_{max}$ and $\overline{\partial}_{min}$ in the literature. In [R9], we denoted the maximal closed extension by $\overline{\partial}_w$ and the minimal closed extension by $\overline{\partial}_s$. In that case, the subscript refers to $\overline{\partial}_w$ as the $\overline{\partial}$-operator in a weak sense and to $\overline{\partial}_s$ as the $\overline{\partial}$-operator in a strong sense. In the present exposition, we will later use the notation $\overline{\partial}_w$ for the localized version of $\overline{\partial}_{max}$ and $\overline{\partial}_s$ for a localized version of $\overline{\partial}_{min}$.

Note that Pardon and Stern use the notation $\overline{\partial}_D$ for $\overline{\partial}_{min}$ and $\overline{\partial}_N$ for $\overline{\partial}_{max}$. Their notation refers to some kind of Dirichlet respectively Neumann boundary conditions (see [PS1]). It is interesting to study under which circumstances the extensions coincide. See the work of Grieser and Lesch [GL], Pardon and Stern [PS2], or Brüning and Lesch [BL] for this topic.

follows from the well-known fact that in this special situation, the L^2-Dolbeault cohomology is canonically isomorphic to the L^2_{loc}-Dolbeault cohomology which in turn is canonically isomorphic to the C^∞-cohomology (cf. [LM], VIII.4).

On the other hand, we can use Serre duality and the L^2-version of Serre duality to deduce that also $H^{n-p,n-q}_{min}(D)$ and $H^{n-q}_{cpt}(D,\Omega^{n-p})$ are canonically isomorphic for $q>0$ in this nice situation. We will explain that more precisely. While it is clear that there is a non-degenerate pairing

$$H^q(D,\Omega^p)\times H^{n-q}_{cpt}(D,\Omega^{n-p})\to\mathbb{C}$$

by the classical Serre duality for the $\overline{\partial}$-operator has closed range on D (cf. [S6]), we will now describe the L^2-Serre duality between $H^{p,q}_{max}(D)$ and $H^{n-p,n-q}_{min}(D)$ because this is a very important tool in the study of the $\overline{\partial}$-operator in the L^2-sense on singular complex spaces.[10]

Analogous to (7), let

$$\vartheta_{cpt}=-\overline{*}\overline{\partial}_{cpt}\overline{*}:A^{p,q+1}_{cpt}(N)\to A^{p,q}_{cpt}(N) \tag{8}$$

be the (formal) adjoint operator to the $\overline{\partial}$-operator acting on smooth forms with compact support. Here, $*$ is the Hodge-$*$-operator and $\overline{*}$ its conjugated version. Then we define as before the maximal closed extension

$$\vartheta_{max}:L^2_{p,q+1}(N)\to L^2_{p,q}(N),$$

i.e. the ϑ-operator in the sense of distributions, and the minimal closed extension

$$\vartheta_{min}:L^2_{p,q+1}(N)\to L^2_{p,q}(N)$$

given as the closure of the graph of ϑ_{cpt}. Since $\overline{\partial}_{max}$ is the $\overline{\partial}$-operator in the sense of distributions, we have that $\overline{\partial}_{max}=\vartheta^*_{cpt}$ by definition, and this yields

$$\overline{\partial}^*_{max}=\vartheta^{**}_{cpt}=\vartheta_{min}=-\overline{*}\overline{\partial}_{min}\overline{*}. \tag{9}$$

Similarly, $\vartheta_{max}=\overline{\partial}^*_{cpt}$ implies

$$\overline{\partial}^*_{min}=(\overline{\partial}^{**}_{cpt})^*=(\vartheta^*_{max})^*=\vartheta_{max}=-\overline{*}\overline{\partial}_{max}\overline{*}. \tag{10}$$

The two relations (9) and (10) together imply that the operator $\overline{*}$ induces an isomorphism between the space of $\overline{\partial}_{max}$-harmonic forms

$$\mathcal{H}^{p,q}_{max}(N):=\{f\in\operatorname{Dom}\overline{\partial}_{max}\cap\operatorname{Dom}\overline{\partial}^*_{max}:\overline{\partial}_{max}f=\overline{\partial}^*_{max}f=0\}$$

and the $\overline{\partial}_{min}$-harmonic forms of conjugate degree

$$\mathcal{H}^{n-p,n-q}_{min}(N):=\{f\in\operatorname{Dom}\overline{\partial}_{min}\cap\operatorname{Dom}\overline{\partial}^*_{min}:\overline{\partial}_{min}f=\overline{\partial}^*_{min}f=0\}.$$

If the $\overline{\partial}$-operators under consideration have closed range such that L^2-cohomology classes have unique harmonic representatives, one can deduce the following version of L^2-Serre duality (see [PS1], Proposition 1.3 or [R9], Theorem 2.3):

[10] The relevance of such kinds of duality has been also realized by Chakrabarti and Shaw [CS].

Theorem 2.3. *Let N be a Hermitian complex manifold of dimension n and let $0 \leq p, q \leq n$. Assume that the $\overline{\partial}$-operators in the sense of distributions*

$$\overline{\partial}_{max}: \quad L^2_{p,q-1}(N) \to L^2_{p,q}(N), \tag{11}$$

$$\overline{\partial}_{max}: \quad L^2_{p,q}(N) \to L^2_{p,q+1}(N) \tag{12}$$

both have closed range (with the usual assumptions for $q = 0$ or $q = n$). Then there exists a non-degenerate pairing

$$\{\cdot, \cdot\} : H^{p,q}_{max}(N) \times H^{n-p,n-q}_{min}(N) \to \mathbb{C}$$

given by

$$\{\eta, \psi\} := \int_N \eta \wedge \psi.$$

Thus, it is always essential to study the closed range property of the $\overline{\partial}$-operators under consideration on singular complex spaces.[11] This problem can be treated in the spirit of our general philosophy to use a resolution of singularities to obtain a regular model for the L^2-cohomology. If this is somehow successful, one can then deduce properties of the $\overline{\partial}$-operator on singular spaces from well-known properties of the $\overline{\partial}$-operator on the desingularization.

This philosophy is the main theme of the joint paper with N. Øvrelid [OR] from where the following example is taken: Let X be a Hermitian complex space of pure dimension n with only isolated singularities. Let

$$\pi : M \to X$$

be a resolution of singularities which exists due to Hironaka [H5], and let σ be any (positive definite) Hermitian metric on M. We denote by $L^2_{p,q}$ the spaces of L^2-forms on $\operatorname{Reg} X = X - \operatorname{Sing} X$, and by $L^{2,\sigma}_{p,q}$ the spaces of L^2-forms on M with respect to σ. Let $\Omega \subset\subset X$ be a relatively compact open subset of X such that the boundary of Ω does not intersect the singular set of X, $b\Omega \cap \operatorname{Sing} X = \emptyset$. Let $\Omega^* := \Omega - \operatorname{Sing} X$ and $\Omega' := \pi^{-1}(\Omega)$.

So, the resolution of singularities has the following nice effect: If the original domain Ω has a 'good' boundary $b\Omega$, then Ω' is a domain in a complex manifold with the same 'good' boundary. One might consider for example a domain Ω with a strongly pseudoconvex boundary, or assume that X is a compact space and $\Omega = X$ (no boundary at all). In both cases we know that the $\overline{\partial}$-equation has compact solution operators on Ω' (modulo the obstructions to solving the equation), and that the $\overline{\partial}$-Neumann operator exists and is compact. It is thus interesting to relate properties of the $\overline{\partial}$-operator on Ω^* (which have to be studied) to properties of the $\overline{\partial}$-operator on Ω' (which are well understood):

Theorem 2.4. ([OR], Theorem 1.1) *Let $q \geq 1$ and either $p + q \neq n$ or $(p, q) = (0, n)$. Under the assumptions above, the $\overline{\partial}$-operator in the sense of distributions*

$$\overline{\partial}_{max} : L^2_{p,q-1}(\Omega^*) \to L^2_{p,q}(\Omega^*)$$

[11] Besides that, recall that the closed range property of appropriate $\overline{\partial}$-operators is also essential for boundedness of the $\overline{\partial}$-Neumann operator.

has closed range of finite codimension in $\ker \overline{\partial} \subset L^2_{p,q}(\Omega^*)$ *exactly if the* $\overline{\partial}$*-operator in the sense of distributions*

$$\overline{\partial}^M_{max} : L^{2,\sigma}_{p,q-1}(\Omega') \to L^{2,\sigma}_{p,q}(\Omega')$$

has closed range of finite codimension in $\ker \overline{\partial}^M_{max} \subset L^{2,\sigma}_{p,q}(\Omega')$.

If this is the case, then there exists a compact $\overline{\partial}$*-solution operator*

$$\mathbf{S} : Im\, \overline{\partial}_{max} \subset L^2_{p,q}(\Omega^*) \to L^2_{p,q-1}(\Omega^*)$$

exactly if there exists a compact $\overline{\partial}$*-solution operator*

$$\mathbf{S}_M : Im\, \overline{\partial}^M_{max} \subset L^{2,\sigma}_{p,q}(\Omega') \to L^{2,\sigma}_{p,q-1}(\Omega').$$

Our main tools in the proof of Theorem 2.4 are the existence of $\overline{\partial}$-solution operators with some gain of regularity at isolated singularities due to Fornæss–Øvrelid–Vassiliadou (see [FOV2], Theorem 1.1 and Theorem 1.2) and a characterization of precompactness in the space of L^2-forms on arbitrary Hermitian manifolds which was given in [R8], Theorem 2.5. This characterization is used to show that the $\overline{\partial}$-solution operators of Fornæss–Øvrelid–Vassiliadou are actually compact. Other ingredients are Hironaka's resolution of singularities and Kohn's subelliptic estimates. For the details, see [OR], Theorem 1.1., and chapter 3 of this exposition.

We have seen that the L^2-Dolbeault cohomology with respect to the $\overline{\partial}_{max}$ and the $\overline{\partial}_{min}$-operator, respectively, do not coincide in general by considering the example $H^{p,q}_{max}(D) \cong H^q(D, \Omega^p)$ and $H^{p,q}_{min}(D) \cong H^q_{cpt}(D, \Omega^p)$ for $0 < q < n$, when D is a smoothly bounded strongly pseudoconvex domain in a complex manifold. In this case, the boundary bD of D is of real codimension 1. Since the $\overline{\partial}_{min}$-operator differs from the $\overline{\partial}_{max}$-operator by a kind of boundary condition at bD, which leads to the fact that the $\overline{\partial}_{min}$-cohomology coincides with the cohomology with compact support in the example described above, one may ask wether this boundary condition looses its effect if the boundary is of higher codimension. This is an interesting phenomenon which we will discuss now briefly.

Let M be a compact Hermitian manifold of dimension n and $A \subset M$ a proper closed analytic subset. Then the $\overline{\partial}$-equation in the L^2-sense of distributions extends over the analytic set A, i.e. let $f \in L^2_{p,q}(M - A)$ and $g \in L^2_{p,q+1}(M - A)$ such that

$$\overline{\partial} f = g$$

on $M - A$ in the sense of distributions (i.e. $\overline{\partial}_{max} f = g$). Let $\widetilde{f}$ and $\widetilde{g}$ be the trivial extensions of f and g to M. Then:

$$\overline{\partial} \widetilde{f} = \widetilde{g}$$

in the sense of distributions on M (i.e. $\overline{\partial}_{max} \widetilde{f} = \widetilde{g}$). A proof in a more general setting which deals also with L^r-forms, $1 \leq r \leq \infty$, and exceptional sets of arbitrary codimension can be found in [R4], Theorem 3.2. This means that there is a canonical isomorphism (induced by trivial extension of forms)

$$H^{p,q}_{max}(M-A) \cong H^{p,q}_{max}(M) \tag{13}$$

for all $0 \leq p,q \leq n$. Since M is a compact manifold, the groups in (13) are of finite dimension. But if $H^{p,q}_{max}(M-A)$ is of finite dimension, then the operator $\overline{\partial}_{max}$ has closed range on $M-A$ by a Banach space argument which can be found in the book of Henkin and Leiterer [HL2], Appendix 2.4. This is true in all degrees $0 \leq p,q \leq n$. So, there is a canonical isomorphism

$$H^{p,q}_{max}(M-A) \cong H^{n-p,n-q}_{min}(M-A) \tag{14}$$

by the L^2-duality Theorem 2.3. On the other hand,

$$H^{p,q}_{max}(M) \cong H^{n-p,n-q}_{max}(M), \tag{15}$$

because there is clearly no difference between the $\overline{\partial}_{max}$ and the $\overline{\partial}_{min}$-cohomology on a compact manifold. But another application of (13) (for forms of degree $(n-p,n-q)$ tells us that also

$$H^{n-p,n-q}_{max}(M) \cong H^{n-p,n-q}_{max}(M-A). \tag{16}$$

Combining the isomorphism (14), (15) and (16), we see that in fact

$$H^{n-p,n-q}_{min}(M-A) \cong H^{n-p,n-q}_{max}(M-A)$$

for all $0 \leq p,q \leq n$.

However, this phenomenon does not persist if we replace M by a singular Hermitian space X and remove the singular set (i.e. consider the different types of the $\overline{\partial}$-equation on $X - \operatorname{Sing} X$). If X is a compact Hermitian space of pure dimension n with only isolated singularities, then actually

$$H^{p,q}_{max}(X - \operatorname{Sing} X) \cong H^{p,q}_{min}(X - \operatorname{Sing} X)$$

for all $0 \leq p,q \leq n$ such that $|p+q-n| > 1$ (see [PS2], Corollary 2.42), but the groups are not isomorphic for $|p+q-n| \leq 1$ as we shall see later in this exposition.[12]

Now that we have somewhat clarified the notion of L^2-Dolbeault cohomology on a singular space, we will return to discuss the arithmetic L^2-genus of singular Hermitian spaces. Our main tool and object to study is the canonical sheaf of Grauert and Riemenschneider.

[12]This phenomenon occurs also in other singular configurations like e.g. if one studies the $\overline{\partial}$-equation on positive currents (see [BS]). Then one can also define maximal and minimal closed extensions of the $\overline{\partial}$-operator and one sees that the operators and their cohomology do not coincide in general.

2.2 The canonical sheaf of Grauert–Riemenschneider $\mathcal{K}_X$

2.2.1 The arithmetic L^2-genus

When we are looking for a notion of an arithmetic L^2-genus which is birationally invariant, the right choice of L^2-cohomology groups to sum up are the cohomology groups $H^{n,q}_{max}$ with respect to the maximal closed extension of the $\overline{\partial}$-operator because these groups themselves are already birationally invariant:

Theorem 2.5. ([R9], Theorem 1.5) *Let X be a Hermitian compact complex space of pure dimension n, $\pi : M \to X$ any resolution of singularities and $0 \leq q \leq n$. Then there is a canonical isomorphism*

$$\pi^* : H^{n,q}_{max}(X - \operatorname{Sing} X) \xrightarrow{\cong} H^{n,q}_{(2)}(M) \tag{17}$$

induced by pull-back of forms under π.[13]

This is the unchallenged prototype to illustrate our general philosophy to use a resolution of singularities to obtain a regular model for the L^2-cohomology on a singular space. One can deduce immediately that the groups $H^{n,q}_{max}(X - \operatorname{Sing} X)$ are of finite dimension and that the operator $\overline{\partial}_{max}$ has closed range for (n, q)-forms on $X - \operatorname{Sing} X$ (as above, one can use the argument in [HL2], Appendix 2.4). The identities (9) and (10) tell us that the $\overline{\partial}_{min}$-operator has closed range on $(0, q)$-forms and that we can use the L^2-version of Serre duality, Theorem 2.3, to deduce that there are canonical isomorphisms

$$H^{0,q}_{(2)}(M) \xrightarrow{\cong} H^{0,q}_{min}(X - \operatorname{Sing} X) \tag{18}$$

for all $0 \leq q \leq n$. We will see later how these maps are induced by push-forward of forms. Note that Theorem 2.5 settles Conjecture 2.2 for the arithmetic L^2-genus defined as

$$\chi_{(2)}(X) := \sum_{q=0}^{n} (-1)^q \dim H^{n,q}_{max}(X - \operatorname{Sing} X).$$

Alternatively, we could take the alternating sum over the dimensions of the groups $H^{0,q}_{min}(X - \operatorname{Sing} X)$. We will now explain the statement of Theorem 2.5 more precisely and sketch the proof which contains many interesting features. The main tools are Hironaka's resolution of singularities [H5], the canonical sheaf of Grauert–Riemenschneider [GR3], Takegoshi's relative vanishing theorem for canonical sheaves [T1], and a local vanishing result which is based on results of Demailly [D1], Donelly–Fefferman [DF], Ohsawa [O2] and which is finally due to Pardon-Stern [PS1]. Pardon and Stern proved Theorem 2.5 for projective varieties in [PS1].

Let us first define the canonical sheaf of Grauert and Riemenschneider which is the key object that we need to study. It is the sheaf of germs of square-integrable n-forms which are holomorphic with respect to the localized version of the $\overline{\partial}$-operator in the sense of distributions which we will denote by $\overline{\partial}_w$ in the following.

[13] Since $\overline{\partial}_{max} = \overline{\partial}_{min}$ on M, we use the notation $H^{p,q}_{(2)}$ for any L^2-cohomology on M.

2.2.2 The weak $\overline{\partial}$-operator $\overline{\partial}_w$ and its L^2-complex

We recall some of the essential constructions from [R9]. Let (X,h) always be a (singular) Hermitian complex space of pure dimension n and $U \subset X$ an open subset. As indicated above, on a singular space, it is most fruitful to consider forms that are square-integrable up to the singular set. Hence, we will use the following concept of locally square-integrable forms:

$$L^{2,loc}_{p,q}(U) := \{f \in L^{2,loc}_{p,q}(U - \operatorname{Sing} X) : f|_K \in L^2_{p,q}(K - \operatorname{Sing} X)\ \forall K \subset\subset U\}.$$

It is easy to check that the presheaves given as

$$\mathcal{L}^{p,q}(U) := L^{2,loc}_{p,q}(U)$$

are already sheaves $\mathcal{L}^{p,q} \to X$. On $L^{2,loc}_{p,q}(U)$, we denote by

$$\overline{\partial}_w(U) : L^{2,loc}_{p,q}(U) \to L^{2,loc}_{p,q+1}(U)$$

the $\overline{\partial}$-operator in the sense of distributions on $U - \operatorname{Sing} X$ which is closed and densely defined. When there is no danger of confusion, we will simply write $\overline{\partial}_w$ for $\overline{\partial}_w(U)$. The subscript refers to $\overline{\partial}_w$ as an operator in a weak sense. Since $\overline{\partial}_w$ is a local operator, i.e.

$$\overline{\partial}_w(U)|_V = \overline{\partial}_w(V)$$

for open sets $V \subset U$, we can define the presheaves of germs of forms in the domain of $\overline{\partial}_w$,

$$\mathcal{C}^{p,q} := \mathcal{L}^{p,q} \cap \overline{\partial}_w^{-1} \mathcal{L}^{p,q+1},$$

given by

$$\mathcal{C}^{p,q}(U) = \mathcal{L}^{p,q}(U) \cap \operatorname{Dom} \overline{\partial}_w(U).$$

These are actually already sheaves because the following is also clear: If $U = \bigcup U_\mu$ is a union of open sets, $f_\mu = f|_{U_\mu}$ and

$$f_\mu \in \operatorname{Dom} \overline{\partial}_w(U_\mu),$$

then

$$f \in \operatorname{Dom} \overline{\partial}_w(U) \quad \text{ and } \quad \big(\overline{\partial}_w(U) f\big)|_{U_\mu} = \overline{\partial}_w(U_\mu) f_\mu.$$

Moreover, it is easy to see that the sheaves $\mathcal{C}^{p,q}$ admit partitions of unity, and so we obtain fine sequences

$$\mathcal{C}^{p,0} \xrightarrow{\overline{\partial}_w} \mathcal{C}^{p,1} \xrightarrow{\overline{\partial}_w} \mathcal{C}^{p,2} \xrightarrow{\overline{\partial}_w} \dots \tag{19}$$

We will see later, when we deal with resolution of singularities, that

$$\mathcal{K}_X := \ker \overline{\partial}_w \subset \mathcal{C}^{n,0}$$

is just the canonical sheaf of Grauert and Riemenschneider because the L^2-property of $(n,0)$-forms remains invariant under modifications of the metric.

The $L^{2,loc}$-Dolbeault cohomology with respect to the $\overline{\partial}_w$-operator on an open set $U \subset X$ is by definition the cohomology of the complex (19) which is denoted by $H^q(\Gamma(U, \mathcal{C}^{p,*}))$. The cohomology with compact support is $H^q(\Gamma_{cpt}(U, \mathcal{C}^{p,*}))$. Note that this is the cohomology of forms with compact support in U, not with compact support in $U - \operatorname{Sing} X$.

It is clearly interesting to study wether the sequence (19) is exact, which is well-known to be the case in regular points of X. In singular points, the situation is quite complicated for forms of arbitrary degree and not completely understood. However, the $\overline{\partial}_w$-equation is locally solvable in the L^2-sense at arbitrary singularities for forms of degree (n,q), $q > 0$:

Theorem 2.6. (Pardon-Stern [PS1], Proposition 2.1) *Let X be a Hermitian complex space of pure dimension n. Then*

$$0 \to \mathcal{K}_X \hookrightarrow \mathcal{C}^{n,0} \xrightarrow{\overline{\partial}_w} \mathcal{C}^{n,1} \xrightarrow{\overline{\partial}_w} \mathcal{C}^{n,2} \xrightarrow{\overline{\partial}_w} \dots \longrightarrow \mathcal{C}^{n,n} \to 0 \tag{20}$$

is a fine resolution. For an open set $U \subset X$, it follows that

$$H^q(U, \mathcal{K}_X) \cong H^q(\Gamma(U, \mathcal{C}^{n,*})) \ , \ H^q_{cpt}(U, \mathcal{K}_X) \cong H^q(\Gamma_{cpt}(U, \mathcal{C}^{n,*})).$$

If X has only isolated singularities, Fornæss–Øvrelid–Vassiliadou showed that the $\overline{\partial}_w$-equation is locally solvable in the L^2-sense for forms of degree (p,q) with $p + q > n$ (see [FOV2], Theorem 1.2).

The main idea for the proof of Theorem 2.6 is as follows. Locally, one can approximate the incomplete metric on $X - \operatorname{Sing} X$ by a sequence of complete Kähler metrics for which one already knows the local vanishing result by a theorem of Donelly and Fefferman [DF]. One obtains a sequence of solutions with a uniform L^2-bound on compact subsets of $X - \operatorname{Sing} X$. By taking the weak limit, we get a solution with an L^2-bound in the original metric. This strategy was used before by Ohsawa in the case when X has only isolated singularities [O2], but also appears in an earlier paper of Demailly [D1]. For the details, we refer to [PS1], Proposition 2.1, or to section 3 and 4 in [R9], where also a generalization to (n,q)-forms with values in a semi-positive line bundle can be found.

A similar local L^2-vanishing result for (n,q)-forms on positive closed currents of bidimension (n,n) has been discovered by Berndtsson and Sibony [BS].

2.2.3 Resolution of $(X, \mathcal{K}_X)$ and its L^2-cohomology

Let $\pi : M \to X$ be a resolution of singularities (which exists due to Hironaka [H5]), i.e. a proper holomorphic surjection such that

$$\pi|_{M-E} : M - E \to X - \operatorname{Sing} X$$

is biholomorphic, where $E = |\pi^{-1}(\operatorname{Sing} X)|$ is the exceptional set. We may assume that E is a divisor with only normal crossings, i.e. the irreducible components of E are regular and meet complex transversely, but we do not need that for the moment. Let $Z := \pi^{-1}(\operatorname{Sing} X)$ be the unreduced exceptional divisor. For the topic of desingularization, we refer to [AHL], [BM] and [H2].

Let

$$\gamma := \pi^* h$$

be the pullback of the Hermitian metric h of X to M. γ is positive semidefinite (a pseudo-metric) with degeneracy locus E.

We give M the structure of a Hermitian manifold with a freely chosen (positive definite) metric σ. Then $\gamma \lesssim \sigma$ and $\gamma \sim \sigma$ on compact subsets of $M - E$. For an open set $U \subset M$, we denote by $L^{p,q}_\gamma(U)$ and $L^{p,q}_\sigma(U)$ the spaces of square-integrable (p,q)-forms with respect to the (pseudo-)metrics γ and σ, respectively.

Since σ is positive definite and γ is positive semi-definite, there exists a continuous function $g \in C^0(M, \mathbb{R})$ such that

$$dV_\gamma = g^2 dV_\sigma. \tag{21}$$

This yields $|g||\omega|_\gamma = |\omega|_\sigma$ if ω is an $(n,0)$-form, and $|\omega|_\sigma \lesssim_U |g||\omega|_\gamma$ on $U \subset\subset M$ if ω is a (n,q)-form, $0 \leq q \leq n$.[14] So, for an (n,q) form ω on $U \subset\subset M$:

$$\int_U |\omega|^2_\sigma dV_\sigma \lesssim_U \int_U g^2 |\omega|^2_\gamma g^{-2} dV_\gamma = \int_U |\omega|^2_\gamma dV_\gamma. \tag{22}$$

Conversely, $|g||\eta|_\gamma \lesssim_U |\eta|_\sigma$ on $U \subset\subset M$ if η is a $(0,q)$-form, $0 \leq q \leq n$.[15] So, for a $(0,q)$ form η on $U \subset\subset M$:

$$\int_U |\eta|^2_\gamma dV_\gamma \lesssim_U \int_U g^{-2} |\eta|^2_\sigma g^2 dV_\sigma = \int_U |\eta|^2_\sigma dV_\sigma. \tag{23}$$

For open sets $U \subset\subset M$ and all $0 \leq q \leq n$, we conclude the relations

$$L^{n,q}_\gamma(U) \subset L^{n,q}_\sigma(U), \tag{24}$$
$$L^{0,q}_\sigma(U) \subset L^{0,q}_\gamma(U). \tag{25}$$

For an open set $\Omega \subset X$, $\Omega^* = \Omega - \operatorname{Sing} X$, $\widetilde{\Omega} := \pi^{-1}(\Omega)$, pullback of forms under π gives the isometry

$$\pi^* : L^2_{p,q}(\Omega^*) \longrightarrow L^{p,q}_\gamma(\widetilde{\Omega} - E) \cong L^{p,q}_\gamma(\widetilde{\Omega}), \tag{26}$$

where the last identification is by trivial extension of forms over the thin exceptional set E. Combining (24) with (26), we see that π^* maps

$$\pi^* : L^2_{n,q}(\Omega^*) \to L^{n,q}_\sigma(\pi^{-1}(\Omega)) \tag{27}$$

if $\Omega \subset\subset X$ is a relatively compact open set. We shall now show how (27) induces the map

$$\pi^* : H^{n,q}_{max}(X - \operatorname{Sing} X) \to H^{n,q}_{(2)}(M) \tag{28}$$

from Theorem 2.5 (where X is compact).

[14]This statement means that $|\omega|_\sigma / |\omega|_\gamma$ is locally bounded on M for (n,q)-forms.
[15]For $(0,q)$-forms, $|\omega|_\gamma / |\omega|_\sigma$ is locally bounded.

It makes sense to explain that from a slightly more general point of view. For that, we need a suitable realization of the L^2-cohomology on M. Let $\mathcal{L}^{p,q}_\sigma$ be the sheaves of germs of forms on M which are locally in $L^{p,q}_\sigma$, and we denote again by $\overline{\partial}_w$ the $\overline{\partial}$-operator in the sense of distributions on such forms because there is no danger of confusion in what follows. We can simply use the definitions from section 2.2.2 with the choice $X = M$ and $\operatorname{Sing} X = \emptyset$. Again, we denote the sheaves of germs in the domain of $\overline{\partial}_w$ by

$$\mathcal{C}^{p,q}_\sigma := \mathcal{L}^{p,q}_\sigma \cap \overline{\partial}_w^{-1} \mathcal{L}^{p,q+1}_\sigma$$

in the sense that

$$\mathcal{C}^{p,q}_\sigma(U) = \mathcal{L}^{p,q}_\sigma(U) \cap \operatorname{Dom} \overline{\partial}_w(U).$$

It is well-known that

$$\mathcal{K}_M := \ker \overline{\partial}_w \subset \mathcal{C}^{n,0}_\sigma$$

is the usual canonical sheaf on M and that

$$0 \to \mathcal{K}_M \hookrightarrow \mathcal{C}^{n,0}_\sigma \xrightarrow{\overline{\partial}_w} \mathcal{C}^{n,1}_\sigma \xrightarrow{\overline{\partial}_w} \mathcal{C}^{n,2}_\sigma \longrightarrow ... \tag{29}$$

is a fine resolution so that

$$H^q(U, \mathcal{K}_M) \cong H^q(\Gamma(U, \mathcal{C}^{n,*}_\sigma)) \ , \ H^q_{cpt}(U, \mathcal{K}_M) \cong H^q(\Gamma_{cpt}(U, \mathcal{C}^{n,*}_\sigma))$$

on open sets $U \subset M$.

Now we can use (27) to see that π^* defines a morphism of complexes

$$\pi^* : (\mathcal{C}^{n,*}, \overline{\partial}_w) \to (\pi_* \mathcal{C}^{n,*}_\sigma, \pi_* \overline{\partial}_w). \tag{30}$$

Let $\Omega \subset X$ be an open set and let $f \in \mathcal{C}^{n,q}(\Omega)$, $g \in \mathcal{C}^{n,q+1}(\Omega)$ such that $\overline{\partial}_w f = g$. By (27), it follows that $\pi^* f \in \mathcal{L}^{n,q}_\sigma(\pi^{-1}(\Omega))$ and $\pi^* g \in \mathcal{L}^{n,q+1}_\sigma(\pi^{-1}(\Omega))$ such that $\overline{\partial}_w \pi^* f = \pi^* g$ on $\pi^{-1}(\Omega) - E$. But then the L^2-extension theorem [R4], Theorem 3.2, tells us that $\overline{\partial}_w \pi^* f = \pi^* g$ on $\pi^{-1}(\Omega)$. So $\pi^* f \in \mathcal{C}^{n,q}_\sigma(\pi^{-1}(\Omega))$, $\pi^* g \in \mathcal{C}^{n,q+1}_\sigma(\pi^{-1}(\Omega))$ and (30) is in fact a morphism of complexes. Including $\mathcal{K}_X = \ker \overline{\partial}_w \subset \mathcal{C}^{n,0}$ and $\mathcal{K}_M = \ker \overline{\partial}_w \subset \mathcal{C}^{n,0}_\sigma$, we obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}_X \arrow[r] \arrow[d, "\pi^*"] & \mathcal{C}^{n,0} \arrow[r, "\overline{\partial}_w"] \arrow[d, "\pi^*"] & \mathcal{C}^{n,1} \arrow[r, "\overline{\partial}_w"] \arrow[d, "\pi^*"] & \mathcal{C}^{n,2} \arrow[r, "\overline{\partial}_w"] \arrow[d, "\pi^*"] & \cdots \\
0 \arrow[r] & \pi_* \mathcal{K}_M \arrow[r] & \pi_* \mathcal{C}^{n,0}_\sigma \arrow[r, "\pi_* \overline{\partial}_w"] & \pi_* \mathcal{C}^{n,1}_\sigma \arrow[r, "\pi_* \overline{\partial}_w"] & \pi_* \mathcal{C}^{n,2}_\sigma \arrow[r, "\pi_* \overline{\partial}_w"] & \cdots
\end{tikzcd} \tag{31}$$

Note that the upper line is exact by Theorem 2.6. It follows that π^* induces a map on the cohomology of the complexes,

$$\pi^* : H^q(\Gamma(\Omega, \mathcal{C}^{n,*})) \longrightarrow H^q(\Gamma(\pi^{-1}(\Omega), \mathcal{C}^{n,*}_\sigma)), \tag{32}$$

for any open set $\Omega \subset X$ and all $q \geq 0$. If X is compact and we choose $\Omega = X$, then the left hand side in (32) is just $H^{n,q}_{max}(X - \operatorname{Sing} X)$ for $\overline{\partial}_w = \overline{\partial}_{max}$ on $X - \operatorname{Sing} X$, and the right hand side is $H^{n,q}_{(2)}(M)$ for $\overline{\partial}_w = \overline{\partial}_{max} = \overline{\partial}_{min}$ on M. This defines (28).

We will now use Takegoshi's vanishing theorem [T1] to show that the lower line in the commutative diagram (31) is also exact. This will yield that (32) is in fact an isomorphism and that implies in particular Theorem 2.5.

Before, we shall mention another implication of the commutative diagram (31). The vertical arrow on the left hand side is an isomorphism because $\mathcal{L}^{n,0} \cong \pi_* \mathcal{L}^{n,0}_\sigma$ and the $\overline{\partial}_w$-equation extends over the exceptional set as described above. So,

$$\pi_* \mathcal{K}_M \cong \mathcal{K}_X. \tag{33}$$

Thus, $\mathcal{K}_X$ is in fact the canonical sheaf of Grauert–Riemenschneider as introduced in [GR3]. It only remains to relate the direct image of the fine resolution (29) of $\mathcal{K}_M$ to the fine resolution (20) of $\mathcal{K}_X$. This can be done by use of Takegoshi's vanishing theorem which tells us that the higher direct image sheaves of $\mathcal{K}_M$ do vanish:

$$R^q \pi_* \mathcal{K}_M = 0, \quad q > 0. \tag{34}$$

This statement also relies on L^2-methods. Recognizing that

$$\big(R^q \pi_* \mathcal{K}_M\big)_x = \varinjlim_{x \in U} H^q(\pi^{-1}(U), \mathcal{K}_M), \tag{35}$$

Takegoshi proves his vanishing theorem by using a priori estimates to deduce that the $\overline{\partial}$-equation is solvable in the L^2-sense for (n,q)-forms on M in domains of the form $\pi^{-1}(U)$ where U is a small strongly pseudoconvex set in X.

Since (29) is exact, (34) and (35) imply that the lower line of the commutative diagram (31) is another fine resolution of $\pi_* \mathcal{K}_M \cong \mathcal{K}_X$ (use also (33)). But then

$$H^q(\Omega, \mathcal{K}_X) \cong H^q(\Omega, \pi_* \mathcal{K}_M) \cong H^q(\pi^{-1}(\Omega), \mathcal{K}_M)$$

and (32) is an isomorphism for all open sets $\Omega \subset X$ and all $q \geq 0$. If X is compact, the choice $\Omega = X$ proves Theorem 2.5.

Note that we have not only proven Theorem 2.5, but also derived statements about the resolution of $L^{2,loc}$-cohomology on non-compact spaces. It is also worthwhile to mention that we can also consider the cohomology with compact support. From that, we get isomorphisms

$$\pi^* : H^q(\Gamma_{cpt}(\Omega, \mathcal{C}^{n,*})) \xrightarrow{\cong} H^q(\Gamma_{cpt}(\pi^{-1}(\Omega), \mathcal{C}^{n,*}_\sigma))$$

for the L^2-Dolbeault cohomology with compact support.

Finally, we should briefly indicate how the dual isomorphism

$$H^{0,q}_{(2)}(M) \xrightarrow{\cong} H^{0,q}_{min}(X - \operatorname{Sing} X)$$

is induced by push-forward of forms (if X is compact). By (25), π induces the maps

$$\psi^* := (\pi|_{M-E}^{-1})^* : L^{0,q}_\sigma(M) \longrightarrow L^2_{0,q}(X - \operatorname{Sing} X).$$

Let f be a smooth representative of a cohomology class $[f] \in H^{0,q}_{(2)}(M)$. Then it is clear that $\overline{\partial}_{max} \psi^* f = 0$. On the other hand, since f is bounded, one can show that f can be approximated in $L^{0,q}_\sigma(M)$ by a sequence of smooth forms $\{f_j\}_j$ with support in $M - E$ such that the sequence $\{\overline{\partial} f_j\}_j$ converges to 0 in $L^{0,q+1}_\sigma(M)$. But then the sequence $\{(\psi^* f_j, \overline{\partial} \psi^* f_j)\}_j$ converges to $(\psi^* f, 0)$ in $L^2_{0,q}(X - \operatorname{Sing} X) \times L^2_{0,q+1}(X - \operatorname{Sing} X)$ and that shows that $\overline{\partial}_{min} \psi^* f = 0$. Hence, $\psi^* f$ defines a class in $H^{0,q}_{min}(X - \operatorname{Sing} X)$.

2.3 The canonical sheaf with boundary condition $\mathcal{K}_X^s$

Similarly to Theorem 2.5, we would now like to understand also the L^2-cohomology in the sense of distributions for $(0,q)$-forms $H^{0,q}_{max}$ on singular spaces. This is far more difficult as we will see soon. One reason is that the $\overline{\partial}$-equation in the L^2-sense of distributions is usually not locally solvable at singularities for $(0,q)$-forms. However, for Hermitian spaces with only isolated singularities, we can now give an almost complete description of the L^2-cohomology in terms of a resolution of singularities.

First steps in this direction had been made by Diederich, Fornæss, Øvrelid and Vassiliadou in [F1], [DFV], [FOV2] who showed that there are only finitely many obstructions to solving the $\overline{\partial}$-equation in the L^2-sense at isolated singularities, but they did not try to identify these obstructions explicitely. The first attempt to find an explicit representation in terms of a resolution of singularities was made in [R3] and [R6] for homogeneous isolated singularities. In this spirit, the complete description for arbitrary isolated singularities has been given recently in [R9], [OV5] and [R10]. The key idea, introduced in [R9], is as follows. We have already seen that one can use statements about the $\overline{\partial}_{max}$-equation on (n,q)-forms and L^2-Serre duality to deduce statements about the $\overline{\partial}_{min}$-equation on $(0,q)$-forms. Conversely, we will now first derive statements about the $\overline{\partial}_{min}$-operator on (n,q)-forms which can be used later to deduce statements about the $\overline{\partial}$-equation in the sense of distributions on $(0,q)$-forms. Since we wish to work in a sheaf-theoretic context as before, we need a localized version of the $\overline{\partial}_{min}$-operator. That is the $\overline{\partial}_s$-operator which was introduced in [R9]. It is the $\overline{\partial}$-operator in the sense of distributions with some Dirichlet boundary condition at the singular set of the variety. The key object to study will then be the canonical sheaf with respect to this operator, denoted by $\mathcal{K}_X^s$, that is the sheaf of germs of square-integrable $(n,0)$-forms which are holomorphic with respect to the $\overline{\partial}_s$-operator, i.e. which satisfy some boundary condition at the exceptional set.

2.3.1 The strong $\overline{\partial}$-operator $\overline{\partial}_s$ and its L^2-complex

We introduce now a suitable local realization of the $\overline{\partial}_{min}$-operator. This is the $\overline{\partial}$-operator with a Dirichlet boundary condition at the singular set $\operatorname{Sing} X$ of X, where X is any Hermitian singular complex space. For any open set $U \subset X$, let

$$\overline{\partial}_s(U) : L^{2,loc}_{p,q}(U) \to L^{2,loc}_{p,q+1}(U)$$

be defined as follows.[16] Let $f \in \operatorname{Dom} \overline{\partial}_w$. We say that f is in the domain of $\overline{\partial}_s$ if there exists a sequence of forms $\{f_j\}_j \subset \operatorname{Dom} \overline{\partial}_w \subset L^{2,loc}_{p,q}(U)$ with essential support away from the singular set,

$$\operatorname{supp} f_j \cap \operatorname{Sing} X = \emptyset,$$

such that

$$f_j \to f \quad \text{in} \quad L^2_{p,q}(K - \operatorname{Sing} X), \tag{36}$$

$$\overline{\partial}_w f_j \to \overline{\partial}_w f \quad \text{in} \quad L^2_{p,q+1}(K - \operatorname{Sing} X) \tag{37}$$

[16] Again, we write simply $\overline{\partial}_s$ for $\overline{\partial}_s(U)$ if there is no danger of confusion.

for each compact subset $K \subset\subset U$.

The subscript refers to $\overline{\partial}_s$ as an extension in a strong sense. Note that we can assume without loss of generality (by use of cut-off functions and smoothing with Dirac sequences) that the forms f_j are smooth with compact support in $U - \operatorname{Sing} X$. This is the equivalent definition that we used in [R9] where we denoted the operator also by $\overline{\partial}_{s,loc}$. It is now clear that

$$\overline{\partial}_s(U)|_V = \overline{\partial}_s(V)$$

for open sets $V \subset U$, and we can define the presheaves of germs of forms in the domain of $\overline{\partial}_s$,

$$\mathcal{F}^{p,q} := \mathcal{L}^{p,q} \cap \overline{\partial}_s^{-1} \mathcal{L}^{p,q+1},$$

given by

$$\mathcal{F}^{p,q}(U) = \mathcal{L}^{p,q}(U) \cap \operatorname{Dom} \overline{\partial}_s(U).$$

Here, we shall check a bit more carefully that these are already sheaves: Let $U = \bigcup U_\mu$ be a union of open sets, $f \in L^{2,loc}_{p,q}(U)$ and $f_\mu = f|_{U_\mu} \in \operatorname{Dom} \overline{\partial}_s(U_\mu)$ for all μ. We claim that $f \in \operatorname{Dom} \overline{\partial}_s(U)$. To see this, we can assume (by taking a refinement if necessary) that the open cover $\mathcal{U} := \{U_\mu\}$ is locally finite, and choose a partition of unity $\{\varphi_\mu\}$ for $\mathcal{U}$. On U_μ choose a sequence $\{f_j^\mu\} \subset L^{p,q}_{loc}(U_\mu)$ as in (36), (37), and consider

$$f_j := \sum_\mu \varphi_\mu f_j^\mu.$$

It is clear that $\{f_j\} \subset L^{p,q}_{loc}(U)$. If $K \subset\subset U$ is compact, then $K \cap \operatorname{supp} \varphi_\mu$ is a compact subset of U_μ for each μ, so that $\{f_j^\mu\}$ and $\{\overline{\partial} f_j^\mu\}$ converge in the L^2-sense to f_μ resp. $\overline{\partial}_w f_\mu$ on $K \cap \operatorname{supp} \varphi_\mu$. But then $\{f_j\}$ and $\{\overline{\partial} f_j\}$ converge in the L^2-sense to f resp. $\overline{\partial}_w f$ on K (recall that the cover is locally finite) and that is what we had to show.

As for $\mathcal{C}^{p,q}$, it is clear that the sheaves $\mathcal{F}^{p,q}$ are fine, and we obtain fine sequences

$$\mathcal{F}^{p,0} \xrightarrow{\overline{\partial}_s} \mathcal{F}^{p,1} \xrightarrow{\overline{\partial}_s} \mathcal{F}^{p,2} \xrightarrow{\overline{\partial}_s} \ldots \tag{38}$$

We can now introduce the sheaf

$$\mathcal{K}_X^s := \ker \overline{\partial}_s \subset \mathcal{F}^{n,0} \tag{39}$$

which we may call the canonical sheaf of holomorphic n-forms with Dirichlet boundary condition. $\mathcal{K}_X^s$ is a coherent analytic sheaf (see Lemma 2.10 below). Our main objective in the following is to compare different representations of the cohomology of $\mathcal{K}_X^s$. One of them will be the $L^{2,loc}$-Dolbeault cohomology with respect to the $\overline{\partial}_s$-operator on open sets $U \subset X$, i.e. the cohomology of the complex (38) which is denoted by $H^q(\Gamma(U, \mathcal{F}^{p,*}))$. The corresponding cohomology with compact support is the cohomology of sections with compact support in U, denoted by $H^q(\Gamma_{cpt}(U, \mathcal{F}^{p,*}))$.

As for the $\overline{\partial}_w$-complex, it is clear that (38) is exact in regular points of X by the L^2-Grothendieck-Dolbeault lemma, but it is an interesting problem to understand the obstructions to exactness in singular points. Clearly, what we would like to have is that the complex $(\mathcal{F}^{n,*}, \overline{\partial}_s)$ is a fine resolution of our canonical sheaf with boundary condition $\mathcal{K}_X^s$. But this is in fact true at least on spaces with only isolated singularities.

Let X be a Hermitian complex space of pure dimension $n \geq 2$ with only isolated singularities.[17] Then the $\overline{\partial}_s$-equation is locally exact on (n,q)-forms for $1 \leq q \leq n-1$ by [R9], Lemma 5.4, and for $q \geq 2$ by [R9], Lemma 6.3. Hence:

Theorem 2.7. *Let X be a Hermitian complex space of pure dimension $n \geq 2$ with only isolated singularities. Then*

$$0 \to \mathcal{K}_X^s \hookrightarrow \mathcal{F}^{n,0} \xrightarrow{\overline{\partial}_s} \mathcal{F}^{n,1} \xrightarrow{\overline{\partial}_s} \mathcal{F}^{n,2} \xrightarrow{\overline{\partial}_s} \dots \longrightarrow \mathcal{F}^{n,n} \to 0 \tag{40}$$

is a fine resolution. For an open set $U \subset X$, it follows that

$$H^q(U, \mathcal{K}_X^s) \cong H^q(\Gamma(U, \mathcal{F}^{n,*})) \ , \ H^q_{cpt}(U, \mathcal{K}_X^s) \cong H^q(\Gamma_{cpt}(U, \mathcal{F}^{n,*})).$$

The most important tool for the proof of Theorem 2.7 are the results of Fornæss, Øvrelid and Vassiliadou [FOV2] on the regularity of the $\overline{\partial}$-equation in the L^2-sense of distributions at isolated singularities. Let V be a strongly pseudoconvex small neighborhood of an isolated singularity in X. In the notation from section 2.2.2, Theorem 1.1 in [FOV2] can be interpreted as

$$H^q(\Gamma_{cpt}(V, \mathcal{C}^{0,*})) = 0$$

for all $0 < q < n$, i.e. the $\overline{\partial}_w$-equation with compact support is solvable for such $(0,q)$-forms. By some kind of mixed duality (a mixture of usual Serre duality and L^2-duality at the singularity), one can deduce that

$$H^q(\Gamma(V, \mathcal{F}^{n,*})) = 0$$

for all $0 < q < n$ (this is Lemma 5.4 in [R9]). That shows that (40) is exact at $\mathcal{F}^{n,q}$ for such q. Exactness at $\mathcal{F}^{n,0}$ is clear for $\mathcal{K}_X^s = \ker \overline{\partial}_s$. On the other hand, consider the $\overline{\partial}_w$-equation $\overline{\partial}_w f = g$ on such a domain V where g is a $\overline{\partial}_w$-closed (n,q)-form. This equation can be solved by the method from Theorem 2.6. But here, in the case of isolated singularities, one can show that for $q \geq 2$, there is a solution f with a gain of regularity such that in fact $\overline{\partial}_s f = g$. This can be deduced from Theorem 1.2 in [FOV2] and appears in [R9] as Lemma 6.3. It follows that (40) is exact at $\mathcal{F}^{n,q}$ for $q \geq 2$.

The strategy for the proof of Theorem 1.1 and Theorem 1.2 in [FOV2] is as follows: on a domain V as above, Fornæss, Øvrelid and Vassiliadou modify the metric so that they obtain a complete Kähler metric on the regular part of V, solve the equation by L^2-estimates and study how the solution behaves in the original metric. In the case of isolated singularities, this can be computed quite explicit, and so one obtains results for (p,q)-forms of degree $p+q \neq n$, not only for (n,q) and $(0,q)$-forms. Moreover, this also yields the gain of regularity that we used above.

[17] For the present exposition, we will omit the case of complex dimension 1 which is much easier to handle and does not require the techniques on which we focus here.

It would be very interesting to know wether the complex (40) is also exact for arbitrary singularities. Unfortunately, the methods of Fornæss, Øvrelid and Vassiliadou from [FOV2] cannot be generalized to non-isolated singularities and the methods we used in the proof of Theorem 2.6 for arbitrary singularities are not strong enough (i.e. no gain of regularity can be deduced from the estimates). Nevertheless, many particular examples suggest that the $\overline{\partial}_s$-equation is locally exact for (n,q)-forms in general. By duality, this would be related to conject that the $\overline{\partial}_w$-equation with compact support is locally solvable for $(0,q)$-forms, $0 < q < n$. This is for example very reasonable on homogeneous varieties as the results from [RZ2] suggest which we will discuss later when we treat integral formulas.

2.3.2 Resolution of $(X, \mathcal{K}_X^s)$

We restrict our considerations from now on to a Hermitian complex space (X,h) of pure dimension $n \geq 2$ with only isolated singularities. As in section 2.2.3, let

$$\pi : M \to X$$

be a resolution of singularities, $\gamma := \pi^* h$ and σ any positive definite metric on M. This time, we require that the exceptional set $E = |\pi^{-1}(\operatorname{Sing} X)|$ has only normal crossings. We need a little refinement of the statement (24) which says that $L^{n,q}_\gamma(U) \subset L^{n,q}_\sigma(U)$ for open sets $U \subset\subset M$ and all $q \geq 0$.

Let $Z := \pi^{-1}(\operatorname{Sing} X)$ be the unreduced exceptional divisor. In the following, we have to deal with forms with values in $\mathcal{O}(|Z| - Z)$. If Z has the multiplicity m on an irreducible component of the exceptional divisor, then $Z - |Z|$ has multiplicity $m-1$ on that component. Sections in $\mathcal{O}(|Z|-Z)$ are the holomorphic functions that vanish to order $m-1$ on that component of the exceptional divisor. Note that it might happen that $|Z| - Z = \emptyset$, for example if X has just a homogeneous singularity which is resolved by a single blow-up.

When we are dealing with forms with values in $\mathcal{O}(|Z| - Z)$, we can adopt two different points of view. First, let $L_{|Z|-Z} \to M$ be the holomorphic line bundle associated to the divisor $|Z| - Z$ such that holomorphic sections of $L_{|Z|-Z}$ correspond to sections of $\mathcal{O}(|Z| - Z)$, and give $L_{|Z|-Z}$ the structure of a Hermitian line bundle by choosing an arbitrary positive definite Hermitian metric. Then, denote by

$$L^{p,q}_\sigma(U, L_{|Z|-Z})\ ,\ L^{p,q}_{\sigma,loc}(U, L_{|Z|-Z})$$

the spaces of (locally) square-integrable (p,q)-forms with values in $L_{|Z|-Z}$ (with respect to the metric σ on M and the chosen metric on $L_{|Z|-Z}$). We can then define the sheaves of germs of square-integrable (p,q)-forms with values in $L_{|Z|-Z}$, $\mathcal{L}^{p,q}_\sigma(L_{|Z|-Z})$, by the assignment

$$\mathcal{L}^{p,q}_\sigma(L_{|Z|-Z})(U) = L^{p,q}_{\sigma,loc}(U, L_{|Z|-Z}).$$

The second point of view is to use the sheaves

$$\mathcal{L}^{p,q}_\sigma \otimes \mathcal{O}(|Z| - Z)$$

which are canonically isomorphic to the sheaves $\mathcal{L}^{p,q}_\sigma(L_{|Z|-Z})$. Let us keep both points of view in mind. As in section 2.2.3, let

$$\mathcal{C}^{p,q}_\sigma(L_{|Z|-Z}) := \mathcal{L}^{p,q}_\sigma(L_{|Z|-Z}) \cap \overline{\partial}_w^{-1} \mathcal{L}^{p,q+1}_\sigma(L_{|Z|-Z}), \tag{41}$$

where $\overline{\partial}_w$ is the $\overline{\partial}$-operator in the sense of distributions for forms with values in $L_{|Z|-Z}$. It is clear that the sheaves $\mathcal{C}^{p,q}_\sigma(L_{|Z|-Z})$ are fine.

Now then, the ordinary Lemma of Dolbeault tells us that

$$0 \to \mathcal{K}_M \otimes \mathcal{O}(|Z|-Z) \hookrightarrow \mathcal{C}^{n,0}_\sigma(L_{|Z|-Z}) \xrightarrow{\overline{\partial}_w} \mathcal{C}^{n,1}_\sigma(L_{|Z|-Z}) \xrightarrow{\overline{\partial}_w} ... \tag{42}$$

is a fine resolution of the sheaf of germs of holomorphic n-forms with values in $\mathcal{O}(|Z|-Z)$, i.e. vanishing to a certain order on the exceptional set.

The refinement of (24) that we need is as follows:

Lemma 2.8. *Let $q \geq 1$. Then*

$$L^{n,q}_\gamma(U) \subset L^{n,q}_\sigma(U, L_{|Z|-Z})$$

for any open set $U \subset\subset M$, i.e.

$$\mathcal{L}^{n,q}_\gamma \subset \mathcal{L}^{n,q}_\sigma(L_{|Z|-Z}).$$

The proof is in [R10], Lemma 5.1, but we will repeat it here because this really illustrates how the divisor $|Z|-Z$ comes into play.

Proof. Since the statement is local, it is enough to consider a point $P \in E$ and a neighborhood U of P such that U is an open set in $\mathbb{C}^n$, that E is the normal crossing $\{z_1 \cdots z_d = 0\}$, and $P = 0$. For σ we can take the Euclidean metric.

Let us investigate the behavior of $(0,1)$-forms under the resolution $\pi : M \to X$ at the isolated singularity $\pi(P)$. We can assume that a neighborhood of $\pi(P)$ is embedded holomorphically into $W \subset\subset \mathbb{C}^L$, $L \gg n$, such that $\pi(P) = 0$, and that $\gamma = \pi^* h$ where h is the Euclidean metric in $\mathbb{C}^L$. Let $w_1, ..., w_L$ be the Cartesian coordinates of $\mathbb{C}^L$. We are interested in the behavior of the forms $\eta_\mu := \pi^* d\overline{w_\mu}$ at the exceptional set. Let $dz_N := dz_1 \wedge \cdots \wedge dz_n$. It follows from the observations in section 2.2.3 that a form α is in $L^{n,q}_\gamma(U)$ exactly if it can be written in multi-index notation[18] as

$$\alpha = \sum_{|K|=q} \alpha_K dz_N \wedge \eta_K = dz_N \wedge \sum_{|K|=q} \alpha_K \eta_K \tag{43}$$

with coefficients $\alpha_K \in L^{0,0}_\sigma(U)$. That can be seen as follows. Let $\alpha \in L^{n,q}_\gamma(U)$ be written in the form (43). Since the forms η_K are orthogonal to dz_N, we have

$$|\alpha|_\gamma = |dz_N|_\gamma \Big| \sum_{|K|=q} \alpha_K \eta_K \Big|_\gamma.$$

[18] $\eta_K = \eta_{k_1} \wedge \cdots \wedge \eta_{k_q}$ for $K = (k_1, ..., k_q)$ and the sum should be over indices of ascending order.

Let g be a function as in section 2.2.3. Since $|dz_N|_\gamma = |g|^{-1}$ and $|\eta_K|_\gamma \leq 1$, there are coefficients α_K in (43) such that

$$|\alpha|_\gamma = |g|^{-1} \sum_{|K|=q} |\alpha_K|.$$

So, α is in $L^{n,q}_\gamma(U)$ exactly if $|\alpha|_\gamma$ is in $L^{0,0}_\gamma(U)$ which is the case exactly if all the $g^{-1}\alpha_K$ are in $L^{0,0}_\gamma(U)$. By use of (21), this is the case exactly if all the α_K are in $L^{0,0}_\sigma(U)$. The representation (43) is not unique.

Let Z have the order $k_j \geq 1$ on $\{z_j = 0\}$, i.e. assume that Z is given by $f = z_1^{k_1} \cdots z_d^{k_d}$. Since $Z = \pi^{-1}(\operatorname{Sing} X)$, each $\pi^* w_\mu$ must vanish of order k_j on $\{z_j = 0\}$. We conclude that $\pi^* w_\mu$ has a factorization

$$\pi^* w_\mu = f g_\mu = z_1^{k_1} \cdots z_d^{k_d} \cdot g_\mu,$$

where g_μ is a holomorphic function on U. So,

$$\eta_\mu = \pi^* d\overline{w_\mu} = d\pi^* \overline{w_\mu} = \left(\overline{z_1}^{k_1-1} \cdots \overline{z_d}^{k_d-1}\right) \cdot \beta_\mu,$$

where the β_μ are $(0,1)$-forms that are bounded with respect to the non-singular metric σ. This means that $\eta_\mu = \pi^* d\overline{w_\mu}$ vanishes at least to the order of $Z - |Z|$ on the exceptional set E.

So, (43) implies that a form α is in $L^{n,q}_\gamma(U)$ exactly if it can be written in multi-index notation as

$$\alpha = \left(\overline{z_1}^{k_1-1} \cdots \overline{z_d}^{k_d-1}\right)^q \sum_{|K|=q} \alpha_K dz_N \wedge \beta_K \tag{44}$$

with coefficients $\alpha_K \in L^{0,0}_\sigma(U)$. We conclude that $\mathcal{L}^{n,q}_\gamma \subset \mathcal{L}^{n,q}_\sigma(L_{|Z|-Z})$ for $q \geq 1$. $\square$

It follows from Lemma 2.8 that pull-back of forms under π defines a map

$$\pi^* : L^2_{n,q}(\Omega) \to L^{n,q}_\sigma(\pi^{-1}(\Omega), L_{|Z|-Z})$$

for any open set $\Omega \subset\subset X$ and $q \geq 1$ (extend the pull-back of forms again trivially over the exceptional set). As in (30), this induces maps

$$\pi^* : \mathcal{F}^{n,q} \to \pi_* \mathcal{C}^{n,q}_\sigma(L_{|Z|-Z}) \tag{45}$$

for all $q \geq 1$: let $\Omega \subset X$ be an open set, $f \in \mathcal{L}^{n,q}(\Omega)$ and $g \in \mathcal{L}^{n,q+1}(\Omega)$ such that $\overline{\partial}_s f = g$. Then $\pi^* f \in \mathcal{L}^{n,q}_\sigma(L_{|Z|-Z})$ and $\pi^* g \in \mathcal{L}^{n,q+1}_\sigma(L_{|Z|-Z})$ by Lemma 2.8 and $\overline{\partial}_w \pi^* f = \pi^* g$ on $\pi^{-1}(\Omega) - E$. But then $\overline{\partial}_w \pi^* f = \pi^* g$ on $\pi^{-1}(\Omega)$ by the L^2-extension theorem for the $\overline{\partial}_w$-equation, and so $\pi^* f \in \mathcal{C}^{n,q}_\sigma(L_{|Z|-Z})(\pi^{-1}(\Omega))$. It follows also that π^* commutes with the $\overline{\partial}$-operator in the sense that

$$\pi^* \circ \overline{\partial}_s = \overline{\partial}_w \circ \pi^*.$$

We need an analogous statement for $(n,0)$-forms, but it is clear that we cannot expect $\mathcal{L}^{n,0}_\gamma \subset \mathcal{L}^{n,0}_\sigma(L_{|Z|-Z})$ because $\mathcal{L}^{n,0}_\gamma \cong \mathcal{L}^{n,0}_\sigma$ and sections of $\mathcal{O}(|Z| - Z)$ are forms that vanish to a certain order on the exceptional set. It turns out that we can bridge this gap by realizing that $(n,0)$-forms in the domain of the $\overline{\partial}_s$-equation must vanish to some order in the singularities. The reason is the boundary condition which is involved in the definition of the $\overline{\partial}_s$-operator.

Lemma 2.9. *For an open set $\Omega \subset X$, pull-back of forms under π defines a map*

$$\pi^* : \mathcal{F}^{n,0}(\Omega) \to \mathcal{C}^{n,0}_\sigma(L_{|Z|-Z})(\pi^{-1}(\Omega)),$$

i.e. π^* gives a morphism of sheaves

$$\pi^* : \mathcal{F}^{n,0} \to \pi_* \mathcal{C}^{n,0}_\sigma(L_{|Z|-Z}).$$

The proof is again in [R10], Lemma 5.1, but we will also repeat this one briefly because it illustrates how the boundary condition of the operator $\overline{\partial}_s$ comes into play.

Proof. We can assume that we are in the same local situation as in Lemma 2.8 with $U = \pi^{-1}(\Omega)$, i.e. consider a small neighborhood U of the point $P = 0 \in \mathbb{C}^n$, assume that $E = \{z_1 \cdots z_d = 0\}$ and that Z is given by $f = z_1^{k_1} \cdots z_d^{k_d}$.

Let $\phi \in \mathcal{F}^{n,0}(\Omega)$ with

$$\overline{\partial}_s \phi = \psi \in \mathcal{L}^{n,1}(\Omega).$$

This means by (24) and Lemma 2.8 (with trivial extension of the forms over the exceptional set) that

$$\pi^* \phi \in \mathcal{L}^{n,0}_\gamma(U) \cong \mathcal{L}^{n,0}_\sigma(U)$$

and

$$\pi^* \psi \in \mathcal{L}^{n,0}_\gamma(U) \subset \mathcal{L}^{n,1}_\sigma(L_{|Z|-Z})(U)$$

such that there exists a sequence of smooth forms ϕ_j with support away from E with

$$\begin{aligned} \phi_j \to \pi^*\phi \quad &\text{in} \quad L^{n,0}_\gamma(V) \cong L^{n,0}_\sigma(V), \\ \overline{\partial}\phi_j \to \pi^*\psi \quad &\text{in} \quad L^{n,1}_\gamma(V) \end{aligned}$$

on suitable open sets $V \subset U$. The considerations above show that convergence in $L^{n,1}_\gamma(V)$ implies convergence in $L^{n,1}_\sigma(V, L_{|Z|-Z})$. By use of the inhomogeneous Cauchy formula, we will show that this implies convergence of $\{\phi_j\}_j$ in $L^{n,0}_\sigma(V, L_{|Z|-Z})$, as well. But then $\pi^*\phi \in \mathcal{L}^{n,0}_\sigma(L_{|Z|-Z})(V)$.

Since we treat a local question at $0 \in \mathbb{C}^n$, it does no harm to work on a suitable neighborhood of the origin and to cut-off $\pi^*\phi$ and the ϕ_j by a real-valued smooth function $\chi \in C^\infty_{cpt}(\mathbb{C})$ satisfying $\chi(z_1) = 1$ for $|z_1| \leq \epsilon$, $\chi(z_1) = 0$ for $|z_1| \geq 2\epsilon$, and $|\chi'| \leq 2\epsilon^{-1}$ for a fixed $\epsilon > 0$ small enough. So, replace $\pi^*\phi(z)$ by $\pi^*\phi(z)\chi(z_1)$ and $\phi_j(z)$ by $\phi_j(z)\chi(z_1)$.

Because the ϕ_j have compact support away from E, we have the representation

$$\phi_j(z) = \frac{z_1^{k_1-1}}{2\pi i} \int_{\mathbb{C}} \frac{\partial \phi_j}{\partial \overline{\zeta_1}}(\zeta_1, z_2, ..., z_n) \frac{d\zeta_1 \wedge d\overline{\zeta_1}}{\zeta_1^{k_1-1}(\zeta_1 - z_1)}, \tag{46}$$

omitting dz_N in the notation for simplicity. Note that k_1 is the order of Z on $\{z_1 = 0\}$. But $\overline{\partial}\phi_j \to \pi^*\psi$ in $L^{n,1}_\sigma(V, L_{|Z|-Z})$ implies that

$$\zeta_1^{-k_1+1} z_2^{-k_2+1} \cdots z_d^{-k_d+1} \overline{\partial}\phi_j(\zeta_1, z_2, ..., z_n) \to \zeta_1^{-k_1+1} z_2^{-k_2+1} \cdots z_d^{-k_d+1} \pi^*\psi(\zeta_1, z_2, ..., z_n)$$

in the L^2-sense with respect to the non-singular metric σ, i.e. in $L^{n,1}_\sigma(V)$. But the Cauchy formula (46) is bounded as an operator $L^2 \to L^2$. Hence, the formula (46) converges to

$$\pi^*\phi(z) = \frac{z_1^{k_1-1} z_2^{k_2-1}\cdots z_d^{k_d-1}}{2\pi i} \int_{\mathbb{C}} (\pi^*\psi)_1(\zeta_1, z_2, ..., z_n) \frac{d\zeta_1 \wedge d\overline{\zeta_1}}{\zeta_1^{k_1-1} z_2^{k_2-1}\cdots z_d^{k_d-1}(\zeta_1 - z_1)}$$

in $L^{n,0}_\sigma$, and the integral on the right-hand side is itself in $L^{n,0}_\sigma$. Here, $(\pi^*\psi)_1$ is the $d\overline{z_1}$-part of $\pi^*\psi$. But then $\pi^*\phi$ is $z_1^{k_1-1}\cdots z_d^{k_d-1}$ multiplied with an $L^{n,1}_\sigma$-form. So $\pi^*\phi \in L^{n,0}_\sigma(V, L_{|Z|-Z})$ with $\overline{\partial}_w \pi^*\phi = \pi^*\psi$ and that completes the proof. □

As above, it follows by extension of the $\overline{\partial}_w$-equation over the exceptional set that the commutator relation

$$\pi^* \circ \overline{\partial}_s = \overline{\partial}_w \circ \pi^*$$

holds also on $(n, 0)$-forms. Hence, π^* defines a morphism of complexes

$$\pi^* : (\mathcal{F}^{n,*}, \overline{\partial}_s) \to (\pi_*\mathcal{C}^{n,*}_\sigma(L_{|Z|-Z}), \pi_*\overline{\partial}_w).$$

Including $\mathcal{K}^s_X = \ker \overline{\partial}_s \subset \mathcal{F}^{n,0}$ and $\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z) = \ker \overline{\partial}_w \subset \mathcal{C}^{n,0}_\sigma(L_{|Z|-Z})$, we obtain the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}^s_X \arrow[r] \arrow[d, "\pi^*"] & \mathcal{F}^{n,0} \arrow[r, "\overline{\partial}_s"] \arrow[d, "\pi^*"] & \mathcal{F}^{n,1} \arrow[r, "\overline{\partial}_s"] \arrow[d, "\pi^*"] & \cdots \\ 0 \arrow[r] & \pi_*(\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)) \arrow[r] & \pi_*\mathcal{C}^{n,0}_\sigma(L_{|Z|-Z}) \arrow[r, "\pi_*\overline{\partial}_w"] & \pi_*\mathcal{C}^{n,1}_\sigma(L_{|Z|-Z}) \arrow[r, "\pi_*\overline{\partial}_w"] & \cdots \end{tikzcd} \quad (47)$$

Note that the upper line of the diagram is exact by Theorem 2.7 since we have restricted our attention to Hermitian spaces of pure dimension $n \geq 2$ with only isolated singularities. It follows from commutativity of the diagram that π^* induces maps on the cohomology of the complexes,

$$\pi^* : H^q(\Gamma(\Omega, \mathcal{F}^{n,*})) \longrightarrow H^q(\Gamma(\pi^{-1}(\Omega), \mathcal{C}^{n,*}_\sigma(L_{|Z|-Z}))), \quad (48)$$
$$\pi^* : H^q(\Gamma_{cpt}(\Omega, \mathcal{F}^{n,*})) \longrightarrow H^q(\Gamma_{cpt}(\pi^{-1}(\Omega), \mathcal{C}^{n,*}_\sigma(L_{|Z|-Z}))), \quad (49)$$

for any open set $\Omega \subset X$ and all $q \geq 0$. If X is compact and we choose $\Omega = X$, then the left hand side in both, (48) and (49), is just $H^{n,q}_{min}(X - \operatorname{Sing} X)$ since then $\overline{\partial}_s = \overline{\partial}_{min}$ on $X - \operatorname{Sing} X$, and the right hand side is just the L^2-cohomology for forms with values in the line bundle $L_{|Z|-Z}$, $H^{n,q}_{(2)}(M, L_{|Z|-Z}))$, since $\overline{\partial}_w = \overline{\partial}_{max} = \overline{\partial}_{min}$ on the compact manifold M.

Analogously to the case of the canonical sheaf of Grauert–Riemenschneider $\mathcal{K}_M$, we would like to express the L^2cohomology of the canonical sheaf with boundary condition $\mathcal{K}^s_X$ in terms of the L^2-cohomology of the sheaf of holomorphic n-forms with values in the line bundle $L_{|Z|-Z}$, $\mathcal{K}_M \otimes (\mathcal{O}(|Z| - Z)$. For this, we can use the commutative diagram (47) because we have:

Lemma 2.10. *Let (X,h) be a Hermitian complex space with only isolated singularities and $\pi : M \to X$ a resolution with only normal crossings as above, $Z = \pi^{-1}(\operatorname{Sing} X)$ the unreduced excveptional divisor. Then*

$$\pi^* : \mathcal{K}_X^s \to \pi_*\big(\mathcal{K}_M \otimes (\mathcal{O}(|Z| - Z)\big)$$

is an isomorphism. This implies that $\mathcal{K}_X^s$ is a coherent analytic sheaf by Grauert's direct image theorem [G1].

This has been proved in [R9], Lemma 6.2. Since this is the last – and again very interesting – step to complete the understanding of $\mathcal{K}_X^s$ on a Hermitian space with isolated singularities and the meaning of the divisor $Z - |Z|$, we also recall the proof of this statement.

Proof. It is clear that π^* is injective since we are talking here about the pull-back of $(n,0)$-forms and not about cohomology classes. It remains to show that

$$\pi_*\big(\mathcal{K}_M \otimes (\mathcal{O}(|Z| - Z)\big) \subset \pi^*\mathcal{K}_M^s.$$

That can be done by a cut-off procedure that we recall from [PS1]. We only have to consider singular points. So, let Ω be a small neighborhood of a singular point $p \in \operatorname{Sing} X$. We can assume that Ω is embedded holomorphically in $\mathbb{C}^L$ such that $p = 0 \in \mathbb{C}^L$. Let $U := \pi^{-1}(\Omega)$ and $\phi \in \Gamma(U, \mathcal{K}_M \otimes \mathcal{O}(|Z| - Z))$. Then $\phi \in \mathcal{L}_\gamma^{n,0}(U)$, and we have to show that there exists a sequence of smooth forms $\{\phi_k\}_k$ with support away from the exceptional set E such that

$$\begin{aligned} \phi_k \to \phi \quad &\text{in} \quad L_\gamma^{n,0}(K) \cong L_\sigma^{n,0}(K), \\ \overline{\partial}\phi_k \to 0 \quad &\text{in} \quad L_\gamma^{n,1}(K) \end{aligned}$$

on compact subsets $K \subset U$. This is sufficient, because then $\phi = \pi^*(\pi|_{U-E}^{-1})^*\phi$ and $(\pi|_{U-E}^{-1})^*\phi \in \mathcal{K}_X^s(\Omega)$ since $\{(\pi|_{U-E}^{-1})^*\phi_k\}_k$ is then a sequence of smooth forms with support away from the singular point $p \in \operatorname{Sing} X$ such that

$$\begin{aligned} (\pi|_{U-E}^{-1})^*\phi_k \to (\pi|_{U-E}^{-1})^*\phi \quad &\text{in} \quad L^2_{n,0}(C - \operatorname{Sing} X), \\ \overline{\partial}(\pi|_{U-E}^{-1})^*\phi_k \to 0 \quad &\text{in} \quad L^2_{n,1}(C - \operatorname{Sing} X) \end{aligned}$$

on compact subsets $C \subset \Omega$.

As in [PS1], Lemma 3.6, let $\rho_k : \mathbb{R} \to [0,1]$, $k \geq 1$, be smooth cut-off functions satisfying

$$\rho_k(x) = \begin{cases} 1 & , \ x \leq k, \\ 0 & , \ x \geq k+1, \end{cases}$$

and $|\rho_k'| \leq 2$. Moreover, let $r : \mathbb{R} \to [0, 1/2]$ be a smooth increasing function with

$$r(x) = \begin{cases} x & , \ x \leq 1/4, \\ 1/2 & , \ x \geq 3/4, \end{cases}$$

and $|r'| \leq 1$. We need a function measuring the distance to the exceptional set E in M. A good choice is just the pull-back of the Euclidean distance in $\mathbb{C}^L$. So, let

$$F := \left(\sum_{j=1}^{L} |w_j|^2 \right)^{1/2},$$

where $w_1, ..., w_L$ are the Cartesian coordinates of $\mathbb{C}^L$. Since the metric h is quasi-isometric to the Euclidean metric in $\mathbb{C}^L$, we have $|\overline{\partial} F|_h \lesssim 1$. As cut-off functions we can use

$$\mu_k := \rho_k(\log(-\log r(\pi^* F)))$$

on M. Thus, we claim that

$$\phi_k := \mu_k \phi$$

is a suitable sequence of smooth forms with support away from E. Let $K \subset U$ be a compact subset. It is clear that $\phi_k \to \phi$ in $L^{n,0}_\gamma(K)$ as $k \to \infty$ and that $\overline{\partial}\phi_k = \overline{\partial}\mu_k \wedge \phi$ (since ϕ is holomorphic). What we have to show is that

$$\overline{\partial}\mu_k \wedge \phi \to 0 \tag{50}$$

in $L^{n,1}_\gamma(K)$ as $k \to \infty$. By definition,

$$\begin{aligned} |\overline{\partial}\mu_k|^2_\gamma &\leq \frac{|\rho_k'(\log(-\log(r(\pi^* F))))|^2}{r^2(\pi^* F)\log^2(r(\pi^* F))} |r'|^2 |\overline{\partial}\pi^* F|^2_\gamma & (51) \\ &\lesssim \frac{\chi_k(\pi^* F)}{(\pi^* F)^2 \log^2(\pi^* F)}, & (52) \end{aligned}$$

where χ_k is the characteristic function of $[e^{-e^{k+1}}, e^{-e^k}]$ because $|\pi^*\overline{\partial} F|_\gamma = |\overline{\partial} F|_h \lesssim 1$ and μ_k is constant outside $[e^{-e^{k+1}}, e^{-e^k}]$. Note that the denominator in (51) equals the denominator in (52) if $\pi^* F$ is small enough.

We may assume that U is an open set in $\mathbb{C}^n$ and that E is just the normal crossing $\{z_1 \cdots z_d = 0\}$. The unreduced exceptional divisor $Z = \pi^{-1}(\{0\})$ is given as the common zero set of the holomorphic functions $\{\pi^* w_1, ..., \pi^* w_L\}$. Let Z have the order $k_j \geq 1$ on $\{z_j = 0\}$, i.e. assume that Z is given by the holomorphic function $f = z_1^{k_1} \cdots z_d^{k_d}$. It follows that $\pi^* F \sim |z_1^{k_1} \cdots z_d^{k_d}|$, and (52) yields

$$|\overline{\partial}\mu_k|_\gamma \lesssim |z_1|^{-k_1} \cdots |z_d|^{-k_d} |\log|^{-1}\big(|z_1 \cdots z_d|\big). \tag{53}$$

The assumption $\phi \in \Gamma(U, \mathcal{K}_M \otimes \mathcal{O}(|Z| - Z))$ implies that

$$z_1^{1-k_1} \cdots z_d^{1-k_d} \phi \in \Gamma(U, \mathcal{K}_M)$$

which does almost compensate the right hand-side of (53). We have to take care of the additional factor

$$\lambda := |z_1 \cdots z_d| |\log||z_1 \cdots z_d|,$$

but this is easy because $z_1^{1-k_1} \cdots z_d^{1-k_d} \phi$ has smooth coefficients and λ^{-1} is locally square-integrable in $\mathbb{C}^n$. Thus

$$\frac{\phi}{z_1^{k_1} \cdots z_d^{k_d} \log|z_1 \cdots z_d|} \in L^{n,0}_\gamma(K) \cong L^{n,0}_\sigma(K)$$

on the compact subset $K \subset U$. Combining this with (53), we see that $\overline{\partial}\mu_k \wedge \phi$ is uniformly bounded in $L^{n,1}_\gamma(K)$, and so

$$\overline{\partial}\mu_k \wedge \phi \to 0 \;\text{ in } L^{n,1}_\gamma(K)$$

by Lebesgue's theorem on dominated convergence because the domain of integration vanishes as $k \to \infty$. $\square$

2.3.3 Resolution of the L^2-cohomology of $\mathcal{K}_X^s$

If X is a Hermitian space of pure dimension $n \geq 2$ with only isolated singularities, we are now finally in the position to represent the L^2-cohomology of the canonical sheaf of holomorphic L^2-forms with boundary condition, $\mathcal{K}_X^s$, in terms of the resolution of singularities of $(X, \mathcal{K}_X^s)$ described in section 2.3.2.

Recall from (47) the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}_X^s & \longrightarrow & \mathcal{F}^{n,0} & \xrightarrow{\overline{\partial}_s} & \mathcal{F}^{n,1} & \xrightarrow{\overline{\partial}_s} & \cdots \\ & & \cong \downarrow \pi^* & & \downarrow \pi^* & & \downarrow \pi^* & & \\ 0 & \longrightarrow & \pi_*(\mathcal{K}_M \otimes \mathcal{O}(|Z|-Z)) & \longrightarrow & \pi_*\mathcal{C}_\sigma^{n,0}(L_{|Z|-Z}) & \xrightarrow{\pi_*\overline{\partial}_w} & \pi_*\mathcal{C}_\sigma^{n,1}(L_{|Z|-Z}) & \xrightarrow{\pi_*\overline{\partial}_w} & \cdots \end{array} \tag{54}$$

Note that the upper line of the diagram is exact by Theorem 2.7 and that the vertical arrow on the left hand side is an isomorphism by Lemma 2.10. This isomorphism

$$\pi^* : \mathcal{K}_X^s \xrightarrow{\cong} \pi_*\big(\mathcal{K}_M \otimes \mathcal{O}(|Z|-Z)\big) \tag{55}$$

is what me mean by resolution of $\mathcal{K}_X^s$.

We also recall that, by commutativity of the diagram (54), π^* induces maps on the cohomology of the complexes,

$$\pi^* : H^q(\Gamma(\Omega, \mathcal{F}^{n,*})) \longrightarrow H^q(\Gamma(\pi^{-1}(\Omega), \mathcal{C}_\sigma^{n,*}(L_{|Z|-Z}))), \tag{56}$$

$$\pi^* : H^q(\Gamma_{cpt}(\Omega, \mathcal{F}^{n,*})) \longrightarrow H^q(\Gamma_{cpt}(\pi^{-1}(\Omega), \mathcal{C}_\sigma^{n,*}(L_{|Z|-Z}))), \tag{57}$$

for any open set $\Omega \subset X$ and all $q \geq 0$. Note that (56) and (57) are isomorphisms for $q = 0$ by use of (55). For $q \geq 1$ this is not necessarily the case as we shall see below. (56) and (57) would be isomorphisms if the lower line of the diagram (54) were exact.

In [R9], we did assume at this point that the invertible sheaf $\mathcal{K}_M \otimes \mathcal{O}(|Z|-Z)$ is locally semi-positive with respect to the base space X, i.e. that any point $x \in X$ has a small neighborhood U_x such that $\mathcal{K}_M \otimes \mathcal{O}(|Z|-Z)$ is semi-positive on $\pi^{-1}(U_x)$ in the sense that the holomorphic line bundle associated to $\mathcal{K}_M \otimes \mathcal{O}(|Z|-Z)$ is semi-positive on $\pi^{-1}(U_x)$. Under this assumption Takegoshi's vanishing theorem (see [T1], Theorem I and Remark 2(a)) yields the vanishing of the higher direct image sheaves

$$R^q\pi_*\big(\mathcal{K}_M \otimes \mathcal{O}(|Z|-Z)\big) = 0 \ , \ q \geq 1. \tag{58}$$

This is also true without any positivity condition if $Z = |Z|$, a situation which occurs for example when we resolve a homogeneous isolated singularity by a single blow-up. Since

$$0 \to \mathcal{K}_M \otimes \mathcal{O}(|Z|-Z) \hookrightarrow \mathcal{C}_\sigma^{n,0}(L_{|Z|-Z}) \xrightarrow{\overline{\partial}_w} \mathcal{C}_\sigma^{n,1}(L_{|Z|-Z}) \xrightarrow{\overline{\partial}_w} \ldots \tag{59}$$

is a fine resolution of $\mathcal{K}_M \otimes \mathcal{O}(|Z|-Z)$, it follows then from (58) analogously to section 2.2.3 that the direct image complex $(\pi_*\mathcal{C}_\sigma^{n,*}(L_{|Z|-Z}), \pi_*\overline{\partial}_w)$ is a fine resolution of $\mathcal{K}_X^s \cong \pi_*\big(\mathcal{K}_M \otimes \mathcal{O}(|Z|-Z)\big)$. So, in this situation, (56) and (57) are in fact isomorphisms. By use of the L^2-version of Serre duality, this also led to a smooth realization of the L^2-cohomology with respect to the $\overline{\partial}$-operator in the sense of distributions for $(0,q)$-forms on a compact Hermitian space (see [R9], Theorem 1.6).

In the present exposition, we explain how the semi-positivity condition can be dropped in general. The first main result in this general situation is as follows:

Theorem 2.11. ([R10], Theorem 1.1) *Let X be a Hermitian complex space of pure dimension $n \geq 2$ with only isolated singularities, and $\pi : M \to X$ a resolution of singularities with only normal crossings. Then:*

$$\mathcal{K}_X^s \cong \pi_*\big(\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)\big),$$

where $\mathcal{K}_X^s$ is the canonical sheaf for the $\overline{\partial}_s$-operator (i.e. the canonical sheaf of holomorphic $(n,0)$-forms with Dirichlet boundary condition), $\mathcal{K}_M$ is the usual canonical sheaf on M and $Z = \pi^{-1}(\operatorname{Sing} X)$ the unreduced exceptional divisor.

The pull-back of forms π^* induces for $q \geq 1$ natural exact sequences

$$\begin{aligned} 0 \to H^q(X, \mathcal{K}_X^s) &\overset{\pi^*}{\longrightarrow} H^q(M, \mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)) \longrightarrow \Gamma(X, \mathcal{R}^q) \to 0, \quad (60)\\ 0 \to H^q_{cpt}(X, \mathcal{K}_X^s) &\overset{\pi^*}{\longrightarrow} H^q_{cpt}(M, \mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)) \longrightarrow \Gamma(X, \mathcal{R}^q) \to 0, \quad (61) \end{aligned}$$

where $\mathcal{R}^q$ is the higher direct image sheaf $R^q\pi_*(\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z))$.

In this statement one can clearly replace X by any open open subset $\Omega \subset X$ so that M has to be replaced by $\pi^{-1}(\Omega)$. Recall that the connection to the mappings (56) and (57) is as follows. Since the upper line in the diagram (54) is a fine resolution, we have

$$\begin{aligned} H^q(\Omega, \mathcal{K}_X^s) &= H^q(\Gamma(\Omega, \mathcal{F}^{n,*})),\\ H^q_{cpt}(\Omega, \mathcal{K}_X^s) &= H^q(\Gamma_{cpt}(\Omega, \mathcal{F}^{n,*})). \end{aligned}$$

On the other hand, (59) also is a fine resolution so that

$$\begin{aligned} H^q(\pi^{-1}(\Omega), \mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)) &= H^q(\Gamma(\pi^{-1}(\Omega), \mathcal{C}_\sigma^{n,*}(L_{|Z|-Z}))),\\ H^q_{cpt}(\pi^{-1}(\Omega), \mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)) &= H^q(\Gamma_{cpt}(\pi^{-1}(\Omega), \mathcal{C}_\sigma^{n,*}(L_{|Z|-Z}))). \end{aligned}$$

The proof of Theorem 2.11 is based on the following observation. If $\mathcal{G}^{n,*}$ is a fine resolution of $\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)$ as in (59), then the non-exactness of the direct image complex $\pi_*\mathcal{G}^{n,*}$ can be expressed by the higher direct image sheaves $\mathcal{R}^q := R^q\pi_*\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)$, $q \geq 1$. These are skyscraper sheaves for X has only isolated singularities. So, they are acyclic. On the other hand, global sections in $\mathcal{R}^q$ can be expressed globally by $L^{2,loc}$-forms, i.e. the canonical map

$$\Gamma(X, \ker \pi_*\overline{\partial}_w \cap \pi_*\mathcal{C}_\sigma^{n,q}(L_{|Z|-Z})) \longrightarrow \Gamma(X, \mathcal{R}^q) = \bigoplus_{x \in \operatorname{Sing} X} \mathcal{R}_x^q \qquad (62)$$

is surjective. These two properties allow to express the cohomology of the canonical sheaf $\mathcal{K}_X^s$ in terms of the cohomology of the direct image complex $\pi_*\mathcal{G}^{n,*}$ modulo global sections in $\mathcal{R}^q$. We will explain that more precisely.

Let us first prove the surjectivity of (62). So, let $[\omega] \in \Gamma(X, \mathcal{R}^q)$. Since $\mathcal{R}^q$ is a skyscraper sheaf as described above, $[\omega]$ is represented by a set of germs $\{\omega_x\}_{x\in \operatorname{Sing} X}$, where each ω_x is given by a $\overline{\partial}_w$-closed (n,q)-form with values in $L_{|Z|-Z}$ in a neighborhood U_x of the component $\pi^{-1}(\{x\})$ of the exceptional set,

$$\omega_x \in \ker \overline{\partial}_w \subset \mathcal{C}^{n,q}_\sigma(L_{|Z|-Z})(U_x).$$

This follows from the facts that

$$\big(\mathcal{R}^q\pi_*(\mathcal{K}_M \otimes \mathcal{O}(|Z|-Z))\big)_x = \varinjlim_{x\in W} H^q(\pi^{-1}(W), \mathcal{K}_M \otimes \mathcal{O}(|Z|-Z))$$

and (59) is a fine resolution. We will show in a moment that we can assume that the forms ω_x have compact support in U_x. Hence, by trivial extension, they give rise to a global form $\omega \in \ker \overline{\partial}_w \subset \mathcal{C}^{n,q}_\sigma(L_{|Z|-Z})(\pi^{-1}(X))$, i.e.

$$\omega \in \Gamma(\pi^{-1}(X), \ker \overline{\partial}_w \cap \mathcal{C}^{n,q}_\sigma(L_{|Z|-Z}))$$

represents $[\omega] \in \Gamma(X, \mathcal{R}^q)$. This representation shows that (62) is in fact surjective.

It remains to show that we can choose ω_x with compact support in U_x. To see that, we can use the fact that $Z-|Z|$ is effective so that ω_x can be interpreted as a $\overline{\partial}$-closed form in $\mathcal{C}^{n,q}_\sigma(U_x)$. But Takegoshi's vanishing theorem (see [T1], Theorem 2.1) tells us that there is a solution $\eta_x \in \mathcal{C}^{n,q-1}_\sigma(V_x)$ to the equation $\overline{\partial}_w \eta_x = \omega_x$ on a smaller neighborhood of the component $\pi^{-1}(\{x\})$ of the exceptional set. Since $(\mathcal{C}^{n,*}_\sigma, \overline{\partial}_w)$ is a fine resolution of the canonical sheaf $\mathcal{K}_M$, this fact is also expressed by the vanishing of the higher direct image sheaves

$$R^q\pi_*\mathcal{K}_M = 0 \ , \ q \geq 1,$$

as we have already seen. Let χ_x be a smooth cut-off function with compact support in V_x that is identically 1 in a neighborhood of $\pi^{-1}(\{x\})$. Then $\overline{\partial}_w(\chi_x\eta_x)$ is the form we were looking for because it has compact support in U_x and equals ω_x in a neighborhood of $\pi^{-1}(\{x\})$ so that it can be considered again as a form with values in $L_{|Z|-Z}$.

We can now prove Theorem 2.11 by the use of some homological algebra:

Theorem 2.12. *Let X, M be paracompact Hausdorff spaces and $\pi : M \to X$ a continuous map. Let $\mathcal{C}$ be a sheaf (of abelian groups) over M and*

$$0 \to \mathcal{C} \hookrightarrow \mathcal{C}^0 \xrightarrow{c_0} \mathcal{C}^1 \xrightarrow{c_1} \mathcal{C}^2 \xrightarrow{c_2} \mathcal{C}^3 \longrightarrow ... \tag{63}$$

a fine resolution. Let $\mathcal{A} \cong \pi_*\mathcal{C}$ be a sheaf on X, isomorphic to the direct image of $\mathcal{C}$, and $0 \to \mathcal{A} \to \mathcal{A}^*$ a fine resolution of $\mathcal{A}$ over X.

Let $\mathcal{B} := \pi_*\mathcal{C}$ be the direct image of $\mathcal{C}$ and $(\mathcal{B}^*, b_*) = (\pi_*\mathcal{C}^*, \pi_*c_*)$ the direct image complex which is again fine but not necessarily exact. Since (63) is a fine resolution, the non-exactness of $0 \to \mathcal{B} \to \mathcal{B}^*$ is measured by the higher direct image sheaves

$$\mathcal{R}^q := R^q\pi_*\mathcal{C} \ , \ q \geq 1.$$

Let

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A} \arrow[r] \arrow[d, "\cong"', "f"] & \mathcal{A}^* \arrow[d, "g"] \\ 0 \arrow[r] & \mathcal{B} \arrow[r] & \mathcal{B}^* \end{tikzcd} \tag{64}$$

be a morphism of complexes, and assume that the direct image sheaves $\mathcal{R}^q$ are acyclic and that the canonical maps $\Gamma(X, \ker b_q) \to \Gamma(X, \mathcal{R}^q)$ are surjective for all $q \geq 1$.

Then g induces for all $q \geq 1$ a natural injective homomorphism

$$H^q(\Gamma(X, \mathcal{A}^*)) \xrightarrow{[g_q]} H^q(\Gamma(X, \mathcal{B}^*))$$

with coker $[g_q] = \Gamma(X, \mathcal{R}^q)$. More precisely, there is a natural exact sequence

$$0 \to H^q(\Gamma(X, \mathcal{A}^*)) \xrightarrow{[g_q]} H^q(\Gamma(X, \mathcal{B}^*)) \longrightarrow \Gamma(X, \mathcal{R}^q) \to 0.$$

In this sequence, one can replace $H^q(\Gamma(X, \mathcal{B}^*))$ by $H^q(\Gamma(M, \mathcal{C}^*))$ because

$$\begin{aligned} \Gamma(X, \mathcal{B}^q) &= \Gamma(\pi^{-1}(X), \mathcal{C}^q) = \Gamma(M, \mathcal{C}^q), \\ b_q(\Gamma(X, \mathcal{B}^q)) &= c_q(\Gamma(M, \mathcal{C}^q)) \end{aligned}$$

by definition for all $q \geq 0$.

For the proof, we refer to [R10], Theorem 2.6. Our Theorem 2.11 here follows now from Theorem 2.12 with the following choices. Let

$$\begin{aligned} (\mathcal{A}, \mathcal{A}^*, a_*) &:= (\mathcal{K}_X^s, \mathcal{F}^{n,*}, \overline{\partial}_s), \\ (\mathcal{C}, \mathcal{C}^*, c_*) &:= (\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z), \mathcal{C}_\sigma^{n,*}(L_{|Z|-Z}), \overline{\partial}_w), \\ (\mathcal{B}, \mathcal{B}^*, b_*) &:= (\pi_*(\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)), \pi_*\mathcal{C}_\sigma^{n,*}(L_{|Z|-Z}), \pi_*\overline{\partial}_w), \end{aligned}$$

and for f, g in (64), we use the maps induced by pull-back of froms under $\pi : M \to X$ so that the diagram (64) is equal to the diagram (54). The higher direct image sheaves

$$\mathcal{R}^q = R^q\pi_*\big(\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)\big)$$

are in fact acyclic for $q \geq 1$ because they are skyscraper sheaves with support in the discrete singular set, and the canonical map $\Gamma(X, \ker b_q) \to \Gamma(X, \mathcal{R}^q)$ is nothing else but the map (62). So, it is in fact surjective for all $q \geq 1$.

Hence, Theorem 2.12 yields exactness of the sequences

$$0 \to H^q(\Gamma(X, \mathcal{F}^{n,*})) \xrightarrow{\pi^*} H^q(\Gamma(M, \mathcal{C}_\sigma^{n,*}(L_{|Z|-Z})) \longrightarrow \Gamma(X, \mathcal{R}^q) \to 0 \tag{65}$$

for all $q \geq 1$. But this gives the statement of Theorem 2.11 since the upper line in the diagram (54) and (59) are fine resolutions of $\mathcal{K}_X^s$ and $\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)$, respectively (see the remarks after Theorem 2.11). The statement about the cohomology with compact support in Theorem 2.11 follows analogously by exactly the same proof (Theorem 2.12 holds also for the cohomology with compact support).

If X is compact, then $\overline{\partial}_s = \overline{\partial}_{min}$ on $X - \operatorname{Sing} X$ so that

$$H^q(\Gamma(X, \mathcal{F}^{n,*})) = H^{n,q}_{min}(X - \operatorname{Sing} X).$$

On the other hand, compactness of M yields in this situation that $\overline{\partial}_w = \overline{\partial}_{max} = \overline{\partial}_{min}$ on M. Thus,

$$H^q(\Gamma(M, \mathcal{C}^{n,*}_\sigma(L_{|Z|-Z}))) = H^{n,q}_{min}(M, L_{|Z|-Z}),$$

where $H^{n,q}_{min}(M, L_{|Z|-Z})$ is the L^2-Dolbeault cohomology for forms with values in the Hermitian line bundle $L_{|Z|-Z}$. Hence, we deduce from (65) that pull-back of forms under $\pi : M \to X$ induces the exact sequence

$$0 \to H^{n,q}_{min}(X - \operatorname{Sing} X) \xrightarrow{\pi^*} H^{n,q}_{min}(M, L_{|Z|-Z}) \longrightarrow \Gamma(X, \mathcal{R}^q) \to 0 \tag{66}$$

for all $q \geq 1$ if X is a compact Hermitian space of pure dimension $n \geq 2$ with only isolated singularities.

This L^2-version of Theorem 2.11 is pretty interesting because it allows to carry over our results to $(0, q)$-forms by use of the L^2-Serre duality Theorem 2.3. We would like to do that also for non-compact spaces. So, we have to invest more work. It is well-known that the the L^2- and the L^2_{loc}-Dolbeault cohomology are naturally isomorphic on strongly pseudoconvex domains in complex manifolds. The isomorphism is given by the natural mapping from L^2 to L^2_{loc}-forms. In other words, if D is a relatively bounded domain with smooth strongly pseudoconvex boundary, then there exists for $q \geq 1$ a natural isomorphism

$$H^{p,q}_{max}(D) \xrightarrow{\cong} H^q(D, \Omega^p), \tag{67}$$

where Ω^p is the sheaf of germs of holomorphic p-forms. By use of L^2-duality on the left-hand side and the classical Serre duality on the right-hand side, (67) is equivalent to the natural isomorphism

$$H^{n-q}_{cpt}(D, \Omega^{n-p}) \xrightarrow{\cong} H^{n-p,n-q}_{min}(D) \tag{68}$$

for all $q \geq 1$. This map is induced by the natural inclusion of forms with compact support into the domain of the $\overline{\partial}_{min}$-operator (forms with compact support vanish close to the boundary of the domain so that there is no problem with the boundary condition). The principles which lead to (67) and (68) can be carried over to spaces with isolated singularities in the sense that we can allow isolated singularities in the interior of such a domain D with strongly pseudoconvex boundary:

Theorem 2.13. ([R10], Theorem 6.8) *Let X be a Hermitian complex space of pure dimension $n \geq 2$ with only isolated singularities and $\Omega \subset\subset X$ a domain with strongly pseudoconvex boundary which does not intersect the singular set, $b\Omega \cap \operatorname{Sing} X = \emptyset$.*

Let $0 \leq q < n$. Then the natural inclusion map

$$\iota : H^q(\Gamma_{cpt}(\Omega, \mathcal{F}^{n,*})) \to H^{n,q}_{min}(\Omega^*)$$

is an isomorphism.

Concerning the proof, we should remark here that this is not a straight forward generalization from the situation on manifolds: the proof requires for example an application of Theorem 2.11 to overcome the additional difficulties.

On the other hand, (68) is also valid for forms with values in holomorphic vector bundles so that the natural inclusion of compact forms into the domain of $\overline{\partial}_{min}$ gives the natural isomorphism

$$H^q(\Gamma_{cpt}(\pi^{-1}(\Omega), \mathcal{C}^{n,*}_\sigma(L_{|Z|-Z})) \xrightarrow{\cong} H^{n,q}_{min}(\pi^{-1}(\Omega), L_{|Z|-Z}) \tag{69}$$

if Ω is an open set as in Theorem 2.13 and $0 \leq q < n$.

With the help of (69) and Theorem 2.13, one can deduce the following L^2-version of Theorem 2.11:

Theorem 2.14. ([R10], Theorem 1.2) *Let (X, h) be a Hermitian complex space of pure dimension $n \geq 2$ with only isolated singularities, $\pi : M \to X$ a resolution of singularities with only normal crossings, and $\Omega \subset\subset X$ a relatively compact domain. Let $0 \leq q < n$, $\widetilde{\Omega} := \pi^{-1}(\Omega)$ and $\Omega^* = \Omega - \operatorname{Sing} X$. Provide $\widetilde{\Omega}$ with a (regular) Hermitian metric which is equivalent to $\pi^* h$ close to the boundary $b\widetilde{\Omega}$.*

Let $Z := \pi^{-1}(\Omega \cap \operatorname{Sing} X)$ be the unreduced exceptional divisor (over Ω) and $\mathcal{K}_M$ the canonical sheaf on M. Let $L_{|Z|-Z} \to M$ be a Hermitian holomorphic line bundle such that holomorphic sections in $L_{|Z|-Z}$ correspond to holomorphic sections in $\mathcal{O}(|Z| - Z)$.

Then the pull-back of forms π^* induces a natural exact sequence

$$0 \to H^{n,q}_{min}(\Omega^*) \xrightarrow{h_q} H^{n,q}_{min}(\widetilde{\Omega}, L_{|Z|-Z}) \longrightarrow \Gamma(\Omega, R^q\pi_*(\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)) \to 0, \tag{70}$$

where the group on the right hand side has to be replaced by 0 if $q = 0$.

Note that we do not require a strongly pseudoconvex boundary of the domain Ω and that singularities in the boundary $b\Omega$ are permitted. Any regular metric on M will do the job if there are no singularities in the boundary of Ω. If $\Omega = X$ is compact, then the case $q = n$ can be included as we have already seen (in that case Theorem 2.11 gives the statement directly, see (66)).

The principles of the proof of Theorem 2.14 are as follows. First, it follows directly from Theorem 2.11 by use of Theorem 2.13 and (69) that the statement is true for a small strongly pseudoconvex neighborhood of an isolated singularity. For the general statement, let $x_1, ..., x_L$ be the (finitely many) isolated singularities in the domain Ω and choose pairwise disjoint strongly pseudoconvex neighborhoods $\Omega_1, ..., \Omega_L$ of the $x_1, ..., x_L$ in Ω. Then the statement of the Theorem is clearly also true on $V := \bigcup_{j=1}^L \Omega_j$. On the other hand, let $U_1, ...U_L$ be smaller neighborhoods of the isolated singularities such that $U_j \subset\subset \Omega_j$ and let $U := \Omega - \big(\bigcup_{j=1}^L \overline{U_j}\big)$. Then the statement of the Theorem is trivially true on U and on $U \cap V$. But there is a Mayer-Vietoris sequence for the $\overline{\partial}_{min}$-operator for the open covering $U \cup V = \Omega$ because in this special constellation we have a partition of unity as follows: let $\chi \in C^\infty_{cpt}(V)$ be identically 1 in a neighborhood of $\bigcup \overline{U_j}$. If ω is a form in the domain of $\overline{\partial}_{min}(U \cup V)$, then $\chi\omega \in \operatorname{Dom} \overline{\partial}_{min}(V)$, $(1 - \chi)\omega \in \operatorname{Dom} \overline{\partial}_{min}(U)$ and $\omega = \chi\omega + (1 - \chi)\omega$. So, the boundary condition of the $\overline{\partial}_{min}$-operator does not cause difficulties for the Mayer-Vietoris sequence and the statement of the Theorem is also true on $\Omega = U \cup V$ by use of the long exact Mayer-Vietoris sequence. This is elaborated in detail in [R10], section 6.6.

2.3.4 L^2-Dolbeault cohomology of low degree

We can now use the L^2-version of Serre duality Theorem 2.3 to derive from Theorem 2.14 also a resolution of the L^2-Dolbeault cohomology with respect to the $\overline{\partial}$-operator in the sense of distributions $\overline{\partial}_{max}$ for $(0,q)$-forms. What we get immediately from the L^2-resolution of the $\overline{\partial}_{min}$-cohomology of (n,q)-forms is as follows:

Theorem 2.15. *In the situation of Theorem 2.14, there is a natural surjective mapping*

$$s_q : H^{0,q}_{max}(\pi^{-1}(\Omega), L_{Z-|Z|}) \to H^{0,q}_{max}(\Omega - \operatorname{Sing} X) \tag{71}$$

for all $0 < q \leq n$, where $L_{Z-|Z|} = L^*_{|Z|-Z}$ is the dual of the bundle from above. The map s_n is an isomorphism.

If $\Omega = X$ is compact, then the case $q = 0$ can be included.

Proof. We could deduce the statement directly from the injectivity of the map h_{n-q} in (70) if L^2-Serre duality (i.e. Theorem 2.3) would apply to the cohomology groups in (70). However, we can not assume that the cohomology groups allow for harmonic representation and Serre duality if Ω is an arbitrary domain. But this problem can be bypassed as in the proof of Theorem 2.14 by use of a Mayer-Vietoris sequence. So, let $V \subset\subset \Omega$ be a neighborhood of the singular set in Ω with smooth strongly pseudoconvex boundary. Then the $\overline{\partial}$-operator in the L^2-sense of distributions $\overline{\partial}_{max}$ has closed range on $V^* = V - \operatorname{Sing} X$ for $(0,q)$-forms (see Theorem 2.4). Hence, there is a non-degenerate pairing

$$\{\cdot,\cdot\}_X : H^{0,q}_{max}(V^*) \times H^{n,n-q}_{min}(V^*) \to \mathbb{C}$$

given by

$$\{[\eta],[\psi]\}_X := \int_{V^*} \eta \wedge \psi$$

according to Theorem 2.3 for all $0 \leq q \leq n$. Let $\widetilde{V} := \pi^{-1}(V)$. Then $\widetilde{V}$ also has a smooth strongly pseudoconvex boundary since there are no singularities in the boundary of V. So, L^2-Serre duality is also valid on $\widetilde{V}$ in M. Here, we need Theorem 2.3 for forms with values in holomorphic line bundles, but this generalization is more or less straight forward (see [R9], Theorem 2.3). So, there is another non-degenerate pairing

$$\{\cdot,\cdot\}_M : H^{0,q}_{max}(\pi^{-1}(V), L_{Z-|Z|}) \times H^{n,n-q}_{min}(\pi^{-1}(V), L_{|Z|-Z}) \to \mathbb{C}$$

given by

$$\{[\eta],[\psi]\}_M := \int_{\pi^{-1}(V)} \eta \wedge \psi$$

So, the map s_q is defined as follows: for a class $[\eta] \in H^{0,q}_{max}(\pi^{-1}(V), L_{Z-|Z|})$, $s_q([\eta])$ is the unique class in $H^{0,q}_{max}(V^*)$ such that

$$\{s_q([\eta]),[\psi]\}_X = \{[\eta],[\pi^*\psi]\}_M$$

for each class $[\psi] \in H^{n,n-q}_{min}(V^*)$. This mapping is surjective since it is the dual morphism to h_{n-q} in (70) because $h_{n-q}([\psi]) = [\pi^*\psi]$. We have thus proved the statement of the Theorem for such a strongly pseudoconvex domain V.

But then we can deduce the general statement as in the remarks on the proof of Theorem 2.14. Let $W \subset\subset V$ be smaller neighborhood of the singular set in Ω and $U := \Omega - \overline{W}$. Then the statement of the Theorem is trivially true on U and on $U \cap V$. But then it follows also on $\Omega = U \cup V$ by use of the Mayer-Vietoris sequence for the $\overline{\partial}_{max}$-equation. $\square$

It remains to identify the kernel of the mapping s_q in Theorem 2.15 for $q < n$. By Theorem 2.14 and the proof of Theorem 2.15, this must be the dual object to

$$\Gamma(\Omega, R^{n-q}\pi_*(\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)).$$

In fact, there is (for $q < n$) a duality between the higher direct image sheaves $R^{n-q}\pi_*\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)$ on the one hand and the flabby cohomology H^q_E of $\mathcal{O}(Z - |Z|)$ with support on the exceptional set $E = |Z|$ on the other hand. This duality is compatible with the L^2-Serre duality pairings that we used above. We will explain that more precisely.

We need some preliminaries on exceptional sets. Let X be a complex space. A compact nowhere discrete, nowhere dense analytic set $A \subset X$ is an exceptional set (in the sense of Grauert [G2], §2.Definition 3) if there exists a proper, surjective map $\pi : X \to Y$ such that $\pi(A)$ is discrete, $\pi : X - A \to Y - \pi(A)$ is biholomorphic and for every open set $D \subset Y$ the map $\pi^* : \Gamma(D, \mathcal{O}_Y) \to \Gamma(\pi^{-1}(D), \mathcal{O}_X)$ is surjective. Exceptional sets are characterized as follows:

Theorem 2.16. *(Grauert [G2], §2.Satz 5) Let X be a complex space and $A \subset X$ a nowhere discrete compact analytic set. Then A is an exceptional set exactly if there exists a strongly pseudoconvex neighborhood $U \subset\subset X$ of A such that A is the maximal compact analytic subset of U.*

An important statement about exceptional sets is as follows:

Theorem 2.17. *(Laufer [L1], Lemma 3.1) Let $\pi : U \to Y$ exhibit A as exceptional set in U with Y a Stein space. If $V \subset U$ with V a holomorphically convex neighborhood of A and $\mathcal{G}$ is a coherent analytic sheaf on U, then the restriction map $\rho : H^i(U, \mathcal{G}) \to H^i(V, \mathcal{G})$ is an isomorphism for $i \geq 1$.*

Let $\pi : M \to X$ be a resolution as in Theorem 2.14 and $E = |\pi^{-1}(\Omega \cap \operatorname{Sing} X)|$ the exceptional set. For a closed subset K of M and a sheaf of abelian groups $\mathcal{G}$, we denote by $H^*_K(M, \mathcal{G})$ the flabby cohomology of $\mathcal{G}$ with support in K. Here, we are interested in the case where K is the exceptional set E. A nice review of cohomology with support on the exceptional set can be found in [OV5], section 3.2, a more extensive treatment in [K1].

Let $\Omega \subset X$ be an open set and $\widetilde{\Omega} := \pi^{-1}(\Omega)$. In the following, we may assume that $E \subset \widetilde{\Omega}$. Since X has only isolated singularities, there exists a (smoothly bounded) strongly pseudoconvex neighborhood V of E in $\widetilde{\Omega}$ which exhibits E as exceptional set in $\widetilde{\Omega}$ in the sense of Theorem 2.16 and Theorem 2.17.

Since E is a compact subset of V in M, $H^*_E(V,\mathcal{G}) = H^*_E(\widetilde{\Omega},\mathcal{G})$ by excision and so we have natural homomorphisms

$$\gamma_j : H^j_E(\widetilde{\Omega},\mathcal{G}) \to H^j_{cpt}(V,\mathcal{G}).$$

Then:

Theorem 2.18. *(Karras [K1], Proposition 2.3) If $\mathcal{G}$ is a coherent analytic sheaf on M such that $\operatorname{depth}_x\mathcal{G} \geq d$ for all $x \in V - E$, then*

$$\gamma_j : H^j_E(\widetilde{G},\mathcal{G}) \to H^j_{cpt}(V,\mathcal{G})$$

is an isomorphism for $j < d$.

On the other hand we have seen that

$$\Gamma(\Omega, R^q\pi_*\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)) = \varinjlim_U H^q(\pi^{-1}(U), \mathcal{K}_M \otimes \mathcal{O}(|Z| - Z))$$

for $q \geq 1$, where the limit is over open neighborhoods of $\operatorname{Sing} X \cap \Omega$. But then the natural maps (induced by restriction of cohomology classes)

$$\alpha_q : H^q(V, \mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)) \to \Gamma(\Omega, R^q\pi_*\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)) \tag{72}$$

are isomorphisms for $q \geq 1$ by Theorem 2.17.

By use of Serre duality, there exists a non-degenerate pairing

$$H^{n-q}_{cpt}(V, \mathcal{O}(Z - |Z|)) \times H^q(V, \mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)) \to \mathbb{C}. \tag{73}$$

Since $\operatorname{depth} \mathcal{O}(Z - |Z|) = n$, we can combine Theorem 2.18 with (72) and (73) and obtain:

Theorem 2.19. *In the situation described above, there is a natural non-degenerate pairing*

$$\{\cdot,\cdot\}_E : H^{n-q}_E(\widetilde{\Omega}, \mathcal{O}(Z - |Z|)) \times \Gamma(\Omega, R^q\pi_*\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)) \to \mathbb{C}$$

for $1 \leq q \leq n$ which is induced by Serre duality and the natural isomorphisms

$$\gamma_{n-q} : \quad H^{n-q}_E(\widetilde{\Omega}, \mathcal{O}(Z - |Z|)) \to H^{n-q}_{cpt}(V, \mathcal{O}(Z - |Z|)), \tag{74}$$
$$\alpha_q : \quad H^q(V, \mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)) \to \Gamma(\Omega, R^q\pi_*\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)). \tag{75}$$

More precisely, for two classes represented by (74) *and* (75),

$$\gamma^{-1}_{n-q}([\eta]) \in H^{n-q}_E(\widetilde{\Omega}, \mathcal{O}(Z - |Z|))$$

and

$$\alpha_q([\psi]) \in \Gamma(\Omega, R^q\pi_*\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z)),$$

we have:

$$\{\gamma^{-1}_{n-q}([\eta]), \alpha_q([\psi])\}_E = \int_V \eta \wedge \psi. \tag{76}$$

Let us now return to the exact sequence

$$0 \to H^{n,q}_{min}(\Omega^*) \xrightarrow{h_q} H^{n,q}_{min}(\widetilde{\Omega}, L_{|Z|-Z}) \xrightarrow{p_q} \Gamma(\Omega, R^q\pi_*(\mathcal{K}_M \otimes \mathcal{O}(|Z|-Z)) \to 0, \quad (77)$$

from Theorem 2.14 (let $0 < q < n$). Then the surjection p_q factors through the isomorphism α_q from Theorem 2.19, i.e. $p_q = \alpha_q \circ r_q$ where

$$H^{n,q}_{min}(\widetilde{\Omega}, L_{|Z|-Z}) \xrightarrow{r_q} H^q(V, \mathcal{K}_M \otimes \mathcal{O}(|Z|-Z)) \xrightarrow{\alpha_q} \Gamma(\Omega, R^q\pi_*(\mathcal{K}_M \otimes \mathcal{O}(|Z|-Z))$$

are the natural mappings (recall that $V \subset\subset \widetilde{\Omega}$), i.e. r_q is the map induced by restriction of forms from $\widetilde{\Omega}$ to V. Combining Theorem 2.15 with Theorem 2.19, we can see that the short sequence dual to (77) is (with $p = n-q$)

$$0 \to H^p_E(\widetilde{\Omega}, \mathcal{O}(Z-|Z|)) \xrightarrow{i_p} H^{0,p}_{max}(\widetilde{\Omega}, L_{Z-|Z|}) \xrightarrow{s_p} H^{0,p}_{max}(\Omega^*) \to 0. \quad (78)$$

Here, the injection i_p factors through the isomorphism γ_p from Theorem 2.19, i.e. $i_p = \iota_p \circ \gamma_p$ where

$$H^p_E(\widetilde{\Omega}, \mathcal{O}(Z-|Z|)) \xrightarrow{\gamma_p} H^p_{cpt}(V, \mathcal{O}(Z-|Z|)) \xrightarrow{\iota_p} H^{0,p}_{max}(\widetilde{\Omega}, L_{Z-|Z|})$$

are the natural mappings, i.e. ι_p is the map induced by trivial extension of forms with compact support in V to $\widetilde{\Omega}$. As in the proof of Theorem 2.15, where we have seen that $s_{n-q} = h_q^*$ in the sense that $\{s_{n-q}([\eta]), [\psi]\}_\Omega = \{[\eta], h_q[\psi]\}_{\widetilde{\Omega}}$, we deduce from (76) that the map i_{n-q} is the map dual to p_q:

$$\begin{aligned}
\{i_{n-q}(\gamma_{n-q}^{-1}([\eta])), [\psi]\}_{\widetilde{\Omega}} &= \{\iota_{n-q}[\eta], [\psi]\}_{\widetilde{\Omega}} = \int_{\widetilde{\Omega}} \iota_{n-q}\eta \wedge \psi = \int_V \eta \wedge r_q\psi \\
&= \{\gamma_{n-q}^{-1}([\eta]), \alpha_q(r_q([\psi]))\}_E = \{\gamma_{n-q}^{-1}([\eta]), p_q([\psi])\}_E
\end{aligned}$$

for all classes $[\eta] \in H^{n-q}_{cpt}(V, \mathcal{O}(Z-|Z|))$ and $[\psi] \in H^{n,q}_{min}(\widetilde{\Omega}, L_{|Z|-Z})$.

Hence it follows that (78) is in fact a short exact sequence induced by natural mappings. Summing up, we conclude another main result:

Theorem 2.20. ([R10], Theorem 1.3) *Let (X,h) be a Hermitian complex space of pure dimension $n \geq 2$ with only isolated singularities, $\pi : M \to X$ a resolution of singularities with only normal crossings, and $\Omega \subset\subset X$ a relatively compact domain.*

Let $\widetilde{\Omega} := \pi^{-1}(\Omega)$, $\Omega^* = \Omega - \operatorname{Sing} X$ and provide $\widetilde{\Omega}$ with a Hermitian metric which is equivalent to π^*h close to the boundary $b\widetilde{\Omega}$. Let $Z := \pi^{-1}(\Omega \cap \operatorname{Sing} X)$ be the unreduced exceptional divisor (over Ω) and $\mathcal{K}_M$ the canonical sheaf on M. Let $L_{Z-|Z|} \to M$ be a Hermitian holomorphic line bundle such that holomorphic sections in $L_{Z-|Z|}$ correspond to holomorphic sections in $\mathcal{O}(Z-|Z|)$.

Let $0 \leq q \leq n$ if $\Omega = X$ is compact and $0 < q \leq n$ otherwise. Then there exists a natural exact sequence

$$0 \to H^q_E(\widetilde{\Omega}, \mathcal{O}(Z-|Z|)) \to H^{0,q}_{max}(\widetilde{\Omega}, L_{Z-|Z|}) \to H^{0,q}_{max}(\Omega^*) \to 0,$$

where H^*_E is the flabby cohomology with support on the exceptional set $E = |Z|$. In case $q = n$, $H^n_E(\widetilde{\Omega}, \mathcal{O}(Z-|Z|))$ has to replaced by 0.

Note again that we do not require a strongly pseudoconvex boundary of the domain Ω, that singularities in the boundary $b\Omega$ are permitted and that any regular metric on M will do the job if there are no singularities in the boundary of Ω. Note also that $H^0_E(\widetilde{\Omega}, \mathcal{O}(Z - |Z|)) = 0$ by the identity theorem.

The idea to identify the kernel of the natural map

$$H^{0,q}_{max}(\widetilde{\Omega}, L_{Z-|Z|}) \to H^{0,q}_{max}(\Omega^*)$$

as the flabby cohomology of $\mathcal{O}(Z - |Z|)$ with support on E is inspired by the work of Øvrelid and Vassiliadou [OV5] who proved Theorem 2.20 recently in the case $q = n - 1$ (see [OV5], Theorem 1.4). Their method is quite different from our approach over Theorem 2.11.[19]

2.3.5 L^2-cohomology of a complex surface

If X is a complex surface with only isolated singularities (e.g. a normal surface) and $\pi : M \to X$ a resolution of singularities as in Theorem 2.20, then Øvrelid and Vassiliadou showed that $H^1_E(M, \mathcal{O}(Z - |Z|)) = 0$ (see [OV5], the proof of Corollary 5.2 and Remark 5.2.4).

Combining this with Theorem 2.11 and Theorem 2.20, we obtain the following result for a Hermitian complex surface like e.g. a normal projective surface which carries the restriction of the Fubini-Study metric:

Theorem 2.21. *Let X be a Hermitian complex surface with only isolated singularities and $\pi : M \to X$ a resolution of singularities with only normal crossings. Then there exist for all $0 \leq q \leq 2$ natural isomorphisms*

$$H^{2,q}_{max}(X - \operatorname{Sing} X) \to H^{2,q}(M).$$

Let $Z := \pi^{-1}(\operatorname{Sing} X)$ be the unreduced exceptional divisor. Then there exist for all $0 \leq q \leq 2$ natural isomorphisms

$$H^{0,q}(M, L_{Z-|Z|}) \to H^{0,q}_{max}(X - \operatorname{Sing} X),$$

where $L_{Z-|Z|} \to M$ is a Hermitian holomorphic line bundle such that holomorphic sections in $L_{Z-|Z}$ correspond to sections in $\mathcal{O}(Z - |Z|)$.

This gives an almost complete description of the L^2-cohomology of a complex surface with isolated singularities. Only the middle cohomology $H^{1,1}$ is missing. Theorem 2.21 was conjectured and mostly proven by Pardon and Stern in [PS1] (there was a difficulty with the critical group $H^{0,1}_{max}$). This difficulty has been first overcome completely now by Øvrelid and Vassiliadou in [OV5].

[19] We may remark that Øvrelid–Vassiliadou also use our representation $\mathcal{K}^s_X = \pi_*(\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z))$ from [R9].

3 The $\overline{\partial}$-Neumann operator

Besides studying the obstructions to solving the $\overline{\partial}$-equation in the L^2-sense on a singular space – what we have done above by providing a smooth model for the L^2-cohomology – it is also very interesting to study the regularity of the $\overline{\partial}$-equation in general. On domains in complex manifolds, the close connection between the regularity of the $\overline{\partial}$-equation on the one hand and the geometry of the domain (and its boundary) on the other hand is one of the central topics of complex analysis. It is an interesting task to establish such connections also between the regularity of the $\overline{\partial}$-equation at singularities and the geometry of the singularities. In the present exposition, we study the existence of compact L^2-solution operators for the $\overline{\partial}$-equation at isolated singularities and compactness of the $\overline{\partial}$-Neumann operator in the presence of such singularities. The material covered here stems mainly from [R8] and [OR].[20]

Compactness of the $\overline{\partial}$-Neumann operator can be seen as a boundary case of subelliptic regularity (when the gain in the subelliptic estimate tends to zero) and is an important property in the study of weakly pseudoconvex domains (see [S10] for a comprehensive discussion of the topic). Moreover, compactness of the $\overline{\partial}$-Neumann operator yields that the corresponding space of L^2-forms has an orthonormal basis consisting of eigenforms of the $\overline{\partial}$-Laplacian $\Box = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$. The eigenvalues of $\Box$ are non-negative, have no finite limit point and appear with finite multiplicity. It might be an interesting question to study wether there is a nice connection between the eigenvalues and the structure of the singularities.

We will derive compactness of the $\overline{\partial}$-Neumann operator not as usually from elliptic or subelliptic estimates, but from the existence of compact L^2-solution operators for the $\overline{\partial}$-equation at isolated singularities. It remains to study wether the complex Laplacian fulfills some subelliptic estimates at isolated singularities. A strong evidence is the gain of regularity of the L^2-solution operators for the $\overline{\partial}_w$-equation of Fornæss, Øvrelid and Vassiliadou [FOV2], but also the Hölder regularity at isolated singularities studied by Ruppenthal and Zeron [RZ1] (which we will discuss later in the context of integral formulas). Given there are some subelliptic estimates, this would lead to the question of which order they are and wether there is a connection between the order of the subelliptic estimates and the type of the singularities. Recall in this context that a domain in $\mathbb{C}^n$ with smooth pseudoconvex boundary is of finite type[21] exactly if the $\overline{\partial}$-Neumann problem is subelliptic, and that there is a deep connection between the type of the boundary and the order of subellipticity (cf. the works of Kohn, Catlin and D'Angelo).

There are other interesting regularity questions concerning the $\overline{\partial}$-Neumann operator, particularly in view of the canonical $\overline{\partial}$-solution operator $\overline{\partial}^* N$. Even if the $\overline{\partial}$-Neumann operator does not gain (a fraction of) a derivative, it could still be globally regular[22], a property which can be deduced from compactness as Kohn and

[20] Many of the main ideas originate from [R8], but the material has been set in a more general context in [OR], so that it seems appropriate to present the more general statements from [OR].

[21] A smoothly bounded domain Ω is called of finite type, if the order of contact of analytic varieties to Ω is bounded form above.

[22] An operator is called globally regular on a domain Ω if it preserves $C^\infty(\overline{\Omega})$

Nirenberg have shown on smoothly bounded domains (see [KN]).

Such questions are important in the following context, closing the circle to the L^2-theory described above. We have seen that the $\overline{\partial}_w$-equation is locally solvable for (n, q)-forms on a Hermitian space X of pure dimension n with arbitrary singularities. When we consider the $\overline{\partial}_s$-equation instead, we had to assume that X has only isolated singularities. But there are many examples which suggest that the $\overline{\partial}_s$-equation is also locally solvable for arbitrary singularities.

One way to obtain a solution of the $\overline{\partial}_s$-equation would be as follows. We know already that the $\overline{\partial}_w$-equation is solvable (and $\overline{\partial}_s$-closed forms are trivially $\overline{\partial}_w$-closed). So, we may consider the canonical $\overline{\partial}_w$ solution operator $\overline{\partial}_w^* N$, where N is the $\overline{\partial}_w$-Neumann operator. If this operator is sufficiently regular, i.e. if it maps smooth forms with compact support away from the singular set to bounded forms, then the solution would be a solution for the $\overline{\partial}_s$-equation.

3.1 The closed range property

Before we discuss more sophisticated properties of the $\overline{\partial}$-Neumann operator, we have to address the problem wether there exists a bounded (i.e. continuous) $\overline{\partial}$-Neumann operator. This question is closely related to the closed range property of the $\overline{\partial}$-operator. Let X be a Hermitian complex space of pure dimension n. Then the $\overline{\partial}$-operator in the sense of distributions has closed range in $L^2_{p,q}$ at isolated singularities of X for $p + q \neq n$ (or $q = n$).

3.1.1 Canonical solution operators and the $\overline{\partial}$-Neumann operator

In this section, we introduce the canonical solution operators for the $\overline{\partial}$-equation and the $\overline{\partial}^*$-equation and show how they can be used to represent the $\overline{\partial}$-Neumann operator on arbitrary Hermitian manifolds (Theorem 3.3). This gives a nice characterization of compactness of the $\overline{\partial}$-Neumann operator. The material here is well-known but it seems appropriate to include it here for convenience of the reader and as a foundation for our later discussion. More details can be found in [OR], section 7.

Let M be a Hermitian complex manifold of dimension n, and let $0 \leq p, q \leq n$ and $q \geq 1$. Let $L^2_*(M)$ be the L^2-forms on M,

$$\overline{\partial}_{p,q} : L^2_{p,q-1}(M) \to L^2_{p,q}(M)$$

the $\overline{\partial}$-operator in the sense of distributions and $\overline{\partial}^*_{p,q}$ its L^2-adjoint. In this chapter, by the $\overline{\partial}$-operator, we always mean the $\overline{\partial}$-operator in the sense of distributions which was also denoted by $\overline{\partial}_w$ or $\overline{\partial}_{max}$ before. It is well known that $\overline{\partial}_{p,q}$ has closed range exactly if $\overline{\partial}^*_{p,q}$ has closed range (see e.g. [H6], Theorem 1.1.1).

We define canonical solution operators for $\overline{\partial}_{p,q}$ and $\overline{\partial}^*_{p,q}$ even if the ranges of these operators are not closed by using the orthogonal decomposition

$$\begin{aligned} L^2_{p,q-1}(M) &= \ker \overline{\partial}_{p,q} \oplus \overline{\operatorname{Im} \overline{\partial}^*_{p,q}}, \\ L^2_{p,q}(M) &= \ker \overline{\partial}^*_{p,q} \oplus \overline{\operatorname{Im} \overline{\partial}_{p,q}}. \end{aligned}$$

Let

$$\begin{aligned}
\mathbf{S}_{p,q}:\ \ &\mathrm{Im}\,\overline{\partial}_{p,q} \to \big(\ker\overline{\partial}_{p,q}\big)^{\perp} = \overline{\mathrm{Im}\,\overline{\partial}^{*}_{p,q}},\\
\mathbf{S}'_{p,q}:\ \ &\mathrm{Im}\,\overline{\partial}^{*}_{p,q} \to \big(\ker\overline{\partial}^{*}_{p,q}\big)^{\perp} = \overline{\mathrm{Im}\,\overline{\partial}_{p,q}},
\end{aligned}$$

be given by the following assignments: for $u \in \mathrm{Im}\,\overline{\partial}_{p,q}$, let $\mathbf{S}_{p,q}u$ be the the unique element in $\overline{\partial}^{-1}_{p,q}(\{u\})$ orthogonal to $\ker\overline{\partial}_{p,q}$, and for $v \in \mathrm{Im}\,\overline{\partial}^{*}_{p,q}$, let $\mathbf{S}'_{p,q}v$ be the unique element in $(\overline{\partial}^{*}_{p,q})^{-1}(\{v\})$ orthogonal to $\ker\overline{\partial}^{*}_{p,q}$.

The operator $\mathbf{S}_{p,q}$ is bounded exactly if $\overline{\partial}_{p,q}$ has closed range. That can be seen as follows: Assume that $\mathbf{S}_{p,q}$ is bounded, then it is clear that $\overline{\partial}_{p,q}$ has closed range because $\overline{\partial}_{p,q}\mathbf{S}_{p,q}u = u$ for all $u \in \mathrm{Im}\,\overline{\partial}_{p,q}$. Assume conversely that $\overline{\partial}_{p,q}$ has closed range. Then we consider the Banach space

$$B_{\overline{\partial}} := \mathrm{Dom}\,\overline{\partial}_{p,q} \cap (\ker\overline{\partial}_{p,q})^{\perp}$$

with the norm

$$\|f\|^2_{B_{\overline{\partial}}} := \|f\|^2_{L^2_{p,q-1}(M)} + \|\overline{\partial}_{p,q}f\|^2_{L^2_{p,q}(M)}.$$

Then $\overline{\partial}_{p,q} : B_{\overline{\partial}} \to \mathrm{Im}\,\overline{\partial}_{p,q}$ is a bounded linear isomorphism and the same holds for the inverse operator $\mathbf{S}_{p,q} : \mathrm{Im}\,\overline{\partial}_{p,q} \to B_{\overline{\partial}}$.

By definition, $\mathbf{S}_{p,q}$ is bounded if there exists a constant $C_{p,q} > 0$ such that

$$\|f\|_{L^2_{p,q-1}(M)} \leq C_{p,q}\|\overline{\partial}_{p,q}f\|_{L^2_{p,q}(M)} \tag{79}$$

for all $f \in B_{\overline{\partial}}$ (see also [H6], Theorem 1.1.1).

Since $\overline{\partial}_{p,q}$ has closed range exactly if $\overline{\partial}^{*}_{p,q}$ has closed range, note that $\mathbf{S}_{p,q}$ is bounded exactly if $\mathbf{S}'_{p,q}$ is bounded, and in that case there also exists a constant $C'_{p,q} > 0$ such that

$$\|u\|_{L^2_{p,q}(M)} \leq C'_{p,q}\|\overline{\partial}^{*}_{p,q}u\|_{L^2_{p,q-1}(M)} \tag{80}$$

for all $u \in \mathrm{Dom}\,\overline{\partial}^{*}_{p,q} \cap (\ker\overline{\partial}^{*}_{p,q})^{\perp}$. We will see later that one can take $C'_{p,q} = C_{p,q}$.

Assume from now on that $\overline{\partial}_{p,q}$ has closed range. Then we extend $\mathbf{S}_{p,q}$ and $\mathbf{S}'_{p,q}$ to bounded operators

$$\begin{aligned}
\mathbf{S}_{p,q}:\ \ &L^2_{p,q}(M) \to \big(\ker\overline{\partial}_{p,q}\big)^{\perp} \subset L^2_{p,q-1}(M),\\
\mathbf{S}'_{p,q}:\ \ &L^2_{p,q-1}(M) \to \big(\ker\overline{\partial}^{*}_{p,q}\big)^{\perp} \subset L^2_{p,q}(M),
\end{aligned}$$

by setting

$$\begin{aligned}
\mathbf{S}_{p,q}u = 0 \qquad &\text{for} \quad u \in (\mathrm{Im}\,\overline{\partial}_{p,q})^{\perp} = \ker\overline{\partial}^{*}_{p,q},\\
\mathbf{S}'_{p,q}v = 0 \qquad &\text{for} \quad v \in (\mathrm{Im}\,\overline{\partial}^{*}_{p,q})^{\perp} = \ker\overline{\partial}_{p,q}.
\end{aligned}$$

Now then, $\mathbf{S}'_{p,q}$ is nothing else but the L^2-adjoint of $\mathbf{S}_{p,q}$, i.e. $\mathbf{S}'_{p,q} = \mathbf{S}^{*}_{p,q}$. That follows by symmetry and shows that one can choose $C'_{p,q} = C_{p,q}$.

Lemma 3.1. $\mathbf{S}_{p,q}$ *is compact exactly if there exists a compact* $\overline{\partial}$*-solution operator*

$$\mathbf{T}_{p,q} : Im\ \overline{\partial}_{p,q} \to L^2_{p,q-1}(M).$$

Proof. Simply compose $\mathbf{T}_{p,q}$ with the (bounded) orthogonal projection onto $(\ker \overline{\partial}_{p,q})^\perp$ and extend this operator by zero to $(\text{Im } \overline{\partial}_{p,q})^\perp$. The other direction is trivial. $\square$

Now, we draw our attention to the $\overline{\partial}$-Neumann operator which we will represent by use of the canonical $\overline{\partial}$-solution operators discussed above. On

$$\text{Dom}\,\square_{p,q} = \{u \in \text{Dom}\,\overline{\partial}_{p,q+1} \cap \text{Dom}\,\overline{\partial}^*_{p,q} : \overline{\partial}_{p,q+1} u \in \text{Dom}\,\overline{\partial}^*_{p,q+1}, \overline{\partial}^*_{p,q} u \in \text{Dom}\,\overline{\partial}_{p,q}\},$$

we define the $\overline{\partial}$-Laplacian

$$\square_{p,q} = \overline{\partial}_{p,q}\overline{\partial}^*_{p,q} + \overline{\partial}^*_{p,q+1}\overline{\partial}_{p,q+1} : \text{Dom}\,\square_{p,q} \subset L^2_{p,q}(M) \to L^2_{p,q}(M).$$

It is well-known that this is a densely defined, closed, self-adjoint operator (see e.g. [R8], Theorem 3.1) such that there is the orthogonal decomposition

$$L^2_{p,q}(M) = \ker \square_{p,q} \oplus \overline{\text{Im } \square_{p,q}}$$

with

$$\ker \square_{p,q} = \ker \overline{\partial}^*_{p,q} \cap \ker \overline{\partial}_{p,q+1},$$

and the orthogonal decomposition

$$\overline{\text{Im } \square_{p,q}} = \overline{\text{Im } \overline{\partial}_{p,q}} \oplus \overline{\text{Im } \overline{\partial}^*_{p,q+1}}. \tag{81}$$

The $\overline{\partial}$-Neumann operator

$$N_{p,q} = \square^{-1}_{p,q} : \text{Im } \square_{p,q} \subset L^2_{p,q}(M) \to \text{Dom}\,\square_{p,q} \cap (\ker \square_{p,q})^\perp \subset L^2_{p,q}(M)$$

is defined as follows: for $u \in \text{Im } \square_{p,q}$, let $v = Nu$ be the unique $v \in \square^{-1}_{p,q}(\{u\})$ which is orthogonal to $\ker \square_{p,q}$.

$N_{p,q}$ is a bounded operator exactly if $\square_{p,q}$ has closed range. That can be seen as follows: Assume that $N_{p,q}$ is bounded, then it is clear that $\square_{p,q}$ has closed range because $\square_{p,q} N_{p,q} u = u$ for all $u \in \text{Im } \square_{p,q}$. Assume conversely that $\square_{p,q}$ has closed range. Then we consider the Banach space

$$B_\square := \text{Dom}\,\square_{p,q} \cap (\ker \square_{p,q})^\perp$$

with the norm

$$\|u\|^2_{B_\square} = \|u\|^2_{L^2_{p,q}(M)} + \|\square_{p,q} u\|^2_{L^2_{p,q}(M)}.$$

Then $\square : B_\square \to \text{Im } \square_{p,q}$ is a bounded linear isomorphism and so the same holds for $N_{p,q} = \square^{-1}_{p,q} : \text{Im } \square_{p,q} \to B_\square$. By definition, $N_{p,q}$ is bounded if there exists a constant $C^\square_{p,q} > 0$ such that

$$\|u\|_{L^{p,q}(M)} \leq C^\square_{p,q} \|\square_{p,q} u\|_{L^{p,q}(M)} \quad \forall u \in B_\square. \tag{82}$$

Assume from now on that $\overline{\partial}_{p,q}$ and $\overline{\partial}_{p,q+1}$ both have closed range. It follows from (79) and (80) that there exist constants $C_{p,q} > 0$ and $C_{p,q+1} > 0$ such that

$$\|u\|_{L^2_{p,q}(M)} \leq C_{p,q} \|\overline{\partial}^*_{p,q} u\|_{L^2_{p,q-1}(M)}$$

for all $u \in \operatorname{Dom} \overline{\partial}^*_{p,q} \cap \operatorname{Im} \overline{\partial}_{p,q}$, and that

$$\|u\|_{L^2_{p,q}(M)} \leq C_{p,q+1} \|\overline{\partial}_{p,q+1} u\|_{L^2_{p,q+1}(M)}$$

for all $u \in \operatorname{Dom} \overline{\partial}_{p,q+1} \cap \operatorname{Im} \overline{\partial}^*_{p,q+1}$. Let

$$u \in B_\square = \operatorname{Dom} \square_{p,q} \cap \overline{\operatorname{Im} \square_{p,q}}.$$

By use of (81) and the assumption that $\overline{\partial}_{p,q}$ and $\overline{\partial}^*_{p,q+1}$ have closed range, u has the orthogonal decomposition $u = u_1 + u_2$ with $u_1 \in \operatorname{Dom} \overline{\partial}^*_{p,q} \cap \operatorname{Im} \overline{\partial}_{p,q}$ and $u_2 \in \operatorname{Dom} \overline{\partial}_{p,q+1} \cap \operatorname{Im} \overline{\partial}^*_{p,q+1}$. So, with

$$C^\square_{p,q} := \max\{C^2_{p,q}, C^2_{p,q+1}\}$$

we obtain:

$$\begin{aligned}
\|u\|^2 &= \|u_1\|^2 + \|u_2\|^2 \leq C^\square_{p,q} \big(\|\overline{\partial}^*_{p,q} u_1\|^2 + \|\overline{\partial}_{p,q+1} u_2\|^2 \big) \\
&= C^\square_{p,q} \big(\|\overline{\partial}^*_{p,q} u\|^2 + \|\overline{\partial}_{p,q+1} u\|^2 \big) \\
&= C^\square_{p,q} (\square_{p,q} u, u) \leq C^\square_{p,q} \|\square_{p,q} u\| \|u\|.
\end{aligned}$$

Hence, $\square_{p,q}$ has closed range and $N_{p,q}$ is bounded by (82).

For the representation of $N_{p,q}$ in terms of $\mathbf{S}_{p,q}$ and $\mathbf{S}_{p,q+1}$, we need:

Lemma 3.2. *Let $\overline{\partial}_{p,q}$ and $\overline{\partial}_{p,q+1}$ have closed range. Then:*

$$\overline{\partial}^*_{p,q} N_{p,q} = \mathbf{S}_{p,q}, \tag{83}$$

$$\overline{\partial}_{p,q+1} N_{p,q} = \mathbf{S}^*_{p,q+1}. \tag{84}$$

Moreover,

$$N_{p,q} \overline{\partial}_{p,q} v = \mathbf{S}^*_{p,q} v \quad \text{for } v \in \operatorname{Dom} \overline{\partial}_{p,q}, \tag{85}$$

$$N_{p,q} \overline{\partial}^*_{p,q+1} f = \mathbf{S}_{p,q+1} f \quad \text{for } f \in \operatorname{Dom} \overline{\partial}^*_{p,q+1}. \tag{86}$$

Proof. For $u \in L^2_{p,q}(M)$ we use the orthogonal decomposition

$$u = u_1 + u_2 + u_3 \in \operatorname{Im} \overline{\partial}_{p,q} \oplus \operatorname{Im} \overline{\partial}^*_{p,q+1} \oplus \ker \square_{p,q}$$

and the fact that $\mathbf{S}_{p,q} u = \mathbf{S}_{p,q} u_1$ is the unique element in $\overline{\partial}^{-1}_{p,q}(\{u_1\})$ which is orthogonal to $\ker \overline{\partial}_{p,q}$, and that $\mathbf{S}^*_{p,q+1} u = \mathbf{S}^*_{p,q+1} u_2$ is the unique element in $(\overline{\partial}^*_{p,q+1})^{-1}(\{u_2\})$ which is orthogonal to $\ker \overline{\partial}^*_{p,q+1}$. On the other hand,

$$\begin{aligned}
u_1 + u_2 &= \square_{p,q} N_{p,q}(u_1 + u_2) = \square_{p,q} N_{p,q} u \\
&= \overline{\partial}_{p,q} \overline{\partial}^*_{p,q} N_{p,q} u + \overline{\partial}^*_{p,q+1} \overline{\partial}_{p,q+1} N_{p,q} u.
\end{aligned}$$

But $u_1 \in \ker \overline{\partial}_{p,q+1} = (\mathrm{Im}\, \overline{\partial}^*_{p,q+1})^\perp$ and $u_2 \in \ker \overline{\partial}^*_{p,q} = (\mathrm{Im}\, \overline{\partial}_{p,q})^\perp$ so that necessarily

$$\begin{aligned} u_1 &= \overline{\partial}_{p,q}\overline{\partial}^*_{p,q} N_{p,q} u, \\ u_2 &= \overline{\partial}^*_{p,q+1}\overline{\partial}_{p,q+1} N_{p,q} u. \end{aligned}$$

This proves (83), (84) since $\overline{\partial}^*_{p,q} N_{p,q} u \in (\ker \overline{\partial}_{p,q})^\perp$ and $\overline{\partial}_{p,q+1} N_{p,q} u \in (\ker \overline{\partial}^*_{p,q+1})^\perp$. Moreover, for all $v \in \mathrm{Dom}\, \overline{\partial}_{p,q}$ and all $u \in L^2_{p,q}(M)$ we have by the use of (83):

$$(N_{p,q}\overline{\partial}_{p,q} v, u) = (\overline{\partial}_{p,q} v, N_{p,q} u) = (v, \overline{\partial}^*_{p,q} N_{p,q} u) = (v, \mathbf{S}_{p,q} u) = (\mathbf{S}^*_{p,q} v, u).$$

This shows (85), and (86) follows analogously. $\square$

This yields:

Theorem 3.3. *Let $\overline{\partial}_{p,q}$ and $\overline{\partial}_{p,q+1}$ have closed range. Then:*

$$N_{p,q} = \mathbf{S}^*_{p,q}\mathbf{S}_{p,q} + \mathbf{S}_{p,q+1}\mathbf{S}^*_{p,q+1}. \tag{87}$$

Hence, the $\overline{\partial}$-Neumann operator $N_{p,q}$ is compact exactly if the canonical $\overline{\partial}$-solution operators $\mathbf{S}_{p,q}$ and $\mathbf{S}_{p,q+1}$ both are compact.

Proof. Let $u \in L^2_{p,q}(M)$. Then $v = N_{p,q} u \in \mathrm{Im}\, \square_{p,q} = (\ker \square_{p,q})^\perp$. For such a v, we have:

$$(N_{p,q}\square_{p,q} v, f) = (v, \square_{p,q} N_{p,q} f) = (v, f_1) = (v, f)$$

for all $f \in L^2_{p,q}(M)$ by use of the orthogonal decomposition

$$f = f_1 + f_2 \in \mathrm{Im}\, \square_{p,q} \oplus \ker \square_{p,q}.$$

Hence $N_{p,q}\square_{p,q} N_{p,q} u = N_{p,q} u$, and so (87) follows from (83) – (86):

$$\begin{aligned} N_{p,q} u &= N_{p,q}\square_{p,q} N_{p,q} u = (N_{p,q}\overline{\partial}_{p,q})(\overline{\partial}^*_{p,q} N_{p,q}) u + (N_{p,q}\overline{\partial}^*_{p,q+1})(\overline{\partial}_{p,q+1} N_{p,q}) u \\ &= \mathbf{S}^*_{p,q}\mathbf{S}_{p,q} u + \mathbf{S}_{p,q+1}\mathbf{S}^*_{p,q+1} u. \end{aligned}$$

For a bounded operator T, it is well known that T is compact exactly if T^* is compact, and this is the case exactly if T^*T is compact.

So, assume that $\mathbf{S}_{p,q}$ and $\mathbf{S}_{p,q+1}$ are compact. Then it follows from (87) that $N_{p,q}$ is compact. Conversely, assume that $N_{p,q}$ is compact. Then $\mathbf{S}^*_{p,q}\mathbf{S}_{p,q}$ and $\mathbf{S}_{p,q+1}\mathbf{S}^*_{p,q+1}$ both are compact by (87) for they are positive. It follows that $\mathbf{S}_{p,q}$ and $\mathbf{S}_{p,q+1}$ both are compact. $\square$

This gives a nice criterion for compactness of the $\overline{\partial}$-Neumann operator on arbitrary Hermitian manifolds which we will apply in the context of isolated singularities, because we can show that there exist compact solution operators for the $\overline{\partial}$-equation at isolated singularities.

3.1.2 Line bundles twisted along the exceptional set

It is our aim to relate properties of the $\overline{\partial}$-operator on a singular space to properties of the $\overline{\partial}$-operator on a resolution of that space where the problems are well understood. As a preparation, we need to study properties of the $\overline{\partial}$-operator for forms with values in holomorphic line bundles which are twisted along the exceptional set of a resolution of isolated singularities.

As before, let X be a Hermitian complex space of pure dimension n with only isolated singularities and $\pi : M \to X$ a resolution of singularities. Let D be a divisor on M with support on the exceptional set E of the resolution $\pi : M \to X$, and denote by $\mathcal{O}(D)$ the sheaf of germs of meromorphic functions f such that $\operatorname{div}(f) + D \geq 0$. We denote by L_D the associated holomorphic line bundle such that sections in $\mathcal{O}(D)$ correspond to sections in L_D. Recall that we used this notation already in the discussion of the L^2-theory in case of the line bundles $L_{Z-|Z|}$ and $L_{|Z|-Z}$, where $Z = \pi^{-1}(\operatorname{Sing} X)$ was the unreduced exceptional divisor and $|Z| = E$ the underlying reduced divisor. The constant function $f \equiv 1$ induces a meromorphic section s_D of L_D such that $\operatorname{div}(s_D) = D$. One can then identify sections in $\mathcal{O}(D)$ with sections in $\mathcal{O}(L_D)$ by $g \mapsto g \otimes s_D$, and we denote the inverse mapping by $g \mapsto g \cdot s_D^{-1}$. If D is an effective divisor, then s_D is a holomorphic section of L_D and

$$\mathcal{O}(-D) \subset \mathcal{O} \subset \mathcal{O}(D).$$

We have used these inclusions before without mentioning s_D explicitly, but in the following we need to jump between different line bundles so that we should be more careful with the notation.

If Y is an effective divisor, then there is the natural inclusion $\mathcal{O}(D) \subset \mathcal{O}(D+Y)$ which induces the inclusion $\mathcal{O}(L_D) \subset \mathcal{O}(L_{D+Y})$ given by $g \mapsto (g \cdot s_D^{-1}) \otimes s_{D+Y}$. This also induces the natural inclusion of smooth sections of vector bundles

$$\Gamma(U, L_D) \subset \Gamma(U, L_{D+Y}) \tag{88}$$

for open sets $U \subset M$.

We give each L_D the structure of a Hermitian holomorphic line bundle by choosing an arbitrary positive definite Hermitian metric on L_D. We denote by

$$L^{p,q}_\sigma(U, L_D)$$

the space of square-integrable (p,q)-forms with values in L_D (with respect to the metric σ on M and the chosen metric on L_D). If $U \subset\subset M$ is relatively compact, then (88) induces the natural inclusion

$$L^{p,q}_\sigma(U, L_D) \subset L^{p,q}_\sigma(U, L_{D+Y}) \tag{89}$$

for any effective divisor Y. This does not depend on the metrics chosen on the line bundles L_D and L_{D+Y} because U is relatively compact in M.

For more details on Hermitian holomorphic line bundles twisted along the exceptional set of the desingularization, we refer to section 2 in [R9] and section 2 in [OV4].

Note that the inclusions (24) and (25) remain valid for forms with values in a line bundle L_D. Moreover, by [R4], Lemma 2.1, or [FOV1], Lemma 3.1, respectively, there exists an effective divisor $\widetilde{D}$ with support on the exceptional set E such that

$$L^{p,q}_\sigma(U, L_{-\widetilde{D}}) \subset L^{p,q}_\gamma(U) \subset L^{p,q}_\sigma(U, L_{\widetilde{D}}) \tag{90}$$

for all $0 \leq p, q \leq n$ and open sets $U \subset\subset M$. This follows from the fact that dV_γ vanishes of a certain order (exactly) on E. For simplicity, we assume that X has only finitely many isolated singularities so that we can choose a fixed positive integer m such that the effective divisor mE satisfies (90):

$$L^{p,q}_\sigma(U, L_{-mE}) \subset L^{p,q}_\gamma(U) \subset L^{p,q}_\sigma(U, L_{mE}) \tag{91}$$

for all $0 \leq p, q \leq n$ and open sets $U \subset\subset M$.

In this chapter, we are generally interested in properties of the $\overline{\partial}$-operator

$$\overline{\partial} : L^{p,q}_\gamma(U - E) \to L^{p,q+1}_\gamma(U - E),$$

which we would like to relate to properties of the $\overline{\partial}$-operator

$$\overline{\partial} : L^{p,q}_\sigma(U) \to L^{p,q+1}_\sigma(U),$$

where U is a neighborhood of the exceptional set. We denote by $\overline{\partial}_D$ the $\overline{\partial}$-operator acting on L^2-forms with values in L_D. All operators have to be understood in the sense of distributions.

As a preparation, we need:

Theorem 3.4. ([OR], Theorem 4.1) *Let $U \subset\subset M$ be a neighborhood of the exceptional set E and D_1, D_2 two divisors with support on E. Then*

$$\overline{\partial}_{D_1} : L^{p,q}_\sigma(U, L_{D_1}) \to L^{p,q+1}_\sigma(U, L_{D_1}) \tag{92}$$

has closed range of finite codimension in $\ker \overline{\partial}_{D_1}$ *exactly if*

$$\overline{\partial}_{D_2} : L^{p,q}_\sigma(U, L_{D_2}) \to L^{p,q+1}_\sigma(U, L_{D_2}) \tag{93}$$

has closed range of finite codimension in $\ker \overline{\partial}_{D_2}$*. If this is the case, then there exists a compact $\overline{\partial}$-solution operator*

$$\mathbf{S}_1 : \mathit{Im}\, \overline{\partial}_{D_1} \subset L^{p,q+1}_\sigma(U, L_{D_1}) \to L^{p,q}_\sigma(U, L_{D_1}) \tag{94}$$

exactly if there exists a compact $\overline{\partial}$-solution operator

$$\mathbf{S}_2 : \mathit{Im}\, \overline{\partial}_{D_2} \subset L^{p,q+1}_\sigma(U, L_{D_2}) \to L^{p,q}_\sigma(U, L_{D_2}). \tag{95}$$

Both implications of this equivalence follow by a similar argument which we will use again later when we actually treat the $\overline{\partial}$-operator on the singular space. We will elaborate the argument once here in detail for convenience of the reader. Later we will only indicate briefly how to adopt the reasoning in different situations.

Proof. It is enough to prove the statement for two divisors D_1, D_2 such that $D_2 - D_1 \geq 0$. The general statement follows then by comparing both bundles, L_{D_1} and L_{D_2}, with $L_{|D_1|+|D_2|}$, because $(|D_1|+|D_2|) - D_j \geq 0$ for $j = 1, 2$.

Since E is compact by assumption (because $E \subset U \subset\subset M$), it consists of k pairwise disjoint components E_μ, $\mu = 1, ..., k$, such that each $a_\mu := \pi(E_\mu)$ is an isolated singularity. For $\mu = 1, ..., k$ choose pairwise disjoint strongly pseudoconvex neighborhoods $V_\mu \subset\subset U$ of the components E_μ. Then it is well known that (for each $\mu = 1, ..., k$) the operators

$$\overline{\partial}_{D_1}|_{V_\mu} : L^{p,q}_\sigma(V_\mu, L_{D_1}) \to L^{p,q+1}_\sigma(V_\mu, L_{D_1})$$

and

$$\overline{\partial}_{D_2}|_{V_\mu} : L^{p,q}_\sigma(V_\mu, L_{D_2}) \to L^{p,q+1}_\sigma(V_\mu, L_{D_2})$$

have closed range of finite codimension in the corresponding kernels of $\overline{\partial}_{D_1}|_{V_\mu}$ and $\overline{\partial}_{D_2}|_{V_\mu}$, respectively, and that there are corresponding compact $\overline{\partial}$-solution operators (see e.g. the proof of Lemma 2.2 in [OV4] which implies that the corresponding $\overline{\partial}$-Neumann operators are compact). We denote by H^μ_1 and H^μ_2 the range of $\overline{\partial}_{D_1}|_{V_\mu}$ and $\overline{\partial}_{D_2}|_{V_\mu}$ in $L^{p,q+1}_\sigma(V_\mu, L_{D_1})$ and $L^{p,q+1}_\sigma(V_\mu, L_{D_2})$, respectively, and by

$$\mathbf{T}^\mu_1 : H^\mu_1 \to L^{p,q}_\sigma(V_\mu, L_{D_1}) \ , \quad \mathbf{T}^\mu_2 : H^\mu_2 \to L^{p,q}_\sigma(V_\mu, L_{D_2})$$

corresponding compact $\overline{\partial}$-solution operators.

Assume first that $\overline{\partial}_{D_1}$ has closed range of finite codimension and that

$$\mathbf{S}_1 : \operatorname{Im} \overline{\partial}_{D_1} \subset L^{p,q+1}_\sigma(U, L_{D_1}) \to L^{p,q}_\sigma(U, L_{D_1}) \tag{96}$$

is a corresponding bounded $\overline{\partial}$-solution operator. Consider the bounded linear map

$$\Phi_2 : \ker \overline{\partial}_{D_2} \subset L^{p,q+1}_\sigma(U, L_{D_2}) \to G_2 := \bigoplus_{\mu=1}^{k} \ker \overline{\partial}_{D_2}|_{V_\mu}$$

given by

$$\Phi_2(f) := (f|_{V_1}, ..., f|_{V_k}).$$

Since $H^1_2 \oplus \cdots \oplus H^k_2$ is a closed subspace of finite codimension in G_2, the same holds for

$$H_2 := \Phi_2^{-1}\big(H^1_2 \oplus \cdots \oplus H^k_2\big) = \big\{ f \in \ker \overline{\partial}_{D_2} : f|_{V_\mu} \in H^\mu_2, \mu = 1, ..., k \big\}$$

in $\ker \overline{\partial}_{D_2}$. Now then, choose a smooth cut-off function χ which has compact support in $V := \bigcup_\mu V_\mu$ and is identically 1 in a smaller neighborhood of the exceptional set E. Then we can define a bounded linear map

$$\Psi_2 : H_2 \to \ker \overline{\partial}_{D_1} \subset L^{p,q+1}_\sigma(U, L_{D_1})$$

by the assignment

$$f \in H_2 \mapsto \Big(\big(f - \sum_{\mu=1}^{k} \overline{\partial}_{D_2}(\chi \mathbf{T}^\mu_2(f|_{V_\mu})) \big) \cdot s^{-1}_{D_2} \Big) \otimes s_{D_1}$$

since

$$f - \sum_{\mu=1}^{k} \overline{\partial}_{D_2}(\chi \mathbf{T}_2^{\mu}(f|_{V_\mu}))$$

is identically zero in a neighborhood of E and $\operatorname{supp} D_1 \subset E$. By assumption, $\overline{\partial}_{D_1}$ has closed range $\mathrm{Im}\ \overline{\partial}_{D_1}$ of finite codimension in $\ker \overline{\partial}_{D_1}$ so that

$$H_2' := \Psi_2^{-1}(\mathrm{Im}\ \overline{\partial}_{D_1})$$

has closed range of finite codimension in H_2. As we have already seen that H_2 in turn is closed of finite codimension in $\ker \overline{\partial}_{D_2}$, it follows that H_2' is a closed subspace of finite codimension in $\ker \overline{\partial}_{D_2}$.

On the other hand, since $\Psi_2(H_2') \subset \mathrm{Im}\ \overline{\partial}_{D_1}$, we can define by use of (96) a $\overline{\partial}$-solution operator

$$\mathbf{S}_2' : H_2' \subset L^{p,q+1}_{\sigma}(U, L_{D_2}) \to L^{p,q}_{\sigma}(U, L_{D_2})$$

by setting

$$\mathbf{S}_2'(f) := \big((\mathbf{S_1} \circ \Psi_2(f)) \cdot s_{D_1}^{-1}\big) \otimes s_{D_2} + \chi \sum_{\mu=1}^{k} \mathbf{T}_2^{\mu}(f|_{V_\mu}).$$

Here, we use the natural inclusion $\Gamma(U, L_{D_1}) \subset \Gamma(U, L_{D_2})$, $g \mapsto (g \cdot s_{D_1}^{-1}) \otimes s_{D_2}$, which exists since $D_2 \geq D_1$ by assumption. Since this inclusion commutes with the $\overline{\partial}$-operator,

$$\begin{aligned}
\overline{\partial}_{D_2}\big(\big((\mathbf{S}_1 \circ \Psi_2(f)) \cdot s_{D_1}^{-1}\big) \otimes s_{D_2}\big) &= \big(\big(\overline{\partial}_{D_1}(\mathbf{S}_1 \circ \Psi_2(f))\big) \cdot s_{D_1}^{-1}\big) \otimes s_{D_2} \\
&= \big(\Psi_2(f) \cdot s_{D_1}^{-1}\big) \otimes s_{D_2} \\
&= f - \sum_{\mu=1}^{k} \overline{\partial}_{D_2}(\chi \mathbf{T}_2^{\mu}(f|_{V_\mu})).
\end{aligned}$$

Hence, $\overline{\partial}_{D_2}\mathbf{S}_2'(f) = f$, so that $H_2' \subset \mathrm{Im}\ \overline{\partial}_{D_2}$. Summing up, we have

$$H_2' \subset \mathrm{Im}\ \overline{\partial}_{D_2} \subset \ker \overline{\partial}_{D_2},$$

where H_2' is closed and of finite codimension in $\ker \overline{\partial}_{D_2}$. But then $\mathrm{Im}\ \overline{\partial}_{D_2}$ is also closed and of finite codimension in $\ker \overline{\partial}_{D_2}$. We can now extend $\mathbf{S}_2'$ easily to a $\overline{\partial}$-solution operator on $\mathrm{Im}\ \overline{\partial}_{D_2}$. Choose a basis $e_1, ..., e_l$ of the complement of H_2' in $\mathrm{Im}\ \overline{\partial}_{D_2}$. Then there exists forms $h_1, ..., h_l \in L^{p,q}_{\sigma}(U, L_{D_2})$ such that $\overline{\partial}_{D_2} h_\nu = e_\nu$ for $\nu = 1, ..., l$. Then each $f \in \mathrm{Im}\ \overline{\partial}_{D_2}$ has a unique representation

$$f = f' + \sum_{\nu=1}^{l} a_\nu e_\nu, \quad f' \in H_2',\ a_1, ..., a_l \in \mathbb{C}, \tag{97}$$

and we define

$$\mathbf{S}_2(f) := \mathbf{S}_2'(f') + \sum_{\nu=1}^{l} a_\nu h_\nu. \tag{98}$$

If in addition $\mathbf{S}_1$ is compact, then $\mathbf{S}_2'$ is compact because all the $\mathbf{T}_2^\mu$, $\mu = 1, ..., k$, are compact. But then $\mathbf{S}_2$ is a compact $\overline{\partial}$-solution operator on Im $\overline{\partial}_{D_2}$.

Recall the principle of the proof in other words. We assume solvability on a space of finite codimension in

$$\ker \overline{\partial}_{D_1} \subset L^{p,q+1}_\sigma(U, L_{D_1})$$

and would like deduce solvability on a space of finite codimension in

$$\ker \overline{\partial}_{D_2} \subset L^{p,q+1}_\sigma(U, L_{D_2}),$$

and we can use the natural inclusion

$$L^*_\sigma(U, L_{D_1}) \subset L^*_\sigma(U, L_{D_2}). \tag{99}$$

So, let $f \in \ker \overline{\partial}_{D_2}$. If we would like to apply the solvability property in $L^{p,q+1}_\sigma(U, L_{D_1})$ to f, we need an inclusion opposite to (99). This can be achieved by solving $\overline{\partial}_{D_2} g = f$ on the strongly pseudoconvex neighborhood V of the exceptional set and considering $f - \overline{\partial}_{D_2}(\chi g)$ instead. This form lies naturally in $\ker \overline{\partial}_{D_1}$ because it vanishes in a neighborhood of the exceptional set. Now we can apply the solvability property in $L^{p,q+1}_\sigma(U, L_{D_1})$ and go back to $L^*_\sigma(U, L_{D_2})$ by use of the natural inclusion (99). All this has to be done on subspaces of finite codimension.

Let us consider now the converse direction of the statement of the theorem which is a bit more complicated. We assume solvability on a space of finite codimension in

$$\ker \overline{\partial}_{D_2} \subset L^{p,q+1}_\sigma(U, L_{D_2})$$

and will deduce solvability on a space of finite codimension in

$$\ker \overline{\partial}_{D_1} \subset L^{p,q+1}_\sigma(U, L_{D_1}),$$

We would like to use the natural inclusion (99) similarly as above, but we need to distinguish the two cases $q \geq 1$ and $q = 0$. Again, all the arguments have to be considered more precisely on appropriate subspaces of forms. For all the details, we refer to the proof of Theorem 4.1 in [OR].

Let us first consider the case $q \geq 1$. Let $f \in \ker \overline{\partial}_{D_1}$. By modifying f on the strongly pseudoconvex set V, we can assume as above that f is vanishing on a neighborhood of the exceptional set (though this may seem superfluous at first sight). By the natural inclusion (99), we can use the solvability property in $L^{p,q+1}_\sigma(U, L_{D_2})$ to consider the equation $\overline{\partial}_{D_2} g = f$. Here, we have the problem that such a solution g cannot be considered as a solution in $L^*_\sigma(U, L_{D_1})$. But we are already prepared for this problem. Since we have arranged the situation so that f is vanishing in a neighborhood of the exceptional set, g is $\overline{\partial}_{D_2}$-closed there. So, we can consider $\overline{\partial}_{D_2} h = g$ on a smaller strongly pseudoconvex neighborhood W of the exceptional set and consider $g - \overline{\partial}_{D_2}(\chi' h)$ instead of g where χ' is another suitable cut-off function. Here we need that $q \geq 1$. So, we can assume that a solution g of the equation $\overline{\partial}_{D_2} g = f$ is vanishing identically in a neighborhood of the exceptional set, and thus it is no problem to deduce $\overline{\partial}_{D_1} g = f$ in $L^{p,q+1}_\sigma(U, L_{D_1})$.

It only remains to treat the case $q = 0$. Since the statement of the theorem is trivial if the dimension of M is 1, we can assume that $\dim_{\mathbb{C}} M \geq 2$. Hence, let $f \in \ker \overline{\partial}_{D_1} \subset L^{p,1}_{\sigma}(U, L_{D_1})$. By the inclusion (99), we can consider f as a form in $\ker \overline{\partial}_{D_2}$. We can assume that f has compact support in the strongly pseudoconvex neighborhood V of the exceptional set by the following argument. Let $\overline{\partial}_{D_2} g = f$ be a solution from the solvability property in $L^{p,1}_{\sigma}(U, L_{D_2})$ and χ a cut-off function with support in V as above. Then we can consider $f - \overline{\partial}_{D_2}\big((1-\chi)g\big) = f - \overline{\partial}_{D_1}\big((1-\chi)g\big)$ instead of f because $(1-\chi)g$ is vanishing identically in a neighborhood of the exceptional set. But then there is (as usually on a suitable subspace) a solution for the equation $\overline{\partial}_{D_1} h = f$ in $L^{p,1}_{\sigma}(V, L_{D_1})$ with compact support in the strongly pseudoconvex set V. By trivial extension, this is clearly also a solution on the whole set U. Here we need that the dimension of M is not 1 because solvability with compact support behaves well only for (r,s)-forms with $0 < s < \dim_{\mathbb{C}} M$. Note that the argument here works for all $q < \dim_{\mathbb{C}} M - 1$, not only for $q = 0$.

As for the opposite direction of the statement of the theorem, it is no problem to involve the statement about compactness also here, because all the solution operators that we use on strongly pseudoconvex domains are well-known to be compact operators. □

3.1.3 Closed range of the $\overline{\partial}$-operator at isolated singularities

We will now explain the principle behind the proof of the first part of Theorem 2.4. Recall that our general philosophy is to use a resolution of singularities to obtain a regular model for the problems that we consider. If this is somehow successful, one can then deduce properties of the $\overline{\partial}$-operator on singular spaces from well-known properties of the $\overline{\partial}$-operator on the resolution. Our strategy is similar to the proof of Theorem 3.4 above. Instead of the exceptional set, we have to deal with isolated singularities. Thus, we need a substitute for the nice properties of the $\overline{\partial}$-operator on a strongly pseudoconvex neighborhood of the exceptional set, namely we need to know something about the $\overline{\partial}$-operator in a neighborhood of isolated singularities. Here, we can use the results of Fornæss, Øvrelid and Vassiliadou [FOV2].

As always, let X be a Hermitian complex space and let $a \in X$ be an isolated singularity. Then, we denote by $d_a(z)$ the distance $\operatorname{dist}_X(z,a)$ of a point z to the singular point a in X. Here, $\operatorname{dist}_X(z,y)$ is the infimum of the length of piecewise smooth curves connecting two points z, y in X.

Theorem 3.5. *Let X be a Hermitian complex space of pure dimension n and $a \in X$ an isolated singularity. Then there exists a (strongly pseudoconvex) neighborhood U of a in X (with $U \cap \operatorname{Sing} X = \{a\}$) such that the following is true: If $p + q < n$, $q \geq 1$, then there exists a closed subspace H of finite codimension in*

$$\ker \overline{\partial}_{max} : L^2_{p,q}(U^*) \to L^2_{p,q+1}(U^*),$$

where $U^* = U - \{a\}$, and a constant $C > 0$ such that for each $f \in H$ there exists $u \in L^2_{p,q-1}(U^*)$ with $\overline{\partial}_{max} u = f$ satisfying

$$\int_{U^*} |u|^2 d_a^{-2} \log^{-4}(1 + d_a^{-1}) dV_X \leq C \int_{U^*} |f|^2 dV_X. \tag{100}$$

If $p+q>n$, there exist constants $c>0$, $C_c>0$ such that for each $f\in\ker\overline{\partial}_{max}\subset L^2_{p,q}(U^*)$ there exists $u\in L^2_{p,q-1}(U^*)$ with $\overline{\partial}_{max}u=f$ satisfying

$$\int_{U^*}|u|^2 d_a^{-2c}dV_X\leq C_c\int_{U^*}|f|^2dV_X. \tag{101}$$

Proof. The statement for $p+q<n$ follows from Theorem 1.1 in [FOV2] by the following observation: There exists a small neighborhood V of a which can be embedded holomorphically in a complex number space $\mathbb{C}^L$ such that $a=0\in\mathbb{C}^L$ and

$$\|z\|\lesssim d_a(z),$$

because the Euclidean distance of a point z to the origin is less or equal to the length of curves connecting z to the origin in X, if the length of a curve is measured with respect to the Euclidean metric. But the restriction of the Euclidean metric to X is isometric to the original Hermitian metric of X.

For U, we can choose the intersection of V with a small ball $U:=V\cap B_r(0)$. So, if the equation $\overline{\partial}_{max}u=f$ is solvable on $U^*=U-\{a\}$ according to Theorem 1.1 from [FOV2], then:

$$\begin{aligned}\int_{U^*}|u|^2d_a^{-2}\log^{-4}(1+d_a^{-1})dV_X &\lesssim \int_{U^*}|u|^2\|z\|^{-2}(-\log\|z\|^2)^{-4}dV_X\\ &\lesssim \int_{U^*}|f|^2dV_X.\end{aligned}$$

By [FOV2], Theorem 1.1, there are only finitely many obstructions to the equation $\overline{\partial}_{max}u=f$ on U^* with that estimate. The space H is closed because we have the estimate (100), and so the $\overline{\partial}$-operator in the sense of distributions has closed range in $L^2_{p,q}(U^*)$.

In the case $p+q>n$, the statement follows analogously from Theorem 1.2 in [FOV2]. Here, the $\overline{\partial}_{max}$-equation is solvable for all $f\in\ker\overline{\partial}_{max}$. $\square$

Theorem 3.5 and Theorem 3.4 are our main ingredients for the proof of Theorem 2.4. Recall the setting of Theorem 2.4. Let X be a Hermitian complex space of pure dimension n with only isolated singularities and $\pi: M\to X$ a resolution of singularities as above. Let σ be any (positive definite) Hermitian metric on M. We denote by $L^2_{p,q}$ the spaces of L^2-forms on $\operatorname{Reg}X=X-\operatorname{Sing}X$, and by $L^{p,q}_\sigma$ the spaces of L^2-forms on M with respect to σ. Let $\Omega\subset\subset X$ be a relatively compact open subset of X such that the boundary of Ω does not intersect the singular set of X, i.e. $b\Omega\cap\operatorname{Sing}X=\emptyset$. Let $\Omega^*:=\Omega-\operatorname{Sing}X$ and $\Omega':=\pi^{-1}(\Omega)$.

Thus, the resolution of singularities has the following nice effect: If the original domain Ω has a 'good' boundary $b\Omega$, then Ω' is a domain in a complex manifold with the same 'good' boundary. One might consider for example a domain Ω with a strongly pseudoconvex boundary, or assume that X is a compact space and $\Omega=X$ (no boundary at all). It is thus interesting to relate properties of the $\overline{\partial}$-operator on Ω^* (which have to be studied) to properties of the $\overline{\partial}$-operator on Ω' (which are well understood):

Theorem 3.6. ([OR], Theorem 1.1) *Let $q \geq 1$ and either $p + q \neq n$ or $(p, q) = (0, n)$. Under the assumptions above, the $\overline{\partial}$-operator in the sense of distributions*

$$\overline{\partial}_{max} : L^2_{p,q-1}(\Omega^*) \to L^2_{p,q}(\Omega^*) \tag{102}$$

has closed range of finite codimension in $\ker \overline{\partial}_{max} \subset L^2_{p,q}(\Omega^*)$ *exactly if the $\overline{\partial}$-operator in the sense of distributions*

$$\overline{\partial}^M_{max} : L^{p,q-1}_\sigma(\Omega') \to L^{p,q}_\sigma(\Omega') \tag{103}$$

has closed range of finite codimension in $\ker \overline{\partial}^M_{max} \subset L^{p,q}_\sigma(\Omega')$.

The case where Ω is compact or has smooth strongly pseudoconvex boundary was already treated in [R8]. We will give a sketch of the proof of Theorem 3.6. All the details can be found in [OR], Theorem 1.1.

Proof. Let $E = \pi^{-1}(\operatorname{Sing} X)$ be the exceptional set of the resolution $\pi : M \to X$. In the following, we can assume that $E = \pi^{-1}(\operatorname{Sing} X \cap \Omega)$ such that $\Omega' = \pi^{-1}(\Omega)$ is a neighborhood of the exceptional set E. We denote by $\{a_1, ..., a_k\}$ the isolated singularities in Ω, so that the exceptional set consists of the components $E_\mu = \pi^{-1}(\{a_\mu\})$, $\mu = 1, ..., k$, which are pairwise disjoint.

Let $\gamma := \pi^* h$ be the pullback of the Hermitian metric h of X to M. γ is positive semidefinite (a pseudo-metric) with degeneracy locus E. We denote by $L^{p,q}_\gamma$ the space of forms which are L^2 on M with respect to the pseudo-metric γ. As in (91), fix a positive integer m such that the effective divisor mE satisfies:

$$L^{p,q}_\sigma(U, L_{-mE}) \subset L^{p,q}_\gamma(U) \subset L^{p,q}_\sigma(U, L_{mE}) \tag{104}$$

for all $0 \leq p, q \leq n$ and open sets $U \subset M$.

Instead of (102), it is equivalent to consider instead the $\overline{\partial}$-operator in the sense of distributions

$$\overline{\partial}_{max} : L^{p,q-1}_\gamma(\Omega' - E) \to L^{p,q}_\gamma(\Omega' - E). \tag{105}$$

By Theorem 3.4, we can replace the conditions on $\overline{\partial}^M_{max}$ by conditions on the $\overline{\partial}$-operator in the sense of distributions for L^2-forms with values in the holomorphic line bundles L_{-mE} or L_{mE}, respectively, which we denote by $\overline{\partial}_{-mE}$ and $\overline{\partial}_{mE}$:

$$\overline{\partial}^M_{max} : L^{p,q-1}_\sigma(\Omega') \to L^{p,q}_\sigma(\Omega') \tag{106}$$

has closed range of finite codimension in $\ker \overline{\partial}^M_{max} \subset L^{p,q}_\sigma(\Omega')$ exactly if

$$\overline{\partial}_{-mE} : L^{p,q-1}_\sigma(\Omega', L_{-mE}) \to L^{p,q}_\sigma(\Omega', L_{-mE}) \tag{107}$$

has closed range of finite codimension in $\ker \overline{\partial}_{-mE} \subset L^{p,q}_\sigma(\Omega', L_{-mE})$ and that in turn is the case exactly if

$$\overline{\partial}_{mE} : L^{p,q-1}_\sigma(\Omega', L_{mE}) \to L^{p,q}_\sigma(\Omega', L_{mE}) \tag{108}$$

has closed range of finite codimension in $\ker \overline{\partial}_{mE} \subset L^{p,q}_\sigma(\Omega', L_{mE})$.

Assume first that $\overline{\partial}^M_{max}$ and thus also $\overline{\partial}_{-mE}$ have closed range of finite codimension. We have to show that $\overline{\partial}_{max}$ has closed range of finite codimension in $\ker \overline{\partial}_{max} \subset L^{p,q}_\gamma(\Omega' - E)$. The case $(p,q) = (0,n)$ is easy because

$$L^{0,n}_\gamma(\Omega' - E) \cong L^{0,n}_\sigma(\Omega')$$

and

$$L^{0,n-1}_\sigma(\Omega') \subset L^{0,n-1}_\gamma(\Omega' - E) \tag{109}$$

by (24) and (25). Consider the continuous linear map $\Psi : L^{0,n}_\gamma(\Omega' - E) \to L^{0,n}_\sigma(\Omega')$. By assumption, $\mathrm{Im}\, \overline{\partial}^M_{max}$ is closed of finite codimension in $L^{0,n}_\sigma(\Omega')$. Thus, $\Psi^{-1}(\mathrm{Im}\, \overline{\partial}^M_{max})$ is closed of finite codimension in $L^{0,n}_\gamma(\Omega'-E)$. But $\Psi^{-1}(\mathrm{Im}\, \overline{\partial}^M_{max}) \subset \mathrm{Im}\, \overline{\partial}_{max}$ because of (109), and so $\mathrm{Im}\, \overline{\partial}_{max}$ is also closed of finite codimension in $L^{0,n}_\gamma(\Omega' - E)$.

Let $p+q \neq n$ and $f \in \ker \overline{\partial}_{max} \subset L^{p,q}_\gamma(\Omega'-E)$. We can proceed as in the proof of Theorem 3.4 and will again omit the details about how to pull-back the closed subspaces of finite codimension under continuous maps. By use of Theorem 3.5, we can solve the equation $\overline{\partial}_{max} g = f$ for such f (in a closed subspace of finite codimension in $\ker \overline{\partial}_{max}$) on a suitable neighborhood of the exceptional set. So, we can consider the forms $f - \overline{\partial}(\chi g)$ instead, where χ is a suitable cut-off function. Now, these forms vanish in a neighborhood of the exceptional set so that they can be considered as $\overline{\partial}_{-mE}$-closed forms in $L^{p,q}_\sigma(\Omega', L_{-mE})$. But if $\overline{\partial}_{-mE} h = f$ in $L^*_\sigma(\Omega', L_{-mE})$, then $\overline{\partial}_{max} h = f$ also in $L^*_\gamma(\Omega' - E)$ because of (104). Hence, the closed range property in $L^*_\sigma(\Omega', L_{-mE})$ carries over to the closed range property in $L^*_\gamma(\Omega' - E)$.

For the converse direction of the statement, assume now that $\overline{\partial}_{max}$ has closed range of finite codimension in $\ker \overline{\partial}_{max} \subset L^{p,q}_\gamma(\Omega'-E)$. We will show that this implies that $\overline{\partial}_{mE}$ has closed range of finite codimension in $\ker \overline{\partial}_{mE} \subset L^{p,q}_\sigma(\Omega', L_{mE})$, which is then equivalent to the same property for the $\overline{\partial}^M_{max}$-operator. For this direction, we can treat the two cases $p + q \neq n$ and $(p,q) = (0,n)$ together.

Let $f \in \ker \overline{\partial}_{mE} \subset L^{p,q}_\sigma(\Omega', L_{mE})$. Let $V \subset\subset \Omega'$ be a strongly pseudoconvex neighborhood of the exceptional set as in the proof of Theorem 3.4. Then the equation $\overline{\partial}_{mE} g = f$ is solvable on V for all f in a closed subspace of finite codimension in $\ker \overline{\partial}_{mE}$. Again, let χ be a smooth cut-off function with support in V and identically 1 in a smaller neighborhood of the exceptional set. Then we can consider the forms $f - \overline{\partial}_{mE}(\chi g)$ instead which are vanishing identically in a neighborhood of the exceptional set. So, such forms are $\overline{\partial}_{max}$-closed forms in $L^{p,q}_\gamma(\Omega' - E)$. But if $\overline{\partial}_{max} h = f$ in $L^*_\gamma(\Omega' - E)$, then also $\overline{\partial}_{mE} h = f$ in $L^*_\sigma(\Omega' - E, L_{mE})$ by (104). Here, the $\overline{\partial}_{mE}$-equation extends over the exceptional set E by the L^2-extension theorem for the $\overline{\partial}$-equation (see Theorem 3.2 in [R4]), so that we obtain $\overline{\partial}_{mE} h = f$ in $L^*_\sigma(\Omega', L_{mE})$. Hence, the closed range property in $L^{p,q}_\gamma(\Omega' - E)$ carries over to the closed range property in $L^{p,q}_\sigma(\Omega', L_{mE})$. □

Note that this proof almost yields the second part of Theorem 2.4 as well (saying that $\overline{\partial}_{max}$ is compact exactly if $\overline{\partial}^M_{max}$ is compact). Assume that $\overline{\partial}_{max}$ is compact, then the proof above shows that $\overline{\partial}_{mE}$ and so $\overline{\partial}^M_{max}$ (by Theorem 3.4) are also compact, because the solution operators on the strongly pseudoconvex neighborhood V of the exceptional set can be chosen to be compact. For the converse direction of the statement, we would need that the solution operators of Fornæss, Øvrelid and Vassiliadou are compact. But this can be achieved after a slight modification as we shall see in the next section.

As an immediate consequence of Theorem 3.6 we obtain (all $\overline{\partial}$-operators have to be understood in the sense of distributions):

Theorem 3.7. *Let X be a Hermitian complex space of pure dimension n with only isolated singularities, and $\Omega \subset\subset X$ with no singularities in the boundary $b\Omega$. Let $\pi : M \to X$ be a resolution of singularities as above and σ any (positive definite) Hermitian metric on M.*

Let $q \geq 1$ and either $p + q \neq n - 1, n$ or $p = 0$. Assume that the $\overline{\partial}$-operators in the sense of distributions with respect to the metric σ on $\Omega' = \pi^{-1}(\Omega)$,

$$\overline{\partial}^M_{p,q} : \quad L^{p,q-1}_\sigma(\Omega') \to L^{p,q}_\sigma(\Omega'), \tag{110}$$

$$\overline{\partial}^M_{p,q+1} : \quad L^{p,q}_\sigma(\Omega') \to L^{p,q+1}_\sigma(\Omega') \tag{111}$$

both have closed range of finite codimension in the corresponding kernels of $\overline{\partial}^M$. Then the following holds:

i.

$$\Box^M_{p,q} = \overline{\partial}^M_{p,q}(\overline{\partial}^M_{p,q})^* + (\overline{\partial}^M_{p,q+1})^*\overline{\partial}^M_{p,q+1}$$

has closed range and the corresponding $\overline{\partial}$-Neumann operator (defined as in section 3.1.1)

$$N^M_{p,q} = (\Box^M_{p,q})^{-1} : L^{p,q}_\sigma(\Omega') \to L^{p,q}_\sigma(\Omega')$$

is bounded.

ii. On $\Omega^* = \Omega - \operatorname{Sing} X$, the operators

$$\overline{\partial}_{p,q} : \quad L^2_{p,q-1}(\Omega^*) \to L^2_{p,q}(\Omega^*),$$

$$\overline{\partial}_{p,q+1} : \quad L^2_{p,q}(\Omega^*) \to L^2_{p,q+1}(\Omega^*)$$

both have closed range of finite codimension in the corresponding kernels of $\overline{\partial}$.

$$\Box_{p,q} = \overline{\partial}_{p,q}\overline{\partial}^*_{p,q} + \overline{\partial}^*_{p,q+1}\overline{\partial}_{p,q+1}$$

has closed range and the corresponding $\overline{\partial}$-Neumann operator

$$N_{p,q} = \Box^{-1}_{p,q} : L^2_{p,q}(\Omega^*) \to L^2_{p,q}(\Omega^*)$$

is bounded.

Proof. Statement i. follows from the discussion in section 3.1.1. Moreover, Theorem 3.6 shows that

$$\overline{\partial}_{p,q} : \quad L^2_{p,q-1}(\Omega^*) \to L^2_{p,q}(\Omega^*),$$

$$\overline{\partial}_{p,q+1} : \quad L^2_{p,q}(\Omega^*) \to L^2_{p,q+1}(\Omega^*)$$

also have closed range of finite codimension in the corresponding kernels of $\overline{\partial}$. So, statement ii. also follows from the discussion in section 3.1.1 $\Box$

3.2 Compact operators on singular spaces

In this section, we will first recall a criterion for precompactness in function spaces on arbitrary Hermitian manifolds from [R8]. This criterion is quite easy to handle and leads to a characterization of compactness of the $\overline{\partial}$-Neumann operator on singular spaces with arbitrary singularities.

The criterion is also useful to see that the $\overline{\partial}$-solution operators of Fornæss, Øvrelid and Vassiliadou at isolated singularities yield compact $\overline{\partial}$-solution operators. This can be incorporated into Theorem 3.6 to see that there are compact $\overline{\partial}$-solution operators on a space with isolated singularities exactly if there are such operators on a suitable resolution. As the $\overline{\partial}$-Neumann operator can be represented in terms of canonical $\overline{\partial}$-solution operators (see Theorem 3.3), we get compactness of the $\overline{\partial}$-Neumann operator on such spaces.

3.2.1 Precompactness on Hermitian manifolds

In this section, we recall some statements about function spaces on Hermitian manifolds from [R8], where more details can be found. Let M be an arbitrary Hermitian manifold. If f is a differential form on M, we denote by $|f|$ its pointwise norm. For a weight function $\varphi \in C^0(M)$, we denote by $L^2_{p,q}(M,\varphi)$ the Hilbert space of (p,q)-forms such that

$$\|f\|^2_{L^2_{p,q}(M,\varphi)} := \int_M |f|^2 e^{-\varphi} dV_M < \infty.$$

Note that we may take different weight functions for forms of different degree. In the following, one can as well consider forms with values in (holomorphic) vector bundles, that does not cause any additional difficulties.

We assume that M is connected. For two points $p,q \in M$, let $\mathrm{dist}_M(p,q)$ be the infimum of the length of curves connecting p and q in M. Let $\Phi : M \to M$ be a diffeomorphism. Then we call

$$\mathrm{md}(\Phi) := \sup_{p\in M} \mathrm{dist}_M(p,\Phi(p))$$

the mapping distance of Φ. If N is another Hermitian manifold and $\Phi : M \to N$ differentiable, the pointwise norm of the tangential map Φ_* is defined by

$$|\Phi_*|(p) := \sup_{\substack{v\in T_pM\\ |v|=1}} |\Phi_*(v)|_{T_{\Phi(p)}N}.$$

This leads to the sup-norm of Φ_*:

$$\|\Phi_*\|_\infty := \sup_{p\in M} |\Phi_*|(p).$$

We also need to measure how far $\Phi_* : TM \to TM$ is from the identity mapping on tangential vectors in the case of a diffeomorphism $\Phi : M \to M$. As the total space TX inherits the structure of a Hermitian manifold, dist_{TM} is also well defined, and we set

$$\|\Phi_* - \mathrm{id}\,\|_\infty \;=\; \sup_{p\in M} \sup_{\substack{v\in T_pM\\ |v|=1}} \mathrm{dist}_{TM}(\Phi_* v, v).$$

Definition 3.8. *Let $\Omega \subset M$ open. A diffeomorphism $\Phi : (\Omega, M) \to (\Omega, M)$ is called a δ-variation of Ω in M if $\Phi|_{M-\Omega}$ is the identity map, mapping distance* $\mathrm{md}(\Phi) < \delta$ *and $\|\Phi_* - \mathrm{id}\,\|_\infty, \|(\Phi^{-1})_* - \mathrm{id}\,\|_\infty < 3\delta$. The set of all δ-variations of Ω in M will be denoted by* $\mathrm{Var}_\delta(\Omega, M)$.

A δ-variation $\Phi \in \mathrm{Var}_\delta(\Omega, M)$ will be called δ-deformation, if it can be connected by a smooth path to the identity map in $\mathrm{Var}_\delta(\Omega, M)$, i.e. if there exists a smooth map

$$\Phi.(\cdot) : [0,1] \times M \to M, \ (t,x) \mapsto \Phi_t(x) \in M,$$

such that $\Phi_t(\cdot) \in \mathrm{Var}_\delta(\Omega, M)$ for all $t \in [0,1]$, $\Phi_0 = \mathrm{id}$, $\Phi_1 = \Phi$ and

$$\left| \frac{\partial}{\partial t} \Phi_t(x) \right| \leq 3\delta \quad \text{for all } t \in [0,1], x \in M. \tag{112}$$

The set of all δ-deformations of Ω in M will be denoted by $\mathrm{Def}_\delta(\Omega, M)$.

Now then, precompact sets in $L^2_{p,q}(M, \varphi)$ can be characterized by:

Theorem 3.9. *Let M be a Hermitian manifold and $\mathcal{A}$ a bounded subset of $L^2_{p,q}(M, \varphi)$. Then $\mathcal{A}$ is precompact if and only if the following two conditions are fulfilled:*

(i) for all $\epsilon > 0$ and all $\Omega \subset\subset M$, there exists $\delta > 0$ such that

$$\|\Phi^* f - f\|_{L^2_{p,q}(M,\varphi)} < \epsilon$$

for all $\Phi \in \mathrm{Def}_\delta(\Omega, M)$ and all $f \in \mathcal{A}$, i.e. the forms in $\mathcal{A}$ behave uniformly under deformations of Ω.

(ii) for all $\epsilon > 0$, there exists $\Omega_\epsilon \subset\subset M$ such that

$$\|f\|_{L^2_{p,q}(M-\Omega_\epsilon,\varphi)} < \epsilon$$

for all $f \in \mathcal{A}$, i.e. the forms in $\mathcal{A}$ satisfy a uniform growth condition 'at the boundary of M'.

For the proof, see [R8], Theorem 2.5. The statement remains true for L^2-forms with values in a Hermitian vector bundle. The proof can be copied without any difficulties, only some notation needs to be added.

Using the Gårding inequality, one can show that condition (i) in Theorem 3.9 is automatically fulfilled for subsets of $L^2_{p,q}(M, \varphi) \cap \mathrm{Dom}\,\overline{\partial}_{max} \cap \mathrm{Dom}\,\overline{\partial}^*_{max}$ which are bounded in the graph norm

$$\|f\|^2_{\Gamma_{p,q}(M,\varphi)} := \|f\|^2_{L^2_{p,q}(M,\varphi)} + \|\overline{\partial}_{max} f\|^2_{L^2_{p,q+1}(M,\varphi)} + \|\overline{\partial}^*_{max} f\|^2_{L^2_{p,q-1}(M,\varphi)}.$$

Here, $\overline{\partial}_{max}$ is the $\overline{\partial}$-operator in the sense of distributions

$$\overline{\partial}_{max} : L^2_{p,q}(M, \varphi) \to L^2_{p,q+1}(M, \varphi),$$

and

$$\overline{\partial}^*_{max} : L^2_{p,q}(M, \varphi) \to L^2_{p,q-1}(M, \varphi)$$

is the Hilbert space adjoint of $\overline{\partial}_{max} : L^2_{p,q-1}(M, \varphi) \to L^2_{p,q}(M, \varphi)$. Note again that we can use different weights $\varphi_{p,q-1}, \varphi_{p,q}, \varphi_{p,q+1} \in C^0(M)$ for forms of different degrees.

Theorem 3.10. *Let M be a Hermitian manifold and let*

$$\mathcal{A} \subset L^2_{p,q}(M,\varphi) \cap \operatorname{Dom} \overline{\partial}_{max} \cap \operatorname{Dom} \overline{\partial}^*_{max}$$

be a set of (p,q)-forms which is bounded in the graph norm $\|\cdot\|_{\Gamma_{p,q}(M,\varphi)}$. Then $\mathcal{A}$ is precompact if and only if the following condition is fulfilled:

(C) for all $\epsilon > 0$, there exists $\Omega_\epsilon \subset\subset M$ such that

$$\|f\|_{L^2_{p,q}(M-\Omega_\epsilon,\varphi)} < \epsilon$$

for all $f \in \mathcal{A}$.

Proof. The proof can be found in [R8], Lemma 3.5. We sketch the essential argument for convenience of the reader.

Let $\epsilon > 0$ and $\Omega \subset\subset M$. Choose Ω_1, Ω_2 such that $\Omega \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset M$ and a smooth cut-off function $\chi \in C^\infty_{cpt}(\Omega_2)$ with $0 \leq \chi \leq 1$ such that $\chi \equiv 1$ on Ω_1. Then there exists a constant $C(\Omega,\chi,\varphi) > 0$ such that

$$\|\Phi^* u - u\|^2_{L^2_{p,q}(M,\varphi)} \leq C(\Omega,\chi,\varphi)\delta^2 \|\chi u\|^2_{W^{1,2}_{p,q}(\Omega_2,\varphi)} \tag{113}$$

for all $u \in C^\infty_{p,q}(\Omega_2)$ and all $\Phi \in \operatorname{Def}_\delta(\Omega, M)$, where we denote by $\|\cdot\|_{W^{1,2}_{p,q}(\Omega_2,\varphi)}$ the Sobolev $W^{1,2}$-norm on (p,q)-forms with respect to our weights.

By use of the Gårding inequality (see e.g. [FK], Theorem 2.2.1), there exits another constant $C'(\Omega_2,\chi,\varphi) > 0$ such that

$$\|\chi u\|^2_{W^{1,2}_{p,q}(\Omega_2,\varphi)} \leq C'(\Omega_2,\chi,\varphi)\|u\|^2_{\Gamma_{p,q}(M,\varphi)} \tag{114}$$

for all $u \in C^\infty_{p,q}(M)$. Estimates (113), (114) and an approximation argument show that the condition (i) in Theorem 3.9 is fulfilled for a set of forms $\mathcal{A}$ which is bounded in the graph norm, because we can achieve

$$\|\Phi^* f - f\|_{L^2_{p,q}(M,\varphi)} < \epsilon$$

for all $f \in \mathcal{A}$ and all $\Phi \in \operatorname{Def}_\delta(\Omega, M)$ by choosing δ small enough in (113). □

As before, Theorem 3.10 generalizes immediately to forms with values in a Hermitian vector bundle. Note that we intend to use Theorem 3.10 with $M = \operatorname{Reg} X$, the regular set of a singular Hermitian space, or an open subset of $\operatorname{Reg} X$.

Theorem 3.10 remains valid for any other closed extension of the $\overline{\partial}$-operator on M, because we only have to prove uniform behavior under deformations in a subset Ω which is relatively compact contained in M.

From Theorem 3.10, one can deduce the following criterion for compactness of the $\overline{\partial}$-Neumann operator (see [R8], Theorem 1.3). Recall that the $\overline{\partial}$-Neumann operator N is defined as follows: for $u \in \operatorname{Im} \Box$, let Nu be the unique form in $\Box^{-1}(\{u\})$ which is orthogonal to $\ker \Box$. According to the remark above, we can consider the $\overline{\partial}$-Neumann operator coming from any closed L^2-extension of the $\overline{\partial}$-operator.

Theorem 3.11. *Let Z be a Hermitian complex space of pure dimension n, $Y \subset Z$ an open Hermitian submanifold and $\overline{\partial}$ a closed L^2-extension of the $\overline{\partial}_{cpt}$-operator on smooth forms with compact support in Y, for example the $\overline{\partial}$-operator in the sense of distributions. Let $0 \leq p, q \leq n$.*

Assume that $\overline{\partial}$ has closed range in $L^2_{p,q}(Y)$ and in $L^2_{p,q+1}(Y)$. Then

$$\Box = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$$

has closed range in $L^2_{p,q}(Y)$ and the following conditions are equivalent:

(i) The $\overline{\partial}$-Neumann operator $N = \Box^{-1} : Im\, \Box \to L^2_{p,q}(Y)$ is compact.

(ii) For all $\epsilon > 0$, there exists $\Omega \subset\subset Y$ such that $\|u\|_{L^2_{p,q}(Y-\Omega)} < \epsilon$ for all

$$u \in \{u \in \mathrm{Dom}(\overline{\partial}) \cap \mathrm{Dom}(\overline{\partial}^*) \cap Im\, \Box : \|\overline{\partial}u\|_{L^2_{p,q+1}} + \|\overline{\partial}^* u\|_{L^2_{p,q-1}} < 1\}.$$

(iii) There exists a smooth function $\psi \in C^\infty(Y, \mathbb{R})$, $\psi > 0$, such that $\psi(z) \to \infty$ as $z \to bY$, and

$$(\Box u, u)_{L^2} \geq \int_Y \psi |u|^2 dV_Y \quad \textit{for all } u \in \mathrm{Dom}\, \Box \cap Im\, \Box \subset L^2_{p,q}(Y).$$

3.2.2 Compact solution operators for the $\overline{\partial}$-equation

We are now in the position to construct compact solution operators for the $\overline{\partial}_{max}$-equation at isolated singularities. Let $a \in \mathrm{Sing}\, X$ be an isolated singularity and U a strongly pseudoconvex neighborhood of a as in Theorem 3.5. Let φ be the weight

$$\varphi = -\log\left(d_a^{-2} \log^{-4}(1 + d_a^{-1})\right)$$

if $p + q < n$ or

$$\varphi = -\log d_a^{-2c}$$

if $p + q > n$. For $p + q \neq n$, $q \geq 1$, let

$$T_1 : L^2_{p,q-2}(U^*) \to L^2_{p,q-1}(U^*, \varphi)$$

and

$$T_2 : L^2_{p,q-1}(U^*, \varphi) \to L^2_{p,q}(U^*)$$

be the $\overline{\partial}$-operators in the sense of distributions (ignore T_1 if $q = 1$).

T_1 and T_2 are closed densely defined operators, $T_2 \circ T_1 = 0$ and T_2 has closed range of finite codimension in $\ker \overline{\partial}_{max} \subset L^2_{p,q}(U^*)$ by Theorem 3.5.

So, the adjoint operators T_1^* and T_2^* are closed densely defined operators with $T_1^* \circ T_2^* = 0$ and T_2^* has closed range. Thus, we can use the orthogonal decomposition

$$L^2_{p,q-1}(U^*, \varphi) = \ker T_2 \oplus \mathcal{R}(T_2^*) \tag{115}$$

to define a bounded solution operator for the $\overline{\partial}$-equation.

Let $H \subset L^2_{p,q}(U^*)$ be the closed subspace from Theorem 3.5 if $p + q < n$ or $H = \ker \overline{\partial}_{max} \subset L^2_{p,q}(U^*)$ if $p + q > n$.

Let
$$L = \{u \in \operatorname{Dom} T_2 : u \perp \ker T_2\}$$
be the Banach space with the norm
$$\|u\|_L^2 := \|u\|^2_{L^2_{p,q-1}(U^*,\varphi)} + \|T_2 u\|^2_{L^2_{p,q}(U^*)}.$$
So, the mapping
$$T_2|_L : L \to \mathcal{R}(T_2),\ u \mapsto T_2 u$$
is a bounded linear isomorphism. Therefore,
$$\mathbf{S} := (T_2|_L)^{-1} : \mathcal{R}(T_2) \subset L^2_{p,q}(U^*) \longrightarrow L \subset \mathcal{R}(T_2^*) \subset L^2_{p,q-1}(U^*,\varphi) \tag{116}$$
is a bounded solution operator for the $\overline{\partial}$-equation and it satisfies
$$T_1^* \circ \mathbf{S} = 0 \tag{117}$$
because $T_1^* \circ T_2^* = 0$.

Since $L^2_{p,q-1}(U^*,\varphi)$ is naturally contained in $L^2_{p,q-1}(U^*)$, we can show by use of the criterion for precompactness Theorem 3.10 that $\mathbf{S}$ is compact as an operator to the latter space. Note that a priori the subspace H from Theorem 3.5 is contained in $\operatorname{Dom}\mathbf{S} = \mathcal{R}(T_2)$, but we can simply assume that $H = \operatorname{Dom}\mathbf{S} = \mathcal{R}(T_2)$.

Theorem 3.12. ([OR], Theorem 5.2) *Let $p+q \neq n$. For $q \geq 2$, the $\overline{\partial}$-solution operator $\mathbf{S}$ is compact as an operator*
$$\mathbf{S} : \operatorname{Dom}\mathbf{S} = H \subset L^2_{p,q}(U^*) \to L^2_{p,q-1}(U^*).$$
For $q = 1$, there exists a bounded operator $P_0 : H \to L^2_{p,0}(U^*)$ such that $\mathbf{S} - P_0$ is a compact $\overline{\partial}$-solution operator
$$\mathbf{S} - P_0 : H \subset L^2_{p,1}(U^*) \to L^2_{p,0}(U^*).$$

Proof. Let
$$\mathcal{L} = \{f \in H : \|f\|_{L^2_{p,q}(U^*)} < 1\}.$$
We will show that $\mathbf{S}(\mathcal{L})$ is precompact in $L^2_{p,q-1}(U^*)$ if $q \geq 2$. To do this, we treat the singular point a and the strongly pseudoconvex boundary bU separately.

So let $\chi \in C^\infty_{cpt}(U)$, $0 \leq \chi \leq 1$, be a smooth cut-off function with compact support in U such that $\chi \equiv 1$ in a neighborhood of the singular point a. Let us first show that
$$\mathcal{L}_1 := \{\chi \mathbf{S}(f) : f \in \mathcal{L}\}$$
is precompact in $L^2_{p,q-1}(U^*)$ by use of the criterion Theorem 3.10 with $M = U^*$.

Since $\mathbf{S}$ is bounded as an operator to $L^2_{p,q-1}(U^*,\varphi)$, there exists a constant $C_S > 0$ such that
$$\|u\|_{L^2_{p,q-1}(U^*,\varphi)} \leq C_S$$
for all $u \in S(\mathcal{L})$.

Let $K = \operatorname{supp}\chi$, $K^* = K - \{a\}$. Now then, let $\epsilon > 0$. Choose $U_\epsilon \subset\subset U^*$ such that

$$e^{-\varphi} \geq 1/\epsilon^2 \tag{118}$$

on $K^* - U_\epsilon$. This is possible because $K^* - U_\epsilon$ is a neighborhood of the point a if U_ϵ is big enough and $e^{-\varphi(z)} \to +\infty$ as z approaches the singular point a. Then

$$\begin{aligned} \epsilon^{-2} \int_{U^*-U_\epsilon} |\chi u|^2 dV_X &= \epsilon^{-2} \int_{K^*-U_\epsilon} |\chi u|^2 dV_X \\ &\leq \int_{K^*-U_\epsilon} |\chi u|^2 e^{-\varphi} dV_X \\ &\leq \int_{U^*} |u|^2 e^{-\varphi} dV_X \leq C_S^2 \end{aligned}$$

for all $u \in \mathbf{S}(\mathcal{L})$ by use of $K = \operatorname{supp}\chi$, (118) and $|\chi| \leq 1$. Hence

$$\|v\|_{L^2_{p,q-1}(U^*-U_\epsilon)} \leq \epsilon C_S$$

for all $v = \chi u \in \mathcal{L}_1$. That proves condition (C) in Theorem 3.10 for the set of forms $\mathcal{L}_1 = \chi\mathbf{S}(\mathcal{L})$. It remains to show that $\mathcal{L}_1$ is bounded in the graph norm $\|\cdot\|_{\Gamma_{p,q}(U^*)}$, but that follows from the construction of the operator $\mathbf{S}$:

For $u \in \mathbf{S}(\mathcal{L})$, we have

$$\|u\|^2_{L^2_{p,q-1}(U^*,\varphi)} + \|T_2 u\|^2_{L^2_{p,q}(U^*)} < C_S^2 + 1$$

and $T_1^* u = 0$. Since χ is constant outside a compact subset of U^*, there exists a constant $C_\chi > 0$ such that

$$\|\chi u\|^2_{L^2_{p,q-1}(U^*,\varphi)} + \|T_2(\chi u)\|^2_{L^2_{p,q}(U^*)} + \|T_1^*(\chi u)\|^2_{L^2_{p,q-2}(U^*)} < C_\chi(C_S^2 + 1). \tag{119}$$

Since $e^{-\varphi}$ is bounded from below by a constant $C_\varphi > 0$ on $\operatorname{supp}\chi$, we also have

$$\|\chi u\|^2_{L^2_{p,q-1}(U^*)} \leq C_\varphi^{-1} \|\chi u\|^2_{L^2_{p,q-1}(U^*,\varphi)},$$

and (119) yields

$$\|v\|_{\Gamma_{p,q}(U^*)} < (1 + C_\varphi^{-1}) C_\chi (C_S^2 + 1)$$

for all $v = \chi u \in \mathcal{L}_1$. This completes the proof of precompactness of $\mathcal{L}_1 = \chi\mathbf{S}(\mathcal{L})$ by use of Theorem 3.10.

The second step is to show that

$$\mathcal{L}_2 = \{(1-\chi)\mathbf{S}(f) : f \in \mathcal{L}\}$$

is precompact in $L^2_{p,q-1}(U^*)$. But this follows from Kohn's theory since U has smooth strongly pseudoconvex boundary and $(1-\chi)$ has support away from the singular point a. We indicate how this can be seen by use of a resolution of singularities.

Let V be an open neighborhood of a in U such that

$$N = \overline{V} \subset\subset \{z \in U : \chi(z) = 1\} \subset\subset U,$$

and let

$$\pi : M \to X$$

be a resolution of singularities as above. Set $U' = \pi^{-1}(U)$ and $N' = \pi^{-1}(N)$. Again, let $\gamma := \pi^* h$ be the pullback of the Hermitian metric h of X to M which is positive semidefinite with degeneracy locus E.

As above, give M the structure of a Hermitian manifold with a freely chosen (positive definite) metric σ. Then

$$\gamma \lesssim \sigma$$

on a neighborhood of $\overline{U'}$ and

$$\gamma \sim \sigma$$

on $U' - N'$ since the degeneracy locus E of γ is compactly contained in $\pi^{-1}(V)$. Since $\gamma = \pi^* h \sim \sigma$ on $U' - N'$, we have

$$L^2_{p,q-1}(U - N) \cong L^{p,q-1}_{\gamma}(U' - N') \cong L^{p,q-1}_{\sigma}(U' - N').$$

But the forms in $\mathcal{L}_2$ have support in $\overline{U} - N$. So, it is enough to show that $\pi^*\mathcal{L}_2$ is precompact in $L^{p,q-1}_{\sigma}(U')$, but this is well-known by Kohn's basic estimate and the Sobolev embedding theorem. For the details of this step, we refer to the proof of [OR], Theorem 5.2.

It remains to consider the case $q = 1$. Let

$$\Pi_0 : L^{p,0}_{\sigma}(U') \to \ker \overline{\partial} \subset L^{p,0}_{\sigma}(U')$$

be the Bergman projection (the orthogonal projection onto $\ker \overline{\partial}$).

We can now define the operator

$$P_0 : H \to \ker \overline{\partial} \subset L^2_{p,0}(U^*)$$

as

$$P_0(f) := (\pi|_{U'-E}^{-1})^* \circ \Pi_0 \circ \pi^* \big((1 - \chi)\mathbf{S}(f)\big).$$

Since $\pi : U' - E \to U^*$ is biholomorphic, it is clear that $\overline{\partial} P_0(f) = 0$, so that $\mathbf{S} - P_0$ remains a solution operator for the $\overline{\partial}$-equation. Since $(1 - \chi) \equiv 0$ on N, it is clear that

$$f \mapsto \Pi_0 \circ \pi^* \big((1 - \chi)\mathbf{S}(f)\big)$$

is a bounded map $H \to \ker \overline{\partial} \subset L^{p,0}_{\sigma}(U')$.

On the other hand, (23) yields (because E is thin):

$$\|(\pi|_{U'-E}^{-1})^* \omega\|_{L^2_{p,0}(U^*)} = \|\omega\|_{L^{p,0}_{\gamma}(U')} \leq \|\omega\|_{L^{p,0}_{\sigma}(U')}.$$

Hence

$$(\pi|_{U'-E}^{-1})^* : L_\sigma^{p,0}(U') \to L^2_{p,0}(U^*) \tag{120}$$

is bounded, and we see that P_0 is a bounded linear map.

It is now easy to see by Kohn's basic estimates that $(1-\chi)\mathbf{S} - P_0$ is compact. Because of (120), it is enough to show that

$$\pi^*\mathcal{L}_2 - \Pi_0\big(\pi^*\mathcal{L}_2\big) \tag{121}$$

is precompact in $L_\sigma^{p,0}(U')$. But this follows as above by Kohn's basic estimates and the Sobolev embedding theorem since U' is a domain with smooth strongly pseudoconvex boundary in the complex manifold M.

On the other hand, the operator $\chi\mathbf{S}$ is compact as in the case $q \geq 2$. That completes the case $q = 1$. $\square$

So, there are compact $\overline{\partial}_{max}$-solution operators $\ker \overline{\partial}_{max} \to L^{p,q-1}(U^*)$ on a small strongly pseudoconvex neighborhood of an isolated singularity if $p+q \neq n$. This leads directly to the proof of the second part of Theorem 2.4.

Theorem 3.13. ([OR], Theorem 1.1) *In the situation of Theorem 3.6, assume that the $\overline{\partial}$-operator in the sense of distributions*

$$\overline{\partial}_{max} : L^2_{p,q-1}(\Omega^*) \to L^2_{p,q}(\Omega^*)$$

has closed range of finite codimension in $\ker \overline{\partial} \subset L^2_{p,q}(\Omega^*)$ *so that the $\overline{\partial}$-operator in the sense of distributions*

$$\overline{\partial}^M_{max} : L_\sigma^{p,q-1}(\Omega') \to L_\sigma^{p,q}(\Omega')$$

also has closed range of finite codimension in $\ker \overline{\partial}^M_{max} \subset L_\sigma^{p,q}(\Omega')$ *(by the equivalence in Theorem 3.6). Recall that $q \geq 1$ and either $p+q \neq n$ or $(p,q) = (0,n)$.*

Then there exists a compact $\overline{\partial}$-solution operator

$$\mathbf{S} : Im\, \overline{\partial}_{max} \subset L^2_{p,q}(\Omega^*) \to L^2_{p,q-1}(\Omega^*)$$

exactly if there exists a compact $\overline{\partial}$-solution operator

$$\mathbf{S}_M : Im\, \overline{\partial}^M_{max} \subset L_\sigma^{p,q}(\Omega') \to L_\sigma^{p,q-1}(\Omega').$$

Proof. The proof follows directly from the proof of Theorem 3.6, see the remark directly after the proof of Theorem 3.6. Note that by Theorem 3.4, one can consider the operators $\overline{\partial}_{mE}$ and $\overline{\partial}_{-mE}$ on the line bundles L_{mE} and L_{-mE} instead of $\overline{\partial}^M_{max}$. Now then, a solution operator $\mathbf{S}_{mE}$ can be constructed as a combination of operators involving $\mathbf{S}$ and compact $\overline{\partial}$-solution operators on a strongly pseudoconvex neighborhood of the exceptional set, so that $\mathbf{S}_{mE}$ is compact if $\mathbf{S}$ is compact. Conversely, $\mathbf{S}$ can be constructed as a combination of $\mathbf{S}_{-mE}$ and the compact solution operators from Theorem 3.12, so that $\mathbf{S}$ is compact if $\mathbf{S}_{-mE}$ is compact. $\square$

3.2.3 Compactness of the $\overline{\partial}$-Neumann operator

It is now easy to treat the $\overline{\partial}$-Neumann operator on a singular space with only isolated singularities because it can be represented by use of canonical $\overline{\partial}$-solution operators.

Theorem 3.14. ([R8], Theorem 1.1, [OR], Theorem 1.1) *Let X be a Hermitian complex space of pure dimension n with only isolated singularities, and $\Omega \subset\subset X$ with no singularities in the boundary $b\Omega$. Let $\pi : M \to X$ be a resolution of singularities as above and σ any (positive definite) Hermitian metric on M.*

Let $q \geq 1$ and either $p + q \neq n - 1, n$ or $p = 0$. Assume that the $\overline{\partial}$-operators in the sense of distributions with respect to the metric σ on $\Omega' = \pi^{-1}(\Omega)$,

$$\overline{\partial}^M_{p,q} : \quad L^{p,q-1}_\sigma(\Omega') \to L^{p,q}_\sigma(\Omega'), \tag{122}$$

$$\overline{\partial}^M_{p,q+1} : \quad L^{p,q}_\sigma(\Omega') \to L^{p,q+1}_\sigma(\Omega') \tag{123}$$

both have closed range of finite codimension in the corresponding kernels of $\overline{\partial}^M$. Let

$$N^M_{p,q} = (\Box^M_{p,q})^{-1} : L^{p,q}_\sigma(\Omega') \to L^{p,q}_\sigma(\Omega')$$

be the $\overline{\partial}$-Neumann operator for $\Box^M_{p,q} = \overline{\partial}^M_{p,q}(\overline{\partial}^M_{p,q})^* + (\overline{\partial}^M_{p,q+1})^*\overline{\partial}^M_{p,q+1}$ which is bounded as seen in section 3.1.1

On $\Omega^* = \Omega - \operatorname{Sing} X$, the operators

$$\overline{\partial}_{p,q} : \quad L^{p,q-1}(\Omega^*) \to L^{p,q}(\Omega^*),$$

$$\overline{\partial}_{p,q+1} : \quad L^{p,q}(\Omega^*) \to L^{p,q+1}(\Omega^*)$$

have closed range of finite codimension in the corresponding kernels (Theorem 3.6). Let

$$N_{p,q} = \Box^{-1}_{p,q} : L^{p,q}(\Omega^*) \to L^{p,q}(\Omega^*)$$

be the $\overline{\partial}$-Neumann operator for $\Box_{p,q} = \overline{\partial}_{p,q}\overline{\partial}^*_{p,q} + \overline{\partial}^*_{p,q+1}\overline{\partial}_{p,q+1}$ which is also bounded.

Then: $N_{p,q}$ is compact on Ω^* exactly if $N^M_{p,q}$ is compact on Ω'.

Proof. By Theorem 3.3, $N^M_{p,q}$ is compact exactly if the canonical $\overline{\partial}^M$-solution operators

$$\mathbf{S}^M_{p,q} : \quad L^{p,q}_\sigma(\Omega') \to L^{p,q-1}_\sigma(\Omega'),$$

$$\mathbf{S}^M_{p,q+1} : \quad L^{p,q+1}_\sigma(\Omega') \to L^{p,q}_\sigma(\Omega')$$

both are compact. By Theorem 3.13 and Lemma 3.1, this is exactly the case if the canonical $\overline{\partial}$-solution operators

$$\mathbf{S}_{p,q} : \quad L^{p,q}(\Omega^*) \to L^{p,q-1}(\Omega^*),$$

$$\mathbf{S}_{p,q+1} : \quad L^{p,q+1}(\Omega^*) \to L^{p,q}(\Omega^*)$$

both are compact. Another application of Theorem 3.3 shows that this in turn is equivalent to compactness of $N_{p,q}$. □

4 Integral formulas

Besides the L^2-theory, which plays a very prominent role in Complex Analysis because of the powerful Hilbert space machinery and its various applications, also other function spaces are of great interest. One may think of L^p, Hölder or C^k-regularity. Here, integral formulas turned out to be extremely fruitful on complex manifolds since the beginning of the 1970s. They do not only give access to such function spaces, but are also very useful to study properties as compactness of operators, and they also yield a broad spectrum of applications as for example Hartogs' extension theorem in $\mathbb{C}^n$ or the approximation of holomorphic functions on strongly pseudoconvex domains due to Henkin, Kerzman and Lieb (see [L2]). Integral formulas have the big advantage that regularity properties can be derived directly and often quite easy from the integral kernels involved as soon as an explicit integral representation is known. For an introduction to the topic, we refer to [R1] and [LM].

We distinguish two kinds of integral formulas on a singular complex spaces X. On the one hand, there are formulas which can be applied to differential forms which are defined just on X itself (i.e. on the regular part of X). We will call such formulas intrinsic. On the other hand, there are formulas which can be applied only to forms which admit continuous or smooth extensions to neighborhoods of X in local embeddings. Such formulas are called extrinsic formulas and are not always applicable because extendibility of functions and forms is a delicate problem. This starts with the difference between weakly and strongly holomorphic functions on the level of holomorphic functions, but is much more complicated in general.

In the present exposition, we focus on intrinsic formulas. Such formulas seem to be more difficult to construct. One reason is that we cannot expect a Bochner-Martinelli-Koppelman formula for forms which are just defined (and maybe L^2) on the regular part of the variety, as there are obstructions to local solvability of the $\overline{\partial}$-equation in the intrinsic sense on $\operatorname{Reg} X$. On the other hand, if we are able to obtain some integral formulas, then they yield very interesting results as we will see below in some cases. We summarize briefly the main results from [R4], [R5], [R6] and from [RZ1], [RZ2]

Finally, we will mention the extrinsic Koppelman formulas of Andersson and Samuelsson [AS1], [AS2], [AS3], which yield a local Grothendieck-Dolbeault lemma for forms which extend smoothly to neighborhoods of the variety in local embeddings. This means that there are no obstructions to $\overline{\partial}$-solvability for intrinsic forms which can be approximated in a suitable sense by extendable forms. This is in some sense related to [R7], where we treat a generalization of Friedrich's extension lemma.

4.1 The Dolbeault lemma at normal crossings

In this section, we present a Dolbeault complex with weights according to normal crossings, which is a useful tool for studying the $\overline{\partial}$-equation on singular complex spaces by resolution of singularities where normal crossings appear naturally. The major difficulty is to prove that this complex is locally exact. This is done by use of a local $\overline{\partial}$-solution operator which involves only Cauchy's integral formula and behaves well for L^p-forms with weights according to normal crossings.

As before, let X be a Hermitian complex space of pure dimension n, and $\pi : M \to X$ a resolution of singularities. So, we may assume that M is a complex manifold of dimension n, π is a proper analytic map which is a biholomorphism outside the exceptional set

$$E = |\pi^{-1}(\operatorname{Sing} X)|,$$

and E consists only of normal crossings. Since we will treat a local question, we can assume that X is an analytic variety in $\mathbb{C}^N$ and that $X^* = \operatorname{Reg} X$ carries the metric induced by the canonical embedding $\iota : X \hookrightarrow \mathbb{C}^N$. We denote by dV_X the volume element on X^*, and by dV_M the volume element on M. We can assume that π preserves orientation.

Let $Q \in E$ be a point on the exceptional set. Then there is a neighborhood U of Q in M with local coordinates $z_1, ..., z_n$ such that we can assume $Q = 0 \in U \subset \mathbb{C}^n$, and

$$E \cap U = \{z \in U : z_1 \cdots z_m = 0\}$$

for a certain integer m, $1 \leq m \leq n$. We can assume furthermore that $z_1, ..., z_n$ are Euclidean coordinates.

Then there exists by Lemma 2.1 in [R4] a holomorphic function $J \in \mathcal{O}(U)$, vanishing exactly on $E \cap U$, such that

$$|J|^2 = \det \operatorname{Jac}_{\mathbb{R}} \pi = \det({}^t\overline{\operatorname{Jac}_{\mathbb{C}}}\pi \cdot \operatorname{Jac}_{\mathbb{C}} \pi) \tag{124}$$

We may write $J = z^w = z_1^{w_1} \cdots z_n^{w_n}$, where $w = (w_1, ..., w_n) \in \mathbb{Z}^n$ is a multi-index with $w_1, ..., w_m \geq 1$ and $w_{m+1} = ... = w_n = 0$. (124) has to be understood in the following sense: $\pi : U \setminus E \to \operatorname{Reg} X$ is a diffeomorphism which has a well defined determinant of the real Jacobian $\operatorname{Jac}_{\mathbb{R}} \pi$ that extends as $|J|^2$ to U. By choosing local holomorphic coordinates on $\operatorname{Reg} X$ one can get the right hand side of (124) where $\operatorname{Jac}_{\mathbb{C}} \pi$ is the complex Jacobian of π as a mapping $\pi : \mathbb{C}^n \to \mathbb{C}^N$. Since $z_1, ..., z_n$ are Euclidean coordinates and X carries the metric induced by the Euclidean metric in $\mathbb{C}^N$, it follows that

$$\pi^* dV_X = (\det \operatorname{Jac}_{\mathbb{R}} \pi)\, dV_M = |J|^2 dV_M. \tag{125}$$

This is a refined version of (21) where we simply used a continuous function g in place of the holomorphic function J. For more details on (124) and (125), see section 2 in [R9].

Now, let $1 \leq p \leq \infty$, and $f \in L^p(G)$ where $G = \pi(U)^* = \operatorname{Reg} \pi(U) \subset X^*$. Then it follows from (125) that

$$\int_{U \setminus E} |\pi^* f|^p |J|^2 dV_M = \int_G |f|^p dV_X,$$

and this yields that (in multi-index notation for $|z|^{2w/p}$):

$$f \in L^p(G) \quad \Leftrightarrow \quad |J|^{2/p} \pi^* f = |z|^{2w/p} \pi^* f \in L^p(U).$$

This gives reason to the following construction:

Definition 4.1. *Let $D \subset \mathbb{C}^n$ be an open set, $1 \leq p \leq \infty$ and $s = (s_1, ..., s_n) \in \mathbb{R}^n$ a real multi-index. Then, we define:*

$$|z|^s L^p_{0,q}(D) := \{f \text{ measurable on } D : |z|^{-s} f \in L^p_{0,q}(D)\}.$$

$|z|^s L^p_{0,q}(D)$ is a Banach space with the norm

$$\|f\|_{|z|^s L^p_{0,q}(D)} := \||z|^{-s} f\|_{L^p_{0,q}(D)}.$$

We use the multi-index notation $|z|^{-s} = |z_1|^{-s_1} \cdots |z_n|^{-s_n}$.

The main objective of the paper [R5] was to study the $\overline{\partial}$-equation on $|z|^s L^p_{0,q}(D)$. But that does not make sense in general for the usual $\overline{\partial}$-operator. It is therefore adequate to introduce the following weighted operator ($\overline{\partial}$ has to be understood always in the sense of distributions):

Definition 4.2. *Let $k = (k_1, ..., k_n) \in \mathbb{Z}^n$ be an integer-valued multi-index, and let f be a measurable $(0, q)$-form on $D \subset \mathbb{C}^n$ such that*

$$z^{-k} f \in L^1_{(0,q),loc}(D) \quad \text{and} \quad \overline{\partial}\big(z^{-k} f\big) \in L^1_{(0,q+1),loc}(D).$$

Then, we set

$$\overline{\partial}_k f := z^k \overline{\partial}\big(z^{-k} f\big) \in |z|^k L^1_{(0,q+1),loc}(D).$$

Note that $\overline{\partial}_k f = 0$ exactly if $\overline{\partial}(z^{-k} f) = 0$. It is clear that $\overline{\partial}_k \circ \overline{\partial}_k = 0$. We will now use the abstract Theorem of de Rham in order to establish a link between the $\overline{\partial}_k$-equation $\overline{\partial}_k g = f$ in $|z|^s L^p_{0,*}(D)$ and certain cohomology groups on D. The right coherent analytic sheaves to look at are the following:

Definition 4.3. *For $1 \leq j \leq n$, let $\mathcal{I}_j = (z_j)$ be the sheaf of ideals of $\{z_j = 0\}$ in $\mathbb{C}^n$. If $k = (k_1, ..., k_n) \in \mathbb{Z}^n$ is an integer-valued multi-index, let*

$$\mathcal{I}^k \mathcal{O} = \mathcal{I}_1^{k_1} \cdots \mathcal{I}_n^{k_n} \mathcal{O}$$

as a subsheaf of the sheaf of germs of meromorphic functions.

Note that we could as well consider the usual $\overline{\partial}$-operator on sections of a holomorphic line bundle $L^k = L_1^{k_1} \otimes \cdots \otimes L_n^{k_n}$ such that $\mathcal{I}^k \mathcal{O} \cong \mathcal{O}(L^k)$, where $\mathcal{O}(L^k)$ is the sheaf of germs of holomorphic sections in L^k. This point of view is equivalent and wouldn't influence the presentation much.

We need to choose the right operator $\overline{\partial}_k$ for given values of p and s. $k = k(p, s)$ should be the maximal value such that $|z|^s L^p_{loc} \subset |z|^k L^1_{loc}$. We will see below (Theorem 4.6) that this is in fact a good choice. So:

Definition 4.4. *Let $1 \leq p \leq \infty$ and s be real numbers. Then we call*

$$k(p, s) := \max\{m \in \mathbb{Z} : |z_1|^s L^p_{loc}(\mathbb{C}) \subset |z_1|^m L^1_{loc}(\mathbb{C})\}$$

the $\overline{\partial}$-weight of (p, s). For $s = (s_1, ..., s_n) \in \mathbb{R}^n$, let $k(p, s) = (k_1, ..., k_n) \in \mathbb{Z}^n$ be given by $k_j := k(p, s_j)$, or, equivalently,

$$k(p, s) := \max\{m \in \mathbb{Z}^n : |z|^s L^p_{loc}(\mathbb{C}^n) \subset |z|^m L^1_{loc}(\mathbb{C}^n)\}.$$

Then, we define for $0 \leq q \leq n$ the sheaves $|z|^s \mathcal{L}^p_{0,q}$ by:

$$|z|^s \mathcal{L}^p_{0,q}(U) := \{f \in |z|^s L^p_{(0,q),loc}(U) : \overline{\partial}_{k(p,s)} f \in |z|^s L^p_{(0,q+1),loc}(U)\}$$

for open sets $U \subset \mathbb{C}^n$ (it is a presheaf which is already a sheaf).

The $\overline{\partial}$-weight can be computed explicitly as follows:

Lemma 4.5. ([R5], Lemma 2.2) *Let $1 \leq p \leq \infty$ and s be real numbers, and $k(p,s)$ the $\overline{\partial}$-weight of (p,s) according to Definition 4.4. Then*

$$k(p,s) = \begin{cases} \max\{m \in \mathbb{Z} : m < 2 + s - 2/p\} & , \ p \neq 1, \\ \max\{m \in \mathbb{Z} : m \leq 2 + s - 2/p\} & , \ p = 1. \end{cases} \tag{126}$$

Now then, the main result from [R5] reads as:

Theorem 4.6. ([R5], Theorem 1.5) *For $1 \leq p \leq \infty$ and $s \in \mathbb{R}^n$, let $k(p,s) \in \mathbb{Z}^n$ be the $\overline{\partial}$-weight according to Definition 4.4. Then:*

$$0 \to \mathcal{I}^k\mathcal{O} \hookrightarrow |z|^s \mathcal{L}^p_{0,0} \xrightarrow{\overline{\partial}_k} |z|^s \mathcal{L}^p_{0,1} \xrightarrow{\overline{\partial}_k} \cdots \xrightarrow{\overline{\partial}_k} |z|^s \mathcal{L}^p_{0,n} \to 0 \tag{127}$$

is an exact (and fine) resolution of $\mathcal{I}^k\mathcal{O}$.

By the abstract Theorem of de Rham, this implies that

$$H^q(U, \mathcal{I}^k\mathcal{O}) \cong \frac{\ker\ (\overline{\partial}_k : |z|^s \mathcal{L}^p_{0,q}(U) \to |z|^s \mathcal{L}^p_{0,q+1}(U))}{\operatorname{Im}\ (\overline{\partial}_k : |z|^s \mathcal{L}^p_{0,q-1}(U) \to |z|^s \mathcal{L}^p_{0,q}(U))}$$

for open sets $U \subset \mathbb{C}^n$. Thus, we can study the equation $\overline{\partial}_k g = f$ on U by investigating the groups $H^q(U, \mathcal{I}^k\mathcal{O})$. Due to the local nature of Theorem 4.6, it is easy to deduce similar statements on complex manifolds, which will be a helpful tool for studying the $\overline{\partial}$-equation on singular spaces as indicated in the beginning. This is done e.g. in the paper [R6] which we will discuss later.

For the proof of Theorem 4.6, one needs to solve the $\overline{\partial}$-equation locally with weights according to normal crossings. It is adequate to do this by using a weighted version of the inhomogeneous Cauchy integral formula in one complex variable and to integrate just over lines parallel to the Cartesian coordinates. Following this idea, we were able to construct a homotopy formula for the $\overline{\partial}_k$-equation on polydiscs which also yields integral solution operators satisfying the regularity properties needed in Theorem 4.6.

Let us explain briefly the connection between our $\overline{\partial}$-weight and the weighted Cauchy formula. Let $D \subset\subset \mathbb{C}$ be a bounded domain in the complex plane and $k \in \mathbb{Z}$. For a measurable function f on D, we define

$$\mathbf{I}^D_k f(z) := \frac{z^k}{2\pi i} \int_D f(\zeta) \frac{d\zeta \wedge d\overline{\zeta}}{\zeta^k (\zeta - z)}, \tag{128}$$

provided, the integral exists. Note that

$$\frac{\partial}{\partial \overline{z}} \mathbf{I}^D_k f = f$$

in the sense of distributions if $f(\zeta)/\zeta^k$ is integrable in ζ over D.

Theorem 4.7. ([R5], Theorem 2.1) *Let $D \subset\subset \mathbb{C}$ be a bounded domain, $1 \leq p \leq \infty$ and s real numbers, and let $k = k(p,s)$ be the $\overline{\partial}$-weight of (p,s) according to Definition 4.4. Then $\mathbf{I}_k^D$ is a bounded linear operator*

$$\mathbf{I}_k^D : |z|^s L^p(D) \to |z|^{s+1-\epsilon} L^p(D) \quad \textit{for all } \epsilon > 0. \tag{129}$$

This statement is the main ingredient in the construction of homotopy formulas for the $\overline{\partial}_k$-equation which is based on a quite sophisticated iteration procedure. Besides that, we can see in Theorem 4.7 that there is a certain gain of regularity which we do not need for the proof of Theorem 4.6. Nevertheless, this gain of regularity can be incorporated in the $\overline{\partial}_k$-homotopy formula (see Theorem 4.4 in [R5]), and it is sometimes necessary to use this extra information to understand the $\overline{\partial}$-equation in the L^p-sense completely on a singular space. The results from [R5] which we will discuss in the next section depend on these more precise statements. Note in this context that the estimate for the boundedness of the operator (129) is not uniform in ϵ so that the case $\epsilon = 0$ cannot be included. There are counterexamples for that kind of regularity.

4.2 L^p-cohomology of cones with isolated singularities

Let $X \subset \mathbb{C}^N$ be a homogeneous variety of pure dimension n with an isolated singularity in the origin $0 \in \mathbb{C}^N$, and let $G \subset\subset X$ be a strongly pseudoconvex neighborhood of 0 in X. One can chose e.g. the intersection of X with a ball centered at the origin. Let $Y \subset \mathbb{CP}^{n-1}$ be the projective variety associated to X, i.e. X is the affine cone over the projective non-singular variety Y. We consider X as a Hermitian space with the restriction of the Euclidean metric. Let $X^* = X - \{0\}$ and $G^* = G - \{0\}$.

In this section, we present the results from [R6] where we studied the obstructions to solvability of the $\overline{\partial}$-equation in the L^p-sense at the isolated singularity for $0 \leq p \leq \infty$. This information is described by the L^p-cohomology groups $H^{0,q}_{(p)}(G^*)$ formed as the space of $\overline{\partial}$-closed L^p-forms modulo the $\overline{\partial}$-L^p-exact forms. By the $\overline{\partial}$-equation, we always mean the $\overline{\partial}$-equation in the sense of distributions. We have chosen the domain G to have a strongly pseudoconvex boundary so that there are no obstructions to solvability of the $\overline{\partial}$-equation in the L^p-sense coming from bG.

To be more precise, let $|\cdot|_X$ and dV_X be the metric and volume form on X^* induced by the restriction of the Euclidean metric from $\mathbb{C}^N$. Now, if $V \subset X^*$ is an open set and ω a measurable (r,q)-form on V, we set

$$\begin{aligned} \|\omega\|^p_{L^p_{r,q}(V)} &:= \int_V |\omega|^p_X dV_X \ , \text{ for } 1 \leq p < \infty, \\ \|\omega\|_{L^\infty_{r,q}(V)} &:= \operatorname*{ess\,sup}_{z \in V} |\omega|_X(z). \end{aligned}$$

As usually, we denote by $L^p_{r,q}(V)$ the space of measurable (r,q)-forms ω on V such that $\|\omega\|_{L^p_{r,q}(V)} < \infty$. We are interested in the following cohomology groups, where the $\overline{\partial}$-equation has to be interpreted in the sense of distributions. Due to the incompleteness of the metric, different extensions of the $\overline{\partial}$-operator on smooth forms may lead to different cohomology groups as we have seen in the case of L^2-cohomology. For an open set $V \subset X^*$, let

$$H^{r,q}_{(p)}(V) := \frac{\{\omega \in L^p_{r,q}(V) : \overline{\partial}\omega = 0\}}{\{\omega \in L^p_{r,q}(V) : \exists f \in L^p_{r,q-1}(V) : \overline{\partial} f = \omega\}}.$$

The key for the understanding of these L^p-cohomology groups lies again in a resolution of singularities which can be achieved in this simple situation by a single blow up of the origin. We will restrict our attention to the cohomology of $(0,q)$-forms. The (r,q)-cohomology can be computed similarly without additional difficulties.

Let $\pi : M \to X$ be the resolution of X obtained by blowing up the origin in $\mathbb{C}^N$. Then the exceptional set $E = \pi^{-1}(\{0\})$ is canonically isomorphic to the non-singular projective variety Y associated to X. M itself is canonically isomorphic to the universal line bundle $U \to Y$ over the projective variety Y. Let $D = \pi^{-1}(G)$ and denote by $\mathcal{I}$ the sheaf of ideals of the exceptional set E in M.

In this situation, we constructed in [R6] integral operators on X that look like the weighted Cauchy formulas in (128) on complex lines through the origin in X (or on the fibers of the universal line bundle $U \to Y$, respectively). The weight one has to choose depends on p and on the degree of forms $(0,q)$. By use of the estimates from [R5] (see Theorem 4.7 above), it is shown in [R6] that

$$H^{0,q}_{(p)}(G^*) \cong H^q(D, \mathcal{I}^k\mathcal{O}), \tag{130}$$

where $k = k(p,q)$ is an integer which is increasing in p, and $\mathcal{I}^k\mathcal{O}$ is the sheaf of germs of holomorphic functions which vanish to the order k on the exceptional set E. We will specify $k(p,q)$ more precisely below.[23] In the case $p = 2$, one has $k(2,q) = 0$ for all $1 \le q < n-1$, a statement which coincides with our L^2-cohomology Theorem 2.14. To understand the connection, note that here $|Z| = Z$ so that $L_{|Z|-Z}$ is the trivial line bundle. But then $\mathcal{K}_M \otimes \mathcal{O}(|Z| - Z) = \mathcal{K}_M$ is just the canonical sheaf on M and $R^s\pi_*\mathcal{K}_M = 0$ for $s \ge 1$ by Takegoshi's vanishing theorem. This is equivalent to

$$H^q_E(D, \mathcal{O}(Z - |Z|)) = H^q_E(D, \mathcal{O}) = 0$$

for $q < n$ (see section 2.3.4). Thus, (130) is for $p = 2$ in fact equivalent to the statement of Theorem 2.14 where we treated the $\overline{\partial}$-equation in the L^2-sense of distributions on Hermitian spaces with isolated singularities.

The motivation to study the groups $H^{0,q}_{(p)}(G^*)$ for $1 \le p \le \infty$ in the paper [R6] was as follows. In view of the large difficulties in computing the L^2-cohomology explicitly, it seems reasonable to gain a broader view and better understanding by also considering L^p-Dolbeault cohomology groups for arbitrary $1 \le p \le \infty$. Besides the L^2-results mentioned above, only the L^∞-case has been addressed in a number of publications: [AZ1], [AZ2], [FG], [R2], [R4], [RZ1], [SZ]. These papers treat Hölder regularity of the $\overline{\partial}$-equation provided the right-hand side of the equation is bounded. This implies the solution of the Cauchy-Riemann equations in the L^∞-sense. In view of those results, the paper [R6] is an attempt to embed the L^2 and L^∞-case into the broader spectrum of an L^p-theory.

[23] Some special cases of p, q have to be excluded from the statement as we will see later.

In this spirit, it is very interesting to understand (130) better. Since D can be interpreted as a strongly pseudoconvex neighborhood of the zero section in the universal bundle $U \to Y$, we can represent the groups $H^q(D, \mathcal{I}^k\mathcal{O})$ as a direct sum of cohomology groups on the compact projective manifold Y by a method which appears in [R4], Theorem 5.1:

$$H^{0,q}_{(p)}(G^*) \cong H^q(D, \mathcal{I}^{k(p,q)}\mathcal{O}) \cong \bigoplus_{\mu \geq k(p,q)} H^q(Y, \mathcal{O}(U^{-\mu})). \tag{131}$$

Since $U^{-1} = U^*$ is a positive holomorphic line bundle[24], we know that the groups on the right hand side of (131) vanish for μ big enough by a well-known theorem of Grauert (see [G2]).

The representation (131) has nice implications. Since Y is a compact complex manifold and $H^q(Y, \mathcal{O}(U^{-\mu})) = 0$ for $\mu \geq \mu_0$ where μ_0 is an index big enough, it follows that there are only finitely many obstructions to solving the $\overline{\partial}$-equation in the L^p-sense. Furthermore, the index $k(p,q)$ is increasing in p so that the number of obstructions is decreasing in p. This observation is in accordance with a result of Fornæss, Øvrelid and Vassiliadou saying that the $\overline{\partial}$-equation is solvable in the L^2-sense at arbitrary singularities for $\overline{\partial}$-closed forms that vanish to an order high enough in the singular set (see [FOV1]). The same is true in the L^p-sense as shown by Andersson and Samuelsson in [AS1], [AS3].

If Y is specified explicitly, then we can use (131) to compute the obstructions explicitly (see the examples in [R6], section 6). E.g. if Y is a compact Riemann surface, then we can compute the dimension of the groups on the right-hand side of (131) by the Riemann-Roch theorem. One can deduce for instance:

Theorem 4.8. ([R6], Theorem 6.3) *Let $Y \subset \mathbb{CP}^{N-1}$ be a compact Riemann surface, X its affine cone in $\mathbb{C}^N$ and $G^* = X \cap B_1(0) - \{0\}$. Then:*

$$H^{0,1}_{(2)}(G^*) = \{0\} \quad \Leftrightarrow \quad Y \cong \mathbb{CP}^1.$$

It remains to explain (130) and the main results from [R6] more precisely, in particular by specifying the exponent $k = k(p,q)$. By use of the isomorphism in (131) on the right-hand side, the first main statement from [R6] reads as:

Theorem 4.9. ([R6], Theorem 1.1) *Let X, Y and U as above, $G \subset\subset X$ strongly pseudoconvex such that $0 \in G$, $G^* = G \setminus \{0\}$, $D = \pi^{-1}(G)$ and $1 \leq p \leq \infty$, $1 \leq q \leq n = \dim X$. Set*

$$a(p,q,n) := \begin{cases} \max\{k \in \mathbb{Z} : k < 1 + q - 2n/p\} & , p \neq 1, \\ \max\{k \in \mathbb{Z} : k \leq 1 + q - 2n/p\} & , p = 1. \end{cases}$$

Then there exists an injective homomorphism

$$H^{0,q}_{(p)}(G^*) \hookrightarrow H^q(D, \mathcal{I}^{a(p,q,n)}\mathcal{O}) \cong \bigoplus_{\mu \geq a(p,q,n)} H^q(Y, \mathcal{O}(U^{-\mu})). \tag{132}$$

[24] The universal bundle of a projective variety is a negative holomorphic line bundle. This follows e.g. by a criterion of Grauert (see [G2]) from the fact that the zero section of U has a strongly pseudoconvex neighborhood.

Note that the right hand side in (132) is finite-dimensional by Grauert's theorem mentioned above because U is a negative holomorphic line bundle. Dimensional reasons imply that $H^{0,n}_{(p)}(G^*) = \{0\}$ for all $1 \leq p \leq \infty$.

The proof of Theorem 4.9 depends on a quite careful analysis of the behavior of L^p-forms under the blow up $\pi : M \to X$ (see [R6], Lemma 2.1) and on a special extension theorem for extension of the $\overline{\partial}$-equation over the exceptional set E (see [R6], Lemma 3.6). Another essential tool is exactness of the $\overline{\partial}_k$-complex in Theorem 4.6 with the weight $k = a(p,q,n)$, and this is the origin of the number $a(p,q,n)$.

The second main statement from [R6] reads as:

Theorem 4.10. ([R6], Theorem 1.2) *Let X, Y and U as above, and let $G \subset\subset X$ be an open set such that $0 \in G$, $G^* = G \setminus \{0\}$, $D = \pi^{-1}(G)$ and $1 \leq p \leq \infty$, $1 \leq q \leq n = \dim X$. Set*

$$c(p,q,n) := \max\{k \in \mathbb{Z} : k \leq 1 + q - 2n/p\}.$$

Then there exists an injective homomorphism

$$\bigoplus_{\mu \geq c(p,q,n)} H^q(Y, \mathcal{O}(U^{-\mu})) \cong H^q(D, \mathcal{I}^{c(p,q,n)}\mathcal{O}) \hookrightarrow H^{0,q}_{(p)}(G^*). \tag{133}$$

Note that

$$a(p,q,n) = c(p,q,n)$$

if $2n/p \notin \mathbb{Z}$ or $p = 1$, and that

$$c(p,q,d) = a(p,q,d) + 1$$

in all other cases. So, there remains a little uncertainness about the contribution of $H^q(Y, \mathcal{O}(U^{-a(p,q,n)}))$, for example if $p = 2$.

On the other hand, this doesn't matter if $a(p,q,n) < 0$ and $q < n-1$ because in that case Kodaira's vanishing theorem implies that $H^q(Y, \mathcal{O}(U^{-a(p,q,n)})) = \{0\}$ because U is a negative holomorphic line bundle (see e.g. [W], Theorem VI.2.4). So, we obtain (130) with $k(p,q) = a(p,q,n) = c(p,q,n)$ in the generic cases where the latter two coincide. For $p = 2$ and $q < n-1$, we can use $k(2,q) = 0$ because Kodaira's vanishing theorem yields $H^q(Y, U^{-\mu}) = \{0\}$ for all $\mu < 0$.

The proof of Theorem 4.10 depends again heavily on the analysis of the behavior of the $\overline{\partial}$-equation in the L^p-sense under the blow-up. Moreover, another crucial tool enters the proof of Theorem 4.10, namely an integration along the fibers of the holomorphic line bundle U. We set up an integral solution formula for $\overline{\partial}$-closed forms with compact support which looks like a weighted Cauchy formula (128) on the fibers. This idea had been used before by E. S. Zeron and the author in [RZ1] to construct an explicit $\overline{\partial}$-integration formula on weighted homogeneous varieties. For the proof of Theorem 4.10, it is essential to exploit the regularity property of the weighted Cauchy formula presented in Theorem 4.7.

The integral formula from the proof of Theorem 4.10 yields as a byproduct:

Theorem 4.11. *Let X and Y be as above, $G \subset\subset X$ an open subset, $G^* = G \setminus \{0\}$, and $1 \leq p \leq \infty$, $1 \leq q \leq \dim X$. Let $\omega \in L^p_{0,q}(G^*) \cap \ker \overline{\partial}$ with compact support in G. Then there exists $\eta \in L^p_{0,q-1}(G^*)$ such that $\overline{\partial}\eta = \omega$.*

Using Theorem 4.11 in case $q = 1$ and Hartogs' extension theorem on normal Stein spaces with isolated singularities, a statement that we will discuss later, it is easy to deduce vanishing of the first cohomology with compact support:

Theorem 4.12. *Let X and Y be as above, $G \subset\subset X$ an open subset, $G^* = G \setminus \{0\}$, and $1 \leq p \leq \infty$. Then:*

$$H^{0,1}_{(p),cpt}(G) := \frac{\{\omega \in L^p_{0,1}(G^*) : \overline{\partial}\omega = 0,\ \operatorname{supp}\omega \subset\subset G\}}{\{\ \omega \in L^p_{0,1}(G^*) : \exists f \in L^p(G^*) : \overline{\partial} f = \omega,\ \operatorname{supp} f \subset\subset G\}} = \{0\}.$$

4.3 Weighted homogeneous varieties

The idea of integrating along complex lines which we used in the proof of Theorem 4.10 can be applied in much more general situations and leads to solutions of the $\overline{\partial}$-equation in the L^p-sense as in Theorem 4.11 on appropriate spaces. This was done by E. S. Zeron and the author in [RZ1] and [RZ2]. The class of spaces which we considered are the so-called weighted homogeneous varieties, and we were able to allow arbitrary singularities.

Definition 4.13. *Let $\beta \in \mathbb{Z}^n$ be a fixed integer vector with strictly positive entries $\beta_k \geq 1$. A holomorphic polynomial $Q(z)$ on $\mathbb{C}^n$ is said to be **weighted homogeneous** of degree $d \geq 1$ with respect to β if the following equality holds for all $s \in \mathbb{C}$ and $z \in \mathbb{C}^n$:*

$$Q(s^\beta * z) \;=\; s^d\, Q(z), \quad \textit{with the action:} \tag{134}$$

$$s^\beta * (z_1, z_2, ..., z_n) \;:=\; (s^{\beta_1} z_1, s^{\beta_2} z_2, ..., s^{\beta_n} z_n). \tag{135}$$

*An algebraic subvariety Σ in $\mathbb{C}^n$ is said to be **weighted homogeneous** with respect to β if Σ is the zero locus of a finite number of weighted homogeneous polynomials $Q_k(z)$ of (maybe different) degrees $d_k \geq 1$, but all of them with respect to the same fixed vector β.*

Let $\Sigma \subset \mathbb{C}^n$ be any subvariety. As above, we consider Σ as a Hermitian complex space with the restriction of the Euclidean metric on $\Sigma^* = \Sigma - \operatorname{Sing}\Sigma$. We denote by dV_Σ the induced volume form and by $|\cdot|_\Sigma$ the induced pointwise norm on the Grassmannian $\Lambda T^*\Sigma^*$. Any Borel-measurable $(0,q)$-form ω on Σ^* admits a representation $\omega = \sum_J f_J d\overline{z_J}$, where the coefficients f_J are Borel-measurable functions on Σ^* which satisfy the inequality $|f_J(z)| \leq |\omega(z)|_\Sigma$ for all points $z \in \Sigma^*$ and multi-indexes $|J| = q$. Note that such a representation is by no means unique. We refer to Lemma 2.2.1 in [R2] for a more detailed treatment of that point. For $1 \leq p < \infty$, we also introduce the L^p-norm of a measurable $(0,q)$-form ω on an open set $U \subset \Sigma^*$ via the formula:

$$\|\omega\|_{L^p_{0,q}(U)} \;:=\; \left(\int_U |\omega|^p_\Sigma \, dV_\Sigma \right)^{1/p}.$$

The main result of the paper [RZ2] is as follows, the $\overline{\partial}$-operator has to be understood again in the sense of distributions:

Theorem 4.14. ([RZ2], Theorem 2) *Let Σ be a weighted homogeneous subvariety of $\mathbb{C}^n$ with respect to a given vector $\beta \in \mathbb{Z}^n$, where $n \geq 2$ and all entries $\beta_k \geq 1$. Consider the class of all $(0,q)$-forms ω given by $\sum_J f_J d\overline{z_J}$, where $q \geq 1$, the coefficients f_J are all Borel-measurable functions in Σ, and $z_1, ..., z_n$ are the Cartesian coordinates of $\mathbb{C}^n$. Let $\sigma \geq -q$ be any fixed integer. The operator $\mathbf{S}^\sigma_q$ below is well defined on Σ for all forms ω which are essentially bounded and have compact support,*

$$\mathbf{S}^\sigma_q \omega(z) := \sum_{|J|=q} \frac{\aleph_J}{2\pi i} \int_{u \in \mathbb{C}} f_J(u^\beta * z)\, \frac{u^\sigma (\overline{u^{\beta_J}}) d\overline{u} \wedge du}{\overline{u}\,(u-1)} \tag{136}$$

$$\textit{with} \quad \aleph_J = \sum_{j \in J,\, K = J \setminus \{j\}} \frac{\beta_j \overline{z_j}\, d\overline{z_K}}{\mathrm{sign}(j,K)} \quad \textit{and} \quad \beta_J = \sum_{j \in J} \beta_j. \tag{137}$$

Observe that the multi-indexes J and K are both ordered in an ascending way and that $\mathrm{sign}(j,K)$ *is the sign of the permutation used for arranging the elements of the q-tuple (j,K) in ascending order. The form $\mathbf{S}^\sigma_q(\omega)$ is a solution of the $\overline{\partial}$-equation $\omega = \overline{\partial} \mathbf{S}^\sigma_q(\omega)$ on the regular part Σ^* of Σ whenever ω is $\overline{\partial}$-closed on Σ^*.*

Theorem 4.14 for $(0,1)$-forms, i.e. the case $q = 1$, had been treated before in [RZ1], Theorem 2. Note that Σ is allowed to have an arbitrary singular locus. The main point of the proof of Theorem 4.14 is to show that $\omega = \overline{\partial} \mathbf{S}^\sigma_q(\omega)$ if ω is $\overline{\partial}$-closed. That is a local statement. So, we cover Σ by charts which we call generalized cones. When one blows up these cones to complex manifolds, one can realize that the integral formula (136) looks essentially like the inhomogeneous Cauchy formula in one complex variable, and one can deduce the statement by use of classical results.

Similar techniques and a slight modification of the equations (136) and (137) can be used to produce a $\overline{\partial}$-solution operator with L^p-estimates on homogeneous affine varieties with arbitrary singular locus.

Theorem 4.15. ([RZ2], Theorem 3) *Let Σ be a pure d-dimensional homogeneous subvariety of $\mathbb{C}^n$ (a cone), where $n \geq 2$ and each entry $\beta_k = 1$ in Definition 4.13. Fix a real number $1 \leq p \leq \infty$ and an integer $1 \leq q \leq d$. Consider the class $L^p_{0,q}(\Sigma)$ of all $(0,q)$-forms ω given by $\sum_J f_J d\overline{z_J}$, where the coefficients f_J are all L^p-integrable functions in Σ, and $z_1, ..., z_n$ are the Cartesian coordinates of $\mathbb{C}^n$. Choose $\sigma \in \mathbb{Z}$ to be the smallest integer such that*

$$\sigma \geq \frac{2d-2}{p} + 1 - q. \tag{138}$$

Then the operator $\mathbf{S}^\sigma_q$ defined below is well defined almost everywhere on Σ for all forms ω which lie in $L^p_{0,q}(\Sigma)$ and have compact support on Σ:

$$\mathbf{S}^\sigma_q \omega(z) := \sum_{|J|=q} \frac{\aleph_J}{2\pi i} \int_{u \in \mathbb{C}} f_J(uz) \frac{u^\sigma\, \overline{u^q}\, d\overline{u} \wedge du}{\overline{u}\,(u-1)}, \tag{139}$$

$$\textit{where} \quad \aleph_J = \sum_{j \in J,\, K = J \setminus \{j\}} \frac{q\, \overline{z_j}\, d\overline{z_K}}{\mathrm{sign}(j,K)}.$$

The form $\mathbf{S}_q^\sigma(\omega)$ is a solution of the $\overline{\partial}$-equation $\omega = \overline{\partial}\mathbf{S}_q^\sigma(\omega)$ on the regular part Σ^* of Σ whenever ω is $\overline{\partial}$-closed on Σ^*.

Finally, if we assume that the support of ω is contained in an open ball B_R of radius $R > 0$ and center at the origin, then there exists a strictly positive constant $C_\Sigma(R,\sigma)$ which does not depend on ω and such that:

$$\left\|\mathbf{S}_q^\sigma(\omega)\right\|_{L^p_{0,q-1}(\Sigma\cap B_R)} \leq C_\Sigma(R,\sigma)\cdot \|\omega\|_{L^p_{0,q}(\Sigma)}. \tag{140}$$

The L^2-version of Theorem 4.15 for $(0,1)$-forms, i.e. the case $q=1$, had been treated before in [RZ1], Theorem 4. The case $p=\infty$ in Theorem 4.15 is a corollary of Theorem 4.14 because the formulas (139) and (136) coincide in the homogeneous case (where all coefficients $\beta_J = q$). The proof of Theorem 4.15 is based on an analysis of the behavior of norms under blowing up the origin and the L^p-regularity of the weighted Cauchy formula.

The essential innovation of Theorem 4.15 is that it gives a solution operator for forms with compact support in the L^p-sense, not only in the L^2-sense, on a homogeneous space with arbitrary singular locus.

4.4 Hölder regularity

Theorem 4.15 is a nice example of how we can derive statements about the $\overline{\partial}$-equation in various function spaces from integral formulas. Besides the statement about the $\overline{\partial}$-equation in the L^p-sense, we can also deduce very interesting statements about Hölder regularity from the integral formulas (136) and (139) in Theorem 4.14 and Theorem 4.15, respectively. If the equation $\overline{\partial}g = \lambda$ is solvable in the L^∞-sense for a $\overline{\partial}$-closed $(0,1)$-form λ at an isolated singularity of a homogeneous variety, then there exists a solution g which is Hölder-α-continuous for all $0<\alpha<1$. We will discuss this result from [RZ1] in this section. In view of what was known about Hölder regularity of the $\overline{\partial}$-equation on singular complex spaces before, this is a surprisingly strong result. It can be interpreted as an indication that there should hold also some subelliptic estimates for the complex Laplacian.

Let us shortly recall what was known about the solution of the $\overline{\partial}$-equation on singular spaces in the L^∞-sense before [RZ1]. Again, let Σ be a singular subvariety of the space $\mathbb{C}^n$ which carries the restriction of the Euclidean metric, and let λ be a bounded $\overline{\partial}$-closed differential form on the regular part of Σ. Fornæss, Gavosto and the author of this thesis have developed a general technique for solving the $\overline{\partial}$-equation $\lambda = \overline{\partial}g$ in the L^∞-sense with some Hölder-α-estimates on the regular part of Σ, which could be applied successfully to varieties of the form $\{z^m = w_1^{k_1}\cdots w_{n-1}^{k_{n-1}}\} \subset \mathbb{C}^n$ (see [FG] and [R2]). For such varieties, the best Hölder regularity α that was obtained is usually $<< 1$. If e.g. $m>1$ and one of the $k_j = 1$, then $\alpha = 1/m$ is the best regularity that was achieved. The method exploits the fact that such a variety can be considered as an m-sheeted analytic covering of $\mathbb{C}^{n-1}$, so that one can project the problem by use of symmetric combinations to that complex number space, solve the $\overline{\partial}$-equation with certain weights in $\mathbb{C}^{n-1}$, and construct the function g from the pull-back of such solutions. There is a certain chance for this strategy to work in general because any locally irreducible complex space can be represented locally as a finitely sheeted analytic covering over a complex number space (cf. [R2] for more).

On the other hand, Acosta, Solís and Zeron have developed an alternative technique for solving the $\overline{\partial}$-equation in the L^∞-sense with some Hölder estimates at all kinds of isolated singularities of hypersurfaces in $\mathbb{C}^3$, i.e. rational double points (see [AZ1], [AZ2] and [SZ]). They use the fact that all such varieties can be represented as quotient varieties in order to pull-back the problem into a complex number space and solve the equation also by use of symmetric combinations. This strategy has the drawback that not all varieties admit such a representation. Here again, the optimal Hölder regularity obtained is $<< 1$.

Then, E. S. Zeron and the author constructed in [RZ1] the following integral formula on weighted homogeneous varieties. All the notations are taken from the previous section.

Theorem 4.16. ([RZ1], Theorem 2) *Let Σ be a weighted homogeneous subvariety of $\mathbb{C}^n$ with respect to a given vector $\beta \in \mathbb{Z}^n$, where $n \geq 2$ and all entries $\beta_k \geq 1$. Consider a (0,1)-form λ given by $\sum_k f_k d\overline{z_k}$, where the coefficients f_k are all Borel-measurable functions on Σ, and $z_1, ..., z_n$ are the Cartesian coordinates of $\mathbb{C}^n$. The following function is well defined for almost all $z \in \Sigma$ if the form λ is essentially bounded and has compact support in Σ:*

$$g(z) := \sum_{k=1}^{n} \frac{\beta_k}{2\pi i} \int_{w \in \mathbb{C}} f_k(w^\beta * z) \frac{\overline{(w^{\beta_k} z_k)}\, dw \wedge d\overline{w}}{\overline{w}\,(w-1)}. \tag{141}$$

If λ is $\overline{\partial}$-closed on the regular part Σ^* of Σ, then the function g is a solution of the $\overline{\partial}$-equation $\lambda = \overline{\partial} g$ on Σ^*.

Later, we gave more general formulas in [RZ2], and Theorem 4.16 reappears then as a special case of Theorem 4.14 as discussed above.

Recall the remarks on the proof of Theorem 4.14. We cover Σ by charts which we call generalized cones. When one blows up these cones to complex manifolds, one can realize that the integral formula (141) looks essentially like the inhomogeneous Cauchy formula in one complex variable, and one can deduce the statement by use of classical results. On the other hand, this gives also a hint towards Hölder regularity. Assume that Σ is homogeneous with an isolated singularity at the origin. Then the formula looks essentially like the Cauchy formula on complex lines through the origin. Since λ is essentially bounded, one can deduce that g is Hölder-α-continuous on complex lines through the origin for all $0 < \alpha < 1$, and this regularity is uniform over all such lines. If we like to deduce Hölder regularity for g on Σ, we also need to investigate the other directions. This can be done conveniently on the generalized cones mentioned above. Here, a generalized cone looks as follows. Let $Y \subset \mathbb{CP}^{n-1}$ be the projective variety associated to Σ. Then a generalized cone is simply the affine cone over an open set in Y which is biholomorphically equivalent to a domain in $\mathbb{C}^{\dim \Sigma - 1}$. On such a cone, we obtain Hölder-α-estimates for all $0 < \alpha < 1$ in directions orthogonal to complex lines through the origin, where the Hölder estimates are bounded by a constant which is vanishing to order 1 when we approach the origin (cf. the anisotropic Hölder estimates [RZ1], Lemma 5). Putting all the estimates together, we obtain the Theorem 4.17 below.

We need to specify the metric on Σ. Given a pair of points z and w in Σ, we define $\mathrm{dist}_\Sigma(z,w)$ to be the infimum of the length of piecewise smooth curves connecting z and w in Σ. It is clear that such curves exist in this situation and that the length of each curve can be measured either in Σ or in the ambient space $\mathbb{C}^n$ because both measures coincide as Σ^* carries the induced norm.

Theorem 4.17. ([RZ1], Theorem 3) *In the situation of Theorem 4.16, suppose that Σ is homogeneous (a cone) and has got only one isolated singularity at the origin of $\mathbb{C}^n$, so that each entry $\beta_k = 1$ in Definition 4.13. Moreover, assume that the support of the form λ is contained in a ball B_R of radius $R > 0$ and center at the origin. Then, for each parameter $0 < \vartheta < 1$, there exists a constant $C_\Sigma(R,\vartheta) > 0$ which does not depend on λ such that the following inequality holds for the function g given in (141) for all points z and w in the intersection $B_R \cap \Sigma$,*

$$|g(z) - g(w)| \leq C_\Sigma(R,\vartheta) \cdot \mathrm{dist}_\Sigma(z,w)^\vartheta \cdot \|\lambda\|_\infty. \tag{142}$$

The notation $\|\lambda\|_\infty$ stands for the essential supremum of $|\lambda(\cdot)|_\Sigma$ on Σ; recall that λ is bounded and has compact support.

We should mention that Theorem 4.17 is a significant improvement of the known results about Hölder regularity. Consider for example $\{z^2 = w_1 w_2\} \subset \mathbb{C}^3$. For this variety, Fornæss-Gavosto and Acosta–Solis–Zeron were only able to prove the statement of Theorem 4.17 for each $\vartheta < 1/2$ (see [AZ1, AZ2, FG]). The author obtained in [R2] also the case $\vartheta = 1/2$ which is still far from the result discussed above. Also the degrees of Hölder regularity that appear in [R2], [AZ1], [AZ2] and [SZ] for other varieties are far away from the result in Theorem 4.17.

4.5 Extrinsic Koppelman formulas and Friedrichs' extension theorem

Finally, some words about extrinsic integral formulas on singular spaces are in order. By an extrinsic formula we mean an integral formula which applies to differential forms which have smooth extensions to neighborhoods of the singular space in local embeddings. Such formulas were first introduced by Henkin and Polyakov for complete intersections in [HP]. A more comprehensive theory has been created recently by Andersson and Samuelsson in [AS1], [AS2], [AS3]. Their results comprise extrinsic Koppelman formulas which yield a local Grothendieck-Dolbeault lemma for forms which extend smoothly to neighborhoods of the variety in local embeddings. So, in contrast to the intrinsic setting (i.e. studying the $\overline{\partial}$-equation in the L^p-sense on the regular part of a Hermitian space), there are no obstructions to solving the $\overline{\partial}$-equation locally in an appropriate extrinsic sense. But, this means also that there are no obstructions to $\overline{\partial}$-solvability for intrinsic forms which can be approximated in a suitable sense by extendable forms. This is in some sense related to [R7], where we treat a generalization of Friedrich's extension lemma.

To explain that more precisely, let us recall the Koppelman formulas of Andersson and Samuelsson from [AS1]. Let X be an analytic space of pure dimension d and let $\mathcal{O}_X$ be the structure sheaf of (strongly) holomorphic functions. Locally X is a subvariety of a domain Ω in $\mathbb{C}^n$ and then $\mathcal{O}_X = \mathcal{O}/\mathcal{J}$, where $\mathcal{J}$ is the sheaf in

Ω of holomorphic functions that vanish on X. In the same way we say that ϕ is a smooth $(0,q)$-form on X, $\phi \in \mathcal{E}_{0,q}(X)$, if given a local embedding, there is a smooth form in a neighborhood in the ambient space such that ϕ is its pull-back to X_{reg}. It is well-known that this defines an intrinsic sheaf $\mathcal{E}^X_{0,q}$. It was proved by Henkin and Polyakov in [HP] that if X is embedded as a reduced complete intersection in a pseudoconvex domain and ϕ is a $\overline{\partial}$-closed smooth form on X, then there is a solution ψ to $\overline{\partial}\psi = \phi$ on X_{reg}. It has been an open question since then wether this holds more generally. This question has been answered in the affirmative by Andersson and Samuelsson for any Stein space X (see [AS1], [AS2] and [AS3]). At this place, we only mention their main formula on pseudoconvex domains in $\mathbb{C}^n$.

Theorem 4.18. ([AS1], Theorem 1.1) *Let Z be an analytic subvariety of pure dimension of a pseudoconvex domain $\Omega \subset \mathbb{C}^n$ and assume that $\omega \subset\subset \Omega$. There are linear operators $\mathcal{K} : \mathcal{E}_{0,q+1}(Z) \to \mathcal{E}_{0,q}(\omega \cap Z_{reg})$ and $\mathcal{P} : \mathcal{E}_{0,0}(Z) \to \mathcal{O}(\omega)$ such that*

$$\phi(z) = \overline{\partial}\mathcal{K}\phi(z) + \mathcal{K}(\overline{\partial}\phi)(z), \quad z \in Z_{reg} \cap \omega,\ \phi \in \mathcal{E}_{0,q}(Z),\ q > 0, \tag{143}$$

and

$$\phi(z) = \mathcal{K}(\overline{\partial}\phi)(z) + \mathcal{P}\phi(z), \quad z \in Z_{reg} \cap \omega,\ \phi \in \mathcal{E}_{0,0}(Z). \tag{144}$$

The operators are given as

$$\mathcal{K}\phi(z) = \int_\zeta K(\zeta, z) \wedge \phi(\zeta), \tag{145}$$

$$\mathcal{P}\phi(z) = \int_\zeta P(\zeta, z) \wedge \phi(\zeta), \tag{146}$$

where K and P are intrinsic kernels on $Z \times (Z_{reg} \cap \omega)$ and $Z \times \omega$, respectively. They are locally integrable with respect to ζ on Z_{reg} and the integrals in (145) and (146) are principal values at Z_{sing}. If ϕ vanishes in a neighborhood of a point x, then $\mathcal{K}\phi$ is smooth at x.

Note that we called (143) and (144) extrinsic formulas because they can only be applied to forms $\phi \in \mathcal{E}_{0,*}(Z)$, i.e. to forms which locally have smooth extensions. Nevertheless, the sheaf $\mathcal{E}^X_{0,*}$ is an intrinsic object on X in the sense that such a form ϕ does depend neither on the choice of the local extensions nor on the local embedding which we choose for an arbitrary singular space X. In this spirit, also the integral operators in (145) and (146) are intrinsic operators in the sense that their value does only depend on ϕ itself, not on the extensions which we can choose for ϕ or on the local embeddings of X which are used.

The knowledge about Theorem 4.18 leads to the natural question wether it is possible to approximate forms which are in some L^p-space on the regular part X_{reg} of X appropriately by forms which extend smoothly to local neighborhoods so that the formulas of Andersson–Samuelsson can be applied. This is for example possible if the L^p-forms vanish to an order high enough in the singular set. For such forms the $\overline{\partial}$-equation can be solved locally in the L^p-sense (see [AS1], Theorem 1.7, or [AS3], Theorem 1.3). In the L^2-case this was shown before by algebraic methods by Fornæss, Øvrelid and Vassiliadou in [FOV1] (see also [OV4]).

This illustrates the importance of extension and approximation results for forms given only on X_{reg}. That is a deep and difficult topic. Such statements depend on the homological dimension of the variety as can seen by methods of analytic geometry. We refer to the results of Scheja [S3] and [S4] at this place where holomorphic and smooth $\overline{\partial}$-closed forms are treated. Clearly, there appear additional difficulties when we are talking about L^p-forms.

There is another reason why it is interesting to study extension and approximation results on singular spaces. When we consider the $\overline{\partial}$-equation in our intrinsic sense, i.e. simply on the regular part of the variety, we have seen in our treatment of the L^2-theory that we have to distinguish different closed L^2-extensions of the $\overline{\partial}$-operator due to the incompleteness of the metric on X_{reg}. This leads to questions like the following. Assume that $\overline{\partial}_w f = g$ in $L^2_*(X_{reg})$, i.e. $\overline{\partial} f = g$ in the sense of distributions on X_{reg}. Under which circumstances does it follow that $\overline{\partial}_s f = g$, i.e. does there exist a sequence of smooth forms $\{f_j\}_j$ with support away from the singular set $\operatorname{Sing} X$ such that $f_j \to f$ and $\overline{\partial} f_j \to g$ in the L^2-sense? Or, if there exists a form f such that $\overline{\partial}_w f = g$ for a $\overline{\partial}_s$-closed form g, does then also exist a form h with $\overline{\partial}_s h = g$?

So, it is interesting to study the connection between the boundary condition of the $\overline{\partial}_s$-operator on the one hand and the approximation by smooth forms on the other hand. One first step in that direction had been made in [R7] where we studied how Friedrichs' extension lemma behaves with respect to boundary values of minimal and maximal closed extensions of differential operators on smoothly bounded manifolds.

Let us recall the results from [R7]. Let D be a relatively compact domain in a Hermitian complex manifold and $\overline{\partial}_\infty : C^\infty_*(\overline{D}) \to C^\infty_*(\overline{D})$ the Cauchy-Riemann operator on smooth forms. For $1 \leq p \leq \infty$, this operator can be considered as a densely defined graph-closable operator on L^p-forms:

$$\overline{\partial}_\infty : \operatorname{Dom}(\overline{\partial}_\infty) = C^\infty_*(\overline{D}) \subset L^p_*(D) \to L^p_*(D)$$

The $\overline{\partial}_\infty$-operator has various closed extensions. The two most important are the minimal closed extension $\overline{\partial}_{min}$ given by the closure of the graph and the maximal closed extension $\overline{\partial}_{max}$, i.e. the $\overline{\partial}$-operator in the sense of distributions. Whereas the two extensions coincide on smoothly bounded domains by Friedrichs' extension lemma (see [F2], [H6]), one has to be very careful when considering non-smooth domains. Especially on regular sets in singular complex spaces, it is crucial to distinguish the different closed extensions of the $\overline{\partial}$-operator for they lead to different Dolbeault cohomology groups as we have seen above. Clearly, the difference between the closed extensions occurs at the boundary of the domain. So, a first step is to study the boundary behavior of $\overline{\partial}_{min}$ and $\overline{\partial}_{max}$ on domains with smooth boundary.

Let $D \subset\subset \mathbb{C}^n$ be a bounded domain with smooth boundary bD, and let $f \in L^p_{0,q}(D)$ with $\overline{\partial} f \in L^p_{0,q+1}(D)$ in the sense of distributions for $1 \leq p < \infty$. Then, we say that f has weak $\overline{\partial}$-boundary values $f_b \in L^p_q(bD)$ in the sense of distributions if

$$\int_D \overline{\partial} f \wedge \phi + (-1)^q \int f \wedge \overline{\partial}\phi = \int_{bD} f_b \wedge \iota^*(\phi) \tag{147}$$

for all $\phi \in C^\infty_{n,n-q-1}(\overline{D})$, where $\iota : bD \hookrightarrow \mathbb{C}^n$ is the embedding of the boundary.

Weak $\overline{\partial}$-boundary values in the sense of distributions are a classical subject of complex analysis and closely related to the investigation of the so-called Hardy spaces (cf. [S9]). Starting from results of Skoda [S9], Harvey and Polking [HP], Schuldenzucker [S5] and Hefer [H3], there has been a considerable progress in the understanding of weak $\overline{\partial}$-boundary values by Hefer in [H4], where boundary values in the sense of distributions are compared to boundary values which arise naturally in the application of integral operators. This is interesting because boundary values defined by restricting the kernel of an integral operator can often be estimated by direct methods, whereas the abstractly given distributional boundary values are less tractable but analytically interesting objects linked to the form on the interior of a domain.

However, in applications the definition of weak $\overline{\partial}$-boundary values by means of the Stokes' formula (147) turns out to be a bit unhandy and it is more convenient to have boundary values in the sense of approximation by smooth forms. In fact, let $f \in \text{Dom}(\overline{\partial}_{max}) \subset L^p_{0,q}(D)$ with weak boundary values $f_b \in L^p_q(bD)$ according to definition (147), and let $r \in C^\infty(\mathbb{C}^n)$ be a smooth defining function for D. Then it is shown in [R7] that there exists a sequence $f_j \in C^\infty_{0,q}(\overline{D})$ such that

$$f_j \to f \text{ in } L^p_{0,q}(D)\ , \quad \overline{\partial} f_j \to \overline{\partial} f \text{ in } L^p_{0,q+1}(D) \tag{148}$$

(the classical Friedrichs' extension lemma) and moreover

$$f_j \wedge \partial r \to f_b \wedge \partial r \text{ on } bD \text{ in } L^p_q(bD), \tag{149}$$

i.e. f has $\overline{\partial}$-boundary values in the sense of approximation ([R7], Theorem 4.4).

This phenomenon is not restricted to the Cauchy-Riemann operator, but holds for arbitrary differential operators of first order with smooth coefficients. So, it is more convenient to adopt a more general point of view. Let M be a smooth, compact Riemannian manifold with smooth boundary, E and F Hermitian vector bundles over M, and $Q : C^\infty(M,E) \to C^\infty(M,F)$ a linear differential operator of first order with C^1-coefficients. Let $1 \leq p < \infty$ and $f \in L^p(M,E)$. We say that $f \in \text{Dom}(Q^p_{min})$ if there exists a sequence $\{f_j\} \subset C^\infty(M,E)$ and a section $g \in L^p(M,F)$ such that

$$f_j \to f \text{ in } L^p(M,E)\ , \quad Qf_j \to g \text{ in } L^p(M,F),$$

and define $Q^p_{min} f := g$. The well-defined operator Q^p_{min} is called the minimal extension of Q because it is the closed extension of Q to an operator $L^p(M,E) \to L^p(M,F)$ with minimal domain of definition. Its graph is simply the closure of the graph of $Q : C^\infty(M,E) \to C^\infty(M,F)$ in $L^p(M,E) \times L^p(M,F)$. Let σ_Q be the principal symbol of Q, ν the outward pointing unit normal to bM, and $\nu^\flat$ the dual cotangent vector. Then, we say that f has boundary values with respect to Q^p_{min} if there exists a sequence $\{f_j\}$ in $C^\infty(M,E)$ such that $\lim_{j\to\infty} f_j = f$ in $L^p(M,E)$, $\lim_{j\to\infty} Qf_j = Q^p_{min} f$ in $L^p(M,F)$, and a section $f_b \in L^p(bM, E|_{bM})$ such that

$$\lim_{j\to\infty} \sigma_Q(\cdot, \nu^\flat(\cdot)) f_j|_{bM} = \sigma_Q(\cdot,\nu^\flat(\cdot)) f_b \quad \text{in} \quad L^p(bM, F|_{bM}).$$

In this case, we call f_b weak Q-boundary values of f with respect to Q^p_{min} (i.e. in the sense of approximation).

Now, we draw our attention to the maximal closed extension of Q, that is the extension of Q in the sense of distributions. We say that $f \in \mathrm{Dom}(Q^p_{max})$ if $Qf = u \in L^p(M,F)$ in the sense of distributions, and set $Q^p_{max} f := u$ in that case. Here again, we can define weak Q-boundary values with respect to Q^p_{max}. We say that f has weak Q-boundary values $f_b \in L^p(bM, E|_{bM})$ with respect to Q^p_{max} (in the sense of distributions), if f_b satisfies the generalized Green-Stokes formula (cf. [T2], Proposition 9.1)

$$(Qf,\phi)_M - (f, Q^*\phi)_M = \frac{1}{i}\int_{bM} \langle \sigma_Q(x,\nu^\flat) f_b, \phi\rangle_{F_x}\, dS(x)$$

for all $\phi \in C^\infty(M,F)$.

The main objective of the paper [R7] is to compare both notions of Q-boundary values. It is easy to see that $\mathrm{Dom}(Q^p_{min}) \subset \mathrm{Dom}(Q^p_{max}) \subset L^p(M,E)$, and that Q^p_{min} is the restriction of Q^p_{max} to $\mathrm{Dom}(Q^p_{min})$. Moreover, it is also clear that weak Q-boundary values in the sense of approximation are weak Q-boundary values in the sense of distributions as well. It is well-known that in fact $Q^p_{min} = Q^p_{max}$ on smooth, compact manifolds with smooth boundary. This result, due to Friedrichs (see [F2], [H6]), is usually called Friedrichs' extension lemma. In [R6], it is observed that the two notions of boundary values coincide as well (see [R7], Theorem 3.3). One might call this Friedrichs' extension lemma with boundary values. In the particular case of the Cauchy-Riemann operator $Q = \overline{\partial}$, we obtain (148), (149).

5 Hartogs' extension theorem

A very nice example for the use of analytic methods on singular complex spaces is Hartogs' extension theorem for $(n-1)$-complete spaces.

The classical Hartogs' extension theorem states that for every open subset $D \subset \mathbb{C}^n$, $n \geq 2$, and $K \subset D$ compact such that $D \setminus K$ is connected, the holomorphic functions on $D \setminus K$ extend to holomorphic functions on D. Whereas first versions of Hartogs' extension theorem were obtained by filling Hartogs' figures with analytic discs (Hartogs' original idea [H1]), no such geometrical proof was known for the general classical theorem in complex number space $\mathbb{C}^n$ for a long time. Proofs of the general theorem in $\mathbb{C}^n$ usually depend on the Bochner-Martinelli-Koppelman kernel or on the solution of the $\overline{\partial}$-equation with compact support (the famous idea due to Ehrenpreis [E], see also [H7]).

Only recently, Merker and Porten were able to fill the gap by giving an involved geometrical proof of Hartogs' extension theorem in $\mathbb{C}^n$ in the spirit of Hartogs' original idea by using a finite number of parameterized families of holomorphic discs and Morse-theoretical tools for the global topological control of monodromy, but no $\overline{\partial}$-theory or integral kernels except the Cauchy kernel (see [MP1]).

Since the key ingredient of this strategy is the existence of a strongly $(n-1)$-convex exhaustion function, it is natural to ask wether the result remains true for $(n-1)$-complete complex spaces.[25] In fact, Hartogs' theorem was generalized to $(n-1)$-complete manifolds by Andreotti and Hill [AH] using cohomological results (the $\overline{\partial}$-method), but no proof was known until now for the more general case of $(n-1)$-complete normal complex spaces. One reason was the lack of global integral kernels or an appropriate $\overline{\partial}$-theory for singular complex spaces. However, Merker and Porten were able to carry over their geometric strategy and to prove Hartogs' extension theorem also for $(n-1)$-complete normal complex spaces (see [MP2]).

In this chapter, we show how one can use $\overline{\partial}$-theoretical considerations for reproducing the result of Merker and Porten on a $(n-1)$-complete complex space X by the simple and striking strategy of Ehrenpreis. More precisely, one can reduce the problem to the solution of a $\overline{\partial}$-equation with compact support on a resolution of singularities $\pi : M \to X$ in the spirit of the $\overline{\partial}$-technique of Ehrenpreis. But the $\overline{\partial}$-equation that we need to consider is actually solvable because $H^1_{cpt}(M, \mathcal{O}) = 0$ if X is cohomologically $(n-1)$-complete. That follows from Takegoshi's vanishing theorem. This solution of the problem was presented in the joint work with M. Colţoiu [CR]. For our strategy, it is enough to assume that X is cohomologically $(n-1)$-complete. Note that $(n-1)$-complete spaces are cohomologically $(n-1)$-complete by the work of Andreotti and Grauert [AG], but the converse is not known.

[25] A complex space of pure dimension n is called $(n-1)$-complete if it has a strictly $(n-1)$-convex exhaustion function $\varphi : X \to \mathbb{R}$, that is an exhaustion function with at least $\dim X + 1 - (n-1) = 2$ strictly positive eigenvalues of the complex Hessian on X. If X is singular, that has to be understood as follows: If X is locally embedded in a $\mathbb{C}^L$, then φ is the restriction of a strictly $(n-1)$-convex function in a neighborhood of X. So, the complex Hessian of the local extension has to have at least $L + 1 - (n-1) = \operatorname{codim} X + 2$ strictly positive eigenvalues. Hence, there will be always two positive directions tangent to X. This is the reason why q-convexity is defined that way.

A first step in that direction had been made in [R3] where the problem was solved before for Stein spaces with isolated singularities. We include that here because it is another, somewhat different, nice example of how to use $\overline{\partial}$-techniques on singular spaces.

For a more detailed introduction to Hartogs' theorem with a full historical record, remarks and references, we refer to [MP1], [MP2] and [OV3]. Though the method of Merker and Porten is technically more involved and harder to reproduce than the $\overline{\partial}$-method, it has the advantage that it works as well for meromorphic functions which is out of scope of the $\overline{\partial}$-method. In fact, Merker and Porten proved the extension theorem even for the extension of meromorphic functions.

5.1 Resolution of singularities

Let X be a connected normal complex space of pure dimension n. Furthermore, let D be a domain in X and $K \subset D$ a compact subset such that $D - K$ is connected. Let $f \in \mathcal{O}(D - K)$. Our aim is to find a holomorphic extension of f to the whole domain D. We can assume that X is non-compact. This assumption is automatically fulfilled if X is a $(n-1)$-complete space.

The assumption about normality implies that X is reduced. Let

$$\pi : M \to X$$

be a resolution of singularities, where M is a complex connected manifold of dimension n, and π is a proper holomorphic surjection. Let $E := \pi^{-1}(\operatorname{Sing} X)$ be the exceptional set of the desingularization. Note that

$$\pi|_{M\setminus E} : M \setminus E \to X \setminus \operatorname{Sing} X \tag{150}$$

is a biholomorphic map. For the topic of desingularization we refer as above to [AHL], [BM] and [H2].

In this section, we observe that the extension problem on X can be reduced to an analogous extension problem on M. Let

$$D' := \pi^{-1}(D),\ K' := \pi^{-1}(K),\ F := f \circ \pi \in \mathcal{O}(D' \setminus K').$$

Clearly, D' is an open set and K' is compact with $K' \subset D'$ since π is a proper holomorphic map. $D \setminus K$ is a connected normal complex space. So, it is connected, reduced and locally irreducible, hence globally irreducible as well (see [GR2]). But then, $D \setminus K \setminus \operatorname{Sing} X$ is still connected. So, the same is true for $D' \setminus K' \setminus E$ because of (150). But then $D' \setminus K'$ and D' are connected, too. That means that the assumptions on D and K behave well under desingularization and it is enough to construct an extension of F to D', because $\pi_* \mathcal{O}_M = \mathcal{O}_X$ for the structure sheaves of M and X by normality of X.

But the existence of such an extension of F follows easily by Ehrenpreis' $\overline{\partial}$-technique (see [H7]) as soon as $H^1_{cpt}(M, \mathcal{O}) = 0$ as we will observe now.

Let

$$\chi \in C^\infty_{cpt}(M)$$

be a smooth cut-off function that is identically one in a neighborhood of K' and has compact support

$$C := \operatorname{supp} \chi \subset\subset D'.$$

Consider

$$G := (1-\chi)F \in C^\infty(D'),$$

which is an extension of F to D', but unfortunately not holomorphic. We have to fix it by the idea of Ehrenpreis. So, let

$$\omega := \overline{\partial} G \in C^\infty_{(0,1),cpt}(D'),$$

which is a $\overline{\partial}$-closed $(0,1)$-form with compact support in D'. We may consider ω as a form on M with compact support. Assume for the moment that $H^1_{cpt}(M, \mathcal{O}) = 0$. So, there exists $g \in C^\infty_{cpt}(M)$ such that

$$\overline{\partial} g = \omega,$$

and g is holomorphic on $M \setminus C$ (where $\omega = \overline{\partial} G = \overline{\partial} F = 0$). Let

$$\widetilde{F} := (1-\chi)F - g \in \mathcal{O}(D'). \tag{151}$$

It remains to show that $\widetilde{F}$ is actually an extension of F. To see that, we have to show that $g \equiv 0$ on an open subset of $D' \setminus C$. Let

$$A := \operatorname{supp} g.$$

Then, $M \setminus \big(A \cup C\big) \neq \emptyset$ because $A \cup C$ is compact but M is not. If M were compact, X would be compact as well. So, there exists a point

$$p \in M \setminus \big(A \cup C\big) \neq \emptyset.$$

Let V be the open connected component of $M \setminus C$ that contains the point p, and $V^c = (M \setminus C) \setminus V$, which is also open. g is holomorphic on V and vanishes in a neighborhood of p, hence $g \equiv 0$ on V. It follows from $C \subset\subset D'$ that $D' \cup (M \setminus C) = D' \cup V \cup V^c = M$. Let $W = D' \cup V^c$. Then $M = V \cup W$ is the union of two open sets. But M is connected. This yields $W \cap V = (D' \cup V^c) \cap V \neq \emptyset$. Thus $D' \cap V \neq \emptyset$. So, $g \equiv 0$ on $V \cap D' \neq \emptyset$, and this implies by (151) that $\widetilde{F} \equiv F$ on $V \cap D'$ (which is not empty). But

$$\emptyset \neq V \cap D' \ \subset \ D' \setminus C \ \subset \ D' \setminus K',$$

where $D' \setminus K'$ is connected, and by the identity theorem we obtain

$$\widetilde{F}|_{D' \setminus K'} \equiv F$$

just as needed.

5.2 Dolbeault cohomology of proper modifications

In its general form, the Grauert-Riemenschneider vanishing theorem (see [GR3], Satz 2.1) states:

Theorem 5.1. *Let X be an n-dimensional compact irreducible reduced complex space with n independent meromorphic functions (Moishezon), and let $\mathcal{S}$ be a quasi-positive coherent analytic sheaf without torsion on X. Then:*

$$H^q(X, \mathcal{S} \otimes \Omega^n_X) = 0, \quad q > 0,$$

where Ω^n_X is the sheaf of holomorphic n-forms on X (the canonical sheaf), defined in the sense of Grauert and Riemenschneider.

This generalization of Kodaira's famous vanishing theorem is also proved by means of harmonic theory. The main point in the proof is ([GR3], Satz 2.3):

Theorem 5.2. *Let X be a projective complex space, $\mathcal{S}$ a quasi-positive coherent analytic sheaf on X without torsion, and let $\pi : M \to X$ be a resolution of singularities, such that $\hat{\mathcal{S}} = \mathcal{S} \circ \pi$ is locally free on M. Then:*

$$R^q\pi_*(\hat{\mathcal{S}} \otimes \Omega^n_M) = 0, \quad q > 0.$$

Here, $\hat{\mathcal{S}} = \mathcal{S} \circ \pi$ denotes the torsion-free preimage sheaf:

$$\mathcal{S} \circ \pi := \pi^*\mathcal{S}/T(\pi^*\mathcal{S}),$$

where $T(\pi^*\mathcal{S})$ is the coherent torsion sheaf of the preimage $\pi^*\mathcal{S}$ (see [G1], p. 61). As a simple consequence of Theorem 5.2, one can deduce:

Corollary 5.3. *Let M be a Moishezon manifold of dimension n, and X a projective variety such that $\pi : M \to X$ is a resolution of singularities. Then:*

$$R^q\pi_*\Omega^n_M = 0, \quad q > 0,$$

where $R^q\pi_*\Omega^n_M$, $q > 0$, are the higher direct image sheaves of Ω^n_M.

Proof. Let F be a positive holomorphic line bundle on $X \subset \mathbb{CP}^L$, and $\mathcal{S}$ the sheaf of sections in F. So, $S \circ \pi$ is a positive locally free sheaf on M ([GR3], Satz 1.4), and as in [GR3], Satz 2.4, it follows from Theorem 5.2 that

$$R^q\pi_*\Omega^n_M \otimes \mathcal{S} = R^q\pi_*(\hat{\mathcal{S}} \otimes \Omega^n_M) = 0$$

for $q > 0$ which implies the statement. □

As Grauert and Riemenschneider mention already in their original paper [GR3], this statement is of local nature and doesn't depend on the projective embedding (whereas their proof does). In fact, the result was generalized later as we have already seen by K. Takegoshi (see [T1], Corollary I; and also [O1]):

Theorem 5.4. *Let M be a complex manifold of dimension n, and X a complex space such that $\pi : M \to X$ is a proper modification. Then:*

$$R^q \pi_* \Omega^n_M = 0, \quad q > 0.$$

The nice proof consists mainly of a vanishing theorem on weakly 1-complete Kähler manifolds which is based on L^2-estimates for the $\overline{\partial}$-operator. As an easy consequence, we obtain:

Theorem 5.5. *Let M be a complex manifold of dimension n, and X a complex space such that $\pi : M \to X$ is a proper modification. Then:*

$$H^q(M, \Omega^n_M) \cong H^q(X, \pi_* \Omega^n_M). \tag{152}$$

Proof. The proof follows directly by the Leray spectral sequence. □

Now, if the space X has nice properties, we can deduce consequences for the Dolbeault cohomology on M. Here, we are particularly interested in q-complete spaces. Recall that a complex space X is q-complete in the sense of Andreotti and Grauert [AG] if it has a strongly q-convex exhaustion function. X is called cohomologically q-complete if $H^k(X, \mathcal{F}) = 0$ for any coherent analytic sheaf $\mathcal{F}$ and all $k \geq q$. Note that q-complete spaces are cohomologically q-complete by the work of Andreotti and Grauert [AG], but the converse is not known.

Theorem 5.6. *Let M be a complex manifold of dimension n, and X a cohomologically q-complete complex space such that $\pi : M \to X$ is a proper modification. Then:*

$$H^{n-k}_{cpt}(M, \mathcal{O}) \cong H^k(M, \Omega^n_M) = 0, \quad k \geq q. \tag{153}$$

Proof. Since X is q-complete, it follows from Theorem 5.5 that

$$H^k(M, \Omega^n_M) \cong H^k(X, \pi_* \Omega^n_M) = 0, \quad k \geq q, \tag{154}$$

because $\pi_* \Omega^n_M$ is coherent by Grauert's direct image theorem (see [G1]). Serre's criterion ([S6], Proposition 6) tells us that we can apply Serre duality [S6] to the cohomology groups in (154), and we get the duality on the left-hand side of (153). □

An immediate consequence is Hartogs' extension theorem:

Theorem 5.7. ([CR], Theorem 1.1) *Let X be a connected normal complex space of dimension $n \geq 2$ which is cohomologically $(n-1)$-complete. Furthermore, let D be a domain in X and $K \subset D$ a compact subset such that $D \setminus K$ is connected. Then each holomorphic function $f \in \mathcal{O}(D \setminus K)$ has a unique holomorphic extension to the whole set D.*

Proof. The proof follows by the procedure described in section 5.1 because here we have in fact $H^1_{cpt}(M, \mathcal{O})$ by Theorem 5.6. Moreover, also $H^0_{cpt}(M, \mathcal{O}) = 0$ so that M cannot be compact. □

5.3 Stein spaces with isolated singularities

There is another way to prove Theorem 5.7 that does not depend on Takegoshi's vanishing theorem if X is a Stein space with isolated singularities. This method was presented in [R3] and we include it here because it is another nice application of $\overline{\partial}$-methods in singular constellations.

Let us return to the situation in section 5.1, i.e. consider the resolution

$$\pi : M \to X$$

and assume that X is a Stein space with isolated singularities. It follows that M is a 1-convex complex manifold, and that there exists a strictly plurisubharmonic exhaustion function

$$\rho : M \to [-\infty, \infty) \ ,$$

such that ρ takes the value $-\infty$ exactly on the exceptional set$E$ (see [CoMi]). We can assume that ρ is real-analytic on $M \setminus E$.

Recall that

$$D' = \pi^{-1}(D),\ K' = \pi^{-1}(K),$$

and that we are looking for an extension of

$$F := f \circ \pi \in \mathcal{O}(D' \setminus K')$$

to the whole domain D'. Choose $\delta > 0$ such that

$$K' \subset W := \{z \in M : \rho(z) < \delta\},$$

which is possible since ρ is an exhaustion function. But ρ is also strictly plurisubharmonic outside E, and it follows that W is strongly pseudoconvex if we choose δ as a regular value of ρ. We will use the fact that

$$\dim H^q(W, \mathcal{S}) < \infty \tag{155}$$

for all coherent analytic sheaves $\mathcal{S}$ and $q \geq 1$.

Since $K' \subset\subset W$, we can choose the cut-off function $\chi \in C^\infty_{cpt}(M)$ from section 5.1 so that it has compact support in $D' \cap W$. Recall that we are looking for a solution g with compact support of the $\overline{\partial}$-equation

$$\overline{\partial} g = \omega = \overline{\partial}\big((1-\chi)F\big), \tag{156}$$

where ω has compact support in $D' \cap W$. Hence we would need that $H^1_{cpt}(W, \mathcal{O}) = 0$. But we only know that W is a strongly pseudoconvex subset of a 1-convex complex manifold. This is now the place to introduce some ideas from the $\overline{\partial}$-theory on singular complex spaces. We will use a result of Fornæss, Øvrelid and Vassiliadou presented in [FOV1], Lemma 2.1.

We must verify their assumptions. So, let

$$\widetilde{W} := \pi(W).$$

That is a strongly pseudoconvex neighborhood of the isolated singularities in X. Hence, it is a holomorphically convex subset of a Stein space by the results of Narasimhan (see [N]), and therefore Stein itself.

Let $\mathcal{J}$ be the sheaf of ideals of the exceptional set E in M. Now, the result of Fornæss, Øvrelid and Vassiliadou reads as:

Theorem 5.8. *Let $\mathcal{S}$ be a torsion-free coherent analytic sheaf on W, and $q > 0$. Then there exists a natural number $T \in \mathbb{N}$ such that*

$$i_q : H^q(W, \mathcal{J}^T\mathcal{S}) \to H^q(W, \mathcal{S})$$

is the zero map, where i_q is the map induced by the natural inclusion $\mathcal{J}^T\mathcal{S} \hookrightarrow \mathcal{S}$.

This statement reflects the fact that the cohomology of M is concentrated along the exceptional set E, and can be killed by putting enough pressure on E. We will now use Theorem 5.8 with the choices $q = n - 1 = \dim X - 1$ and $\mathcal{S} = \Omega^n_W$ the canonical sheaf on W. So, there exists a natural number $\mu > 0$ such that

$$i_{n-1} : H^{n-1}(W, \mathcal{J}^\mu\Omega^n_W) \to H^{n-1}(W, \Omega^n_W) \tag{157}$$

is the zero map. Note that for $\nu \in \mathbb{Z}$ and $0 \leq p \leq n$, the sheaf $\mathcal{J}^\nu\Omega^p_W$ is isomorphic to the sheaf of germs of holomorphic p-forms with values in $L^{\otimes\nu}_{-E}$, where L_{-E} is the holomorphic line bundle corresponding to the divisor $-E$ such that $\mathcal{O}(L_{-E}) \cong \mathcal{O}(-E) \cong \mathcal{J}\mathcal{O}_W$

We will use Serre Duality (cf. [S6]) to change (157) to the dual statement. But, can we apply Serre-Duality to the non-compact manifold W? The answer is yes, because higher cohomology groups are finite-dimensional on W by the result of Grauert (155), and we can use Serre's criterion ([S6], Proposition 6). So, we deduce:

$$i_c : H^1_{cpt}(W, \mathcal{O}_W) \to H^1_{cpt}(W, \mathcal{J}^{-\mu}\mathcal{O}_W) \tag{158}$$

is the zero map, where i_c is induced by the natural inclusion $\mathcal{O}_W \hookrightarrow \mathcal{J}^{-\mu}\mathcal{O}_W$. This statement means that we can have a solution for the $\overline{\partial}$-equation (156) with compact support in W that has a pole of order μ (at most) along E. Let us make that precise.

$\mathcal{J}^{-\mu}\mathcal{O}_W$ is a subsheaf of the sheaf of germs of meromorphic functions $\mathcal{M}_W$. We will now construct a fine resolution for $\mathcal{J}^{-\mu}\mathcal{O}_W$ analogously to the $\overline{\partial}_k$-complex (127) from Theorem 4.6. Let $\mathcal{C}^\infty_{0,q}$ denote the sheaf of germs of smooth $(0,q)$-forms on W. We consider $\mathcal{J}^{-\mu}\mathcal{C}^\infty_{0,q}$ as subsheaves of the sheaf of germs of differential forms with measurable coefficients. Now, we define a weighted $\overline{\partial}$-operator on $\mathcal{J}^{-\mu}\mathcal{C}^\infty_{0,q}$. Let $f \in (\mathcal{J}^{-\mu}\mathcal{C}^\infty_{0,q})_z$ for a point $z \in M$. Then f can be written as $f = h^{-\mu}f_0$, where $h \in (\mathcal{O}_W)_z$ generates $\mathcal{J}_z$ and $f_0 \in (\mathcal{C}^\infty_{0,q})_z$. Let

$$\overline{\partial}_{-\mu} f := h^{-\mu}\overline{\partial} f_0 = h^{-\mu}\overline{\partial}(h^\mu f).$$

We obtain the sequence

$$0 \to \mathcal{J}^{-\mu}\mathcal{O}_W \hookrightarrow \mathcal{J}^{-\mu}\mathcal{C}^\infty_{0,0} \xrightarrow{\overline{\partial}_{-\mu}} \mathcal{J}^{-\mu}\mathcal{C}^\infty_{0,1} \xrightarrow{\overline{\partial}_{-\mu}} \cdots \xrightarrow{\overline{\partial}_{-\mu}} \mathcal{J}^{-\mu}\mathcal{C}^\infty_{0,d} \to 0,$$

which is exact by the Grothendieck-Dolbeault Lemma and well-known regularity results. It is a fine resolution of $\mathcal{J}^{-\mu}\mathcal{O}_W$ since the $\mathcal{J}^{-\mu}\mathcal{C}^\infty_{0,q}$ are closed under multiplication by smooth cut-off functions. Therefore, the abstract Theorem of de Rham implies that:

$$H^q(W, \mathcal{J}^{-\mu}\mathcal{O}_W) \cong \frac{\ker\ (\overline{\partial}_{-\mu} : \mathcal{J}^{-\mu}\mathcal{C}^\infty_{0,q}(W) \to \mathcal{J}^{-\mu}\mathcal{C}^\infty_{0,q+1}(W))}{\mathrm{Im}\ (\overline{\partial}_{-\mu} : \mathcal{J}^{-\mu}\mathcal{C}^\infty_{0,q-1}(W) \to \mathcal{J}^{-\mu}\mathcal{C}^\infty_{0,q}(W))},$$

and we have the analogous statement for forms and cohomology with compact support. Recall that

$$\omega \in C^\infty_{0,1}(W)$$

is $\overline{\partial}$-closed with compact support in W. By the natural inclusion, we have that

$$\omega \in \mathcal{J}^{-\mu}C^\infty_{0,1}(W),$$

too, and it is in fact $\overline{\partial}_{-\mu}$-closed. But then (158) tells us that there exists a solution $g \in \mathcal{J}^{-\mu}\mathcal{C}^\infty(W)$ such that

$$\overline{\partial}_{-\mu} g = \omega,$$

and g has compact support in W. So, $g \in C^\infty(M - E)$ with support in W, and

$$\overline{\partial} g = \omega = \overline{\partial}\big((1-\chi)F\big) \quad \text{on } D' - E.$$

It follows as in section 5.1 that

$$\widetilde{F} := (1-\chi)F - g \in \mathcal{O}(D' - E)$$

is a holomorphic extension of F to $D' - E$. But then

$$\widetilde{f} := \big((1-\chi)F - g\big) \circ \pi^{-1} \ \in \mathcal{O}(D - \operatorname{Sing} X)$$

is a holomorphic extension of f to $D - \operatorname{Sing} X$. But then $\widetilde{f}$ has an extension to D by Riemann's extension theorem for normal spaces since $\dim X \geq 2$ and $\dim \operatorname{Sing} X = 0$ (see e.g. [GR2]). The extension $\widetilde{f}$ is unique because $D \setminus K$ is connected and X is globally and locally irreducible (see again [GR2]). That completes the proof of Theorem 5.7 if X is a Stein space with isolated singularities.

In the meantime, Øvrelid and Vassiliadou have used a similar $\overline{\partial}$-method to show the theorem also for Stein spaces with arbitrary singular set in [OV3]. Moreover, they extended the results from [FOV1] (which are about Stein spaces) also to q-complete spaces in [OV4]. This gives a statement similar to Theorem 5.8 also on $(n-1)$-complete spaces and leads to another proof of Theorem 5.7 by a method similar to the procedure described in this section (see [OV4], Theorem 1.3).

On the other hand, the Koppelman formulas of Andersson and Samuelsson yield another proof of Theorem 5.7 for Stein spaces (see [AS3], Theorem 1.4). This is then a proof by means of some integral formulas.

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