TU Kaiserslautern

Master Thesis

# The Navarro Refinement of McKay's Conjecture for Groups of Lie Type

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# Introduction

In representation theory one investigates a finite group $G$ by representing it as a group of matrices. More concretely a *linear representation* of a group $G$ is a group homomorphism $\mathrm{D} : G \to \mathrm{GL}_l(k)$ from the group $G$ into the group of $l \times l$ invertible matrices over some field $k$. In the case that the field $k = \mathbb{C}$ is the complex numbers, the information of a representation is completely encoded in its character. The *character* $\chi$ of the representation D is defined as its trace function, i.e. as a map $\chi : G \to k$ with

$$\chi(g) = \mathrm{tr}(\mathrm{D}(g)) \text{ for } g \in G.$$

Any character can be expressed as a linear combination with positive integer coefficients of the so-called *irreducible characters*. Hence, in order to study linear representations over the complex numbers, it suffices to consider the set of complex irreducible characters of the group $G$, which we denote by $\mathrm{Irr}(G)$.

Recently, a conjecture by McKay has attracted a lot of attention. Let us fix a prime number $p$. Let us write $\mathrm{Irr}_{p'}(G)$ for the set of irreducible characters $\chi$ such that $p$ does not divide $\chi(1)$, the *degree* of $\chi$. We will refer to the characters of this set as the *$p'$-characters* of $G$. Then the McKay conjecture is as follows:

**Conjecture 0.1.** *Let $G$ be a finite group. Let $P$ be a Sylow $p$-subgroup of $G$. Then there exists a bijection between $\mathrm{Irr}_{p'}(\mathrm{N}_G(P))$ and $\mathrm{Irr}_{p'}(G)$.*

A more detailed overview of the McKay conjecture and its history can be found in the introduction of [10]. The McKay conjecture was verified for many different groups, but until today there is no "canonical" bijection known. This means that for different groups bijections were constructed using distinct methods. However, Navarro suggests that there should exist a bijection of $\mathrm{Irr}_{p'}(\mathrm{N}_G(P))$ to $\mathrm{Irr}_{p'}(G)$ which should commute with certain Galois automorphisms. Let $m$ be the order of the finite group $G$. By Brauer's Theorem (see [8, Theorem 10.3]) every character $\chi$ of $G$ can be afforded by a representation D with entries in the cyclotomic field $\mathbb{Q}_m$, the field obtained from the rational numbers by adjoining the $m$-th roots of unity. We may thus define $\chi^\sigma$ to be the character afforded by the representation

$$\mathrm{D}^\sigma : G \to \mathrm{GL}_l(\mathbb{Q}_m) \text{ with } g \mapsto \mathrm{D}(g)^\sigma.$$

Consequently, the Galois group $\mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$ acts on the set of irreducible characters $\mathrm{Irr}(G)$ by permutation. If $X \subseteq \mathrm{Irr}(G)$ is a subset of the set of irreducible characters of $G$ we denote by $X^\sigma$ the set of characters which are

$\sigma$-invariant. In his conjecture, Navarro considers the following class of Galois automorphisms:

**Definition 0.2.** Let $e$ be a nonnegative integer and $p$ be a prime number. Then a Galois automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$ is called an $(e,p)$-*Galois automorphism* if $\sigma$ sends any $p'$-root of unity $\zeta \in \mathbb{Q}_m$ to $\zeta^{p^e}$.

Navarro proposes the following refinement of the McKay Conjecture (see [16, Conjecture A]).

**Conjecture 0.3.** *Let $G$ be a finite group of order $m$ and $p$ be a prime. Let $P$ be a Sylow $p$-subgroup of $G$. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$ be an $(e,p)$-Galois automorphism for a nonnegative integer $e$. Then there exists a bijection between $\mathrm{Irr}_{p'}(\mathrm{N}_G(P))^\sigma$ and $\mathrm{Irr}_{p'}(G)^\sigma$.*

For the original McKay conjecture a reduction theorem was proved by Isaacs, Malle and Navarro (see [9, Theorem B]). This theorem implies that the McKay conjecture is true for all finite groups and the prime $p$ if all nonabelian simple groups are "good" for the prime $p$. One important step to show that a simple group $G$ is good for a prime is the construction of an automorphism-equivariant bijection for the universal covering group of $G$ which respects central characters (see [9, Section 10]). For simple groups of Lie type in *defining characteristic*, i.e. in the case where the prime of the conjecture coincides with the characteristic of the field defining the simple group of Lie type, this intermediate result was achieved by Maslowski in his dissertation (see [15, Theorem 1]). Later Späth used Maslowski's result to prove that these groups are good for the prime $p$ (see [17, Theorem 1.1]). In this thesis we use Maslowski's results in order to prove Conjecture 0.3 for most quasi-simple groups of Lie type in defining characteristic.

Let $\mathbf{G}$ be a simple algebraic group of simply connected type defined over an algebraic closure $\mathbf{k}$ of $\mathbb{F}_p$. Let $F : \mathbf{G} \to \mathbf{G}$ be a Frobenius endomorphism of the algebraic group $\mathbf{G}$. The finite group $G = \mathbf{G}^F$ of fixed points of $\mathbf{G}$ under the action of the Frobenius endomorphism $F$ is called a group of Lie type. Apart from a few exceptions $G$ is the universal central extension of the simple group $G/\mathrm{Z}(G)$, which is a so-called finite simple group of Lie type. In order to use our methods, we need to impose some conditions on the algebraic group $\mathbf{G}$.

**Assumption 0.4.** *Let $\mathbf{G}$ be a simple algebraic group of simply connected type. Suppose that $\mathbf{G}$ is not of type $D_n$ if $n$ is even and that $p$ is a good prime for $\mathbf{G}$.*

We assume that the root system of $\mathbf{G}$ is not of type $D_n$ if $n$ is even in order to avoid too many technical obstacles. In this case the center of $\mathbf{G}$ is not cyclic which is the reason why we would have to change many definitions to adapt to this particular case. However, the author is quite convinced that the assumption can be removed using the construction in [15] for this type of root system. We also did not consider Suzuki and Ree groups. In order to include these groups one would first have to generalize the methods of Maslowski to these groups (see [15, Introduction] and [15, Example 9.5] for more information).

Maslowski associates a certain regular embedding $i : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ to the group $\mathbf{G}$ such that $\tilde{\mathbf{G}}$ has connected center. The character theory of the $p'$-characters of $\tilde{\mathbf{G}}^F$ is much simpler to describe than the character theory of $\mathbf{G}^F$. This is due to the fact that the Deligne–Lusztig theory is remarkably easier for $\tilde{\mathbf{G}}^F$ since the center of $\tilde{\mathbf{G}}$ is connected. Maslowski defines a labeling for both the $p'$-character of $\tilde{\mathbf{B}}^F$, the normalizer of the Sylow $p$-subgroup $\mathbf{U}^F$ of $\tilde{\mathbf{G}}^F$, and for the $p'$-characters of $\tilde{\mathbf{G}}^F$. He proves that both sets of labels coincide which implies that this labeling gives rise to a bijection $\tilde{f}$ : $\mathrm{Irr}_{p'}(\tilde{\mathbf{B}}^F) \to \mathrm{Irr}_{p'}(\tilde{\mathbf{G}}^F)$ by mapping a character of $\tilde{\mathbf{B}}^F$ to the unique character of $\tilde{\mathbf{G}}^F$ with the same label. We show that this bijection is compatible with the action of the Galois automorphisms from Conjecture 0.3.

**Theorem 0.5.** *Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$, where $m = |\tilde{\mathbf{G}}^F|$, and $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$ be an $(e,p)$-Galois automorphism for a nonnegative integer $e$. Suppose that $\mathbf{G}$ satisfies Assumption 0.4. Then the Maslowski bijection*

$$\tilde{f} : \mathrm{Irr}_{p'}(\tilde{\mathbf{B}}^F) \to \mathrm{Irr}_{p'}(\tilde{\mathbf{G}}^F)$$

*is $\sigma$-equivariant, i.e. $\tilde{f}(\psi^\sigma) = \tilde{f}(\psi)^\sigma$ for any character $\psi \in \mathrm{Irr}_{p'}(\tilde{\mathbf{B}}^F)$.*

If $\psi$ is a character in $\mathrm{Irr}_{p'}(\tilde{\mathbf{B}}^F)$ we consider the characters $\vartheta \in \mathrm{Irr}(\mathbf{B}^F)$ below $\psi$, i.e. irreducible characters which are constituents of $\psi_{\mathbf{B}^F}$. We find necessary and sufficient conditions for the characters $\vartheta$ to be $\sigma$-invariant. For this we use the fact that the $\tilde{\mathbf{T}}^F$-conjugates of linear characters of $\mathbf{U}^F$ are parametrized by certain characters $\phi_S \in \mathrm{Irr}(\mathbf{U}^F)$ for $S \subseteq \{1, \ldots, r\}$. The exact statement is as follows.

**Lemma 0.6.** *Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$, where $m = |\tilde{\mathbf{G}}^F|$, be any Galois automorphism. Suppose that $\mathbf{G}$ satisfies Assumption 0.4. Let $\psi \in \mathrm{Irr}_{p'}(\tilde{\mathbf{B}}^F)$ and $\vartheta \in \mathrm{Irr}(\mathbf{B}^F)$ be an irreducible constituent of $\psi_{\mathbf{B}^F}$. Furthermore, let $S$ be the unique subset of $\{1, \ldots, r\}$ such that $\psi \in \mathrm{Irr}(\tilde{\mathbf{B}}^F \mid \phi_S)$. Then $\vartheta$ is $\sigma$-invariant if and only if $\psi_{\mathbf{B}^F}$ is $\sigma$-invariant and there exists an element $t \in \mathbf{T}$ such that $\phi_S^\sigma = \phi_S^t$.*

The condition of Lemma 0.6 is very explicit. Indeed, with this characterization of $\sigma$-invariant characters we are (at least theoretically) able to compute the number of $\sigma$-invariant $p'$-characters of $\mathbf{B}^F$. We explicitly compute this number if $\mathbf{G}$ is of type $C_n$ in Example 3.14 and Example 3.15.

In order to relate the $p'$-characters of $\tilde{\mathbf{G}}^F$ and $\mathbf{G}^F$ we use the theory of Gelfand–Graev characters. This allows us to prove a similar statement as in the previous lemma for this situation. As a consequence we deduce the following theorem:

**Theorem 0.7.** *Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$, where $m = |\tilde{\mathbf{G}}^F|$, be an $(e,p)$-Galois automorphism for a nonnegative integer $e$. Suppose that $\mathbf{G}$ satisfies Assumption 0.4. Let $\psi \in \mathrm{Irr}_{p'}(\tilde{\mathbf{B}}^F)$. Then all irreducible constituents of $\psi_{\mathbf{B}^F}$ are $\sigma$-invariant if and only if all irreducible constituents of $\tilde{f}(\psi)_{\mathbf{G}^F}$ are $\sigma$-invariant.*

Using a result of Maslowski, namely that the map $\tilde{f} : \mathrm{Irr}(\tilde{\mathbf{B}}^F) \to \mathrm{Irr}(\tilde{\mathbf{G}}^F)$ preserves the underlying central characters, we are able to prove our main theorem.

**Main Theorem 0.8.** *Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$, where $m = |\mathbf{G}^F|$, be an $(e,p)$-Galois automorphism. Suppose that $\mathbf{G}$ satisfies Assumption 0.4. Then there exists a bijection*

$$f : \mathrm{Irr}_{p'}(\mathbf{B}^F)^\sigma \to \mathrm{Irr}_{p'}(\mathbf{G}^F)^\sigma.$$

*Moreover, for every central character $\lambda \in \mathrm{Irr}(\mathrm{Z}(\tilde{\mathbf{G}})^F)$ the map $f$ restricts to a bijection $\mathrm{Irr}_{p'}(\mathbf{B}^F \mid \lambda_{\mathrm{Z}(\mathbf{G})^F})^\sigma \to \mathrm{Irr}_{p'}(\mathbf{G}^F \mid \lambda_{\mathrm{Z}(\mathbf{G})^F})^\sigma$.*

This result shows that Conjecture 0.3 holds for most quasi-simple groups of Lie type in defining characteristic.

# Summary of contents

In Chapter 1 we describe the necessary background material. In Section 1.1 we recall the Clifford theory of finite groups. From Section 1.2 to Section 1.6 we describe some of the character theory of finite groups of Lie type. First we discuss the Deligne–Lusztig theory of $\mathbf{G}^F$ if $\mathbf{G}$ has connected center. In this case we obtain a nice description of the $p'$-characters of $\mathbf{G}^F$ if the prime $p$ is good for $\mathbf{G}$. This is done using Gelfand-Graev characters and the duality functor. If the center of $\mathbf{G}$ is not connected we consider an extension $i : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ of $\mathbf{G}$ by a central torus such that $\tilde{\mathbf{G}}$ has connected center. Then we relate the $p'$-characters of $\tilde{\mathbf{G}}^F$ with the $p'$-characters of $\mathbf{G}^F$. In Section 1.6 we specialize to the case where $\mathbf{G}$ is a simple algebraic group of simply connected type and discuss the Steinberg presentation of $\mathbf{G}$. In Section 1.8 and Section 1.9 we consider a certain regular embedding $i : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ and describe the structure of the finite groups $\mathbf{G}^F$ and $\tilde{\mathbf{G}}^F$. In Section 1.10 we construct the dual group $\tilde{\mathbf{G}}^*$ as an extension of $\mathbf{G}^\vee$ by a central torus, where $\mathbf{G}^\vee$ is the simple algebraic group of simply connected type with root system dual to the root system of $\mathbf{G}$.

In Chapter 2 we describe the Maslowski bijection $\tilde{f} : \mathrm{Irr}_{p'}(\tilde{\mathbf{B}}^F) \to \mathrm{Irr}_{p'}(\tilde{\mathbf{G}}^F)$. For this we recall in Section 2.1 Maslowski's labeling of the $p'$-characters of $\tilde{\mathbf{B}}^F$, using an explicit description of the linear characters of $\mathbf{U}^F$. In Section 2.2 we describe a labeling of the $p'$-characters of $\tilde{\mathbf{G}}^F$ as follows: Any $p'$-character $\chi \in \mathrm{Irr}(\tilde{\mathbf{G}}^F)$ lies in a unique Lusztig series which corresponds to a semisimple $F^*$-stable conjugacy class $(\tilde{s})$ of $\tilde{\mathbf{G}}^*$. The label of the $p'$-character $\chi$ is defined as $\tilde{\pi}(\tilde{s})$, where $\tilde{\pi}$ is a modified Steinberg map as introduced by Maslowski. We conclude this chapter by stating some properties of the map $\tilde{f}$.

We prove our main results in Chapter 3. In Section 3.1 we show that $\tilde{f} : \mathrm{Irr}_{p'}(\tilde{\mathbf{B}}^F) \to \mathrm{Irr}_{p'}(\tilde{\mathbf{G}}^F)$ is $\sigma$-equivariant for an $(e, p)$-Galois automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|\tilde{\mathbf{G}}^F|}/\mathbb{Q})$ (see Theorem 0.5). In Section 3.2 we consider the favorable case that the Galois automorphism $\sigma$ fixes the $p$-th roots of unity of $\mathbb{Q}_{|\tilde{\mathbf{G}}^F|}$. In the following sections we then drop this assumption and generalize our methods. In Section 3.3 we relate the characters of $\tilde{\mathbf{B}}^F$ and $\mathbf{B}^F$. This can be done by purely elementary methods using Clifford theory and the explicit description of the linear characters of $\mathbf{U}^F$. We give a sufficient and necessary criterion for the $p'$-characters of $\mathbf{B}^F$ to be $\sigma$-invariant (see Lemma 0.6). We use this criterion to compute the number of $\sigma$-invariant $p'$-characters in the case that $\mathbf{G}$ is of type $C_n$. In Section 3.4 we relate the $p'$-characters of $\tilde{\mathbf{G}}^F$ and $\mathbf{G}^F$. We reduce the problem to find a criterion for a $p'$-character of $\mathbf{G}^F$ to be $\sigma$-invariant to the evaluation of certain characters of $\mathrm{Irr}(\tilde{\mathbf{G}}^F \mid 1_{\mathbf{G}^F})$ at elements of the torus $\tilde{\mathbf{T}}^F$. Using the explicit construction of the map $\tilde{f}$ we are able to compute the values of these characters, which allows us to prove a similar criterion as for the $p'$-characters of $\mathbf{B}^F$. In Section 3.5 we use the results of the previous two sections to prove Theorem 0.7. This allows us to construct a bijection $f : \mathrm{Irr}_{p'}(\mathbf{B}^F)^\sigma \to \mathrm{Irr}_{p'}(\mathbf{G}^F)^\sigma$ which preserves central characters. This finally proves Theorem 0.8.

# Chapter 1

# Basics

In this chapter we describe the tools needed for this thesis. First we discuss the representation theory of finite groups of Lie type. In particular we discuss some results of Deligne–Lusztig theory. Then we recall the Steinberg presentation of a simple algebraic group $\mathbf{G}$ of simply connected type, introduced by Steinberg in [19]. We discuss a certain regular embedding $i : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ constructed by Maslowski in [15, Section 2].

## 1.1 Clifford theory

We assume that the reader is familiar with the basic notions of character theory like restriction and induction of characters, Frobenius reciprocity and conjugation of characters. A good introduction can be found in [8]. Most of the material presented in this section can be found in [8, Chapter 5] and [8, Chapter 6].

In this section we recall some results of Clifford theory which will be crucial for the remainder of this thesis. The situation is as follows. Let $G$ be a finite group and $N$ a normal subgroup of $G$. Let $\vartheta \in \mathrm{Irr}(N)$. We write $\mathrm{Irr}(G \mid \vartheta)$ for the set of characters $\chi \in \mathrm{Irr}(G)$ for which $\vartheta$ is a constituent of $\chi_N$. Moreover, we write $\mathrm{Irr}(N \mid \chi)$ for the set of characters $\vartheta \in \mathrm{Irr}(N)$ which are constituents of $\chi_N$ for a fixed character $\chi \in \mathrm{Irr}(G)$. In this situation we say that the character $\vartheta$ *lies below* $\chi$ or equivalently that the character $\chi$ *lies above* $\vartheta$. We denote by $( \, , \, )$ the usual scalar product of characters.

**Theorem 1.1** (Clifford's Theorem)**.** *Let $N$ be a normal subgroup of $G$ and $\chi \in \mathrm{Irr}(G)$. Let $\vartheta \in \mathrm{Irr}(N \mid \chi)$ be a character below $\chi$ and suppose that $\vartheta = \vartheta_1, \vartheta_2, ..., \vartheta_t$ are the distinct $G$-conjugates of $\vartheta$. Then $\chi_N = e \sum_{i=1}^{t} \vartheta_i$ with $e = (\chi_N, \vartheta)$.*

*Proof.* [8, Theorem 6.2]. □

The following corollary is a simple consequence of Clifford's Theorem.

**Corollary 1.2.** *Let $N$ be a normal subgroup of a finite group $G$ and $\chi \in \mathrm{Irr}(G)$. Furthermore, we let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$ be a Galois automorphism, where $m = |G|$. Let $\vartheta \in \mathrm{Irr}(N)$ be a character below $\chi$. If $\vartheta$ is $\sigma$-invariant then all irreducible constituents of $\chi_N$ are $\sigma$-invariant.*

*Proof.* By Clifford's Theorem (see Theorem 1.1) we know that any irreducible constituent of $\chi_N$ is given by $\vartheta^g$ for some $g \in G$. Since $\vartheta$ is $\sigma$-invariant it follows that

$$(\vartheta^g(n))^\sigma = \sigma(\vartheta(gng^{-1})) = \vartheta(gng^{-1}) = \vartheta^g(n),$$

for all $n \in N$. Hence, the character $\vartheta^g$ is $\sigma$-invariant as well. □

Let $N$ be a normal subgroup of a finite group $G$. For a character $\vartheta \in \mathrm{Irr}(N)$ we let $\mathrm{I}_G(\vartheta) = \{g \in G \mid \vartheta^g = \vartheta\}$ be the *inertia group* of $\vartheta$ in $G$. We now relate the set $\mathrm{Irr}(G \mid \vartheta)$ to the set $\mathrm{Irr}(\mathrm{I}_G(\vartheta) \mid \vartheta)$. This can be done using the so-called Clifford correspondence.

**Theorem 1.3** (Clifford correspondence)**.** *Let $N$ be a normal subgroup of a finite group $G$ and $\vartheta \in \mathrm{Irr}(N)$. Then the map $\mathrm{Irr}(\mathrm{I}_G(\vartheta) \mid \vartheta) \to \mathrm{Irr}(G \mid \vartheta)$ with $\psi \mapsto \psi^G$ is a bijection.*

*Proof.* [8, Theorem 6.11]. □

The Clifford correspondence can be refined in the following way.

**Lemma 1.4.** *Let $N$ be a normal subgroup of a finite group $G$. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$, where $m = |G|$, and $\vartheta \in \mathrm{Irr}(N)$ be a $\sigma$-invariant character. Then Clifford correspondence restricts to a bijection*

$$\mathrm{Irr}(\mathrm{I}_G(\vartheta) \mid \vartheta)^\sigma \to \mathrm{Irr}(G \mid \vartheta)^\sigma.$$

*Proof.* If $\psi \in \mathrm{Irr}(\mathrm{I}_G(\vartheta) \mid \vartheta)$ is $\sigma$-invariant then the character $\psi^G$ is obviously $\sigma$-invariant again. Now suppose that $\chi \in \mathrm{Irr}(G \mid \vartheta)$ is $\sigma$-invariant. We let $\psi \in \mathrm{Irr}(\mathrm{I}_G(\vartheta) \mid \vartheta)$ such that $\psi^G = \chi$. Then $\psi^\sigma \in \mathrm{Irr}(\mathrm{I}_G(\vartheta) \mid \vartheta)$, since $\vartheta$ is $\sigma$-invariant. Furthermore,

$$(\psi^\sigma)^G = (\psi^G)^\sigma = \chi^\sigma = \chi = \psi^G,$$

since Galois automorphisms commute with induction of characters. Hence, both $\psi$ and $\psi^\sigma$ are Clifford correspondents of $\chi$. This implies $\psi = \psi^\sigma$ by Theorem 1.3. □

**Theorem 1.5.** *Let $N$ be a normal subgroup of a finite group $G$ and $\vartheta \in \operatorname{Irr}(N)$ be an irreducible character of $N$. If $\psi \in \operatorname{Irr}(\mathrm{I}_G(\vartheta))$ is an extension of $\vartheta$ then $\operatorname{Irr}(\mathrm{I}_G(\vartheta) \mid \vartheta) = \{\eta\psi \mid \eta \in \operatorname{Irr}(\mathrm{I}_G(\vartheta) \mid 1_N)\}$. Furthermore, the characters $\eta\vartheta$ are all distinct for distinct $\eta$'s.*

*Proof.* [8, Corollary 6.17]. □

We are now able to prove the following important and well-known lemma.

**Lemma 1.6.** *Let $N$ be a normal subgroup of a finite group $G$ such that the factor group $G/N$ is abelian. Let $\chi \in \operatorname{Irr}(G)$ be a character lying above $\vartheta \in \operatorname{Irr}(N)$. Suppose that $\vartheta$ extends to $\mathrm{I}_G(\vartheta)$. Then $\operatorname{Irr}(G \mid \vartheta) = \{\lambda\chi \mid \lambda \in \operatorname{Irr}(G \mid 1_N)\}$. Furthermore $|\operatorname{Irr}(G \mid \vartheta)| = |\mathrm{I}_G(\vartheta) : N|$.*

*Proof.* By Clifford correspondence (see Theorem 1.3) induction defines a bijection $\operatorname{Irr}(\mathrm{I}_G(\vartheta) \mid \vartheta) \to \operatorname{Irr}(G \mid \vartheta)$. Let $\psi \in \operatorname{Irr}(\mathrm{I}_G(\vartheta) \mid \vartheta)$ with $\psi^G = \chi$. Since $\vartheta$ extends to its inertia group, it follows that $\psi_N = \vartheta$ and $\operatorname{Irr}(\mathrm{I}_G(\vartheta) \mid \vartheta) = \{\eta\psi \mid \eta \in \operatorname{Irr}(\mathrm{I}_G(\vartheta) \mid 1_N)\}$ by Theorem 1.5. Let $\eta \in \operatorname{Irr}(\mathrm{I}_G(\vartheta) \mid 1_N)$ be arbitrary. Since $G/N$ is abelian there exists a character $\lambda \in \operatorname{Irr}(G \mid 1_N)$ such that $\lambda_{\mathrm{I}_G(\vartheta)} = \eta$ (see [8, Corollary 5.5]). By [8, Problem 5.3] it follows that $(\psi\eta)^G = \lambda\chi$. Hence, it follows easily that $\operatorname{Irr}(G \mid \vartheta) = \{\lambda\chi \mid \lambda \in \operatorname{Irr}(G \mid 1_N)\}$. Since Clifford correspondence is a bijection we have $|\operatorname{Irr}(G \mid \vartheta)| = |\operatorname{Irr}(\mathrm{I}_G(\vartheta) \mid \vartheta)|$. By Theorem 1.5 it holds that $|\operatorname{Irr}(\mathrm{I}_G(\vartheta) \mid \vartheta)| = |\mathrm{I}_G(\vartheta) : N|$. □

Suppose that the factor group $G/N$ is cyclic in the situation of the lemma above. Then every irreducible character $\vartheta \in \operatorname{Irr}(N)$ extends to its inertia group $\mathrm{I}_G(\vartheta)$ (see [8, Corollary 11.22]). Thus, the following corollary follows immediately.

**Corollary 1.7.** *Let $N$ be a normal subgroup of a finite group $G$ such that $G/N$ is cyclic. Let $\chi \in \operatorname{Irr}(G)$ be a character lying above $\vartheta \in \operatorname{Irr}(N)$. Then $\operatorname{Irr}(G \mid \vartheta) = \{\lambda\chi \mid \lambda \in \operatorname{Irr}(G \mid 1_N)\}$ and $|\operatorname{Irr}(G \mid \vartheta)| = |\mathrm{I}_G(\vartheta) : N|$.*

We often have to consider the action of Galois automorphisms on linear characters. For convenience, we state the following lemma.

**Lemma 1.8.** *Let $G$ be a finite group of order $m$ and let $p$ be a prime. Let $\sigma \in \operatorname{Gal}(\mathbb{Q}_m/\mathbb{Q})$ and $\xi \in \mathbb{Q}_m$ be a primitive $m$-th root of unity. Let $k$ be a natural number such that $\sigma(\xi) = \xi^k$. Then $\lambda^\sigma = \lambda^k$ for any linear character $\lambda \in \operatorname{Irr}(H)$ of a subgroup $H$ of $G$.*

*Proof.* The values of linear characters of $H$ are $m$-th roots of unity. Since the Galois automorphism $\sigma$ acts on $m$-th roots of unity by taking them to the $k$-th power, the assertion of the lemma follows easily. □

## 1.2 Deligne–Lusztig theory

In this section we summarize some of the main results of Deligne–Lusztig theory which are needed in this thesis. A more detailed and comprehensive introduction on this subject can be found in [2] or [5]. We assume that basic notions from the theory of linear algebraic groups as can be found in [14, Part 1] are known.

We fix the following notations throughout this thesis. Let $p$ be a prime and $q$ be an integral power of $p$. We let $\mathbf{k}$ be an algebraic closure of $\mathbb{F}_p$. Let $\mathbf{G}$ be a connected reductive algebraic group defined over $\mathbb{F}_q$ with corresponding Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$. We fix a maximal $F$-stable torus $\mathbf{T}$ of $\mathbf{G}$ which is *maximally split*, i.e. there exists an $F$-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ such that $\mathbf{T}$ is contained in $\mathbf{B}$ (see [14, Definition 21.13]). Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. We denote by $\Phi$ the root system of $\mathbf{G}$ with respect to the torus $\mathbf{T}$ and by $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ the set of simple roots of $\Phi$ with respect to $\mathbf{T} \subseteq \mathbf{B}$ (see [14, Definition 11.2]). Let $\Phi^+$ be the set of *positive roots*, i.e. the subset of the set of roots $\Phi$ consisting of the roots which can be written as a linear combination of the simple roots with natural numbers as coefficients. Furthermore we denote by $\Phi^\vee$ the set of coroots of $\Phi$. We will later specify our general setup in Section 1.7.

Let $\mathbf{T}'$ be a maximal $F$-stable torus of $\mathbf{G}$ and $\theta \in \mathrm{Irr}(\mathbf{T}'^F)$. The pair $(\mathbf{T}', \theta)$ defines a generalized character $R^{\mathbf{G}}_{\mathbf{T}'}(\theta) \in \mathbb{Z}\,\mathrm{Irr}(\mathbf{G}^F)$ as in [5, Definition 11.1]. We call $R^{\mathbf{G}}_{\mathbf{T}'}(\theta)$ a *Deligne–Lusztig character*. Note that $R^{\mathbf{G}}_{\mathbf{T}''}(\theta'') = R^{\mathbf{G}}_{\mathbf{T}'}(\theta')$ whenever $(\mathbf{T}', \theta')$ and $(\mathbf{T}'', \theta'')$ are $\mathbf{G}^F$*-conjugate*, i.e. if there exists some $g \in \mathbf{G}^F$ such that ${}^g\mathbf{T}' = \mathbf{T}''$ and ${}^g\theta' = \theta''$ (see remark below [5, Corollary 11.15]).

With this notation we can now state the following useful character formula for Deligne–Lusztig characters.

**Lemma 1.9.** *(Character formula for Deligne–Lusztig characters) Let $g = su$ be the Jordan decomposition of some element $g \in \mathbf{G}^F$. Then*

$$R^{\mathbf{G}}_{\mathbf{T}'}(\theta)(g) = |\mathbf{T}'^F|^{-1} |\,\mathrm{C}^\circ_{\mathbf{G}}(s)|^{-1} \sum_{h \in \{s \in \mathbf{G}^F | s \in {}^h\mathbf{T}'\}} \sum_{v \in \mathrm{C}^\circ_{{}^h\mathbf{T}'}(s)^F_u} Q^{\mathrm{C}^\circ_{\mathbf{G}}(s)}_{\mathrm{C}^\circ_{{}^h\mathbf{T}'}(s)}(u, v^{-1})^h\theta(sv),$$

*where $Q^{\mathrm{C}^\circ_{\mathbf{G}}(s)}_{\mathrm{C}^\circ_{\mathbf{T}'}(s)}$ is the Green function as defined in [5, Definition 12.1].*

*Proof.* This is [5, Proposition 12.2]. □

With Lemma 1.9 we can prove the following corollary.

**Corollary 1.10.** *Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$ with $m = |\mathbf{G}^F|$ and $\sigma(\xi) = \xi^k$ for a primitive $m$-th root of unity $\xi \in \mathbb{Q}_m$. Then $R^{\mathbf{G}}_{\mathbf{T}'}(\theta)^\sigma = R^{\mathbf{G}}_{\mathbf{T}'}(\theta^k)$.*

*Proof.* Note that the Green function has values in the integers (see [5, Corollary 10.6]). By Lemma 1.8 it follows that $\theta^\sigma = \theta^k$. Thus, the claim of the corollary follows with Lemma 1.9. □

Let $\mathbf{T}'$ be a maximal $F$-stable torus of $\mathbf{G}$ with character group $X(\mathbf{T}')$ and cocharacter group $Y(\mathbf{T}')$ (see [14, Definition 3.4]). We recall the following definition (see [5, Definition 13.10]).

**Definition 1.11.** Two connected reductive algebraic groups $\mathbf{G}$ and $\mathbf{G}^*$ are in *duality* if there exists a maximal torus $\mathbf{T}'$ of $\mathbf{G}$ and a maximal torus $\mathbf{T}'^*$ of $\mathbf{G}^*$ together with an isomorphism $\delta : Y(\mathbf{T}') \to X(\mathbf{T}'^*)$ which sends the coroots of $\mathbf{T}'$ to the roots of $\mathbf{T}'^*$. Suppose that $\mathbf{G}$ and $\mathbf{G}^*$ are defined over $\mathbb{F}_q$ with respective Frobenius endomorphisms $F$ and $F^*$. If $\mathbf{T}'$ and $\mathbf{T}'^*$ are $F$- resp. $F^*$-stable tori and if $\delta$ commutes with the action of $F$ and $F^*$ then the pair $(\mathbf{G}, F)$ is called *dual to* the pair $(\mathbf{G}^*, F^*)$.

Note that if $(X(\mathbf{T}'), Y(\mathbf{T}'), \Phi, \Phi^\vee)$ is the root datum of a connected reductive group $\mathbf{G}$ then $(Y(\mathbf{T}'), X(\mathbf{T}'), \Phi^\vee, \Phi)$ is a root datum as well. Therefore, by Chevalley's classification of reductive algebraic groups (see [5, Theorem 0.45]) there exists a reductive algebraic group $\mathbf{G}^*$ with (abstract root datum) $(Y(\mathbf{T}'), X(\mathbf{T}'), \Phi^\vee, \Phi)$, which is a dual group for $\mathbf{G}$. So in particular every connected reductive group $\mathbf{G}$ has a dual group $\mathbf{G}^*$. Moreover, if $F : \mathbf{G} \to \mathbf{G}$ is a Frobenius endomorphism of $\mathbf{G}$ then there exists a Frobenius endomorphism $F^* : \mathbf{G}^* \to \mathbf{G}^*$ of $\mathbf{G}^*$ such that $(\mathbf{G}, F)$ and $(\mathbf{G}^*, F^*)$ are dual (see remark below [2, Theorem 4.4.6]).

For the remainder of this section we suppose that $(\mathbf{G}, F)$ and $(\mathbf{G}^*, F^*)$ are in duality via a duality map $\delta : Y(\mathbf{T}) \to X(\mathbf{T}^*)$, where $\mathbf{T}$ is the maximal $F$-stable torus of $\mathbf{G}$ which we fixed at the beginning of this section and $\mathbf{T}^*$ is a maximal $F^*$-stable torus of $\mathbf{G}^*$.

Let us denote by $\langle\ ,\ \rangle : X(\mathbf{T}) \times Y(\mathbf{T}) \to \mathbb{Z}$ the perfect pairing between $X(\mathbf{T})$ and $Y(\mathbf{T})$ as in [14, Proposition 3.6]. The duality isomorphism $\delta$ gives rise to a dual map as explained in the following remark.

**Remark 1.12.** The existence of a map $\delta : Y(\mathbf{T}) \to X(\mathbf{T}^*)$ with properties as above yields the existence of a dual map $\delta^\vee : X(\mathbf{T}) \to Y(\mathbf{T}^*)$ with similar properties (see [2, Definition 4.3.1]) defined by the property $\langle \delta(\gamma), \delta^\vee(\chi) \rangle = \langle \chi, \gamma \rangle$ for all $\chi \in X(\mathbf{T})$ and $\gamma \in Y(\mathbf{T})$.

**Definition 1.13** (Norm map)**.** Let $\mathbf{T}'$ be a maximal $F$-stable torus of $\mathbf{G}$. We let $w$ be the order of the automorphism $\tau$ of $X(\mathbf{T})$ that is induced by the

Frobenius endomorphism $F$ as explained in the remark preceding [5, Theorem 3.17]. Then we define the *norm map* $N_{F^w/F} : \mathbf{T}' \to \mathbf{T}'$ by $t \mapsto \prod_{i=0}^{w-1} F^i(t)$.

Let us now, once and for all, choose an embedding $\mathbf{k}^\times \to \mathbb{C}^\times$. For all positive integers $w$ let us fix a primitive $(q^w - 1)$-th root of unity $\mu \in \mathbf{k}^\times$, i.e., a generator of $\mathbb{F}_{q^w}^\times$ as multiplicative group.

**Lemma 1.14.** *The set of $\mathbf{G}^F$-conjugacy classes of pairs $(\mathbf{T}', \theta)$ where $\mathbf{T}'$ is an $F$-stable maximal torus of $\mathbf{G}$ and $\theta \in \operatorname{Irr}(\mathbf{T}'^F)$ is in bijection with the set of $\mathbf{G}^{*F^*}$-conjugacy classes of pairs $(\mathbf{T}'^*, s)$ where $s \in \mathbf{G}^{*F^*}$ is a semisimple element and $\mathbf{T}'^*$ is an $F^*$-stable maximal torus with $s \in \mathbf{T}'^*$.*

*Proof.* A proof of this lemma is given in [5, Proposition 13.13]. We briefly sketch the idea of the proof since we need the explicit construction later.

Recall that we fixed maximal tori $\mathbf{T}$ of $\mathbf{G}$ and $\mathbf{T}^*$ of $\mathbf{G}^*$ which define the duality of $(\mathbf{G}, F)$ and $(\mathbf{G}^*, F^*)$ via the duality map $\delta : Y(\mathbf{T}) \to X(\mathbf{T}^*)$. Suppose that $(\mathbf{T}', \theta)$ is a pair as in the statement of the lemma. The $\mathbf{G}^F$-conjugacy classes of such pairs $(\mathbf{T}', \theta)$ are parametrized by the $F$-conjugacy classes of the Weyl group $\mathrm{W}(\mathbf{T}) = \mathrm{N}_{\mathbf{G}}(\mathbf{T})/\mathrm{N}^\circ_{\mathbf{G}}(\mathbf{T})$ of the torus $\mathbf{T}$ (see [5, Application 3.23]). We let $g \in \mathbf{G}$ such that ${}^g\mathbf{T}' = \mathbf{T}$. Let $x := g^{-1}F(g)\mathbf{T} \in \mathrm{W}(\mathbf{T})$ denote the image of $g^{-1}F(g)$ in $\mathrm{W}(\mathbf{T})$. Consider the anti-isomorphism $\mathrm{W}(\mathbf{T}) \to \mathrm{W}(\mathbf{T}^*)$ mapping $x \mapsto x^*$ as defined in [2, Proposition 4.2.3]. We let $g^* \in \mathbf{G}^*$ be such that ${g^*}^{-1}F^*(g^*)\mathbf{T}^* = x^*$ in $\mathrm{W}(\mathbf{T}^*)$ and define $\mathbf{T}'^* = {}^{g^*}\mathbf{T}^*$. Then the map $\delta^\vee : X(\mathbf{T}) \to Y(\mathbf{T}^*)$ gives rise to a map $\delta'^\vee : X(\mathbf{T}') \to Y(\mathbf{T}'^*)$ defined by $\delta'^\vee({}^g\chi) = {}^{g^*}\delta^\vee(\chi)$ for $\chi \in X(\mathbf{T})$. Consider the endomorphism $F' := xF$ of $\mathbf{T}'$ and the endomorphism $F'^* := F^*x^*$ of $\mathbf{T}'^*$. Then the map $\delta'^\vee$ defines a duality of $(\mathbf{G}, F')$ and $(\mathbf{G}^*, F'^*)$.

By [5, Proposition 13.11] we have an isomorphism $\operatorname{Irr}(\mathbf{T}'^{F'}) \to {\mathbf{T}'^*}^{F'^*}$ which is constructed as follows: Let $w$ be the order of the automorphism $\tau$ of $X(\mathbf{T})$ induced by $F$. Using the embedding $\mathbf{k}^\times \to \mathbb{C}^\times$ we can assume that $\theta \in \operatorname{Irr}(\mathbf{T}'^{F'})$ has values in $\mathbf{k}^\times$. By [5, Proposition 13.7] we can extend the character $\theta \in \operatorname{Irr}(\mathbf{T}'^{F'})$ to an element $\hat{\theta} \in X(\mathbf{T}')$ of the character group of the torus $\mathbf{T}'$. By applying the duality isomorphism $\delta'^\vee : X(\mathbf{T}') \to Y(\mathbf{T}'^*)$ we have $\delta'^\vee(\hat{\theta}) \in Y(\mathbf{T}'^*)$. Hence, we may define $s = N_{F'^{*w}/F'^*}(\delta^\vee(\hat{\theta})(\mu))$, where $\mu \in \mathbf{k}^\times$ is the $(q^w - 1)$-th root of unity as chosen before Lemma 1.14. Then we define the pair corresponding to $(\mathbf{T}', \theta)$ as $(\mathbf{T}'^*, s)$. □

If $(\mathbf{T}', \theta)$ maps to $(\mathbf{T}'^*, s)$ under the bijection of Lemma 1.14 we say that $(\mathbf{T}, \theta)$ *is in duality* with $(\mathbf{T}'^*, s)$. Note that everything in the construction of $s = N_{F'^{*w}/F'^*}(\delta^\vee(\hat{\theta})(\mu))$ is multiplicative. Thus, we obtain that bijection constructed in Lemma 1.14 is multiplicative in the following sense. If $(\mathbf{T}', \theta_1)$ is in duality with $(\mathbf{T}', s_1)$ and $(\mathbf{T}', \theta_2)$ is in duality with $(\mathbf{T}'^*, s_2)$ then it follows that $(\mathbf{T}', \theta_1\theta_2)$ is in duality with $(\mathbf{T}'^*, s_1 s_2)$.

In the following, we write $R^{\mathbf{G}}_{\mathbf{T}'^*}(s)$ for the character $R^{\mathbf{G}}_{\mathbf{T}'}(\theta)$ if $(\mathbf{T}', \theta)$ is in duality with $(\mathbf{T}'^*, s)$. We want to consider the irreducible constituents of $R^{\mathbf{G}}_{\mathbf{T}'^*}(s)$. For this we state the following definition.

**Definition 1.15** (Lusztig series)**.** Let $(s)$ be the ${\mathbf{G}^*}^{F^*}$-conjugacy class of a semisimple element $s \in {\mathbf{G}^*}^{F^*}$. We write $\mathcal{E}(\mathbf{G}^F, (s))$ for the set of irreducible constituents of $R^{\mathbf{G}}_{\mathbf{T}'^*}(s)$, where $\mathbf{T}'^*$ is a maximal $F^*$-stable torus of $\mathbf{G}^*$ with $s \in \mathbf{T}'^*$.

**Corollary 1.16.** *Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$ with $m = |\mathbf{G}^F|$ and $\sigma(\xi) = \xi^k$ for a primitive $m$-th root of unity $\xi \in \mathbb{Q}_m$. If $\chi \in \mathcal{E}(\mathbf{G}^F, (s))$ then $\chi^\sigma \in \mathcal{E}(\mathbf{G}^F, (s^k))$.*

*Proof.* Let $\mathbf{T}'^*$ be a maximal $F^*$-stable torus of $\mathbf{G}^*$ such that $\chi$ is a constituent of $R^{\mathbf{G}}_{\mathbf{T}'^*}(s)$. Suppose that $(\mathbf{T}', \theta)$ is in duality with $(\mathbf{T}'^*, s)$. Then the character $\chi$ is a constituent of $R^{\mathbf{G}}_{\mathbf{T}'}(\theta)$. By Corollary 1.10 we conclude that $\chi^\sigma$ is a constituent of $R^{\mathbf{G}}_{\mathbf{T}'}(\theta^k)$. Since the bijection of Lemma 1.14 is multiplicative (see remark below Lemma 1.14) we obtain that $(\mathbf{T}', \theta^k)$ is in duality with $(\mathbf{T}'^*, s^k)$. This shows that $\chi$ is a constituent of $R^{\mathbf{G}}_{\mathbf{T}'^*}(s^k)$. Consequently, we have $\chi^\sigma \in \mathcal{E}(\mathbf{G}^F, (s^k))$. □

The following remark is important.

**Remark 1.17.** $\mathcal{E}(\mathbf{G}^F, (s))$ is called a *rational Lusztig series.* In the case that the center of $\mathbf{G}$ is connected it coincides with the geometric Lusztig series defined in [5, Definition 13.16] (see remark preceding [5, Proposition 14.41]). This is due to the fact that semisimple $F$-stable conjugacy classes of $\mathbf{G}^*$ are precisely the conjugacy classes of semisimple elements of ${\mathbf{G}^*}^{F^*}$ if the center of $\mathbf{G}$ is connected (see [5, Remark 13.15(ii)]). Note that these notions differ in general if the center of $\mathbf{G}$ is not connected.

We will now see that the Deligne–Lusztig series partition the characters of $\mathbf{G}^F$ if the center of $\mathbf{G}$ is connected.

**Lemma 1.18.** *Let $\mathbf{G}$ be a connected reductive algebraic group with connected center. Then we have a partition*

$$\mathrm{Irr}(\mathbf{G}^F) = \bigcup_{(s)} \mathcal{E}(\mathbf{G}^F, (s))$$

*where $(s)$ runs over the ${\mathbf{G}^*}^{F^*}$-conjugacy classes $(s)$ of semisimple elements $s \in {\mathbf{G}^*}^{F^*}$.*

*Proof.* This is [5, Proposition 14.41]. □

Moreover, the partition above gives a particularly nice description of the $p'$-characters if $\mathbf{G}$ is a simple algebraic group and the root system of $\mathbf{G}$ is not of a certain type.

**Theorem 1.19.** *Let $\mathbf{G}$ be a simple algebraic group with connected center and not of type $B_n$, $C_n$, $G_2$ or $F_4$ if $q = 2$, or $G_2$ if $q = 3$. Then we have $|\operatorname{Irr}_{p'}(\mathbf{G}^F) \cap \mathcal{E}(\mathbf{G}^F, (s))| = 1$ for any semisimple conjugacy class $(s)$ of ${\mathbf{G}^*}^{F^*}$.*

*Proof.* This follows by [13, Theorem 6.8], using [5, Theorem 13.23] together with [5, Remark 13.24]. □

This result will be crucial for the construction of Maslowski (see Construction 2.10). We need to describe the $p'$-characters in more detail. For this we define as follows:

**Definition 1.20.** Let $(s)$ be a semisimple conjugacy class of ${\mathbf{G}^*}^{F^*}$. We define a classfunction $\chi_{(s)}$ of the finite group $\mathbf{G}^F$ by

$$\chi_{(s)} = |\operatorname{C}_{\mathbf{G}^*}(s)/\operatorname{C}^\circ_{\mathbf{G}^*}(s)| \sum_{\mathbf{T}'^*} (R^{\mathbf{G}}_{\mathbf{T}'^*}(s), R^{\mathbf{G}}_{\mathbf{T}'^*}(s))^{-2} \varepsilon_{\mathbf{G}} \varepsilon_{\mathbf{T}'^*} R^{\mathbf{G}}_{\mathbf{T}'^*}(s),$$

where the sum is over the $\operatorname{C}_{\mathbf{G}^*}(s)^{F^*}$-conjugacy classes of $F^*$-stable maximal tori $\mathbf{T}'^*$ of $\operatorname{C}_{\mathbf{G}^*}(s)$ and $\varepsilon_{\mathbf{G}} = (-1)^{\text{rel. rank } \mathbf{G}}$ defined as in [2, Section 6.5].

The class function $\chi_{(s)}$ is indeed a character of $\mathbf{G}^F$ by [5, Proposition 14.48]. If the center of $\mathbf{G}$ is connected then $\chi_{(s)}$ is even an irreducible character of $\mathbf{G}^F$ by [5, Corollary 14.47(a)]. Hence, by definition of $\chi_{(s)}$ we have $\chi_{(s)} \in \mathcal{E}(\mathbf{G}^F, (s))$.

**Lemma 1.21.** *Let $\mathbf{G}$ be a connected reductive group with connected center. Let $\lambda \in \mathcal{E}(\mathbf{G}^F, (z))$ for some $z \in \operatorname{Z}({\mathbf{G}^*}^{F^*})$ and suppose that $\lambda$ is a linear character. Then $\chi_{(sz)} = \lambda\chi_{(s)}$ if and only if $(sz) = (s)$.*

*Proof.* Suppose that $(\mathbf{T}', \theta)$ is in duality with $(\mathbf{T}'^*, s)$. By Lemma 1.9 it follows that $\lambda R^{\mathbf{G}}_{\mathbf{T}'}(\theta) = R^{\mathbf{G}}_{\mathbf{T}'}(\lambda_{\mathbf{T}'^F}\theta)$. By [5, Proposition 13.30] it follows that $(\mathbf{T}', \lambda_{\mathbf{T}'^F})$ is in duality with $(\mathbf{T}'^*, z)$. By the remark below Lemma 1.14 we conclude that $(\mathbf{T}', \lambda_{\mathbf{T}'^F}\theta)$ is in duality with $(\mathbf{T}'^*, sz)$. This shows $\lambda R^{\mathbf{G}}_{\mathbf{T}'}(s) = R^{\mathbf{G}}_{\mathbf{T}'}(sz)$. From Definition 1.20 it follows that $\chi_{(sz)} = \lambda\chi_{(s)}$. Hence, multiplication with the character $\lambda$ fixes $\chi_{(s)}$ if and only if $\chi_{(sz)} = \chi_{(s)}$. As explained above, we have $\chi_{(sz)} \in \mathcal{E}(\mathbf{G}^F, (sz))$ and $\chi_{(s)} \in \mathcal{E}(\mathbf{G}^F, (s))$. Since the rational Lusztig series form a partition of the irreducible characters of $\mathbf{G}^F$ by Lemma 1.18, we have $\chi_{(sz)} = \lambda\chi_{(s)}$ if and only if $(sz) = (s)$. □

## 1.3 The duality functor

In this section we introduce the notion of the duality functor from [2, Section 8.2]. We need the following definition.

**Definition 1.22.** Let $G$ be a finite group with normal subgroup $N$. For a generalized character $\chi \in \mathbb{Z}\operatorname{Irr}(G)$ we define by

$$\mathrm{T}_{G/N}(\chi)(g) = \frac{1}{|N|}\sum_{n\in N}\chi(ng), \text{ for } g \in G,$$

the *truncation* $\mathrm{T}_{G/N}(\chi)$ with respect to $N$ of the generalized character $\chi$. Note that $\mathrm{T}_{G/N}(\chi) \in \mathbb{Z}\operatorname{Irr}(G)$ is again a generalized character of $G$ (see remark preceding [2, Lemma 8.1.6])

For a subset $J \subseteq \Delta$ of the simple roots of the root system $\Phi$, we denote by $\mathbf{P}_J$ the standard parabolic subgroup of $\mathbf{G}$ associated to $J$ and we let $\mathbf{L}_J$ be the standard Levi complement of $\mathbf{P}_J$ (see [14, Definition 12.3] and [14, Definition 12.7]). In particular, we have a decomposition $\mathbf{P}_J = \mathbf{U}_J \rtimes \mathbf{L}_J$, where $\mathbf{U}_J$ denotes the unipotent radical of the parabolic subgroup $\mathbf{P}_J$.

**Definition 1.23** (Duality functor)**.** For a generalized character $\chi \in \mathbb{Z}\operatorname{Irr}(\mathbf{G}^F)$ we define the *dual generalized character* of $\chi$ by

$$\mathrm{D}_{\mathbf{G}}(\chi) = \sum_J (-1)^{|J'|}(\mathrm{T}_{\mathbf{P}_J^F/\mathbf{U}_J^F}(\chi))^{\mathbf{G}^F},$$

where the sum runs over all $\tau$-stable subsets $J$ of $\Delta$ and $J'$ is the set of $\tau$-orbits on $J$. The map $\mathrm{D}_{\mathbf{G}} : \mathbb{Z}\operatorname{Irr}(\mathbf{G}^F) \to \mathbb{Z}\operatorname{Irr}(\mathbf{G}^F)$ is called *duality functor.*

This functor has useful properties, which we will study in more detail.

**Theorem 1.24.** *Let $\psi$, $\chi \in \mathbb{Z}\operatorname{Irr}(\mathbf{G}^F)$ be two generalized characters of $\mathbf{G}^F$. Then the following holds.*
*(a)* $(\mathrm{D}_{\mathbf{G}} \circ \mathrm{D}_{\mathbf{G}})(\chi) = \chi$ *and*
*(b)* $(\mathrm{D}_{\mathbf{G}}(\chi), \psi) = (\chi, \mathrm{D}_{\mathbf{G}}(\psi))$.

*Proof.* The theory of the duality functor is discussed in [5, Chapter 8]. In particular, part (a) follows from [5, Corollary 8.14] and part (b) is [5, Proposition 8.10]. □

From Lemma 1.24 it follows for two generalized characters $\chi, \psi \in \mathbb{Z}\operatorname{Irr}(\mathbf{G}^F)$ that

$$(\mathrm{D}_{\mathbf{G}}(\chi), \mathrm{D}_{\mathbf{G}}(\psi)) = (\chi, (\mathrm{D}_{\mathbf{G}} \circ \mathrm{D}_{\mathbf{G}})(\psi)) = (\chi, \psi),$$

which shows that $\mathrm{D}_{\mathbf{G}}$ is an isometry. Thus, the dual $\mathrm{D}_{\mathbf{G}}(\chi)$ of an irreducible character $\chi \in \operatorname{Irr}(\mathbf{G}^F)$ is again an irreducible character up to a sign $\pm 1$. We show now that the duality functor is compatible with Galois automorphisms.

**Lemma 1.25.** *Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$ be a Galois automorphism, where $m = |\mathbf{G}^F|$. Then the character $\chi$ is $\sigma$-invariant if and only if $D_{\mathbf{G}}(\chi)$ is $\sigma$-invariant, for any $\chi \in \mathbb{Z}\,\mathrm{Irr}(\mathbf{G}^F)$.*

*Proof.* By definition of the truncation functor in Definition 1.22, we have

$$\mathrm{T}_{\mathbf{P}_J^F/\mathbf{U}_J^F}(\chi^\sigma)^{\mathbf{G}^F} = (\mathrm{T}_{\mathbf{P}_J^F/\mathbf{U}_J^F}(\chi)^{\mathbf{G}^F})^\sigma.$$

This clearly implies $\mathrm{D}_{\mathbf{G}}(\chi^\sigma) = \mathrm{D}_{\mathbf{G}}(\chi)^\sigma$ by Definition 1.23. So if $\chi$ is $\sigma$-invariant then $\mathrm{D}_{\mathbf{G}}(\chi)$ is $\sigma$-invariant. The converse follows from the fact that $\mathrm{D}_{\mathbf{G}} \circ \mathrm{D}_{\mathbf{G}}$ is the identity (see Theorem 1.24): If $\mathrm{D}_{\mathbf{G}}(\chi)$ is $\sigma$-invariant then $\mathrm{D}_{\mathbf{G}}(\mathrm{D}_{\mathbf{G}}(\chi)) = \chi$ is $\sigma$-invariant as well.

□

## 1.4 Galois cohomology

In this section we give different interpretations of the first Galois cohomology group $H^1(F, \mathrm{Z}(\mathbf{G}))$. This cohomology group will become important in Section 1.5 since it parametrizes the different Gelfand–Graev characters of $\mathbf{G}$. The main references for this section and the subsequent section are the articles [4] and [12]. We remark that the author of [12] assumes that $\mathbf{G}$ is a connected semisimple algebraic group, but all results we use in this thesis are applied in [4], where the authors only assume that $\mathbf{G}$ is a connected reductive group. Thus, these results still hold in this more general setup.

First, let us recall some well known results of Galois cohomology. Let $\mathbf{H}$ be an algebraic group with Frobenius endomorphism $F$. We denote by $H^i(F, \mathbf{H})$ the $i$-th Galois cohomology group. Recall that $H^0(F, \mathbf{H}) = \mathbf{H}^F$ and that the functor $H^0(F, -)$ is left-exact (see [6, Theorem 6.3.1]). This implies that if $\mathbf{K}$ is an $F$-stable normal subgroup of $\mathbf{H}$ we have that the exact sequence

$$1 \to \mathbf{K} \to \mathbf{H} \to \mathbf{H}/\mathbf{K} \to 1$$

of algebraic groups induces a long exact sequence

$$1 \to H^0(F, \mathbf{K}) \to H^0(F, \mathbf{H}) \to H^0(F, \mathbf{H}/\mathbf{K}) \to H^1(F, \mathbf{K}) \to H^1(F, \mathbf{H}) \to \dots$$

of Galois cohomology groups (see Theorem [6, 4.6.1]). We will use these facts in the subsequent section without further reference. The following lemma is used in many situations.

**Lemma 1.26.** *Let $\mathbf{H}$ be an algebraic group with Frobenius endomorphism $F$ and let $\mathbf{K}$ be a closed connected $F$-stable normal subgroup of $\mathbf{H}$. Then the natural map $\mathbf{H}^F/\mathbf{K}^F \to (\mathbf{H}/\mathbf{K})^F$ is an isomorphism.*

*Proof.* This is a consequence of the Lang–Steinberg Theorem (see [5, Corollary 3.22]). □

The previous lemma shows that we have an exact sequence

$$1 \to H^0(F, \mathbf{K}) \to H^0(F, \mathbf{H}) \to H^0(F, \mathbf{H}/\mathbf{K}) \to 1,$$

if the algebraic group $\mathbf{K}$ is connected. More generally, it is true that $H^1(F, \mathbf{K}) = 1$ if $\mathbf{K}$ is a connected algebraic group (see remark preceding [5, Proposition 14.23]).

Let us denote by $\mathscr{L} : \mathbf{G} \to \mathbf{G} : g \mapsto F(g)g^{-1}$ the Lang map of the connected reductive group $\mathbf{G}$ with Frobenius map $F$. Since the maximal torus $\mathbf{T}$ of $\mathbf{G}$ is $F$-stable we can consider the restriction $\mathscr{L}_{\mathbf{T}} : \mathbf{T} \to \mathbf{T}$ of the Lang map $\mathscr{L}$ to the torus $\mathbf{T}$.

**Lemma 1.27.** *There is a natural isomorphism*

$$H^1(F, \mathrm{Z}(\mathbf{G})) \cong \mathscr{L}_{\mathbf{T}}^{-1}(\mathrm{Z}(\mathbf{G}))/\mathrm{Z}(\mathbf{G})\mathbf{T}^F.$$

*Proof.* We have an exact sequence of algebraic groups

$$1 \to \mathrm{Z}(\mathbf{G}) \to \mathbf{T} \to \mathbf{T}/\mathrm{Z}(\mathbf{G}) \to 1.$$

Since the torus $\mathbf{T}$ is connected it follows that $H^1(F, \mathbf{T}) = 1$ by the remark below Lemma 1.26. Therefore, the long exact sequence of Galois cohomology becomes

$$1 \to H^0(F, \mathrm{Z}(\mathbf{G})) \to H^0(F, \mathbf{T}) \to H^0(F, \mathbf{T}/\mathrm{Z}(\mathbf{G})) \to H^1(F, \mathrm{Z}(\mathbf{G})) \to 1.$$

Thus, we have the exact sequence

$$1 \to \mathrm{Z}(\mathbf{G})^F \to \mathbf{T}^F \to (\mathbf{T}/\mathrm{Z}(\mathbf{G}))^F \to H^1(F, \mathrm{Z}(\mathbf{G})) \to 1,$$

which induces an exact sequence

$$1 \to \mathbf{T}^F/\mathrm{Z}(\mathbf{G})^F \to (\mathbf{T}/\mathrm{Z}(\mathbf{G}))^F \to H^1(F, \mathrm{Z}(\mathbf{G})) \to 1.$$

For $t \in \mathbf{T}$ we have $t\,\mathrm{Z}(\mathbf{G}) \in (\mathbf{T}/\mathrm{Z}(\mathbf{G}))^F$ if and only if $\mathscr{L}(t) \in Z(\mathbf{G})$. This shows that $(\mathbf{T}/\mathrm{Z}(\mathbf{G}))^F = \mathscr{L}_{\mathbf{T}}^{-1}(\mathrm{Z}(\mathbf{G}))/\mathrm{Z}(\mathbf{G})$. So we obtain $H^1(F, \mathrm{Z}(\mathbf{G})) \cong \mathscr{L}_{\mathbf{T}}^{-1}(\mathrm{Z}(\mathbf{G}))/\mathrm{Z}(\mathbf{G})\mathbf{T}^F$, which shows the result of the lemma. □

We will from now on identify $H^1(F, \mathrm{Z}(\mathbf{G}))$ and $\mathscr{L}_{\mathbf{T}}^{-1}(\mathrm{Z}(\mathbf{G}))/\mathrm{Z}(\mathbf{G})\mathbf{T}^F$ under the natural isomorphism constructed in the proof of Lemma 1.27.

Let us now consider an extension $i : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ of $\mathbf{G}$ by a central torus such that the center of $\tilde{\mathbf{G}}$ is connected. An extension with these properties always exists (see remark preceding [12, Lemma 1.3]). We can give a different characterization of the first Galois cohomology group $H^1(F, \mathrm{Z}(\mathbf{G}))$ using such an extension:

**Lemma 1.28.** *Let $i : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be as above. Then there exists an isomorphism $\beta : \tilde{\mathbf{G}}^F/\mathbf{G}^F\,\mathrm{Z}(\tilde{\mathbf{G}})^F \to \mathscr{L}_{\mathbf{T}}^{-1}(\mathrm{Z}(\mathbf{G}))/\,\mathrm{Z}(\mathbf{G})\mathbf{T}^F$.*

*Proof.* Let $\tilde{g} \in \tilde{\mathbf{G}}^F$. Then there exists some $z_{\tilde{g}} \in \mathrm{Z}(\tilde{\mathbf{G}})$ such that $\tilde{g}z_{\tilde{g}} \in \mathbf{G}$. Hence,

$$\mathscr{L}(\tilde{g}z_{\tilde{g}}) = F(z_{\tilde{g}})z_{\tilde{g}}^{-1} = \mathscr{L}(z_{\tilde{g}}) \in \mathbf{G} \cap \mathrm{Z}(\tilde{\mathbf{G}}) = \mathrm{Z}(\mathbf{G}).$$

Thus, we can define a map $\tilde{\mathbf{G}}^F \to \mathrm{Z}(\mathbf{G})/\mathscr{L}(\mathrm{Z}(\mathbf{G}))$ by mapping $\tilde{g} \mapsto \mathscr{L}(z_{\tilde{g}})\mathscr{L}(\mathrm{Z}(\mathbf{G}))$, which yields by [12, Proposition 1.6] an exact sequence

$$1 \to \mathbf{G}^F\,\mathrm{Z}(\tilde{\mathbf{G}})^F \to \tilde{\mathbf{G}}^F \to \mathrm{Z}(\mathbf{G})/\mathscr{L}(\mathrm{Z}(\mathbf{G})) \to 1.$$

The Lang map $\mathscr{L}_{\mathbf{T}} : \mathbf{T} \to \mathbf{T}$ restricts to a map $\mathscr{L}_{\mathbf{T}}^{-1}(\mathrm{Z}(\mathbf{G})) \to \mathrm{Z}(\mathbf{G})$. By [4, Corollary 1.3'] this map gives an exact sequence

$$1 \to \mathrm{Z}(\mathbf{G})\mathbf{T}^F \to \mathscr{L}_{\mathbf{T}}^{-1}(\mathrm{Z}(\mathbf{G})) \to \mathrm{Z}(\mathbf{G})/\mathscr{L}(\mathrm{Z}(\mathbf{G})) \to 1.$$

The isomorphisms $\tilde{\mathbf{G}}^F/\mathbf{G}^F\,\mathrm{Z}(\tilde{\mathbf{G}})^F \to \mathrm{Z}(\mathbf{G})/\mathscr{L}(\mathrm{Z}(\mathbf{G}))$ and $\mathscr{L}_{\mathbf{T}}^{-1}(\mathrm{Z}(\mathbf{G}))/\,\mathrm{Z}(\mathbf{G})\mathbf{T}^F \to \mathrm{Z}(\mathbf{G})/\mathscr{L}(\mathrm{Z}(\mathbf{G}))$, coming from the two exact sequences above, yield an isomorphism

$$\beta : \tilde{\mathbf{G}}^F/\mathbf{G}^F\,\mathrm{Z}(\tilde{\mathbf{G}})^F \to \mathscr{L}_{\mathbf{T}}^{-1}(\mathrm{Z}(\mathbf{G}))/\,\mathrm{Z}(\mathbf{G})\mathbf{T}^F,$$

as desired. □

We will need the proof of Lemma 1.28 in Remark 1.33 below.

## 1.5 Gelfand–Graev characters

As before, we let $\mathbf{G}$ be a connected reductive group with Frobenius $F : \mathbf{G} \to \mathbf{G}$. The aim of this section is to introduce the Gelfand–Graev characters of $\mathbf{G}$. The action of the Frobenius endomorphism $F$ induces an automorphism $\tau$ on the character group $X(\mathbf{T})$ of the torus $\mathbf{T}$. Since $\mathbf{T}$ is maximally split, it follows by [14, Proposition 22.2] that $\tau$ stabilizes the set of positive roots $\Phi^+$ and the set of simple roots $\Delta$. Hence, $\tau$ acts naturally on the index set of $\Delta = \{\alpha_1, \dots \alpha_n\}$. Thus, we have a partition

$$\{1, \dots, n\} = A_1 \cup \cdots \cup A_r$$

of the index set of $\Delta$ into its $\tau$-orbits. For $\alpha \in \Phi$ we denote by $\mathbf{U}_\alpha$ the root subgroup of $\mathbf{G}$ (with respect to $\mathbf{T}$) associated to the root $\alpha \in \Phi$. We denote by $\mathbf{U}_{A_i}$, $i = 1, \ldots, r$, the product in $\mathbf{U}/[\mathbf{U}, \mathbf{U}]$ of the root subgroups $\mathbf{U}_{\alpha_j}$ for $j \in A_i$. By [4, Lemma 2.2] we have

$$\mathbf{U}^F/[\mathbf{U}, \mathbf{U}]^F \cong \prod_{i=1}^{r} \mathbf{U}_{A_i}^F.$$

By Lemma 1.26 we conclude that $\mathbf{U}^F/[\mathbf{U}, \mathbf{U}]^F \cong (\mathbf{U}/[\mathbf{U}, \mathbf{U}])^F$, which implies that $\mathbf{U}^F/[\mathbf{U}, \mathbf{U}]^F$ is abelian. Apart from a few exceptional cases, more is known about the structure of $\mathbf{U}^F/[\mathbf{U}, \mathbf{U}]^F$ if $\mathbf{G}$ is a simple algebraic group:

**Remark 1.29.** If $\mathbf{G}$ is simple and $\mathbf{G}^F$ is not of type $B_2(2)$, $F_4(2)$ or $G_2(3)$ then $[\mathbf{U}, \mathbf{U}]^F$ is even equal to $[\mathbf{U}^F, \mathbf{U}^F]$ (see [7, Lemma 7]). This means that in this case the irreducible characters of $\mathbf{U}^F/[\mathbf{U}, \mathbf{U}]^F$ correspond via inflation precisely to the linear characters of $\mathbf{U}^F$. In order to exploit this fact we will later assume that we are in this situation.

We introduce the notion of regular characters.

**Definition 1.30.** A linear character $\phi \in \mathrm{Irr}(\mathbf{U}^F)$ is called *regular* if $\phi$ is trivial on $[\mathbf{U}, \mathbf{U}]^F$ and if $\phi$ is nontrivial on each $\mathbf{U}_{A_i}^F$.

In order to define the Gelfand–Graev characters of $\mathbf{G}$ we proceed as in the proof of [4, Theorem 2.4]. For each $\alpha \in \Phi$ we have an isomorphism $x_\alpha : (\mathbf{k}, +) \to \mathbf{U}_\alpha$, which satisfies $F(x_\alpha(a)) = x_{\tau\alpha}(a^q)$ for all $a \in \mathbf{k}$ and all $\alpha \in \Phi$. At the beginning of Section 1.7 we will fix such maps $x_\alpha$. These maps induces an isomorphism $x_i : (\mathbb{F}_{q^{|A_i|}}, +) \to \mathbf{U}_{A_i}^F$ given by

$$x_i(a) = \prod_{k=0}^{|A_i|-1} x_{\tau^k \alpha_i}(a^{q^k}),$$

for all $i = 1, \ldots, r$. Now fix a character $\phi_0 \in \mathrm{Irr}((\mathbb{F}_{q^N}, +))$, where $|A_i|$ divides $N$ for all $i = 1, \ldots, r$, such that the restriction of $\phi_0$ to $(\mathbb{F}_q, +)$ is nontrivial. Then any character $\phi_i \in \mathrm{Irr}(\mathbf{U}_{A_i}^F)$ is given by

$$\phi_i(x_i(a)) = \phi_0(c_i a)$$

for all $a \in \mathbb{F}_{q^{|A_i|}}$ and some $c_i \in \mathbb{F}_{q^{|A_i|}}$. Consequently, any irreducible character $\phi \in \mathrm{Irr}((\mathbf{U}/[\mathbf{U}, \mathbf{U}])^F)$ is of the form $\phi = \prod_{i=1}^r \phi_i$ since $(\mathbf{U}/[\mathbf{U}, \mathbf{U}])^F$ is isomorphic to $\prod_{i=1}^r \mathbf{U}_{A_i}^F$. Thus, we have the following lemma:

**Lemma 1.31.** *The map* $\kappa : \mathrm{Irr}((\mathbf{U}/[\mathbf{U}, \mathbf{U}])^F) \to \prod_{i=1}^r \mathbb{F}_{q^{|A_i|}}$ *given by* $\phi = \prod_{i=1}^r \phi_i \mapsto (c_1, ..., c_r)$, *where* $c_i \in \mathbb{F}_{q^{|A_i|}}$ *is such that* $\phi_i(x_i(a)) = \phi_0(c_i a)$ *for all* $a \in \mathbb{F}_{q^{|A_i|}}$, *is a bijection.*

The regular characters of $\mathbf{U}^F$ are precisely those characters which correspond to tuples with non zero entries under this bijection. Note that the parametrization $\kappa$ of the irreducible characters of $\mathbf{U}^F/[\mathbf{U},\mathbf{U}]^F$ depends on the choice of the character $\phi_0$. For the purpose of this thesis we do not have to specify the choice of $\phi_0$.

By [4, 2.4.5] we have $\mathrm{N}_{\mathbf{T}}((\mathbf{U}/[\mathbf{U},\mathbf{U}])^F) = \mathscr{L}_{\mathbf{T}}^{-1}(\mathrm{Z}(\mathbf{G}))$. Therefore $\mathscr{L}_{\mathbf{T}}^{-1}(\mathrm{Z}(\mathbf{G}))$ acts on the set of characters of $(\mathbf{U}/[\mathbf{U},\mathbf{U}])^F$ by conjugation. Moreover, $\mathscr{L}_{\mathbf{T}}^{-1}(\mathrm{Z}(\mathbf{G}))$ acts transitively on the set of regular characters of $\mathbf{U}^F$ (see [4, 2.4.10]).

There exists a unique character $\psi_1$, which corresponds to the tuple $(1, ..., 1)$ under the map $\kappa : \mathrm{Irr}((\mathbf{U}/[\mathbf{U},\mathbf{U}])^F) \to \prod_{i=1}^r \mathbb{F}_{q^{|A_i|}}$. More concretely, the character $\psi_1$ is given by

$$\psi_1 = \prod_{i=1}^r \phi_i, \text{ with } \phi_i(x_i(a)) = \phi_0(a) \text{ for } a \in \mathbb{F}_{q^{|A_i|}}.$$

For $z \in H^1(F, \mathrm{Z}(\mathbf{G})) = \mathscr{L}_{\mathbf{T}}^{-1}(\mathrm{Z}(\mathbf{G}))/\mathrm{Z}(\mathbf{G})\mathbf{T}^F$ we take a representative $t_z \in \mathscr{L}_{\mathbf{T}}^{-1}(\mathrm{Z}(\mathbf{G}))$ and define $\psi_{t_z} = {}^{t_z}\psi_1$. We identify $\psi_{t_z} \in \mathrm{Irr}(\mathbf{U}^F/[\mathbf{U},\mathbf{U}]^F)$ with its inflation $\psi_{t_z} \in \mathrm{Irr}(\mathbf{U}^F)$.

**Definition 1.32.** For $z \in H^1(F, \mathrm{Z}(\mathbf{G}))$ we define the *Gelfand–Graev character* $\Gamma_z$ by $\Gamma_z = \psi_{t_z}^{\mathbf{G}^F}$.

For a fixed element $z \in H^1(F, \mathrm{Z}(\mathbf{G}))$, all $\psi_{t_z}$ are $\mathbf{T}^F$-conjugate. Thus, the definition of the Gelfand–Graev character $\Gamma_z$ depends only on $z$ and not on the representative $t_z$ chosen in the definition of $\Gamma_z$. Note that the characters $\Gamma_z$ are distinct for distinct $z \in H^1(F, \mathrm{Z}(\mathbf{G}))$ (see [4, Scholium 3.6]). Therefore we constructed a complete set of representatives of characters of $\mathbf{G}^F$ which are induced by regular characters of $\mathbf{U}^F$.

**Remark 1.33.** By Lemma 1.28 we have an isomorphism

$$\beta : \tilde{\mathbf{G}}^F/\mathbf{G}^F\,\mathrm{Z}(\tilde{\mathbf{G}})^F \to \mathscr{L}_{\mathbf{T}}^{-1}(\mathrm{Z}(\mathbf{G}))/\mathrm{Z}(\mathbf{G})\mathbf{T}^F.$$

We let $\tilde{g} \in \tilde{\mathbf{G}}^F$ be an element of $\tilde{\mathbf{G}}^F$ such that the coset of $\tilde{g}$ in $\tilde{\mathbf{G}}^F/\mathbf{G}^F\,\mathrm{Z}(\tilde{\mathbf{G}})\mathbf{T}^F$ is mapped to $z \in H^1(F, \mathrm{Z}(\mathbf{G})) = \mathscr{L}_{\mathbf{T}}^{-1}(\mathrm{Z}(\mathbf{G}))/\mathrm{Z}(\mathbf{G})\mathbf{T}^F$ via the map $\beta$. Let us define $g_z := \tilde{g}$. For $\tilde{g} \in \tilde{\mathbf{G}}^F$ we find $z_{\tilde{g}} \in \mathrm{Z}(\tilde{\mathbf{G}})$ such that $\tilde{g}z_{\tilde{g}} \in \mathbf{G}$, as in the proof of Lemma 1.28. Moreover, since the torus $\mathbf{T}$ is connected, it follows by [14, Theorem 21.7] that the Lang map $\mathscr{L}_{\mathbf{T}} : \mathbf{T} \to \mathbf{T}$ is surjective. Since $\mathscr{L}(z_{\tilde{g}}) \in \mathrm{Z}(\mathbf{G})$ by the proof of Lemma 1.28, we find some $t_z \in \mathscr{L}_{\mathbf{T}}^{-1}(\mathrm{Z}(\mathbf{G}))$ such that $\mathscr{L}(z_{\tilde{g}}) = \mathscr{L}(t_z)$. By the construction of the map $\beta : \tilde{\mathbf{G}}^F/\mathbf{G}^F\,\mathrm{Z}(\tilde{\mathbf{G}})^F \to \mathscr{L}_{\mathbf{T}}^{-1}(\mathrm{Z}(\mathbf{G}))/\mathrm{Z}(\mathbf{G})\mathbf{T}^F$ in Lemma 1.28 it follows that

$\beta$ maps the coset of $g_z$ to the coset of $t_z$. Since $\mathscr{L}(z_{\tilde{g}}) = \mathscr{L}(t_z)$, we conclude that $t_z z_{\tilde{g}}^{-1} \in \tilde{\mathbf{G}}^F$. We have $t_z z_{\tilde{g}}^{-1} z_{\tilde{g}} = t_z \in \mathbf{T} \subseteq \mathbf{G}$. Thus, it follows by the definition of the map $\tilde{\mathbf{G}}^F \to \mathrm{Z}(\mathbf{G})/\mathscr{L}(\mathrm{Z}(\mathbf{G}))$ in the proof of Lemma 1.28 that $t_z z_{\tilde{g}}^{-1}$ and $g_z = \tilde{g}$ define the same coset in $\tilde{\mathbf{G}}^F/\mathbf{G}^F\,\mathrm{Z}(\tilde{\mathbf{G}})^F$. Note that $\mathbf{G}^F\,\mathrm{Z}(\tilde{\mathbf{G}})^F \subseteq \tilde{\mathbf{G}}^F$ acts trivial by conjugation on characters of $\tilde{\mathbf{G}}^F$. As a consequence we conclude that

$$\Gamma_z = \psi_{t_z}^{\mathbf{G}^F} = {}^{t_z}\Gamma_1 = {}^{t_z z_{\tilde{g}}^{-1}}\Gamma_1 = {}^{g_z}\Gamma_1.$$

Hence, we can define Gelfand–Graev characters using the isomorphism $\beta : \tilde{\mathbf{G}}^F/\mathbf{G}^F\,\mathrm{Z}(\tilde{\mathbf{G}})^F \to \mathscr{L}_{\mathbf{T}}^{-1}(\mathrm{Z}(\mathbf{G}))/\,\mathrm{Z}(\mathbf{G})\mathbf{T}^F = H^1(F, \mathrm{Z}(\mathbf{G}))$.

## 1.6 The $p'$-characters of $\mathbf{G}^F$

We want to describe the $p'$-characters of $\mathbf{G}^F$. For this, let us temporarily assume that the center of the algebraic group $\mathbf{G}$ is connected. In this case we have $H^1(F, \mathrm{Z}(\mathbf{G})) = 1$ by the remark below Lemma 1.26. Thus, the group $\mathbf{G}$ has a unique Gelfand–Graev character, which we will denote by $\Gamma$.

Moreover, we call a character $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ *semisimple* if $(\chi, \mathrm{D}_{\mathbf{G}}(\Gamma)) \neq 0$ (see proof of [2, Proposition 8.3.7]). We recall the definition of a good prime (see remark below [5, Proposition 14.17]).

**Definition 1.34.** A prime $p$ is called *bad* (i.e., *not good*) for $\mathbf{G}$ if the root system of $\mathbf{G}$ is of type $B_n$, $C_n$ or $D_n$ if $p = 2$, or of type $G_2$, $F_4$, $E_6$, $E_7$ if $p = 2, 3$ and of type $E_8$ if $p = 2, 3, 5$.

The semisimple characters are precisely the $p'$-characters if the prime $p$ is good for the algebraic group $\mathbf{G}$.

**Theorem 1.35** (Green, Lehrer, Lusztig)**.** *Let $\mathbf{G}$ be a connected reductive group with connected center. Let $p$ be a good prime for the group $\mathbf{G}$. Then the character $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ is semisimple if and only if it is of $p'$-degree.*

*Proof.* This is a consequence of [2, Proposition 8.3.4]. □

Using this theorem, we can prove the following lemma.

**Lemma 1.36.** *Let $\mathbf{G}$ be a connected reductive algebraic group with connected center. Let $p$ be a good prime for the group $\mathbf{G}$. Then $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ is a $p'$-character if and only if $\chi = \varepsilon\,\mathrm{D}_{\mathbf{G}}(\chi_{(\tilde{s})})$ for some semisimple conjugacy class $(\tilde{s})$ of ${\mathbf{G}^*}^{F^*}$ and some sign $\varepsilon \in \{\pm 1\}$.*

*Proof.* Since $\mathbf{G}$ has connected center, the unique Gelfand–Graev character $\Gamma$ of $\mathbf{G}$ can be written as

$$\Gamma = \sum_{(s)} \chi_{(s)},$$

where $(s)$ runs over the semisimple conjugacy classes of $\mathbf{G}^{*F^*}$. This is a decomposition of the Gelfand–Graev character into distinct irreducible characters (see [5, Corollary 14.47]). By applying the duality map $\mathrm{D}_{\mathbf{G}}$, we obtain

$$\mathrm{D}_{\mathbf{G}}(\Gamma) = \sum_{(s)} \mathrm{D}_{\mathbf{G}}(\chi_{(s)}).$$

Let $\chi$ be an irreducible constituent of $\mathrm{D}_{\mathbf{G}}(\Gamma)$. Then there exists a semisimple conjugacy class $(s)$ of $\mathbf{G}^{*F^*}$ such that $(\mathrm{D}_{\mathbf{G}}(\chi_{(s)}), \chi) \neq 0$. Since $\chi_{(s)}$ is irreducible it follows that $\mathrm{D}_{\mathbf{G}}(\chi_{(s)})$ is an irreducible character up to a sign (see remark below Theorem 1.24). Consequently we must have $\chi = \varepsilon\, \mathrm{D}_{\mathbf{G}}(\chi_{(s)})$, where $\varepsilon \in \{\pm 1\}$. By Theorem 1.35 the irreducible constituents of $\mathrm{D}_{\mathbf{G}}(\Gamma)$ are precisely the irreducible $p'$-characters of $\mathbf{G}^F$. □

Thus, we have a description for the $p'$-characters of $\mathbf{G}^F$ if the center of $\mathbf{G}$ is connected and $p$ is a good prime for $\mathbf{G}$.

Let us now assume that the center of $\mathbf{G}$ is not necessarily connected. As in Section 1.4 we consider an extension $i : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ of $\mathbf{G}$ by a central torus such that the center of $\tilde{\mathbf{G}}$ is connected. The embedding $i : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ gives rise to a dual morphism $i^* : \tilde{\mathbf{G}}^* \to \mathbf{G}^*$. The map $i^*$ is surjective and has central kernel (see [1, Section 15.1]). We have a map from semisimple conjugacy classes of $\tilde{\mathbf{G}}^{*F^*}$ to semisimple conjugacy classes of $\mathbf{G}^{*F^*}$. More concretely, if $(\tilde{s})$ is a semisimple conjugacy class of $\tilde{\mathbf{G}}^{*F^*}$, it follows that $(s)$ is a semisimple conjugacy class of $\mathbf{G}^{*F^*}$, where $s := i^*(\tilde{s})$.

By [4, Proposition 3.10], we have $(\chi_{(s)}, \Gamma_z) = 1$, for all $z \in H^1(F, \mathrm{Z}(\mathbf{G}))$. Thus, there exists a unique common irreducible constituent of $\Gamma_z$ and $\chi_{(s)}$, which we will denote by $\chi_{(s),z}$. Moreover, by [4, 3.15.1] we have

$$(\chi_{(\tilde{s})})_{\mathbf{G}^F} = \chi_{(s)} = \sum_{z} \chi_{(s),z},$$

where the sum is over the distinct characters $\chi_{(s),z}$ for $z \in H^1(F, \mathrm{Z}(\mathbf{G}))$.

An important fact is that the duality functor commutes with restriction of characters in our situation. For later reference we state this in the following lemma.

**Lemma 1.37.** *Let $\mathbf{G}$ be a connected reductive group and $i : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ an extension of $\mathbf{G}$ by a central torus such that $\tilde{\mathbf{G}}$ has connected center. Then $\mathrm{D}_{\tilde{\mathbf{G}}}(\chi)_{\mathbf{G}^F} = \mathrm{D}_{\mathbf{G}}(\chi_{\mathbf{G}^F})$ for any $\chi \in \mathbb{Z}\,\mathrm{Irr}(\tilde{\mathbf{G}}^F)$.*

*Proof.* This is mentioned at the beginning of page 172 of [4]. □

We can now prove a similar result as in Lemma 1.36.

**Lemma 1.38.** *Let* $\mathbf{G}$ *be a connected reductive group. Let* $i : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ *and* $i^* : \tilde{\mathbf{G}}^* \to \mathbf{G}^*$ *be as above. Let* $p$ *be a good prime for* $\tilde{\mathbf{G}}$ *such that any* $p'$*-character* $\chi \in \mathrm{Irr}_{p'}(\tilde{\mathbf{G}}^F)$ *is of the form* $\chi = \varepsilon\,\mathrm{D}_{\tilde{\mathbf{G}}}(\chi_{(\tilde{s})})$ *for some semisimple conjugacy class* $(\tilde{s})$ *of* $\tilde{\mathbf{G}}^{*F^*}$ *and some sign* $\varepsilon \in \{\pm 1\}$*. Then the irreducible constituents of* $\chi_{\mathbf{G}^F}$ *are precisely the characters* $\psi_{(s),z} := \varepsilon\,\mathrm{D}_{\mathbf{G}}(\chi_{(s),z})$ *for* $s = i^*(\tilde{s})$ *and* $z \in H^1(F, \mathrm{Z}(\mathbf{G}))$*.*

*Proof.* Recall from the beginning of this section that we have

$$\chi_{(\tilde{s})_{\mathbf{G}^F}} = \chi_{(s)} = \sum_z \chi_{(s),z},$$

where the sum is over a complete set of distinct characters $\chi_{(s),z}$, $z \in H^1(F, Z(\mathbf{G}))$. Since the duality functor commutes with restriction by Lemma 1.37 we have

$$\chi_{\mathbf{G}^F} = (\varepsilon\,\mathrm{D}_{\tilde{\mathbf{G}}}(\chi_{(\tilde{s})}))_{\mathbf{G}^F} = \varepsilon\,\mathrm{D}_{\mathbf{G}}(\chi_{(s)}) = \sum_z \varepsilon\,\mathrm{D}_{\mathbf{G}}(\chi_{(s),z}).$$

Using the fact that $\mathrm{D}_{\mathbf{G}}$ is an isometry (see remark below Theorem 1.24) it follows that this is a decomposition into distinct irreducible characters. □

## 1.7 Steinberg presentation

In the remaining part of Chapter 1 we will introduce the special setup which we want to consider in Chapter 2 and Chapter 3. Thus, we keep the notations and definitions introduced in the following sections for the remainder of this thesis. In this section we describe the Steinberg presentation of a simple algebraic group of simply connected type as introduced in [19].

Let $\Phi$ be an abstract indecomposable root system. Let $\Delta = \{\alpha_1, ..., \alpha_n\}$ be a base of the root system $\Phi$. We denote by $\Phi^+$ the set of positive roots, i.e. the subset of the root system $\Phi$ which consists of the roots which can be written as a linear combination of the simple roots with natural numbers as coefficients. We write $\Phi^\vee$ for the set of coroots of $\Phi$ with base given by $\Delta^\vee = \{\alpha_1^\vee, ..., \alpha_n^\vee\}$. We assume that $\Phi$ has at least rank 2 (i.e., $\Phi$ is not of type $A_1$). We consider the group $\mathbf{G}$ generated by the set of symbols $\{x_\alpha(t) \mid \alpha \in \Phi, t \in \mathbf{k}\}$ subject to the following relations:

1. $x_\alpha(t_1)x_\alpha(t_2) = x_\alpha(t_1 + t_2)$ for all $t_1$, $t_2 \in \mathbf{k}$ and $\alpha \in \Phi$.

2. Let $\alpha, \beta \in \Phi$ with $\alpha + \beta \neq 0$. Then
$$[x_\alpha(t_1), x_\beta(t_2)] = \prod_{i,j>0, i\alpha+j\beta\in\Phi} x_{i\alpha+j\beta}(c_{ij\alpha\beta} t_2^i t_1^j),$$
where the product is taken over a fixed order of the roots $\Phi$ and $c_{i,j,\alpha,\beta} \in \{\pm 1, \pm 2, \pm 3\}$ are as in [19, Lemma 15] (where the $c_{i,j,\alpha,\beta}$ possibly depend on the chosen order).

3. $h_\alpha(t_1)h_\alpha(t_2) = h_\alpha(t_1 t_2)$ for all $t_1$, $t_2 \in \mathbf{k}^\times$, where $h_\alpha(t) := w_\alpha(t)w_\alpha(-1)$ and $w_\alpha(t) := x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$ for $t \in \mathbf{k}^\times$.

Steinberg shows that the group $\mathbf{G}$ is the universal Chevalley group constructed from $\Phi$ and $\mathbf{k}$ (see [19, Theorem 8]). Furthermore, he shows that $\mathbf{G}$ can be given the structure of a simple algebraic group in a natural way such that $\mathbf{G}$ becomes a linear algebraic group over $\mathbf{k}$ of simply connected type with root system (isomorphic to) $\Phi$ (see [19, Theorem 6] and the Existence Theorem in [19, Chapter 5]). We summarize the structural information for the algebraic group $\mathbf{G}$ in the following lemma.

**Lemma 1.39.** *Let $\mathbf{G}$ be defined as above. Then*
*(a)* $\mathbf{T} = \{h_\alpha(t) \mid \alpha \in \Phi^+, t \in \mathbf{k}^\times\}$ *is a maximal torus of* $\mathbf{G}$.
*(b)* $\mathbf{U}_\alpha = \{x_\alpha(t) \mid t \in \mathbf{k}\}$ *for* $\alpha \in \Phi$ *are the root subgroups of* $\mathbf{G}$ *(relative to the* $\mathbf{T}$*). In particular, we have an isomorphism* $x_\alpha : (\mathbf{k}, +) \to \mathbf{U}_\alpha$ *of algebraic groups given by* $t \mapsto x_\alpha(t)$.
*(c)* $\mathbf{U} = \prod_{\alpha\in\Phi^+} \mathbf{U}_\alpha$ *is a maximal connected unipotent subgroup of* $\mathbf{G}$.
*(d)* $\mathbf{B} = \mathbf{TU} = \mathrm{N}_{\mathbf{G}}(\mathbf{U})$ *is a Borel subgroup of* $\mathbf{G}$.

*Proof.* See [15, Section 1]. □

Furthermore we write (by abuse of notation) $\Phi \subseteq X(\mathbf{T})$ for the set of roots with respect to the torus $\mathbf{T}$. Moreover $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ will denote the base of $\Phi$ with respect to $\mathbf{T} \subseteq \mathbf{B}$.

## 1.8 A regular embedding

Fix an indecomposable root system $\Phi$. Let $\mathbf{G}$ be the simply connected group of root system $\Phi$ defined over the field $\mathbf{k}$ of characteristic $p$, as in the previous section. Note that the center of $\mathbf{G}$ is a finite subgroup $\mathrm{Z}(\mathbf{G}) = \bigcap_{i=1}^n \ker(\alpha_i)$ (see [14, Theoren 8.17]). We let $d_p$ be the minimal number of generators of $\mathrm{Z}(\mathbf{G})$. Note that $d_p = 0$ if and only if $\mathrm{Z}(\mathbf{G})$ is trivial which is precisely the case when the center of $\mathbf{G}$ is connected (see [14, Proposition 1.13 (c)]). For a root system $\Phi$ we let $d$ be the maximal $d_p$ occurring for any prime $p$. We list the possible numbers for $d$ in the following remark.

**Remark 1.40.** By [14, Proposition 9.15] the center $\mathrm{Z}(\mathbf{G})$ of a simple algebraic group $\mathbf{G}$ of simply connected type with root system $\Phi$ is isomorphic to the $p'$-part of the fundamental group $\Lambda(\Phi)$. The fundamental group for the various types of root systems is listed in [14, Table 9.2]. Note that $\Lambda(\Phi)$ is cyclic except in case where the root system $\Phi$ is of type $D_n$ and $n$ is even. In this case the fundamental group of $\Phi$ is $\Lambda(\Phi) \cong C_2 \times C_2$, where $C_2$ denotes the cyclic group of order 2. Thus, we have $d = 2$ if the root system of $\mathbf{G}$ is of type $D_n$ and $n$ is even. In the remaining cases we have either $d = 1$ or $d = 0$. Note that $d = 0$ only occurs in a few exceptional cases.

We will restrict ourselves from now on to the case that $d = 1$ or $d = 0$. This way, we avoid a lot of technical issues later.

**Assumption 1.41.** *We assume from now on that the root system $\Phi$ of $\mathbf{G}$ is not of type $D_n$ if $n$ is even.*

Let $\mathbf{S} = (\mathbf{k}^\times)^d$ be a torus of rank $d$. Let $\rho : \mathrm{Z}(\mathbf{G}) \to \mathbf{S}$ be an injective group homomorphism and define a group $\tilde{\mathbf{G}}$ by

$$\tilde{\mathbf{G}} = \mathbf{G} \times_\rho \mathbf{S} = (\mathbf{G} \times \mathbf{S})/\{(z, \rho(z)^{-1}) \mid z \in \mathrm{Z}(\mathbf{G})\}.$$

We have natural embeddings $i : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ and $j : \mathbf{S} \hookrightarrow \tilde{\mathbf{G}}$. As such it is often convenient to identify $\mathbf{G}$ and $\mathbf{S}$ with their images in $\tilde{\mathbf{G}}$ under these embeddings. Under this identification $\tilde{\mathbf{G}} = \mathbf{GS}$ has connected center $\mathrm{Z}(\tilde{\mathbf{G}}) = \mathbf{S}$. Moreover, $\tilde{\mathbf{G}}$ is an extension of $\mathbf{G}$ by the central torus $\mathbf{S}$. Again we list some structural information on the group $\tilde{\mathbf{G}}$ in the following lemma.

**Lemma 1.42.** *Let $\tilde{\mathbf{G}}$ be defined as above. Then the following statements hold.*
*(a)* $\tilde{\mathbf{T}} = \mathbf{TS}$ *is a maximal torus of* $\tilde{\mathbf{G}}$.
*(b)* $\mathbf{U} = \prod_{\alpha \in \Phi^+} \mathbf{U}_\alpha$ *is a maximal connected unipotent subgroup of* $\tilde{\mathbf{G}}$.
*(c)* $\tilde{\mathbf{B}} = \tilde{\mathbf{T}}\mathbf{U} = \mathrm{N}_{\tilde{\mathbf{G}}}(\mathbf{U})$ *is a Borel subgroup of* $\tilde{\mathbf{G}}$.

*Proof.* These facts are mentioned in [15, Section 2] □

Note that the construction of $\tilde{\mathbf{G}}$ depends on the choice of $\rho$. This will become crucial in the next section when we discuss extensions of Frobenius automorphisms of $\mathbf{G}$ to the group $\tilde{\mathbf{G}}$. In [15, Section 6] explicit choices are made which we assume to be taken in this thesis.

## 1.9 Frobenius endomorphism

As in [15, Section 3] we want to explicitly construct the Frobenius endomorphisms of $\mathbf{G}$. Note that we do not consider twisted Frobenius endomorphisms, i.e. endomorphisms of $\mathbf{G}$ which are not Frobenius endomorphisms of $\mathbf{G}$, but whose square or cube is a Frobenius endomorphism. This means that we will not consider Ree or Suzuki groups in this thesis.

We define $F_q : \mathbf{G} \to \mathbf{G}$ by $x_\alpha(t) \mapsto x_\alpha(t^q)$ for all $\alpha \in \Phi$ and $t \in \mathbf{k}$. This defines a homomorphism of algebraic groups which we call the *standard Frobenius map*. Let $\tau$ be a symmetry of the Dynkin diagram associated to the base $\Delta$ of the root system $\Phi$. Then $\tau$ can be used to realize a graph automorphism $\gamma : \mathbf{G} \to \mathbf{G}$. We can choose $\gamma$ such that $\gamma(x_{\alpha_i}(t)) = x_{\tau(\alpha_i)}(t)$ for $\alpha_i \in \Delta$ and $t \in \mathbf{k}$. In particular the order of $\gamma$ coincides with the order $w$ of the symmetry $\tau$. Any Frobenius endomorphism for the group $\mathbf{G}$ with respect to an $\mathbb{F}_q$-structure is then, up to inner automorphisms of $\mathbf{G}$, given by a map $F : \mathbf{G} \to \mathbf{G}$ such that $F = F_q \circ \gamma$ for a standard Frobenius map $F_q$ and a graph automorphism $\gamma$ of $\mathbf{G}$ (see [14, Theorem 22.5]). We now state that $\tilde{\mathbf{G}}$ is a regular embedding in the sense of [1, Hypothesis 15.1].

**Lemma 1.43.** *Let $i : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be chosen as in [15, Section 6]. Then $\mathbf{G}$ is a regular embedding in the sense of [1, Hypothesis 15.1]. This means $\tilde{\mathbf{G}}$ is a connected algebraic group with connected center and $[\tilde{\mathbf{G}}, \tilde{\mathbf{G}}] = \mathbf{G}$ and for every Frobenius morphism $F$ of $\mathbf{G}$ there exists a Frobenius morphism of $\tilde{\mathbf{G}}$ which is an extension of $F$.*

**Definition 1.44.** Write $g \in \tilde{\mathbf{G}}$ as $g = xz$ with $x \in \mathbf{G}$ and $z \in \mathbf{S}$. We define the *determinant map* $\det : \tilde{\mathbf{G}} \to \mathbf{k}^\times$ to be the map $\det(xz) = z^{|\Lambda(\mathbf{G})|}$ where $\Lambda(\mathbf{G})$ is the fundamental group of $\mathbf{G}$. Note that the map det is a well-defined homomorphism of algebraic groups (see the remark below [15, Definition 7.2]).

The following lemma illustrates an application of Lemma 1.26.

**Lemma 1.45.** *The map* det *induces isomorphisms* $\tilde{\mathbf{B}}^F/\mathbf{B}^F \cong \tilde{\mathbf{G}}^F/\mathbf{G}^F \cong \mathbf{S}^F$.

*Proof.* The map $\det : \tilde{\mathbf{G}}/\mathbf{G} \to (\mathbf{k}^\times)^d = \mathbf{S}$ is an isomorphism. Moreover, the restriction of det to $\tilde{\mathbf{B}}$ induces an isomorphism $\det : \tilde{\mathbf{B}}/\mathbf{B} \to (\mathbf{k}^\times)^d = \mathbf{S}$ as well (see remark below [15, Proposition 11.3]). The groups $\mathbf{B}$ and $\mathbf{G}$ are connected $F$-stable normal subgroups of $\tilde{\mathbf{B}}$ resp. $\tilde{\mathbf{G}}$. Thus, it follows by Lemma 1.26 that these isomorphisms carry over to the fixed points under the Frobenius endomorphism $F$. □

Note that $\mathbf{S}^F$ is cyclic as finite subgroup of $\mathbf{S} = \mathbf{k}^\times$. Moreover, the group $\mathbf{S}^F$ consists of semisimple elements, which implies that every element of $\mathbf{S}^F$ has order coprime to $p$ (see [5, Proposition 3.18]). This implies that $p \nmid |\tilde{\mathbf{G}}^F : \mathbf{G}^F|$ and $p \nmid |\tilde{\mathbf{B}}^F : \mathbf{B}^F|$ by Lemma 1.45. We list some more properties of the finite groups $\mathbf{G}^F$ and $\tilde{\mathbf{G}}^F$ in the following lemma.

**Lemma 1.46.** *(a)* $\mathbf{U}^F$ *is a Sylow* $p$*-subgroup of* $\mathbf{G}^F$ *and* $\tilde{\mathbf{G}}^F$.
*(b)* $\mathbf{B}^F = \mathrm{N}_{\mathbf{G}^F}(\mathbf{U}^F)$ *and* $\tilde{\mathbf{B}}^F = \mathrm{N}_{\tilde{\mathbf{G}}^F}(\mathbf{U}^F)$.

*Proof.* The proof of part (a) is given in [15, Proposition 3.4]. Part (b) follows by [14, Corollary 24.11]. □

## 1.10 The dual group

Let $\mathbf{G}$ be a simple algebraic group of simply connected type with root system $\Phi$. Furthermore, let $i : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be the regular embedding of $\mathbf{G}$ constructed in Section 1.8. We give an explicit construction of the dual algebraic group of $\tilde{\mathbf{G}}$, following the construction in [15, Section 7]. Let $\mathbf{G}^\vee$ be the simple algebraic group of simply connected group with root system $\Phi^\vee$ (see Section 1.7). We denote by $\mathbf{T}^\vee$ the maximal torus of $\mathbf{G}^\vee$ as in Lemma 1.39. Note that $\mathbf{G}$ and $\mathbf{G}^\vee$ are not necessarily dual to each other (the dual algebraic group of a simply connected group is not simply connected in general). Let $\mathbf{S}^\vee$ be the torus associated to $\mathbf{G}^\vee$ as in Section 1.8. Then we choose an injective group homomorphism $\rho^\vee : \mathrm{Z}(\mathbf{G}^\vee) \to \mathbf{S}^\vee$ as in [15, Section 7]. We denote by $\tilde{\mathbf{G}}^*$ the resulting linear algebraic group

$$\tilde{\mathbf{G}}^* = \mathbf{G}^\vee \times_{\rho^\vee} \mathbf{S}^\vee = (\mathbf{G}^\vee \times \mathbf{S}^\vee)/\{(z, \rho^\vee(z)^{-1}) \mid z \in \mathrm{Z}(\mathbf{G}^\vee)\}.$$

As in Section 1.8 there are embeddings $i^\vee : \mathbf{G}^\vee \hookrightarrow \tilde{\mathbf{G}}^*$ and $j^\vee : \mathbf{S}^\vee \hookrightarrow \tilde{\mathbf{G}}^*$ such that we may write $\tilde{\mathbf{G}}^* = \mathbf{G}^\vee \mathbf{S}^\vee$. Let us furthermore denote by $\tilde{\mathbf{T}}^*$ the maximal torus $\tilde{\mathbf{T}}^* = \mathbf{T}^\vee \mathbf{S}^\vee$ of $\tilde{\mathbf{G}}^*$ as constructed in Lemma 1.42. The notation $\tilde{\mathbf{G}}^*$ is justified in the next lemma.

**Lemma 1.47.** *Let* $\tilde{\mathbf{G}}$ *and* $\tilde{\mathbf{G}}^*$ *be defined as above. Then there exists a duality isomorphism* $\delta : Y(\tilde{\mathbf{T}}) \to X(\tilde{\mathbf{T}}^*)$. *Furthermore, for any Frobenius map* $F$ *of* $\tilde{\mathbf{G}}$ *there exists a Frobenius map* $F^*$ *of* $\tilde{\mathbf{G}}^*$ *such that the pair* $(\tilde{\mathbf{G}}, F)$ *is dual to the pair* $(\tilde{\mathbf{G}}^*, F^*)$ *(via the duality isomorphism* $\delta$*).*

*Proof.* This is [15, Proposition 7.6]. □

The proof makes use the so-called fundamental and dual fundamental weights. We will need this construction later so we will give a brief summary of how the duality isomorphism $\delta$ is constructed. Let us first introduce the notion of fundamental weights in the following lemma.

**Lemma 1.48.** *Let* $\mathbf{G}$ *be a simple algebraic group of simply connected type. Then the coroots* $\omega_i \in Y(\mathbf{T})$ *defined in [15, Section 1] satisfy* $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$ *for all* $i = 1, ..., n$ *and* $j = 1, ..., n$. *The* $\omega_i$ *are called the fundamental weights of* $\mathbf{G}$.

*Proof.* This is discussed in [15, Section 1]. □

Recall that $\tilde{\mathbf{T}} = \mathbf{TS}$. Since $\mathbf{S}$ is a central subgroup of $\tilde{\mathbf{G}}$ we may define $\tilde{\alpha} \in X(\tilde{\mathbf{T}})$ to be the unique extensions of a root $\alpha \in X(\mathbf{T})$ which is trivial on $\mathbf{S}$. Note that the lifts $\tilde{\omega}_i$ of $\omega_i$ satisfy the equation $\langle \tilde{\omega}_i, \tilde{\alpha}_j \rangle = \delta_{ij}$. Let us furthermore denote the extensions of the fundamental weights of $\mathbf{G}^\vee$ by $\tilde{\omega}_i^* \in X(\tilde{\mathbf{T}}^*)$.

An important property of the algebraic group $\tilde{\mathbf{G}}$ is that it has so-called dual fundamental weights which we introduce in the following lemma.

**Lemma 1.49.** *Then the coroots* $\tau_i \in Y(\tilde{\mathbf{T}})$ *defined in [15, Definition 6.10] satisfy* $\langle \tilde{\alpha}_i, \tau_j \rangle = \delta_{ij}$ *for all* $i = 1, \ldots, n$ *and* $j = 1, \ldots, n$. *The elements* $\tau_i \in Y(\tilde{\mathbf{T}})$ *are called dual fundamental weights of* $\tilde{\mathbf{G}}$.

*Proof.* The proof is given in [15, Proposition 6.11]. □

We denote by $\mathbf{z} : \mathbf{k}^\times \to \tilde{\mathbf{T}}$ the map $j : \mathbf{S} = (\mathbf{k}^\times)^d \hookrightarrow \tilde{\mathbf{G}}$ with codomain $\tilde{\mathbf{T}}$. The map $\mathbf{z}$ is an element of the cocharacter group $Y(\tilde{\mathbf{T}})$. Recall the determinant map $\det : \tilde{\mathbf{T}} \to \mathbf{k}^\times$ from Definition 1.44. We denote by $\det^* : \tilde{\mathbf{T}}^* \to \mathbf{k}^\times$ the corresponding map of the dual group $\tilde{\mathbf{G}}^* = \mathbf{G}^\vee \mathbf{S}^\vee$. These definitions allow us to state the definition of the duality isomorphism $\delta : Y(\tilde{\mathbf{T}}) \to X(\tilde{\mathbf{T}}^*)$.

**Remark 1.50.** The duality isomorphism $\delta : Y(\tilde{\mathbf{T}}) \to X(\tilde{\mathbf{T}}^*)$ in Lemma 1.47 is given by $\tau_i \mapsto \tilde{\omega}_i^*$ for $i = 1, \ldots, n$ and by $\mathbf{z} \mapsto \det^*$.

# Chapter 2

# The Maslowski bijection

In this chapter we describe the Maslowski bijection $\tilde{f} : \mathrm{Irr}_{p'}(\tilde{\mathbf{B}}^F) \to \mathrm{Irr}_{p'}(\tilde{\mathbf{G}}^F)$ constructed in [15, Theorem 15.1]. This bijection is given by parameterizing both sets of $p'$-characters of $\tilde{\mathbf{G}}^F$ and $\tilde{\mathbf{B}}^F$ by the same set of labels. In Chapter 3 we use the Maslowski bijection in order to construct a bijection $f : \mathrm{Irr}_{p'}(\mathbf{B}^F)^\sigma \to \mathrm{Irr}_{p'}(\mathbf{G}^F)^\sigma$.

## 2.1 A labeling for the $p'$-characters of $\tilde{\mathbf{B}}^F$

Let us first recall some notation and definitions which we introduced in Section 1.7 to Section 1.10. Let $\mathbf{G}$ be a simple algebraic group of simply connected type. Let $\Phi$ be the root system of $\mathbf{G}$ with respect to the torus $\mathbf{T}$ (where $\mathbf{T}$ is the maximal torus of $\mathbf{G}$ defined in Lemma 1.39). As in Section 1.9 we let $F = F_q \circ \gamma : \mathbf{G} \to \mathbf{G}$ be a Frobenius endomorphism of $\mathbf{G}$. The torus $\mathbf{T}$ is $F$-stable and is contained in the $F$-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$, i.e. the torus $\mathbf{T}$ is maximally split. Furthermore, we denote by $\Delta$ the base of the root system $\Phi$ with respect to $\mathbf{T} \subseteq \mathbf{B}$. Note that $\mathbf{U} = \prod_{\alpha \in \Phi^+} \mathbf{U}_\alpha$ is the unipotent radical of $\mathbf{B}$. Suppose that the graph automorphism $\gamma : \mathbf{G} \to \mathbf{G}$ comes from the symmetry $\tau$ of the Dynkin diagram associated to the base $\Delta$ of the root system $\Phi$. The symmetry $\tau$ acts on the set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. This action induces a partition

$$\{1, \ldots, n\} = A_1 \cup \cdots \cup A_r$$

of the index set of $\Delta$. We now recall some results from Section 1.5. We have an isomorphism

$$\mathbf{U}^F/[\mathbf{U}, \mathbf{U}]^F \cong \prod_{i=1}^{r} \mathbf{U}^F_{A_i},$$

where $\mathbf{U}_{A_i}$, $i = 1, \dots, r$ is the product in $\mathbf{U}/[\mathbf{U}, \mathbf{U}]$ of the root subgroups $\mathbf{U}_{\alpha_j}$ for $j \in A_i$. We assume furthermore that $[\mathbf{U}^F, \mathbf{U}^F] = [\mathbf{U}, \mathbf{U}]^F$, i.e. we exclude the groups mentioned in Remark 1.29. Consequently, in this case we have

$$\mathbf{U}^F/[\mathbf{U}^F, \mathbf{U}^F] \cong \prod_{i=1}^{r} \mathbf{U}_{A_i}^F.$$

Let us briefly recall the construction of the bijection $\kappa$ as explained before Lemma 1.31. We fix a character $\phi_0 \in \mathrm{Irr}((\mathbb{F}_{q^N}, +))$, where $|A_i|$ divides $N$ for all $i = 1, \dots, r$, such that the restriction of $\phi_0$ to $(\mathbb{F}_q, +)$ is nontrivial. For $\alpha \in \Phi$, recall the isomorphism $x_\alpha : (\mathbf{k}, +) \to \mathbf{U}_\alpha$ from Lemma 1.39. These maps yield isomorphisms $x_i : (\mathbb{F}_{q^{|A_i|}}, +) \to \mathbf{U}_{A_i}^F$ for all $i = 1, \dots, r$. For any character $\phi_i \in \mathrm{Irr}(\mathbf{U}_{A_i}^F)$ there exists some $c_i \in \mathbb{F}_{q^{|A_i|}}$ such that $\phi_i(x_i(a)) = \phi_0(c_i a)$ for all $a \in \mathbb{F}_{q^{|A_i|}}$. By Lemma 1.31 we obtain a bijection

$$\kappa : \mathrm{Irr}(\mathbf{U}^F/[\mathbf{U}^F, \mathbf{U}^F]) \to \prod_{i=1}^{r} (\mathbb{F}_{q^{|A_i|}}, +) \text{ given by } \phi = \prod_{i=1}^{r} \phi_i \mapsto (c_1, ..., c_r).$$

Let $S$ be a subset of $\{1, \dots, r\}$. We denote $S^c = \{0, 1, \dots, r\} \backslash S$. Define the character $\phi_S$ of $\mathbf{U}^F/[\mathbf{U}^F, \mathbf{U}^F]$ to be the character which corresponds under $\kappa$ to the tuple $(c_1, ..., c_r)$ with

$$c_i = \begin{cases} 0 & \text{if } i \notin S, \\ 1 & \text{if } i \in S. \end{cases}$$

For simplicity let us identify $\phi_S \in \mathrm{Irr}(\mathbf{U}^F/[\mathbf{U}^F, \mathbf{U}^F])$ with its inflation $\phi_S \in \mathrm{Irr}(\mathbf{U}^F)$. Note that with this notation the character $\psi_1$ introduced before Definition 1.32 is precisely the character $\phi_S$ for $S = \{1, \dots, r\}$.

For each $i = 1, \dots, r$ we choose a fixed representative $a_i \in A_i$. Let $\mu$ be the primitive $(q^w - 1)$-th root of unity of $\mathbf{k}$ as chosen before Lemma 1.14. Furthermore, recall the notion of dual fundamental weights $\tau_i \in Y(\tilde{\mathbf{T}})$ from Lemma 1.49. Exploiting the Steinberg presentation for the group $\mathbf{G}$, Maslowski gives generators for the torus $\tilde{\mathbf{T}}^F$:

**Lemma 2.1.** *The elements $t_i = N_{F^w/F}(\tau_{a_i}(\mu))$ for $i = 1, \dots, r$ together with $t_0 = N_{F^w/F}(\mathbf{z}(\mu))$ if $d = 1$, and $t_0 = 1$ if $d = 0$, generate the torus $\tilde{\mathbf{T}}^F$.*

*Proof.* The proof is given in [15, Proposition 8.1]. □

**Remark 2.2.** If $F = F_q$ then the elements $t_0, \dots, t_n$ have all order $q - 1$ (unless $t_0 = 1$ which occurs if and only if $d = 0$) by [15, Proposition 8.1]. This is not necessarily true if $F$ is not a standard Frobenius. However, by [15, Remark 10.3] there exist integers $r_i$ such that $t_0^{r_i} = t_i^{q^{|A_i|}-1}$ for $i = 1, \dots r$.

The action of $\tilde{\mathbf{T}}^F$ on the characters of $\mathbf{U}^F$ can be explicitly described and one obtains the following result.

**Lemma 2.3.** *The characters* $\{\phi_S \in \mathrm{Irr}(\mathbf{U}^F) \mid S \subseteq \{1, \ldots, r\}\}$ *form a complete set of representatives for the* $\tilde{\mathbf{B}}^F$*-orbits on the linear characters of* $\mathbf{U}^F$.

*Proof.* This is [15, Proposition 8.4]. □

We consider the action of a Galois automorphism on linear characters of $\mathbf{U}^F$.

**Lemma 2.4.** *Let* $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$ *where* $m$ *is the order of* $\mathbf{G}^F$. *Then* $\phi_S^\sigma = \phi_S^{\tilde{t}}$ *for some* $\tilde{t} \in \tilde{\mathbf{T}}^F$.

*Proof.* By the uniqueness statement of Lemma 2.3 we have $\phi_S^\sigma = \phi_{S'}^{\tilde{t}}$ for some $S' \subseteq \{1, \ldots, r\}$ and some $\tilde{t} \in \tilde{\mathbf{T}}^F$. Recall that the subgroup $\mathbf{U}_{A_i}^F$ are stabilized by the $\tilde{\mathbf{T}}^F$-action. Thus, we have

$$\prod_{i \in S'} \phi_i^{\tilde{t}} = \phi_{S'}^{\tilde{t}} = \phi_S^\sigma = \prod_{i \in S} \phi_i^\sigma.$$

Since the characters $\phi_i \in \mathrm{Irr}(\mathbf{U}_{A_i}^F)$ are nontrivial this implies $S = S'$ and $\phi_S^\sigma = \phi_S^t$.

□

The following statement is crucial for the construction of the labeling.

**Lemma 2.5.** *Any character* $\phi_S \in \mathrm{Irr}(\mathbf{U}^F)$ *extends to its inertia group* $\mathrm{I}_{\tilde{\mathbf{B}}^F}(\phi_S)$. *In particular every character in the set* $\mathrm{Irr}(\mathrm{I}_{\tilde{\mathbf{B}}^F}(\phi_S) \mid \phi_S)$ *is linear.*

*Proof.* This is [15, Lemma 8.5]. □

We can now describe the labels for the $p'$-characters of $\tilde{\mathbf{B}}^F$.

**Construction 2.6.** Let $\psi \in \mathrm{Irr}_{p'}(\tilde{\mathbf{B}}^F)$. Since $\mathbf{U}^F$ is a normal $p$-subgroup of $\tilde{\mathbf{B}}^F$ and $\psi$ has $p'$-degree it follows by Theorem 1.1 that $\psi$ lies above a linear character of $\mathbf{U}^F$. Hence, by Lemma 2.3 there exists a uniquely determined subset $S \subseteq \{1, \ldots, r\}$ such that $\psi$ lies above $\phi_S \in \mathrm{Irr}(\mathbf{U}^F)$. By Clifford correspondence (see Theorem 1.3) there exists a unique character $\lambda \in \mathrm{Irr}(\mathrm{I}_{\tilde{\mathbf{B}}^F}(\phi_S) \mid \phi_S)$ with $\lambda^{\tilde{\mathbf{B}}^F} = \psi$. Note that $\mathrm{I}_{\tilde{\mathbf{B}}^F}(\phi_S) = \langle t_i \mid i \in S^c \rangle \mathbf{U}^F$ by the remark preceding [15, Lemma 8.5]. We define the image of $\psi$ under the map $g : \mathrm{Irr}_{p'}(\tilde{\mathbf{B}}^F) \to \mathbb{C}^{r+1}$ by

$$g(\psi)_i = \begin{cases} \lambda(t_i) & \text{if } i \in S^c, \\ 0 & \text{if } i \in S. \end{cases}$$

For $i \in S^c$ we note that the order of $t_i \in \tilde{\mathbf{T}}^F \subseteq \tilde{\mathbf{T}}^{F_{q^w}}$ divides $(q^w - 1)$ by Remark 2.2. Thus, for $i \in S^c$ the values $\lambda(t_i)$ of the linear character $\lambda$ are $(q^w - 1)$-th roots of unity. Therefore $g(\psi)_i$ is either a $(q^w - 1)$-th root of unity or $g(\psi)_i = 0$. Consider the embedding $\mathbb{F}_{q^w}^\times \to \mathbb{C}^\times$ chosen before Lemma 1.14. Thus, we may consider $g(\psi)_i$ as an element of $\mathbb{F}_{q^w}$ and we have a map $g : \mathrm{Irr}_{p'}(\tilde{\mathbf{B}}^F) \to \mathbb{F}_{q^w}^{r+1}$.

**Theorem 2.7.** *Let $w$ be the order of the graph automorphism $\gamma$ of the Frobenius endomorphism $F = F_q \circ \gamma$. We define $\mathcal{A} = \{(a_0, \ldots, a_r) \in (\mathbb{F}_{q^w}^\times)^d \times \mathbb{F}_{q^w}^r \mid a_i^{q^{|A_i|}-1} = a_0^{r_i}\}$, where the integers $r_i$ are defined as in Remark 2.2. Then the map $g : \mathrm{Irr}_{p'}(\tilde{\mathbf{B}}^F) \to \mathcal{A}$ is a bijection.*

*Proof.* This is [15, Theorem 8.6] in the untwisted case (i.e, $w = 1$) and [15, Theorem 10.8] in the twisted case. □

If $F = F_q$, then we may choose $r_i = 0$ for the integers in Remark 2.2. Thus, in this case we obtain $\mathcal{A} = (\mathbb{F}_q^\times)^d \times \mathbb{F}_q^n$.

## 2.2 A labeling for the $p'$-characters of $\tilde{\mathbf{G}}^F$

In this section we introduce the modified Steinberg map as in [15, Section 14], which separates the semisimple conjugacy of $\tilde{\mathbf{G}}^*$. We use this map to define a labeling for the $p'$-characters of $\tilde{\mathbf{G}}^F$ as introduced by Maslowski in [15, Section 14]. For this, we first need to recall some definitions from the representation theory of algebraic groups (see [14, Chapter 15]).

A character $\lambda \in X(\mathbf{T})$ is called a *dominant weight* if $\langle \lambda, \alpha_j^\vee \rangle \geqslant 0$ for all $j = 1, \ldots, n$. Since $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ it follows that all fundamental weights $\omega_i \in X(\mathbf{T})$ of $\mathbf{G}$ are dominant.

By a theorem of Chevalley (see [14, Theorem 15.17]) there exists a rational irreducible $\mathbf{kG}$-module $V$ which is a *highest weight module* of *highest weight* $\lambda$. This means that there exists a vector $v^+ \in V$ generating $V$ as $\mathbf{kG}$-module and satisfying $t.v^+ = \lambda(t).v^+$ for all $t \in \mathbf{T}$.

Let $\mathrm{R}_i : \mathbf{G} \to \mathrm{GL}(V_i)$ denote the representation associated to a highest weight module of highest weight $\omega_i \in X(\mathbf{T})$. We let

$$\pi_i : \mathbf{G} \to \mathbf{k} \text{ with } g \mapsto \mathrm{tr}(\mathrm{R}_i(g))$$

be the trace function of the representation $\mathrm{R}_i$. We define the *Steinberg map* $\pi : \mathbf{G} \to \mathbf{k}^n$ by $\pi(g) := (\pi_1(g), \ldots, \pi_n(g))$. A fundamental property of the Steinberg map is that two elements of $\mathbf{G}$ are $\mathbf{G}$-conjugate if and only if they have the same image under the Steinberg map (see [18, Corollary 6.7]).

We need a further property of the Steinberg map which is a slight generalization of [15, Lemma 14.1] and follows the proof given there.

**Lemma 2.8.** *Let $s \in \mathbf{G}$ be a semisimple element. Then $\pi_i(s^p) = \pi_i(s)^p$.*

*Proof.* Let $\mathrm{R}_i : \mathbf{G} \to \mathrm{GL}(V_i)$ be a representation affording the trace function $\pi_i$. The semisimple element $s \in \mathbf{G}$ is contained in a maximal torus of $\mathbf{G}$ by [14, Corollary 6.11]. Since by [14, Corollary 6.5] all maximal tori of $\mathbf{G}$ are $\mathbf{G}$-conjugate, we may assume that $s \in \mathbf{T}$. Since $\mathrm{R}_i(\mathbf{T})$ is a set of pairwise commuting endomorphisms we may diagonalize the set $\mathrm{R}_i(\mathbf{T})$ simultaneously. Since $\mathbf{k}$ is a field of characteristic $p$, the map $\mathbf{k} \to \mathbf{k}$ with $x \mapsto x^p$ for $x \in \mathbf{k}$ is a field automorphism. This implies

$$\pi_i(s^p) = \mathrm{tr}(\mathrm{R}_i(s^p)) = \mathrm{tr}(\mathrm{R}_i(s)^p) = \mathrm{tr}(\mathrm{R}_i(s))^p = \pi_i(s)^p,$$

since $\mathrm{R}_i(s)$ is a diagonal matrix. □

The fundamental idea of Maslowski is to consider a modification of the Steinberg map for the group $\tilde{\mathbf{G}}^* = \mathbf{G}^\vee \mathbf{S}^\vee$. Let us denote by $\pi^\vee : \mathbf{G}^\vee \to \mathbf{k}^n$ with $\pi^\vee(x) := (\pi_1^\vee(x), \ldots, \pi_n^\vee(x))$ for $x \in \mathbf{G}^\vee$, the Steinberg map of $\mathbf{G}^\vee$. We can write any element $g \in \tilde{\mathbf{G}}^*$ (not necessary unique) as $g = xz$ with $x \in \mathbf{G}^\vee$ and $z \in \mathbf{S}^\vee$. In [15, Section 14] Maslowski defines the map $\tilde{\pi} : \tilde{\mathbf{G}}^* \to (\mathbf{k}^\times)^d \times \mathbf{k}^n$ by

$$g = xz \mapsto (\det{}^*(xz), \pi_1^\vee(x)\tilde{\omega}_1^*(z), \ldots, \pi_n^\vee(x)\tilde{\omega}_n^*(z)).$$

Similar to the result of Steinberg mentioned above Maslowski shows in [15, Proposition 14.2] the following result:

**Lemma 2.9.** *The map $\tilde{\pi}$ separates semisimple conjugacy classes of $\tilde{\mathbf{G}}^*$. Moreover, if $F = F_q$ is the standard Frobenius then the conjugacy classes $(xz)$ of $\tilde{\mathbf{G}}^*$ with image $\tilde{\pi}(xz)$ in $(\mathbb{F}_q^\times)^d \times \mathbb{F}_q^n$ are precisely the $(q-1)^d q^n$ different $F^*$-stable semisimple conjugacy classes of $\tilde{\mathbf{G}}^*$.*

As discussed in Remark 1.17 the $F_q^*$-stable semisimple conjugacy classes of $\tilde{\mathbf{G}}^*$ are precisely the conjugacy classes of semisimple elements of ${\tilde{\mathbf{G}}^*}{}^{F_q^*}$. Therefore we have a bijection between these conjugacy classes and $(\mathbb{F}_q^\times)^d \times \mathbb{F}_q^n$. This allows us to make the following construction.

**Construction 2.10.** We first consider the case that $F = F_q$ is a standard Frobenius map. Let $\chi \in \mathrm{Irr}_{p'}(\tilde{\mathbf{G}}^F)$ be a $p'$-character. Then there exists a unique $F^*$-stable conjugacy class $(\tilde{s})$ of $\tilde{\mathbf{G}}^*$ such that $\chi \in \mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{s}))$ (see Theorem 1.18). We then define the label of $\chi$ by $\tilde{\pi}(\tilde{s}) = (b_0, \ldots, b_n)$.

Now suppose that $F = F_q \circ \gamma$. Let $w$ denote the order of $\gamma$. As in the untwisted case, we associate the $F^*$-stable semisimple conjugacy class $(\tilde{s})$ to the $p'$-character $\chi \in \mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{s}))$. Since ${\tilde{\mathbf{G}}^*}{}^{F^*} \subseteq {\tilde{\mathbf{G}}^*}{}^{F_{q^w}^*}$ it is clear that $(\tilde{s})$ is a semisimple conjugacy class of $\tilde{\mathbf{G}}^*$ which is $F^*_{q^w}$-stable. In particular $(\tilde{s})$ has a label $\tilde{\pi}(\tilde{s}) = (b_0, \dots, b_n) \in \mathbb{F}^\times_{q^w} \times \mathbb{F}^n_{q^w}$. We define the label of $\chi$ by $(b_0, b_{a_1}, \dots, b_{a_r})$, where $a_i \in A_i$ are the fixed representatives of the orbits of the $\tau$-action on $\{1, \dots, n\}$ as chosen before Lemma 2.1.

The possible labels which occur in Construction 2.10 consist precisely of the elements of $\mathcal{A} \subseteq (\mathbb{F}^\times_{q^w})^d \times \mathbb{F}^r_{q^w}$, where $\mathcal{A}$ is defined as in Theorem 2.7, as the following theorem shows.

**Theorem 2.11.** *The map $h : \mathrm{Irr}_{p'}(\tilde{\mathbf{G}}^F) \to \mathcal{A}$ which maps a character $\chi \in \mathrm{Irr}_{p'}(\tilde{\mathbf{G}}^F)$ to its label $(b_0, b_{a_1}, \dots, b_{a_r})$ as in Construction 2.10 is a bijection.*

*Proof.* If $F$ is a standard Frobenius then $\mathcal{A} = (\mathbb{F}^\times_q)^d \times \mathbb{F}^n_q$ by the remark below Theorem 2.7. In this case the theorem is a consequence of Lemma 2.9 together with Theorem 1.19. If $F$ is a twisted Frobenius one has to observe which labels occur for the various $p'$-characters. This has been done in [15, Proposition 14.4] and [15, Proposition 14.5]. □

The previous theorem together with Theorem 2.7 implies the following result.

**Theorem 2.12.** *The map $\tilde{f} = h^{-1} \circ g : \mathrm{Irr}_{p'}(\tilde{\mathbf{B}}^F) \to \mathrm{Irr}_{p'}(\tilde{\mathbf{G}}^F)$ is a bijection.*

We will state some properties of $\tilde{f}$. First we observe that the map $\tilde{f}$ is compatible with the multiplication by linear characters:

**Theorem 2.13.** *The map $\tilde{f}$ defines by restriction a bijective map $\mathrm{Irr}(\tilde{\mathbf{B}}^F \mid 1_{\mathbf{B}^F}) \to \mathrm{Irr}(\tilde{\mathbf{G}} \mid 1_{\mathbf{G}^F})$. Moreover, $\tilde{f}(\eta\psi) = \tilde{f}(\eta)\tilde{f}(\psi)$ for $\psi \in \mathrm{Irr}_{p'}(\tilde{\mathbf{B}}^F)$ and $\eta \in \mathrm{Irr}(\tilde{\mathbf{B}}^F \mid 1_{\mathbf{B}^F})$.*

*Proof.* This is [15, Theorem 15.3]. □

As a consequence of Theorem 2.13 we obtain that the bijective map $\mathrm{Irr}(\tilde{\mathbf{B}}^F \mid 1_{\mathbf{B}^F}) \to \mathrm{Irr}(\tilde{\mathbf{G}} \mid 1_{\mathbf{G}^F})$, which we obtained by restricting $\tilde{f}$, is an isomorphism of abelian groups. In particular we have $\tilde{f}(1_{\tilde{\mathbf{B}}^F}) = 1_{\tilde{\mathbf{G}}^F}$. The label of $1_{\tilde{\mathbf{G}}^F}$ is $h(1_{\tilde{\mathbf{G}}^F}) = (1, \pi^\vee_{a_1}(1), \dots, \pi^\vee_{a_r}(1))$. Thus, we obtain $\pi^\vee_{a_i}(1) = 1$ for all $i = 1, \dots, r$, since $g(1_{\tilde{\mathbf{B}}^F}) = (1, 1, \dots, 1)$.

**Lemma 2.14.** *The following two statements hold.*

*(a) Let $\psi \in \mathrm{Irr}_{p'}(\tilde{\mathbf{B}}^F)$ and $\vartheta \in \mathrm{Irr}(\mathbf{B}^F \mid \psi)$. Then $\mathrm{Irr}(\tilde{\mathbf{B}}^F \mid \vartheta) = \{\chi\eta \mid \eta \in \mathrm{Irr}(\tilde{\mathbf{B}}^F \mid 1_{\mathbf{B}^F})\}$.*

*(b) Let $\chi \in \mathrm{Irr}_{p'}(\tilde{\mathbf{G}}^F)$ and $\phi \in \mathrm{Irr}(\mathbf{G}^F \mid \chi)$. Then $\mathrm{Irr}(\tilde{\mathbf{G}}^F \mid \phi) = \{\chi\lambda \mid \lambda \in \mathrm{Irr}(\tilde{\mathbf{G}}^F \mid 1_{\mathbf{G}^F})\}$.*

*Moreover, the number of different $\tilde{\mathbf{B}}^F$-conjugates of $\vartheta$ and the number of different $\tilde{\mathbf{G}}^F$-conjugates of $\phi$ coincide.*

*Proof.* Note that $\tilde{\mathbf{B}}^F/\mathbf{B}^F \cong \tilde{\mathbf{G}}^F/\mathbf{G}^F \cong \mathbf{S}^F$ by Lemma 1.45 and $\mathbf{S}^F$ is cyclic. Therefore the result follows directly from Corollary 1.7. □

The previous lemma is true in more generality.

**Remark 2.15.** If the root system of $\tilde{\mathbf{G}}$ is of type $D_n$ with $n$ even then the statement of Lemma 2.14 is still true. In this case one has to show that any $p'$-character of $\mathbf{G}^F$ (resp. $\mathbf{B}^F$) extends to its inertia group in $\tilde{\mathbf{G}}^F$ (resp. $\tilde{\mathbf{B}}^F$) in order to apply Lemma 1.6. This is proved in [15, Proposition 11.3] for $\tilde{\mathbf{B}}^F$ and in [1, Theorem 15.11] for $\tilde{\mathbf{G}}^F$.

In Chapter 3 we use the following corollary for the construction of a bijection $f : \mathrm{Irr}_{p'}(\mathbf{B}^F)^\sigma \to \mathrm{Irr}_{p'}(\mathbf{G}^F)^\sigma$.

**Corollary 2.16.** *Let $\psi \in \mathrm{Irr}_{p'}(\tilde{\mathbf{B}}^F)$ and $\vartheta \in \mathrm{Irr}(\mathbf{B}^F \mid \psi)$. Furthermore we let $\phi \in \mathrm{Irr}(\mathbf{G}^F)$ be a constituent of $\tilde{f}(\psi)_{\mathbf{G}^F}$. Then the map $\tilde{f}$ induces by restriction a bijection $\mathrm{Irr}(\tilde{\mathbf{B}}^F \mid \vartheta) \to \mathrm{Irr}(\tilde{\mathbf{G}}^F \mid \phi)$.*

*Proof.* This follows from Lemma 2.14 and Theorem 2.13. □

# Chapter 3

# The McKay Conjecture and Galois automorphisms

In this chapter we prove that the bijection $\tilde{f} : \mathrm{Irr}_{p'}(\tilde{\mathbf{B}}^F) \to \mathrm{Irr}_{p'}(\tilde{\mathbf{G}}^F)$ constructed in Chapter 2 is compatible with the action of $(e,p)$-Galois automorphisms. Then we will relate the $p'$-characters of $\tilde{\mathbf{G}}^F$ (resp. $\tilde{\mathbf{B}}^F$) with the $p'$-characters of $\mathbf{G}^F$ (resp. $\mathbf{B}^F$). This, together with the bijection $\tilde{f}$, allows us to construct a bijection $f : \mathrm{Irr}_{p'}(\mathbf{B}^F)^\sigma \to \mathrm{Irr}_{p'}(\mathbf{G}^F)^\sigma$ which is compatible with central characters.

## 3.1 Compatibility of the character bijection with Galois automorphisms

In the previous chapters we had to impose several conditions on $(\mathbf{G}, F)$. In order to apply all results from the previous chapters we have to make some assumptions. For the convenience of the reader we will recall them now.

**Assumption 3.1.** *Let* $\mathbf{G}$ *be a simple algebraic group of simply connected type. We assume from now on that* $\mathbf{G}$ *is not of type* $D_n$ *if* $n$ *is even and that* $p$ *is a good prime for* $\mathbf{G}$.

We will frequently write $H = \mathbf{H}^F$ for the set of fixed points under the Frobenius endomorphism $F$ of an algebraic group $\mathbf{H}$. For example, $\tilde{G} = \tilde{\mathbf{G}}^F$, $G = \mathbf{G}^F$ and $\tilde{B} = \tilde{\mathbf{B}}^F$, $B = \mathbf{B}^F$. Let us recall from Definition 0.2 the notion of an $(e,p)$-Galois automorphism.

**Definition 3.2.** Let $e$ be a nonnegative integer and $p$ be a prime number. Then a Galois automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$ is called an $(e,p)$-*Galois automorphism* if $\sigma$ sends any $p'$-root of unity $\zeta \in \mathbb{Q}_m$ to $\zeta^{p^e}$.

For a natural number $m$ we write $m_p$ for the highest $p$-power dividing $m$ and $m_{p'} = \frac{m}{m_p}$ for the $p'$-part of $m$. We first show some elementary properties of $(e,p)$-Galois automorphisms:

**Lemma 3.3.** *Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$ and $k$ be an integer such that $\sigma(\xi) = \xi^k$ for a primitive $m$-th root of unity $\xi \in \mathbb{Q}_m$.*
*(a) Then $\sigma$ is an $(e,p)$-Galois automorphism if and only if $k \equiv p^e \mod m_{p'}$.*
*(b) Let $\tilde{m}$ be a multiple of $m$. Then any $(e,p)$-Galois automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$ extends to an $(e,p)$-Galois automorphism $\tilde{\sigma} \in \mathrm{Gal}(\mathbb{Q}_{\tilde{m}}/\mathbb{Q})$.*

*Proof.* For the proof of (a) we let $\zeta \in \mathbb{Q}_m$ be an $m_{p'}$-th root of unity. Then $\zeta = \xi^u$ for some natural number $u$. Thus, $\sigma(\zeta) = \sigma(\xi^u) = \xi^{uk} = \zeta^k$ which implies that $\sigma(\zeta) = \zeta^{p^e}$ if and only if $\zeta^{p^e} = \zeta^k$. Since the order of $\zeta$ is a divisor of $m_{p'}$ this holds true if $k \equiv p^e \mod m_{p'}$. Conversely, if we take $\zeta \in \mathbb{Q}_m$ as primitive $m_{p'}$-th root of unity, the equation $\zeta^{p^e} = \zeta^k$ implies that $k \equiv p^e \mod m_{p'}$.

We now prove part (b). Let $\sigma_1 \in \mathrm{Gal}(\mathbb{Q}_{m_p}/\mathbb{Q})$ and $\sigma_2 \in \mathrm{Gal}(\mathbb{Q}_{m_{p'}}/\mathbb{Q})$ be the restriction of $\sigma$ to $\mathbb{Q}_{m_p}$ resp. $\mathbb{Q}_{m_{p'}}$. Choose a Galois automorphism $\tilde{\sigma}_1 \in \mathrm{Gal}(\mathbb{Q}_{\tilde{m}_p}/\mathbb{Q})$ extending $\sigma_1$. Define $\tilde{\sigma}_2 \in \mathrm{Gal}(\mathbb{Q}_{\tilde{m}_{p'}}/\mathbb{Q})$ by $\tilde{\sigma}_2(\zeta) = \zeta^{p^e}$ for a primitive $\tilde{m}_{p'}$-th root of unity $\zeta \in \mathbb{Q}_{\tilde{m}_{p'}}$. Clearly, $\tilde{\sigma}_2$ is an extension of $\sigma_2$.

By [11, Chapter VI, Theorem 1.14] we find a unique Galois automorphism $\tilde{\sigma} \in \mathrm{Gal}(\mathbb{Q}_{\tilde{m}}/\mathbb{Q})$ such that $\tilde{\sigma}$ restricts to $\tilde{\sigma}_1$ and $\tilde{\sigma}$ restricts to $\tilde{\sigma}_2$. We conclude that $\tilde{\sigma}$ is an extension of $\sigma$. Moreover, since $\tilde{\sigma}$ restricts to $\tilde{\sigma}_2$ it follows that $\tilde{\sigma} \in \mathrm{Gal}(\mathbb{Q}_{\tilde{m}}/\mathbb{Q})$ is an $(e,p)$-Galois automorphism. □

Note that part (b) of Lemma 3.3 implies that any $(e,p)$-Galois automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$ extends to an $(e,p)$-Galois automorphism $\tilde{\sigma} \in \mathrm{Gal}(\mathbb{Q}_{|\tilde{G}|}/\mathbb{Q})$. This means that if we want to prove Conjecture 0.3 for the finite group $G$ and an $(e,p)$-Galois automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$, we may assume without loss of generality that $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|\tilde{G}|}/\mathbb{Q})$.

We now show that the bijection $\tilde{f} : \mathrm{Irr}_{p'}(\tilde{B}) \to \mathrm{Irr}_{p'}(\tilde{G})$ is $\sigma$-equivariant for $(e,p)$-Galois automorphisms $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|\tilde{G}|}/\mathbb{Q})$.

**Theorem 3.4.** *Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$, with $m = |\tilde{G}|$, be an $(e,p)$-Galois automorphism. Then the bijection*

$$\tilde{f} : \mathrm{Irr}_{p'}(\tilde{B}) \to \mathrm{Irr}_{p'}(\tilde{G})$$

*is compatible with $\sigma$, i.e. $\tilde{f}(\psi^\sigma) = \tilde{f}(\psi)^\sigma$ for any character $\psi \in \mathrm{Irr}_{p'}(\tilde{B})$. Moreover, if we denote the label of $\psi$ by $g(\psi) = (c_0, \dots, c_r)$ then the label of $\psi^\sigma$ is given by $g(\psi^\sigma) = (c_0^{p^e}, \dots, c_r^{p^e})$.*

*Proof.* We proceed in several steps. Let $\psi \in \mathrm{Irr}_{p'}(\tilde{B})$ with label $g(\psi) = (c_0, \ldots, c_r)$. In a first step we compute the label $g(\psi^\sigma)$ of the character $\psi^\sigma$.

In a second step we let $\chi = \tilde{f}(\psi)$ be the $p'$-character of $\tilde{G}$ which has the same label as $\psi$. Then we prove that $h(\chi^\sigma) = g(\psi^\sigma)$. This implies that $\tilde{f}(\psi^\sigma) = \chi^\sigma$, since $\tilde{f} = h^{-1} \circ g$.

**First step:** Since the character $\psi \in \mathrm{Irr}_{p'}(\tilde{B})$ has $p'$-degree and $U$ is a normal $p$-subgroup of $\tilde{B}$, any irreducible constituent of $\psi_U$ is linear. By Lemma 2.3 there exists a unique $S \subseteq \{1, \ldots, r\}$ such that $\psi$ lies above the character $\phi_S \in \mathrm{Irr}(U)$. By Clifford correspondence (see Theorem 1.3) there exists a unique character $\lambda \in \mathrm{Irr}(\mathrm{I}_{\tilde{B}}(\phi_S) \mid \phi_S)$ such that $\lambda^{\tilde{B}} = \psi$.

Since $\psi$ lies above $\phi_S$ it follows that $\psi^\sigma$ lies above the character $\phi_S^\sigma$. By Lemma 2.4 we have $\phi_S^\sigma = \phi_S^{\tilde{t}}$ for some $\tilde{t} \in \tilde{T}$. The character $\lambda^\sigma$ lies above the character $\phi_S^\sigma = \phi_S^{\tilde{t}}$. Since the factor group $\tilde{B}/U \cong \tilde{T}$ is abelian the character $(\lambda^\sigma)^{\tilde{t}^{-1}}$ is well-defined. Consequently, $(\lambda^\sigma)^{\tilde{t}^{-1}}$ lies above the character $\phi_S$ and $((\lambda^\sigma)^{\tilde{t}^{-1}})^{\tilde{B}} = \psi^\sigma$. Note that $\lambda$ is a linear character by Lemma 2.5. We let $\xi$ be a primitive $m$-th root of unity and we let $k$ be an integer such that $\sigma(\xi) = \xi^k$. Then we have $\lambda^\sigma = \lambda^k$ by Lemma 1.8. By definition of the map $g$ in Construction 2.6 we obtain

$$g(\psi^\sigma)_i = (\lambda^k)^{\tilde{t}^{-1}}(t_i) = (\lambda^k)(\tilde{t} t_i \tilde{t}^{-1}) = \lambda^k(t_i)$$

for every $i \in S^c$. For $i \in S$ we get $g(\psi^\sigma)_i = 0$. Consequently, the label of $\psi^\sigma$ is given by $g(\psi^\sigma) = (c_0^k, c_1^k, \ldots, c_r^k)$.

By Lemma 3.3 we have $k \equiv p^e \mod m_{p'}$ since $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$ is an $(e,p)$-Galois automorphism. By [14, Table 24.1] it follows that $(q^w - 1)$ divides $m_{p'}$, which implies that $k \equiv p^e \mod (q^w - 1)$. Since $(c_0, \ldots, c_r) \in \mathbb{F}_{q^w}^{r+1}$ we have

$$g(\psi^\sigma) = (c_0^k, \ldots, c_r^k) = (c_0^{p^e}, \ldots, c_r^{p^e}).$$

**Second step:** We have $\chi \in \mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{s}))$ for some semisimple conjugacy class $(\tilde{s})$ of the dual group $\tilde{\mathbf{G}}^{*F^*}$. By Corollary 1.16 we deduce that $\chi^\sigma \in \mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{s}^k))$. Note that $\tilde{s} \in \tilde{\mathbf{G}}^{*F^*}$ is semisimple and $|\tilde{\mathbf{G}}^F| = |\tilde{\mathbf{G}}^{*F^*}|$ since $(\tilde{\mathbf{G}}, F)$ and $(\tilde{\mathbf{G}}^*, F^*)$ are in duality (see [2, Proposition 4.4.4]). By [5, Proposition 3.18] it follows that the order of $\tilde{s}$ is a divisor of $m_{p'}$. Since $k \equiv p^e \mod m_{p'}$ by Lemma 3.3, this shows that $\tilde{s}^{p^e} = \tilde{s}^k$. Hence, we have $\chi^\sigma \in \mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{s}^{p^e}))$.

Let us now first assume that $F = F_q$ is a standard Frobenius map. We may write $\tilde{s} \in \tilde{\mathbf{G}}^*$ (not necessarily unique) as $\tilde{s} = xz$ where $x \in \mathbf{G}^\vee$ and $z \in \mathbf{S}^\vee$. By Construction 2.10 the label of the character $\chi^\sigma \in \mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{s}^{p^e}))$ is given by

$$\tilde{\pi}(\tilde{s}^{p^e}) = \tilde{\pi}((xz)^{p^e}) = (\det{}^*((xz)^{p^e}), \pi_1^\vee(x^{p^e})\tilde{\omega}_1^*(z^{p^e}), \ldots, \pi_n^\vee(x^{p^e})\tilde{\omega}_n^*(z^{p^e})).$$

Note that $\det^*((xz)^{p^e}) = \det^*(xz)^{p^e}$ since $\det^* \in X(\tilde{\mathbf{T}}^*)$. We claim that $\tilde{\pi}_i(\tilde{s}^{p^e}) = \tilde{\pi}_i(\tilde{s})^{p^e}$ for all $i = 1, \ldots, n$. We obtain

$$\tilde{\pi}_i(\tilde{s}^{p^e}) = \tilde{\pi}_i(x^{p^e} z^{p^e}) = \pi_i^\vee(x^{p^e})\tilde{\omega}_i^*(z^{p^e})$$

by definition of the modified Steinberg map in Section 2.2. By Lemma 2.8 we have $\pi_i^\vee(x^{p^e}) = \pi_i^\vee(x)^{p^e}$. Moreover, we have $\tilde{\omega}_i^*(z^{p^e}) = \tilde{\omega}_i^*(z)^{p^e}$ since $\tilde{\omega}_i^* \in X(\tilde{\mathbf{T}}^*)$. Therefore we obtain

$$\tilde{\pi}_i(\tilde{s}^{p^e}) = \pi_i^\vee(x)^{p^e}\tilde{\omega}_i^*(z)^{p^e} = (\pi_i^\vee(x)\tilde{\omega}_i^*(z))^{p^e} = \tilde{\pi}_i(\tilde{s})^{p^e}.$$

Hence, we obtain $\tilde{\pi}_i(\tilde{s}^{p^e}) = \tilde{\pi}_i(\tilde{s})^{p^e}$ as claimed above. Since $\tilde{\pi}(xz) = (c_0, \ldots, c_n)$ we have $\tilde{\pi}((xz)^{p^e}) = (c_0^{p^e}, \ldots, c_n^{p^e})$ and therefore the label of $\chi^\sigma$ is given by $h(\chi^\sigma) = (c_0^{p^e}, \ldots, c_n^{p^e}) = g(\psi^\sigma)$ and we have $\tilde{f}(\psi^\sigma) = \tilde{f}(\psi)^\sigma$, as desired.

Let us now assume that $F = F_q \circ \gamma$. Let $w$ be the order of $\gamma$ and let $\tilde{\pi}(\tilde{s}) = (b_0, \ldots, b_n)$ be the image of the $F_{q^w}$-stable conjugacy class $(\tilde{s})$ of $\tilde{\mathbf{G}}^*$ under the modified Steinberg map. Then, as we have shown above, the image of $(\tilde{s}^{p^e})$ under the modified Steinberg map is given by $\tilde{\pi}(\tilde{s}^{p^e}) = (b_0^{p^e}, \ldots, b_n^{p^e})$. By construction of the labeling, the label of the character $\chi \in \mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{s}))$ is given by $h(\chi) = (b_{a_0}, \ldots, b_{a_r})$ and the label of $\chi^\sigma \in \mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{s}^{p^e}))$ is $h(\chi^\sigma) = (b_{a_0}^{p^e}, \ldots, b_{a_r}^{p^e})$ (see Construction 2.10). We have $g(\psi) = (c_0, \ldots, c_r) = (b_{a_0}, \ldots, b_{a_r}) = h(\chi)$ since $\tilde{f}(\psi) = \chi$. Thus, we conclude that

$$g(\psi^\sigma) = (c_0^{p^e}, \ldots, c_r^{p^e}) = (b_{a_0}^{p^e}, \ldots, b_{a_r}^{p^e}) = h(\chi^\sigma).$$

This proves $\tilde{f}(\psi^\sigma) = \chi^\sigma$, as desired. □

As an immediate consequence of Theorem 3.4 we have the following corollary:

**Corollary 3.5.** *Under the assumptions of the previous theorem the bijective map $\tilde{f}$ restricts to a bijection $\mathrm{Irr}_{p'}(\tilde{B})^\sigma \to \mathrm{Irr}_{p'}(\tilde{G})^\sigma$.*

In the following example we compute the number of $\sigma$-invariant $p'$-characters of $\tilde{B} = \tilde{\mathbf{B}}^F$ if $F = F_q$ is a standard Frobenius.

**Example 3.6.** Let $F = F_q$ be a Standard Frobenius map. Then we have a bijection $g : \mathrm{Irr}_{p'}(\tilde{\mathbf{B}}^F) \to \mathcal{A}$ with $\mathcal{A} = (\mathbb{F}_q^\times)^d \times \mathbb{F}_q^n$ (see remark below Theorem 2.7). Let $\psi \in \mathrm{Irr}_{p'}(\tilde{B})$ with label $g(\psi) = (c_0, \ldots, c_n)$. By Theorem 3.4 the character $\psi$ is $\sigma$-invariant if and only if $(c_0^{p^e}, \ldots, c_n^{p^e}) = (c_0, \ldots, c_n)$. Let $q = p^f$ and let $s \in \{1, \ldots, f\}$ be defined as $s = \gcd(e, f)$. Then $\psi$ is $\sigma$-invariant if and only if $(c_0, \ldots, c_n) \in (\mathbb{F}_{p^s}^\times)^d \times \mathbb{F}_{p^s}^n$. Consequently we have $|\mathrm{Irr}_{p'}(\tilde{B})^\sigma| = (p^s - 1)^d p^{sn}$.

## 3.2 Relating the characters for a special class of Galois automorphisms

For this section only we will furthermore assume that $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$, where $m = |\tilde{G}|$, is a Galois automorphism which fixes all $p$-th roots of unity of $\mathbb{Q}_m$. We will later drop this assumption. In particular, this assumption implies that all linear characters of $U$ are $\sigma$-invariant, as we are about to show.

**Lemma 3.7.** *Under the assumptions as above, any linear character of $U$ is $\sigma$-invariant.*

*Proof.* The linear characters of $U$ are precisely the inflation of characters of $U/[U,U]$. We have $U/[U,U] \cong \prod_{i=1}^{r} \mathbf{U}_{A_i}^F$ and $\mathbf{U}_{A_i}^F \cong (\mathbb{F}_{q^{|A_i|}}, +)$ (see Section 2.1). As additive group, the group $(\mathbb{F}_{q^{|A_i|}}, +)$ is isomorphic to a product of cyclic groups of order $p$. We conclude that the character values of characters of $U/[U,U]$ are $p$-th roots of unity. These roots are fixed by the Galois automorphism $\sigma$ by assumption. Thus, every linear character of $U$ is $\sigma$-invariant. □

**Lemma 3.8.** *Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$, where $m$ is the order of $\tilde{G}$. Suppose that $\sigma$ fixes all $p$-th roots of unity of $\mathbb{Q}_m$. Let $\chi \in \mathrm{Irr}_{p'}(\tilde{G})$ such that $\chi_G$ is $\sigma$-invariant. Then $\psi \in \mathrm{Irr}_{p'}(G)^\sigma$ for all $\psi \in \mathrm{Irr}(G \mid \chi)$.*

*Proof.* By Lemma 1.36 we have $\chi = \varepsilon\, \mathrm{D}_{\tilde{\mathbf{G}}}(\chi_{(\tilde{s})})$ for some $\varepsilon \in \{\pm 1\}$ and a semisimple conjugacy class $(\tilde{s})$ of $\tilde{\mathbf{G}}^{*F^*}$. We let $i^* : \tilde{\mathbf{G}}^* \to \mathbf{G}^*$ be the dual morphism corresponding to the regular embedding $i : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ as in Section 1.6. The characters $\psi_{(s),z} = \varepsilon\, \mathrm{D}_{\mathbf{G}}(\chi_{(s),z}) \in \mathrm{Irr}_{p'}(G)$ for $s = i^*(\tilde{s})$ and $z \in H^1(F, \mathrm{Z}(\mathbf{G}))$ are by Lemma 1.38 the irreducible constituents of $\chi_G$. Recall that $\chi_{(s),z}$ is by definition the unique common constituent of $\chi_{(s)}$ and $\Gamma_z$. The Gelfand–Graev character $\Gamma_z = \psi_{t_z}^G$ is the induced character of the linear character $\psi_{t_z} \in \mathrm{Irr}(U)$ (see Definition 1.32 and Remark 1.29). Thus, we have $\Gamma_z^\sigma = \Gamma_z$ since the Galois automorphism $\sigma$ fixes any linear character of $U$ by Lemma 3.7.

As in the proof of Lemma 1.38 we see that $\chi_G = \varepsilon\, \mathrm{D}_{\mathbf{G}}(\chi_{(s)})$. Since $\chi_G$ is $\sigma$-invariant we conclude by Lemma 1.25 that $\chi_{(s)}$ is $\sigma$-invariant as well. Therefore $\chi_{(s),z}^\sigma$ is the unique common constituent of $\Gamma_z^\sigma = \Gamma_z$ and $\chi_{(s)}^\sigma = \chi_{(s)}$. This shows $\chi_{(s),z}^\sigma = \chi_{(s),z}$. Using Lemma 1.25 again we deduce that $\psi_{(s),z}^\sigma = \psi_{(s),z}$. □

**Lemma 3.9.** *Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$, where $m$ is the order of $\tilde{G}$. Suppose that $\sigma$ fixes all $p$-th roots of unity of $\mathbb{Q}_m$. Let $\psi \in \mathrm{Irr}_{p'}(\tilde{B})$ such that $\psi_B$ is $\sigma$-invariant. Then $\vartheta \in \mathrm{Irr}_{p'}(B)^\sigma$ for all $\vartheta \in \mathrm{Irr}(B \mid \psi)$.*

*Proof.* Since $U$ is a normal $p$-subgroup of $\tilde{B}$ the $p'$-character $\psi$ lies above a linear character of $U$. By Lemma 2.3 we know that the character $\psi$ lies above a character $\phi_S$ for some $S \subseteq \{1, \ldots, r\}$. By Theorem 1.3 there exists a character $\lambda \in \operatorname{Irr}(\tilde{I} \mid \phi_S)$, where $\tilde{I} = \mathrm{I}_{\tilde{B}}(\phi_S)$, such that $\lambda^{\tilde{B}} = \psi$. By Lemma 2.5 the character $\phi_S$ extends to its inertia group $\tilde{I}$ and the linear character $\lambda \in \operatorname{Irr}(\tilde{I} \mid \phi_S)$ is an extension of $\phi_S$. Note that $\lambda_I \in \operatorname{Irr}(I \mid \phi_S)$, where $I = \mathrm{I}_B(\phi_S)$. By Theorem 1.3 we conclude that $\vartheta := (\lambda_I)^B$ is an irreducible character of $B$. Moreover, we have

$$(\psi_B, \vartheta) = (\psi_B, (\lambda_I)^B) = (\psi, (\lambda_I)^{\tilde{B}}) = (\lambda^{\tilde{B}}, (\lambda_I)^{\tilde{B}}) = (\lambda, ((\lambda_I)^{\tilde{B}})_{\tilde{I}}).$$

By Mackey's Theorem (see [8, Problem 5.6]) we can conclude that

$$(\lambda, ((\lambda_I)^{\tilde{B}})_{\tilde{I}}) \geqslant (\lambda, (\lambda_I)^{\tilde{I}}) = (\lambda_I, \lambda_I) = 1 \neq 0.$$

Combining the two formulas above, we deduce that the character $\vartheta = (\lambda_I)^B$ is an irreducible character of $B$ which lies below the character $\psi$.

We need to prove that $\vartheta = (\lambda_I)^B$ is $\sigma$-invariant. By Lemma 1.4 we know that $(\lambda_I)^B$ is $\sigma$-invariant if and only if $\lambda_I$ is $\sigma$-invariant. Since $\psi_B$ is $\sigma$-invariant, we have $0 \neq (\psi_B, \vartheta) = ((\psi^\sigma)_B, \vartheta)$. Therefore $\psi^\sigma \in \operatorname{Irr}(\tilde{B} \mid \vartheta)$. By Lemma 2.14 there exists a character $\eta \in \operatorname{Irr}(\tilde{B} \mid 1_B)$ such that $\psi^\sigma = \psi\eta$. The character $\lambda$ lies above $\phi_S \in \operatorname{Irr}(U)$. Now $\phi_S^\sigma = \phi_S$ implies that $\lambda^\sigma$ lies above the character $\phi_S$. We conclude that

$$(\lambda\eta_{\tilde{I}})^{\tilde{B}} = \lambda^{\tilde{B}}\eta = \psi\eta = \psi^\sigma = (\lambda^{\tilde{B}})^\sigma = (\lambda^\sigma)^{\tilde{B}}.$$

This means $(\lambda\eta_{\tilde{I}})^{\tilde{B}} = (\lambda^\sigma)^{\tilde{B}} = \psi^\sigma \in \operatorname{Irr}(\tilde{B} \mid \phi_S)$. The characters $\lambda\eta_{\tilde{I}}$ and $\lambda^\sigma$ are lying above $\phi_S$ which implies $\lambda\eta_{\tilde{I}} = \lambda$ by Theorem 1.3. Since $\eta \in \operatorname{Irr}(\tilde{B} \mid 1_B)$ we have in particular $\eta_I = 1_I$. Hence, we get

$$\lambda_I = \lambda_I\eta_I = (\lambda\eta_{\tilde{I}})_I = (\lambda^\sigma)_I = (\lambda_I)^\sigma.$$

Thus, $\lambda_I$ is $\sigma$-invariant which proves that $\vartheta = (\lambda_I)^B$ is $\sigma$-invariant. Therefore the character $\vartheta \in \operatorname{Irr}_{p'}(B)$ is an irreducible $\sigma$-invariant constituent of the character $\psi_B$. Thus, by Corollary 1.2 all irreducible constituents of $\psi_B$ are $\sigma$-invariant. □

Note that the following lemma is true without the additional assumption that the Galois automorphism $\sigma \in \operatorname{Gal}(\mathbb{Q}_{|\tilde{G}|}/\mathbb{Q})$ fixes the $p$-th roots of unity.

**Lemma 3.10.** *Let $\sigma \in \operatorname{Gal}(\mathbb{Q}_m/\mathbb{Q})$, where $m = |\tilde{G}|$, be an $(e, p)$-Galois automorphism. Let $\psi \in \operatorname{Irr}_{p'}(\tilde{B})$ and $\chi = \tilde{f}(\psi)$. Then the character $\psi_B$ is $\sigma$-invariant if and only if $\chi_G$ is $\sigma$-invariant.*

*Proof.* Suppose that $\psi_B$ is $\sigma$-invariant. Then $\psi^\sigma = \psi\eta$ for some $\eta \in \mathrm{Irr}(\tilde{B} \mid 1_B)$ by Lemma 2.14. Using Theorem 2.13 we obtain

$$\tilde{f}(\psi^\sigma) = \tilde{f}(\psi\eta) = \tilde{f}(\psi)\tilde{f}(\eta),$$

with $\tilde{f}(\eta) \in \mathrm{Irr}(\tilde{G} \mid 1_G)$. By Theorem 3.4, we have $\tilde{f}(\psi)^\sigma = \tilde{f}(\psi^\sigma)$. Therefore

$$\chi_G^\sigma = (\tilde{f}(\psi)_G)^\sigma = \tilde{f}(\psi^\sigma)_G = \tilde{f}(\psi)_G\tilde{f}(\eta)_G = \tilde{f}(\psi)_G = \chi_G,$$

which shows that $\chi_G$ is $\sigma$-invariant. An analogous argument shows that if $\chi_G$ is $\sigma$-invariant then $\psi_B$ is $\sigma$-invariant. This shows that $\psi_B$ is $\sigma$-invariant if and only if $\chi_G$ is $\sigma$-invariant. □

Using the results of this section we can draw the following conclusion.

**Corollary 3.11.** *Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$, where $m = |G|$, be an $(e,p)$-Galois automorphism which fixes the p-th roots of unity. Let $\psi \in \mathrm{Irr}_{p'}(\tilde{B})$ and let $\chi = \tilde{f}(\psi)$. Then all irreducible constituents of $\psi_B$ are $\sigma$-invariant if and only if all irreducible constituents of $\chi_G$ are $\sigma$-invariant.*

*Proof.* If $\psi_B$ has an irreducible $\sigma$-invariant constituent then all irreducible constituents of $\psi_B$ are $\sigma$-invariant by Corollary 1.2. This implies that $\psi_B$ is $\sigma$-invariant. The same argument shows that if $\chi_G$ has an irreducible $\sigma$-invariant constituent then $\chi_G$ is $\sigma$-invariant. Now the claim of the corollary follows directly by Lemma 3.10, together with Lemma 3.8 and Lemma 3.9. □

This corollary allows us to define a bijection $f : \mathrm{Irr}_{p'}(B)^\sigma \to \mathrm{Irr}_{p'}(G)^\sigma$ (see Theorem 3.26 below). In the subsequent sections we will drop the assumption that the Galois automorphism $\sigma$ fixes the $p$-th roots of unity. In order to show an analogous statement as in Corollary 3.11 for this more general case we need to find appropriate generalizations of Lemma 3.8 and Lemma 3.9.

## 3.3 Relating the $p'$-characters of $\tilde{B}$ and $B$

In this section we generalize the results of the previous section by allowing a larger class of Galois automorphisms. In a first step we generalize Lemma 3.9.

**Lemma 3.12.** *Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$ be a Galois automorphism, where $m = |\tilde{G}|$. Let $\psi \in \mathrm{Irr}_{p'}(\tilde{B})$ and $\vartheta$ be an irreducible constituent of $\psi_B$. Furthermore, we let $S$ be the unique subset of $\{1,\ldots,r\}$ such that $\psi \in \mathrm{Irr}(\tilde{B} \mid \phi_S)$. Then $\vartheta$ is $\sigma$-invariant if and only if $\psi_B$ is $\sigma$-invariant and there exists an element $t \in B$ such that $\phi_S^\sigma = \phi_S^t$.*

*Proof.* We first assume that $\vartheta$ is $\sigma$-invariant. By Corollary 1.2 all irreducible constituents of $\psi_B$ are $\sigma$-invariant. This implies that $\psi_B$ is $\sigma$-invariant as well. The character $\phi_S$ is a constituent of some $\tilde{B}$-conjugate of $\vartheta$, such that we may assume that $\phi_S$ is below $\vartheta$. Since $\vartheta$ is $\sigma$-invariant we conclude that $\phi_S^\sigma$ is a constituent of $\vartheta_U$ again. By Theorem 1.1 it follows that $\phi_S^\sigma$ is $B$-conjugate to $\phi_S$.

Now assume conversely that $\psi_B$ is $\sigma$-invariant and that there exists some $t \in B$ such that $\phi_S^\sigma = \phi_S^t$. Let $I = \mathrm{I}_B(\phi_S)$ and $\tilde{I} = \mathrm{I}_{\tilde{B}}(\phi_S)$. We let $\lambda \in \mathrm{Irr}(\tilde{I} \mid \phi_S)$ be the character such that $\lambda^{\tilde{B}} = \psi$ (see Theorem 1.3). As in the proof of Lemma 3.9 we see that $\lambda_I^B$ is an irreducible constituent of $\psi_B$. By Corollary 1.2 we may assume that $\vartheta = (\lambda_I)^B$. Since $\psi_B$ is $\sigma$-invariant, we have $0 \neq (\psi_B, \vartheta) = ((\psi^\sigma)_B, \vartheta)$. Therefore $\psi^\sigma \in \mathrm{Irr}(\tilde{B} \mid \vartheta)$. By Lemma 2.14 there exists a character $\eta \in \mathrm{Irr}(\tilde{B} \mid 1_B)$ such that $\psi^\sigma = \psi\eta$. Since $\lambda$ lies above $\phi_S$, it follows that $\lambda^\sigma$ lies above $\phi_S^\sigma = \phi_S^t$. Thus, the character $(\lambda^\sigma)^{t^{-1}}$ lies above $\phi_S$. This implies that

$$((\lambda^\sigma)^{t^{-1}})^{\tilde{B}} = \psi^\sigma = \psi\eta = (\lambda\eta_{\tilde{I}})^{\tilde{B}}.$$

By Theorem 1.3, we have $(\lambda^\sigma)^{t^{-1}} = \lambda\eta_{\tilde{I}}$. Since $\eta \in \mathrm{Irr}(\tilde{B} \mid 1_B)$ this implies $\lambda_I^\sigma = \lambda_I^t$. This shows that $\vartheta = (\lambda_I^t)^B = (\lambda_I^\sigma)^B = \vartheta^\sigma$. □

Recall that the character $\psi_1$ introduced before Definition 1.32 is the character $\phi_S$ for $S = \{1, \ldots, r\}$ (see remark preceding Lemma 2.1). Moreover, by Lemma 2.4 there exists an element $\tilde{t} \in \tilde{T}$ such that $\psi_1^\sigma = \psi_1^{\tilde{t}}$. As a consequence of Lemma 3.12 we obtain the following corollary.

**Corollary 3.13.** *Let $\psi \in \mathrm{Irr}_{p'}(\tilde{B})$. Suppose that $\psi \in \mathrm{Irr}(\tilde{B} \mid \phi_S)$ for $S \subseteq \{1, \ldots, r\}$. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$, where $m = |\tilde{G}|$, be a Galois automorphism. We let $\tilde{t} \in \tilde{T}$ such that $\psi_1^\sigma = \psi_1^{\tilde{t}}$. Then every irreducible constituent of $\psi_B$ is $\sigma$-invariant if and only if $\psi_B$ is $\sigma$-invariant and $\tilde{t} \in \mathrm{I}_{\tilde{B}}(\phi_S)B$.*

*Proof.* Since the conjugation action of $\tilde{\mathbf{T}}^F$ stabilizes the subgroups $\mathbf{U}_{A_i}^F$, we obtain

$$\psi_1^\sigma = \phi_S^\sigma \times \phi_{S^c}^\sigma = \phi_S^{\tilde{t}} \times \phi_{S^c}^{\tilde{t}} = \psi_1^{\tilde{t}}.$$

This shows that $\phi_S^\sigma = \phi_S^{\tilde{t}}$. By Lemma 3.12 every irreducible constituent of $\psi_B$ is $\sigma$-invariant if and only if $\psi_B$ is $\sigma$-invariant and $\phi_S^\sigma = \phi_S^{\tilde{t}} = \phi_S^t$ for some $t \in B$. This is equivalent to saying that $\tilde{t}t^{-1} \in \mathrm{I}_{\tilde{B}}(\phi_S)$. The latter statement is equivalent to $\tilde{t} \in \mathrm{I}_{\tilde{B}}(\phi_S)B$. □

We now show on an example how one can compute the number of $\sigma$-invariant characters of $\mathrm{Irr}_{p'}(B)$. For simplicity let $F = F_q$ be a standard

Frobenius and suppose that $d \neq 0$. Let us assume that $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$ is an $(e,p)$-Galois automorphism. Furthermore we let $\xi$ be a primitive $m$-th root of unity and $k$ be an integer such that $\sigma(\xi) = \xi^k$.

We need to compute $\psi_1^\sigma$. Let $(a_1, \ldots, a_n) \in \mathbb{F}_q^n$ be arbitrary. Then we obtain

$$\psi_1(x_1(a_1), \ldots, x_n(a_n))^\sigma = \prod_{i=1}^n \phi_i(x_i(a_i))^\sigma = \prod_{i=1}^n \phi_i(x_i(a_i))^k = \prod_{i=1}^n \phi_0(ka_i).$$

by Lemma 1.8. Let $\mu \in \mathbf{k}^\times$ be the fixed $(q-1)$-th root of unity chosen before Lemma 1.14. We let $c$ be an integer such that $\mu^c = k$ in $\mathbb{F}_q^\times$ and define $\tilde{t} = \prod_{i=1}^n t_i^c$. Then we obtain

$$\psi_1(x_1(a_1), \ldots, x_n(a_n))^{\tilde{t}} = \prod_{i=1}^n \phi_i(x_i(a_i))^{\tilde{t}} = \prod_{i=1}^n \phi_i(x_i(\mu^c a_i)) = \prod_{i=1}^n \phi_0(ka_i),$$

thanks to [15, Proposition 8.1]. This shows $\psi_1^\sigma = \psi_1^{\tilde{t}}$. We want to find out in which cases $\tilde{t} \in \mathrm{I}_{\tilde{B}}(\phi_S)B$ for a subset $S$ of $\{1, \ldots, n\}$.

Note that $\mathrm{I}_{\tilde{B}}(\phi_S) = \langle t_i \mid i \in S^c \rangle U$ by the remark before [15, Lemma 8.5]. Thus, we have $\tilde{t} \in \mathrm{I}_{\tilde{B}}(\phi_S)B$ if and only if there exists some $x \in \langle t_i \mid i \in S^c \rangle$ such that $\tilde{t}x^{-1} \in B$. The homomorphism $\det : \tilde{\mathbf{B}} \to \mathbf{k}^\times$ gives rise to an isomorphism $\det : \tilde{B}/B \to \mathrm{Z}(\tilde{G})$ (see Lemma 1.45). Hence, $\tilde{t}x^{-1} \in B$ if and only if $\det(\tilde{t}) = \det(x)$. We have $\det(t_i) = t_0^{e_i}$ where the nonnegative integers $e_i$ are defined as in [15, Proposition 11.4]. Writing $x = \prod_{i \in S^c} t_i^{d_i}$, we have

$$\det(x) = \prod_{i \in S^c} t_0^{d_i e_i} = t_0^{\sum_{i \in S^c} d_i e_i} \text{ and } \det(\tilde{t}) = \prod_{i=1}^n t_0^{ce_i} = t_0^{c \sum_{i=1}^n e_i}.$$

Now the element $t_0 \in \tilde{T}$ has order $q-1$ since $d \neq 0$ (see Remark 2.2). Thus, the equation $\det(\tilde{t}) = \det(x)$ is equivalent to

$$\sum_{i \in S^c} d_i e_i \equiv c \sum_{i=1}^n e_i \mod (q-1).$$

**Example 3.14.** We now consider the special case that $\mathbf{G}$ is of type $C_n$ and that $q$ is odd. In this case we have $e_i = 2$ for $i < n$ and $e_n = 1$ (see [15, Example 11.7]). If $n \in S^c$ the equation $\sum_{i \in S^c} d_i e_i \equiv c \sum_{i=1}^n e_i \mod (q-1)$ becomes

$$\sum_{i \in S^c \setminus \{n\}} 2d_i + d_n \equiv c(2(n-1)+1) \mod (q-1).$$

A solution to this is clearly given by $d_n = c(2(n-1)+1)$ and $d_i = 0$ for $i \in S^c \backslash \{n\}$. If $n \notin S^c$ we obtain the equation

$$\sum_{i \in S^c} 2d_i \equiv c(2(n-1)+1) \mod (q-1).$$

Now this is equivalent to the equation $\sum_{i \in S^c} d_i \equiv \frac{c}{2}(2(n-1)+1) \mod \frac{(q-1)}{2}$ if $c \equiv 0 \mod 2$ and has no solution if $c \equiv 1 \mod 2$. In the first case $d_0 = \frac{c}{2}(2(n-1)+1)$ and $d_i = 0$ for all $i \in S^c \backslash \{0\}$ is a solution to this equation. We conclude that $\tilde{t} \in T$ if $c \equiv 0 \mod 2$. On the other hand, if $c \equiv 1 \mod 2$ then $\tilde{t} \in \mathrm{I}_{\tilde{B}}(\phi_S)$ if and only if $n \in S^c$.

For $\psi \in \mathrm{Irr}_{p'}(\tilde{B})$ we want to give sufficient and necessary conditions for $\psi_B$ to be $\sigma$-invariant. We let $(c_0, \dots, c_n)$ be the label of $\psi$. Suppose that $\phi_S \in \mathrm{Irr}(U)$ is below $\psi$. By Lemma 2.14 we conclude that $\psi_B$ is $\sigma$-invariant if and only if $\psi^\sigma = \psi\eta$ for some $\eta \in \mathrm{Irr}(\tilde{B} \mid 1_B)$. The labels of characters $\eta \in \mathrm{Irr}(\tilde{B} \mid 1_B)$ are given by $(\lambda^{e_0}, \dots, \lambda^{e_n})$ for $\lambda \in \mathbb{F}_q^\times$ by [15, Proposition 11.4]. Recall from Theorem 3.4 that the label of $\psi^\sigma$ is $g(\psi^\sigma) = (c_0^{p^e}, \dots, c_n^{p^e})$. Hence, $\psi^\sigma = \psi\eta$ if and only if $c_i^{p^e-1} = \lambda^{e_i}$ for all $i \in S^c$. Thus, we conclude that $\psi_B$ is $\sigma$-invariant if and only if there exists some $\lambda \in \mathbb{F}_q^\times$ such that $c_i^{p^e-1} = \lambda^{e_i}$ for all $i \in S^c$. Let us now continue Example 3.14.

**Example 3.15.** Let us again consider the case that the root system of $\mathbf{G}$ is of type $C_n$ and that $q \equiv 1 \mod 2$. Let $q = p^f$ and let $s \in \{1, \dots, f\}$ be defined as $s = \gcd(e, f)$. Note that this definition occurred previously in Example 3.6. Let $\psi \in \mathrm{Irr}_{p'}(\tilde{B})$ and suppose that $\phi_S$ is below $\psi$. Let $g(\psi) = (c_0, \dots, c_n)$ be the label of $\psi$. As we have seen above, the character $\psi_B$ is $\sigma$-invariant if and only if there exists some $\lambda \in \mathbb{F}_q^\times$ such that $c_i^{p^e-1} = \lambda^{e_i}$ for all $i \in S^c$. Thus, by Theorem 2.7 we need to find all tuples $(c_0, \dots, c_n) \in \mathcal{A}$, where $\mathcal{A} = (\mathbb{F}_q^\times)^d \times \mathbb{F}_q^n$, such that there exists some $\lambda \in \mathbb{F}_q^\times$ satisfying the system of equations

$$c_i^{p^e-1} = \lambda^{e_i} \text{ for all } i \in S^c,$$

where $S = \{i \mid c_i = 0\}$ and $S^c = \{0, \dots, n\} \backslash S$.

If $c_n \neq 0$ or equivalently $n \in S^c$ then $\lambda := c_n^{\frac{p^e-1}{2}}$ satisfies $c_n^{p^e-1} = \lambda^2$. We conclude that the label $(c_0, \dots, c_n)$ satisfies $c_i^{p^e-1} = \lambda^{e_i}$ for all $i \in S^c$ if and only if $c_i = s_i c_n^2$ for all $i \in S^c \backslash \{n\}$ and some $s_i \in \mathbb{F}_{p^s}^\times$. Consequently, there are precisely $(q-1)p^{s(n-1)}(p^s-1)$ labels $(c_0, \dots, c_n) \in \mathcal{A}$ with $c_n \neq 0$ satisfying this system of equations.

If $c_n = 0$ we obtain the system of equations $c_i^{p^e-1} = \lambda$ for all $i \in S^c$. Hence, the label $(c_0, \dots, c_n)$ satisfies this system of equation if and only if

$c_i = s_i c_0$ for all $i \in S^c \backslash \{0\}$ and $s_i \in \mathbb{F}_{p^s}^\times$. Thus, in this case we have precisely $(q-1)p^{s(n-1)}$ possible labels.

We are now able to compute the number of $\sigma$-invariant $p'$-characters of $B$. Let $\vartheta \in \mathrm{Irr}(B)$ below $\psi \in \mathrm{Irr}_{p'}(\tilde{B})$. Suppose that $c \equiv 0 \mod 2$ (where $c$ is defined as in Example 3.14). We write $M_\psi = \mathrm{Irr}(\tilde{B} \mid \vartheta)$ for the set of characters above $\vartheta$ and $N_\psi = \mathrm{Irr}(B \mid \psi)$ for the set of characters below $\psi$. For a set of representatives $\Omega \subseteq \mathrm{Irr}_{p'}(\tilde{B})$ of the $\tilde{B}$-orbits of $\mathrm{Irr}_{p'}(\tilde{B})$ we have by Clifford theory (see also [15, Section 11]) a partition

$$\mathrm{Irr}_{p'}(\tilde{B}) = \bigcup_{\psi \in \Omega} M_\psi$$

and a corresponding partition

$$\mathrm{Irr}_{p'}(B) = \bigcup_{\psi \in \Omega} N_\psi.$$

Let us denote by $N_\psi^\sigma$ the subset of $\sigma$-invariant characters of $N_\psi$. Furthermore, we let $\Omega' \subseteq \Omega$ be the subset of $\Omega$ consisting of the characters $\psi \in \Omega$ such that $\psi_B$ is $\sigma$-invariant. Using Corollary 3.13 we obtain

$$\mathrm{Irr}_{p'}(B)^\sigma = \bigcup_{\psi \in \Omega'} N_\psi^\sigma.$$

Since $c \equiv 0 \mod 2$ we have $\tilde{t} = \prod_{i=1}^n t_i^c \in T$ by Example 3.14. Since $\psi^\sigma = \psi^{\tilde{t}}$ as shown in Example 3.14 this implies $N_\psi^\sigma = N_\psi$ for all $\psi \in \Omega'$ by Corollary 3.13. By [15, Example 11.7] we have $|M_\psi| = q-1$, if $c_n = 0$, and $|M_\psi| = \frac{q-1}{2}$, if $c_n \neq 0$. Moreover using [15, Proposition 11.6] we have $|N_\psi| = 1$, if $c_n = 0$, and $|N_\psi| = 2$, if $c_n \neq 0$. As we have seen above, $\mathrm{Irr}_{p'}(\tilde{B})$ consists of $(q-1)p^{s(n-1)}(p^s-1)$ characters with label $c_n \neq 0$ and $(q-1)p^{s(n-1)}$ characters with label $c_n = 0$.

Thus, we conclude that there are precisely $p^{s(n-1)}(p^s-1)$ elements of $\Omega'$ with label $c_n \neq 0$ and $2p^{s(n-1)}$ elements of $\Omega'$ with label $c_n = 0$. As a consequence we conclude that

$$|\mathrm{Irr}_{p'}(B)^\sigma| = p^{s(n-1)}(p^s-1) + 4p^{s(n-1)} = p^{sn} + 3p^{s(n-1)}.$$

If $c \equiv 1 \mod 2$ an analogous computation proves that

$$|\mathrm{Irr}_{p'}(B)^\sigma| = p^{s(n-1)}(p^s-1).$$

Note that if $\psi \in \Omega'$ (i.e., $\psi_B$ is $\sigma$-invariant) and $|N_\psi| = 2$ it follows that $\sigma$ permutes the two characters of $N_\psi$ if $c \equiv 1 \mod 2$.

Let us single out a special case. If $\sigma = id$ then $s = f$ and $c \equiv 0 \mod 2$. This implies that $|\mathrm{Irr}_{p'}(B)| = p^{fn} + 3p^{f(n-1)} = q^n + 3q^{n-1}$ and we recover as a special case the result obtained by Maslowski in [15, Example 11.7].

## 3.4 Relating the $p'$-characters of $\tilde{G}$ and $G$

Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$, where $m = |\tilde{G}|$, be any Galois automorphism. Let $\chi \in \mathrm{Irr}_{p'}(\tilde{G})$ be an irreducible $p'$-character of $\tilde{G}$. In this section we give sufficient and necessary conditions for the irreducible constituents of $\chi_G$ to be $\sigma$-invariant.

We fix some notation for the remainder of this section. By Lemma 1.36, we know that $\chi = \pm \mathrm{D}_{\tilde{\mathbf{G}}}(\chi_{(\tilde{s})})$ for some semisimple conjugacy class $(\tilde{s})$ of $\tilde{\mathbf{G}}^{*F^*}$. As in Section 1.6 we let $i^* : \tilde{\mathbf{G}}^* \to \mathbf{G}^*$ be the dual map corresponding to the regular embedding $i : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$. We define $s := i^*(\tilde{s})$. By Lemma 1.38 the characters $\psi_{(s),z} = \pm \mathrm{D}_{\mathbf{G}}(\chi_{(s),z})$, $z \in H^1(F, \mathrm{Z}(\mathbf{G}))$, are precisely the irreducible constituents of $\chi_G$. Note that by Clifford's Theorem the condition that $\chi_G$ is $\sigma$-invariant is necessary (see Corollary 1.2). So we may assume that $\chi_G$ is $\sigma$-invariant. We have $\chi_G = \pm \mathrm{D}_{\mathbf{G}}((\chi_{(\tilde{s})})_G)$ since restriction commutes with the duality functor by Lemma 1.37. Thus, by Lemma 1.25 we conclude that $\chi_{(s)} = (\chi_{(\tilde{s})})_G$ is $\sigma$-invariant. Again by Lemma 1.25 it follows that the character $\psi_{(s),z}$ is $\sigma$-invariant if and only if $\chi_{(s),z}$ is $\sigma$-invariant.

We study the action of the Galois automorphism $\sigma$ on the Gelfand–Graev characters more closely. Let $\Gamma_z = ({}^{t_z}\psi_1)^G$ be the Gelfand–Graev character corresponding to $z \in H^1(F, \mathrm{Z}(\mathbf{G}))$. By Lemma 2.4 we know that $\psi_1^\sigma = \psi_1^{\tilde{t}}$ for some suitable $\tilde{t} \in \tilde{T}$. As in Remark 1.33 we consider the isomorphism $\beta : \tilde{\mathbf{G}}^F/\mathbf{G}^F \mathrm{Z}(\tilde{\mathbf{G}})^F \to \mathscr{L}_{\mathbf{T}}^{-1}(\mathrm{Z}(\mathbf{G}))/\mathrm{Z}(\mathbf{G})\mathbf{T}^F = H^1(F, Z(\mathbf{G}))$. Suppose that the coset of $\tilde{t}^{-1}$ in $\tilde{\mathbf{G}}^F/\mathbf{G}^F \mathrm{Z}(\tilde{\mathbf{G}})^F$ maps to $z' \in H^1(F, \mathrm{Z}(\mathbf{G}))$ via the map $\beta$. We define $g_{z'} = \tilde{t}^{-1}$. Recall from Remark 1.33 that $\Gamma_{z'} = {}^{g_{z'}}\Gamma_1$. It follows that

$$\Gamma_z^\sigma = ({}^{t_z}(\psi_1^\sigma))^G = (\psi_1^{\tilde{t}t_z^{-1}})^G = \Gamma_z^{\tilde{t}} = {}^{g_{z'}}\Gamma_z.$$

Moreover, we have

$$\chi_{(s)}^\sigma = \chi_{(s)} = (\chi_{(\tilde{s})})_G = ({}^{g_{z'}}\chi_{(\tilde{s})})_G = {}^{g_{z'}}\chi_{(s)}.$$

By definition of $\chi_{(s),z}$, we know that the character $\chi_{(s),z}^\sigma$ is the unique common irreducible constituent of $\Gamma_z^\sigma = {}^{g_{z'}}\Gamma_z$ and $\chi_{(s)}^\sigma = {}^{g_{z'}}\chi_{(s)}$. We conclude that $\chi_{(s),z}^\sigma = {}^{g_{z'}}\chi_{(s),z}$. This shows that the irreducible constituent $\psi_{(s),z}$ of $\chi_G$ is $\sigma$-invariant if and only if $\chi_{(s),z} = {}^{g_{z'}}\chi_{(s),z}$. The subsequent lemma gives a sufficient and necessary condition for the character $\chi_{(s),z}$ to be $\sigma$-invariant. Note that we identify characters which correspond to each other under the inflation map $\mathrm{Irr}(\tilde{\mathbf{G}}^F/\mathbf{G}^F \mathrm{Z}(\tilde{\mathbf{G}})^F) \to \mathrm{Irr}(\tilde{\mathbf{G}}^F \mid 1_{\mathbf{G}^F Z(\tilde{\mathbf{G}})^F})$.

**Lemma 3.16.** *Let $z' \in H^1(F, \mathrm{Z}(\mathbf{G})) = \tilde{\mathbf{G}}^F/\mathbf{G}^F \mathrm{Z}(\tilde{\mathbf{G}})^F$ with representative $g_{z'} \in \tilde{\mathbf{G}}^F$. Then ${}^{g_{z'}}\chi_{(s),z} = \chi_{(s),z}$ if and only if $\lambda(z') = 1$ for all characters $\lambda \in \mathrm{Irr}(\tilde{\mathbf{G}}^F/\mathbf{G}^F \mathrm{Z}(\tilde{\mathbf{G}})^F)$ with $\lambda\chi_{(\tilde{s})} = \chi_{(\tilde{s})}$.*

*Proof.* This follows from the proof of [4, Proposition 3.12 (ii)]. Since this lemma is central for our argumentation we will reproduce the proof. First note that ${}^{g_{z'}}\chi_{(s),z}$ is the unique common constituent of ${}^{g_{z'}}\Gamma_{z'} = \Gamma_{z'z}$ and ${}^{g_{z'}}\chi_{(s)} = \chi_{(s)}$. Therefore we have ${}^{g_{z'}}\chi_{(s),z} = \chi_{(s),zz'}$. Let us now define

$$H(s) = \{\lambda \in \mathrm{Irr}(\tilde{\mathbf{G}}^F/\mathbf{G}^F\,\mathrm{Z}(\tilde{\mathbf{G}})^F) \mid \lambda\chi_{(\tilde{s})} = \chi_{(\tilde{s})}\}$$

and

$$K(s) = \{z' \in \tilde{\mathbf{G}}^F/\mathbf{G}^F\,\mathrm{Z}(\tilde{\mathbf{G}})^F \mid {}^{g_{z'}}\chi_{(s),z} = \chi_{(s),z}\}.$$

Recall from Section 1.6 that $(\chi_{(\tilde{s})})_{\mathbf{G}^F} = \chi_{(s)} = \sum_z \chi_{(s),z}$ where the sum is over the distinct characters $\chi_{(s),z}$ for $z \in H^1(F, \mathrm{Z}(\mathbf{G}))$. Thus, the restriction of $\chi_{(\tilde{s})}$ to $\mathbf{G}^F$ is multiplicity free. Now [12, Proposition 2.1] shows that this implies

$$K(s) = \{z' \in \tilde{\mathbf{G}}^F/\mathbf{G}^F\,\mathrm{Z}(\tilde{\mathbf{G}})^F \mid \lambda(z') = 1 \text{ for all } \lambda \in H(s)\}.$$

But this is precisely the claim of the lemma. □

This leads to the following corollary which will be of importance.

**Corollary 3.17.** *Let $\tilde{t} \in \tilde{T}$ such that $\psi_1^\sigma = \psi_1^{\tilde{t}}$. Let $\chi = \pm\,\mathrm{D}_{\tilde{\mathbf{G}}}(\chi_{(\tilde{s})}) \in \mathrm{Irr}_{p'}(\tilde{G})$. Then all irreducible constituents of $\chi_G$ are $\sigma$-invariant if and only if $\chi_G$ is $\sigma$-invariant and $\lambda(\tilde{t}) = 1$ for all $\lambda \in \mathrm{Irr}(\tilde{G} \mid 1_G)$ satisfying $\lambda\chi_{(\tilde{s})} = \chi_{(\tilde{s})}$.*

*Proof.* Recall from the discussion of the beginning of this section that the condition that $\chi_G$ is $\sigma$-invariant is necessary for the irreducible constituents of $\chi_G$ to be $\sigma$-invariant. Moreover, if $\chi_G$ is $\sigma$ invariant, then the irreducible constituents of $\chi_G$ are $\sigma$-invariant if and only if $\chi_{(s),z} = {}^{g_{z'}}\chi_{(s),z}$ for $z \in H^1(F, \mathrm{Z}(\mathbf{G}))$, where $g_{z'}^{-1} = \tilde{t}$. Now if $\lambda \in \mathrm{Irr}(\tilde{G} \mid 1_G)$ is a character such $\lambda\chi_{(\tilde{s})} = \chi_{(\tilde{s})}$ then automatically $\mathbf{G}^F Z(\tilde{\mathbf{G}})^F \subseteq \ker(\lambda)$ by [12, Proposition 2.4 (iv)]. Thus, the claim of the corollary follows by Lemma 3.16. □

**Example 3.18.** In the situation of the previous corollary suppose that $\psi_1$ is $\sigma$-invariant. This is for example the case if $\sigma$ fixes the $p$-th roots of unity. In this case, we obtain $z' = 1$ and $\lambda(z') = 1$ for all $\lambda \in \mathrm{Irr}(\tilde{G} \mid 1_G)$. Thus, it follows that the irreducible constituents of $\chi_G$ are $\sigma$-invariant if and only if $\chi_G$ is $\sigma$-invariant. Hence, Corollary 3.17 is a generalization of Lemma 3.8.

We will now investigate the bijection $\tilde{f} : \mathrm{Irr}(\tilde{B} \mid 1_B) \to \mathrm{Irr}(\tilde{G} \mid 1_G)$ in more detail in order to determine the values of the linear characters $\lambda \in \mathrm{Irr}(\tilde{G} \mid 1_G)$ at the elements $t_i$.

For the subsequent lemma recall that if $\lambda \in \mathrm{Irr}(\tilde{G} \mid 1_G)$ then $\lambda \in \mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{z}))$ for some central element $\tilde{z} \in \mathrm{Z}(\tilde{\mathbf{G}}^{*F^*})$ (see [15, Proposition 13.3]). Moreover by [5, Proposition 13.30] we see that $(\tilde{\mathbf{T}}, \lambda_{\tilde{T}})$ is in duality with $(\tilde{\mathbf{T}}^*, \tilde{z})$.

**Lemma 3.19.** *Let $\lambda \in \mathrm{Irr}(\tilde{G} \mid 1_G)$ lie in the Lusztig series $\mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{z}))$. Then $\lambda(t_0) = \det^*(\tilde{z})$ and $\lambda(t_i) = \tilde{\omega}^*_{a_i}(\tilde{z})$ for all $i = 1, \dots, r$.*

*Proof.* This was shown in [15, Proposition 13.4] for $t_0$. We prove the remaining cases by replacing $\det^*$ and $t_0$ by $\tilde{\omega}^*_i$ and $t_i$ in the proof of [15, Proposition 13.4].

Let $\mu \in \mathbf{k}^\times$ be the primitive $(q^w - 1)$-th root of unity chosen before Lemma 1.14. Let $\theta := \lambda_{\tilde{T}} \in \mathrm{Irr}(\tilde{T})$ and $\hat{\theta} \in X(\tilde{\mathbf{T}})$ be the lift of $\theta$ as in the proof of Lemma 1.14. As we have indicated above, $(\tilde{\mathbf{T}}, \theta)$ is in duality with $(\tilde{\mathbf{T}}^*, \tilde{z})$. Let us define $t = \delta^\vee(\hat{\theta})(\mu)$. Recall from Remark 1.50 that $\delta(\tau_i) = \tilde{\omega}^*_i$ for all $i = 1, \dots, n$. Then using Remark 1.12 we obtain

$$\tilde{\omega}^*_{a_i}(t) = \tilde{\omega}^*_{a_i}(\delta^\vee(\hat{\theta}(\mu))) = \mu^{\langle \tilde{\omega}^*_{a_i}, \delta^\vee(\hat{\theta})\rangle} = \mu^{\langle \delta(\tau_{a_i}), \delta^\vee(\hat{\theta})\rangle} = \mu^{\langle \hat{\theta}, \tau_{a_i}\rangle} = \hat{\theta}(\tau_{a_i}(\mu)).$$

Since the pair $(\tilde{\mathbf{G}}, F)$ is dual to the pair $(\tilde{\mathbf{G}}^*, F^*)$ via the duality isomorphism $\delta$ it follows that $\delta$ commutes with the action of $F$ and $F^*$ (see Definition 1.11). Using Remark 1.12 again we obtain

$$\tilde{\omega}^*_{a_i}(F^*(t)) = \mu^{\langle \tilde{\omega}^*_{a_i} \circ F^*, \delta^\vee(\hat{\theta})\rangle} = \mu^{\langle \delta(F \circ \tau_{a_i}), \delta^\vee(\hat{\theta})\rangle} = \mu^{\langle \hat{\theta}, F \circ \tau_{a_i}\rangle} = \hat{\theta}(F(\tau_{a_i}(\mu))).$$

Recall that $\tilde{z} = N_{F^{*w}/F^*}(t)$ since $(\tilde{\mathbf{T}}^*, \tilde{z})$ is in duality with $(\tilde{\mathbf{T}}, \theta)$ (see proof of Lemma 1.14). The two equations above imply that

$$\tilde{\omega}^*_{a_i}(\tilde{z}) = \tilde{\omega}^*_{a_i}(N_{F^{*w}/F^*}(t)) = \prod_{i=0}^{w-1} \tilde{\omega}^*_{a_i}(F^{*i}(t)) = \hat{\theta}(N_{F^w/F}(\tau_{a_i}(\mu))) = \theta(t_i).$$

As $\theta$ is the restriction of the character $\lambda$ to $\tilde{T}$ we have $\theta(t_i) = \lambda(t_i)$. This shows $\lambda(t_i) = \tilde{\omega}^*_{a_i}(\tilde{z})$ as desired. □

**Corollary 3.20.** *Let $\lambda \in \mathrm{Irr}(\tilde{G} \mid 1_G)$ lie in the Lusztig series $\mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{z}))$. Then $\tilde{f}^{-1}(\lambda) = \lambda_{\tilde{B}}$.*

*Proof.* Using Lemma 3.19 we see that the character $\lambda_{\tilde{B}} \in \mathrm{Irr}(\tilde{B} \mid 1_B)$ has label

$$g(\lambda_{\tilde{B}}) = (\det^*(\tilde{z}), \tilde{\omega}^*_{a_1}(\tilde{z}), \dots, \tilde{\omega}^*_{a_r}(\tilde{z})).$$

Moreover, the label of the character $\lambda$ is given by

$$h(\lambda) = (\det^*(\tilde{z}), \pi^\vee_{a_1}(1)\tilde{\omega}^*_{a_1}(\tilde{z}), \dots, \pi^\vee_{a_r}(1)\tilde{\omega}^*_{a_r}(\tilde{z}))$$

by definition of the labeling in Construction 2.10. By the remark below Theorem 2.13 we obtain $h(\lambda) = (\det^*(\tilde{z}), \tilde{\omega}^*_{a_1}(\tilde{z}), \ldots, \tilde{\omega}^*_{a_r}(\tilde{z}))$. By definition of $\tilde{f}$ in Theorem 2.12, it follows that the character $\tilde{f}^{-1}(\lambda)$ has the same label as $\lambda$. Thus, the labels of $\tilde{f}^{-1}(\lambda)$ and $\lambda_{\tilde{B}}$ coincide and we must have $\tilde{f}^{-1}(\lambda) = \lambda_{\tilde{B}}$. □

Using Corollary 3.20 we get the following result.

**Corollary 3.21.** *Restriction of characters defines a bijection* $\mathrm{Irr}(\tilde{G} \mid 1_G) \to \mathrm{Irr}(\tilde{B} \mid 1_B)$.

*Proof.* The map $\tilde{f}^{-1} : \mathrm{Irr}(\tilde{G} \mid 1_G) \to \mathrm{Irr}(\tilde{B} \mid 1_B)$ defines a bijection by Theorem 2.13. By Corollary 3.20 this map is given by restriction of characters. □

For $\chi = \pm \mathrm{D}_{\tilde{\mathbf{G}}}(\chi_{(\tilde{s})})$ with label $(b_0, \ldots, b_r)$ we define $S = \{i \mid b_i = 0\}$. Note that $\chi = \pm \mathrm{D}_{\tilde{\mathbf{G}}}(\chi_{(\tilde{s})}) \in \mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{s}))$ by [3, Theorem 6]. Thus, we have $S = \{i \mid \tilde{\pi}_{a_i}(\tilde{s}) = 0\}$. If $\psi \in \mathrm{Irr}_{p'}(\tilde{B})$ with $\tilde{f}(\psi) = \chi$ then $\psi$ has label $g(\psi) = (b_0, \ldots, b_r)$. Consequently, by Construction 2.6 it follows that $\psi$ lies above the character $\phi_S$. Thus, the definition of $S$ for $\chi \in \mathrm{Irr}_{p'}(\tilde{G})$ is consistent with our previous use of $S$.

**Lemma 3.22.** *Let $\chi = \pm \mathrm{D}_{\tilde{\mathbf{G}}}(\chi_{(\tilde{s})})$ and $S$ be defined as above. Let $\lambda \in \mathrm{Irr}(\tilde{G} \mid 1_G)$ lie in the Lusztig series $\mathcal{E}(\tilde{\mathbf{G}}^F, (\tilde{z}))$. Then $\lambda\chi_{(\tilde{s})} = \chi_{(\tilde{s})}$ if and only if $\lambda(t_i) = 1$ for all $i \in S^c$.*

*Proof.* By Lemma 1.21 we conclude that $\lambda\chi_{(\tilde{s})} = \chi_{(\tilde{s})}$ if and only if $(\tilde{s}\tilde{z}) = (\tilde{s})$. By Lemma 2.9 this is the case if and only if $\tilde{\pi}(\tilde{z}\tilde{s}) = \tilde{\pi}(\tilde{s})$.

Let us now write $\tilde{s} = xz$ with $x \in \mathbf{G}^\vee$ and $z \in \mathbf{S}^\vee$. By definition of the modified Steinberg map we have

$$\tilde{\pi}_j(\tilde{z}\tilde{s}) = \tilde{\pi}_j(xz\tilde{z}) = \pi_j^\vee(x)\tilde{\omega}_j^*(z\tilde{z}) = \tilde{\pi}_j(\tilde{s})\tilde{\omega}_j^*(\tilde{z})$$

for all $j = 1, \ldots, n$ and $\det^*(\tilde{s})\det^*(\tilde{z}) = \det^*(\tilde{s})$. Using [15, Proposition 14.4] we conclude that multiplication with $\lambda$ fixes $\chi_{(\tilde{s})}$ if and only if $\tilde{\omega}^*_{a_i}(\tilde{z}) = 1$ for all $i \notin S$ and $\det^*(\tilde{z}) = 1$. The claim of the lemma follows by Lemma 3.19. □

Recall from Lemma 2.3 that there exists $\tilde{t} \in \tilde{T}$ such that $\psi_1^\sigma = \psi_1^{\tilde{t}}$.

**Lemma 3.23.** *Suppose that $\chi = \pm \mathrm{D}_{\tilde{\mathbf{G}}}(\chi_{(\tilde{s})})$ with label $(b_0, \ldots, b_r)$. We let $S = \{i \mid b_i = 0\}$. Let $\tilde{t} \in \tilde{T}$ such that $\psi_1^\sigma = \psi_1^{\tilde{t}}$. Then all constituents of $\chi_G$ are $\sigma$-invariant if and only if $\chi_G$ is $\sigma$-invariant and $\tilde{t} \in \mathrm{I}_{\tilde{B}}(\phi_S)B$.*

*Proof.* By Corollary 3.17 we know that all constituents of $\chi_G$ are $\sigma$-invariant if and only if $\lambda(\tilde{t}) = 1$ for all $\lambda \in \mathrm{Irr}(\tilde{G} \mid 1_G)$ satisfying $\lambda\chi_{(\tilde{s})} = \chi_{(\tilde{s})}$. By Lemma 3.22 we have $\lambda\chi_{(\tilde{s})} = \chi_{(\tilde{s})}$ if and only if $\lambda(t_i) = 1$ for all $i \in S^c$. Restriction of characters defines a bijection $\mathrm{Irr}(\tilde{G} \mid 1_G) \to \mathrm{Irr}(\tilde{B} \mid 1_B)$ by Corollary 3.21. We conclude that all constituents of $\chi_G$ are $\sigma$-invariant if and only if $\lambda(\tilde{t}) = 1$ for all $\lambda \in \mathrm{Irr}(\tilde{B} \mid 1_B)$ satisfying $\lambda(t_i) = 1$. Note that $\tilde{I} := \mathrm{I}_{\tilde{B}}(\phi_S) = \langle t_i \mid i \in S^c \rangle U$ by the remark preceding [15, Lemma 8.5]. So the characters $\lambda \in \mathrm{Irr}(\tilde{B} \mid 1_B)$ with $\lambda(t_i) = 1$ for $i \in S^c$ are precisely the characters in $\mathrm{Irr}(\tilde{B} \mid 1_{B\tilde{I}})$. Now the natural map $^-\colon \mathrm{Irr}(\tilde{B} \mid 1_{B\tilde{I}}) \to \mathrm{Irr}(\tilde{B}/B\tilde{I})$ defines a bijection of characters. Moreover, since $\tilde{B}/B\tilde{I}$ is an abelian group, it follows that $\tilde{t} \in B\tilde{I}$ if and only if $\lambda(\tilde{t}) = \bar{\lambda}(\tilde{t}) = 1$ for all $\bar{\lambda} \in \mathrm{Irr}(\tilde{B}/B\tilde{I})$. We conclude that all constituents of $\chi_G$ are $\sigma$-invariant if and only if $\tilde{t} \in \mathrm{I}_{\tilde{B}}(\phi_S)B$. □

In the next section we will show that the results obtained so far are sufficient to construct a bijection $f : \mathrm{Irr}_{p'}(B)^\sigma \to \mathrm{Irr}_{p'}(G)^\sigma$.

## 3.5 A character bijection for $\sigma$-invariant characters

In this section we prove our main results. First, we use the results of the two previous sections to prove the following theorem.

**Theorem 3.24.** *Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$, where $m = |G|$, be an $(e, p)$-Galois automorphism. Let $\psi \in \mathrm{Irr}_{p'}(\tilde{B})$ and let $\chi = \tilde{f}(\psi)$. All irreducible constituents of $\psi_B$ are $\sigma$-invariant if and only if all irreducible constituents of $\chi_G$ are $\sigma$-invariant.*

*Proof.* By Lemma 3.10 it follows that $\psi_B$ is $\sigma$-invariant if and only if $\chi_G$ is $\sigma$-invariant. Now the claim of the theorem follows immediately by Corollary 3.13 and Lemma 3.23. □

Let us write $\tilde{Z} = \mathrm{Z}(\tilde{\mathbf{G}})^F$ and $Z = \mathrm{Z}(\mathbf{G})^F$. Note that $\tilde{Z} = \mathrm{Z}(\tilde{\mathbf{G}})^F = \mathbf{S}^F \subseteq \tilde{\mathbf{B}}^F$ by Lemma 1.42. Thanks to a result of Maslowski we know that the map $\tilde{f}$ respects central characters:

**Lemma 3.25.** *If $\lambda \in \mathrm{Irr}(\tilde{Z})$ is the unique character below $\psi \in \mathrm{Irr}_{p'}(\tilde{B})$ then $\lambda$ is the unique character of $\tilde{Z}$ below $\tilde{f}(\psi)$.*

*Proof.* See [15, Proposition 15.2]. □

Now we can show our main result.

**Theorem 3.26.** *Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$, where $m = |G|$, be an $(e,p)$-Galois automorphism. Suppose that* $\mathbf{G}$ *satisfies Assumption 3.1. Then there exists a bijection*

$$f : \mathrm{Irr}_{p'}(B)^\sigma \to \mathrm{Irr}_{p'}(G)^\sigma.$$

*Moreover, for every central character $\lambda \in \mathrm{Irr}(\tilde{Z})$ the map $f$ restricts to a bijection $\mathrm{Irr}_{p'}(B \mid \lambda_Z)^\sigma \to \mathrm{Irr}_{p'}(G \mid \lambda_Z)^\sigma$.*

*Proof.* Let us first note that by the remark below Lemma 1.45 we have $p \nmid |\tilde{G} : G|$ and $p \nmid |\tilde{B} : B|$. In particular, we have by Theorem 1.1 that every $p'$-character of $B$ resp. $G$ lies below a $p'$-character of $\tilde{B}$ resp. $\tilde{G}$. Let $\vartheta \in \mathrm{Irr}_{p'}(B)^\sigma$ and $\psi \in \mathrm{Irr}_{p'}(\tilde{B})$ such that $(\psi_B, \vartheta) \neq 0$. By Theorem 3.24 all irreducible constituents of $\tilde{f}(\psi)_G$ are $\sigma$-invariant.

Let $\phi$ be an irreducible constituent of $\tilde{f}(\psi)_G$. The number of $\tilde{B}$-conjugates of $\vartheta$ is given by $|\tilde{B} : \mathrm{I}_{\tilde{B}}(\vartheta)|$ and the number of $\tilde{G}$-conjugates of $\phi$ is given by $|\tilde{G} : \mathrm{I}_{\tilde{G}}(\phi)|$. By Corollary 2.16 the restriction of $\tilde{f}$ gives a bijection $\mathrm{Irr}(\tilde{B} \mid \vartheta) \to \mathrm{Irr}(\tilde{G} \mid \phi)$. Using Lemma 1.6 we see that the number of $\tilde{B}$-conjugates of $\vartheta$ and the number of $\tilde{G}$-conjugates of $\phi$ coincide. Thus, we can define a map $f : \mathrm{Irr}_{p'}(B)^\sigma \to \mathrm{Irr}_{p'}(G)^\sigma$ by sending the set of characters $\{\vartheta^{\tilde{b}} \mid \tilde{b} \in \tilde{B}\}$ bijectively to the set $\{\phi^{\tilde{g}} \mid \tilde{g} \in \tilde{G}\}$. This is possible since these sets have the same cardinality by the previous considerations. The choice of a character $\psi \in \mathrm{Irr}(\tilde{B} \mid \vartheta)$ as above, is unique up to multiplication of $\psi$ by a linear character in $\mathrm{Irr}(\tilde{B} \mid 1_B)$. By Theorem 2.13 this implies that the image $f(\vartheta)$ of $\vartheta$ is determined by $\vartheta$ up to $\tilde{G}$-conjugation. Therefore, it follows that the map $f$ is injective.

Let $\varphi \in \mathrm{Irr}_{p'}(G)^\sigma$ and suppose that $\chi \in \mathrm{Irr}(\tilde{G} \mid \varphi)$. Then by Theorem 3.24 every irreducible constituent of $\tilde{f}^{-1}(\chi)_B$ is $\sigma$-invariant. By construction of $f : \mathrm{Irr}_{p'}(B)^\sigma \to \mathrm{Irr}_{p'}(G)^\sigma$ there exists an irreducible constituent $\nu \in \mathrm{Irr}_{p'}(B)$ of $\tilde{f}^{-1}(\chi)_B$ such that $f(\nu) = \varphi$. Thus, the map $f : \mathrm{Irr}_{p'}(B)^\sigma \to \mathrm{Irr}_{p'}(G)^\sigma$ is surjective as well.

Now we use Lemma 3.25: Let $\lambda \in \mathrm{Irr}(\tilde{Z})$ be the unique central character below $\psi \in \mathrm{Irr}_{p'}(B)$. Since the character $\vartheta$ is below $\psi$, it follows that $\lambda_Z$ is below $\vartheta$. By Lemma 3.25 it follows that $\lambda$ is the unique central character below $\tilde{f}(\psi)$. Since $f(\vartheta)$ is below $\tilde{f}(\psi)$ it follows that $\lambda_Z$ is below $f(\vartheta)$. As a consequence it follows that $\lambda_Z$ is below $f(\psi)$. This shows that the map $f$ restricts to a bijective map $\mathrm{Irr}_{p'}(B \mid \lambda_Z)^\sigma \to \mathrm{Irr}_{p'}(G \mid \lambda_Z)^\sigma$.

□

As a special case we obtain the validity of Conjecture 0.3 for most simple groups of Lie type in defining characteristic.

**Corollary 3.27.** *Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$, where $m = |G|$, be an $(e,p)$-Galois automorphism. Suppose that $\mathbf{G}$ satisfies Assumption 3.1. Then there exists a bijection*

$$f : \mathrm{Irr}_{p'}(B/\mathrm{Z}(G))^{\sigma} \to \mathrm{Irr}_{p'}(G/\mathrm{Z}(G))^{\sigma}.$$

*Proof.* Take $\lambda = 1_{\tilde{Z}}$ in Theorem 3.26. □

Finally, let us discuss how one could possibly remove some assumptions from Theorem 3.26. As we already remarked in the introduction of this thesis the assumption that $\mathbf{G}$ is not of type $D_n$ if $n$ is even can probably removed using the construction of Maslowski in [15] for this root system type. Only minor changes seem to be necessary in order to cover this case as well.

The assumption that $p$ is a good prime for $\mathbf{G}$ can be weakened for most statements occurring in Chapter 3. Usually, we only need to make the weaker assumption that $(\mathbf{G}, F)$ does not occur in [15, Table 13.2] and that $\mathbf{G}$ is not of type $D_n$ if $n$ is even. The assumption that $p$ is a good prime becomes crucial in our description of the $p'$-characters of $\mathbf{G}^F$ using Theorem 1.35. More concretely, if Lemma 3.23 holds under these weaker assumptions then it follows immediately that Theorem 3.26 holds as well under these assumptions. However, the author is unaware whether Lemma 3.23 still holds in this case. We leave this question open to further research.

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