PhD Seminar: Normal Inverse Gaussian Process for Commodities Modeling- and Risk-Management

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The must important step in (energy)risk management is to find a good model for the analyses and estimate the risk measures of energy derivatives. The normal distribution is the most used tool for the modeling of financial (log)return, but the observation of empirical data show that Commodity returns have distributions with some stylized features like:

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Commodity markets: Basis Concept

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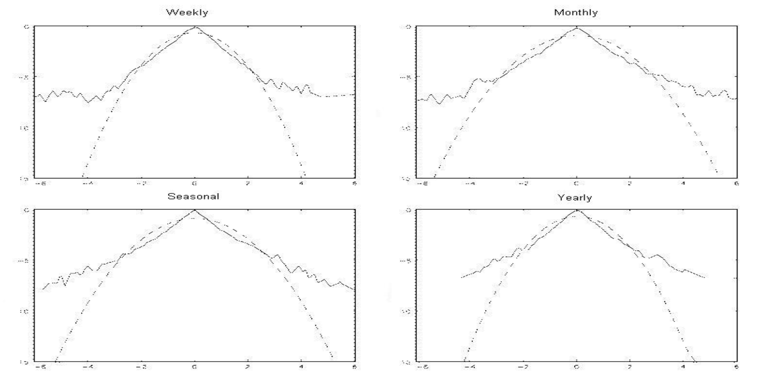
Gaussian vs Empirical

Compare the empirical distribution of electricity forwards log-returns to the normal fitting



Motivation of NIG

from figure 1 we can see:

- No normality
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Motivation of NIG

The Normal distributions is not a good model for commodity since, since it is not possible to model the fundamentals stylized features of the empirical commodity data (i.e. electricity data) like , jump sand heavy tail with a Gaussian distribution.
The NIG is a good alternative to the normal distribution since:

- Its distribution can model the heavy tails, kurtosis, and jump.
- The parameters of NIG distribution can be solved in a closed form

The Normal Variance-Mean Mixture Distribution

Barndorff and Neilsen defined in [?] the normal inverse gaussian (NIG) distribution as a special case of a normal variance-mean mixture distribution.

Definition: Normal Variance-Mean Mixture

In probability theory and statistics, a normal variance-mean mixture with mixing probability density g is the continuous probability distribution of a random variable Y of the form

$$Y = \alpha + \beta V + \sigma \sqrt{V} X, \tag{1}$$

where α and β are real numbers and $\sigma > 0$. The random variables X and V are independent, X is normal distributed with mean zero and variance one, and V is continuously distributed on the positive half-axis with probability density function g

The Normal Variance-Mean Mixture Distribution

The probability density function of a normal variance-mean mixture with mixing probability density g is

$$f(x) = \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2 v}} \exp\left(-(x-\alpha-\beta v)^2/(2\sigma^2 v)\right) g(v)dv \tag{2}$$

and its moment generating function is

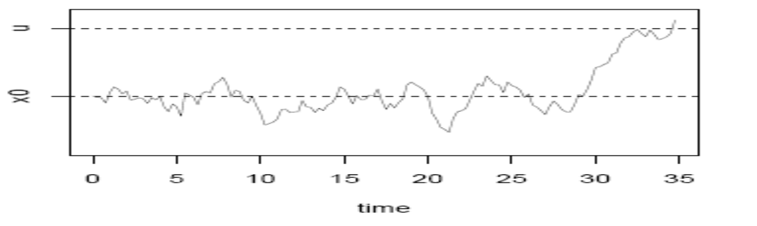
$$M(s) = \exp(\alpha s)\, M_g\left(\beta s + \frac{1}{2}\sigma^2 s^2\right), \tag{3}$$

where M_g is the moment generating function of the probability distribution with density function g, i.e.

$$M_g(s) = E\left(\exp(sV)\right) = \int_0^\infty \exp(sv)g(v)dv. \tag{4}$$

The Inverse Gaussian distribution

Barndorff and Neilsen defined the NIG distribution as a normal variance-mean mixtures when the mixture distribution is a inverse Gaussian distribution.
The inverse Gaussian describes the distribution of first passage time of a Brownian motion to a fixed level $u > 0$.(Recall that the Gaussian describes a Brownian Motion's level at a fixed time)



The Inverse Gaussian distribution

Let be $\{W(t),\ t > 0\}$ be a Wiener process in one dimension with positive drift μ and variance σ^2, with $W(0) = x_0$. Then the time required for $W(t)$ to reach the value $u > x_0$ for the first time (first passage time), is a random variable with inverse Gaussian (IG) distribution. Its probability density function is given by: (with $t > 0$)

$$f\,(t; u, \mu, \sigma, x_0) = \frac{u - x_0}{\sigma\sqrt{2\pi t^3}} \exp \frac{-(u - x_0 - \mu t)^2}{2\sigma^2 t}. \qquad (5)$$

alternatively

$$f\,(t; \nu, \lambda) = \left[\frac{\lambda}{2\pi t^3}\right]^{1/2} \exp \frac{-\lambda(t - \nu)^2}{2\nu^2 t}. \qquad (6)$$

In (6), the distribution of the IG has expectation ν and variance $\frac{\nu^3}{\lambda}$

The Normal Inverse Gaussian distribution

A random variable X follows a Normal Inverse Gaussian (NIG) distribution with parameters $\mu, \alpha, \beta, \delta$ if

$$X|Z = z \sim N(\mu + \beta z, z) \;\; and \;\; Z \sim IG\left(\delta, \sqrt{\alpha^2 - \beta^2}\right)$$

with parameters satisfying the following conditions $\alpha \geq |\beta|$, $\delta \geq 0$.
The density function of the NIG distribution with parameters $\mu, \alpha, \beta, \delta$ is explicitly given as

$$f(x) = \frac{\alpha\delta}{\pi} e^{\left\{\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)\right\}} \frac{K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{\sqrt{\delta^2 + (x-\mu)^2}} \quad (7)$$

where the function $K_1(x) = \frac{1}{2}\int_0^\infty \exp\left(-\frac{1}{2}x\left(t + t^{-1}\right)\right) dt$ is the modified Bessel function of the third kind and index 1.

The Normal Inverse Gaussian distribution:

Interpretation and Visualization

Each parameter of the normal inverse gaussian distribution can be interpreted as having a different effect on the distribution:

- α controls the behavior of the tails.
  Indeed, the steepness of the NIG increases monotonically with an increasing α This also has implications for the tail behavior, by the fact that large values of α implies light tails, while smaller values of α implies heavier tails as illustrated in the following figure.

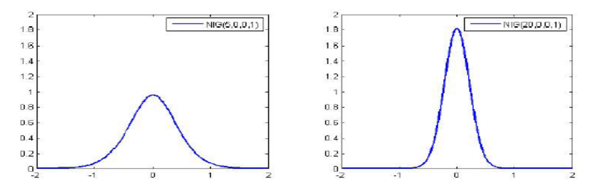


Abbildung: Example for different alphas

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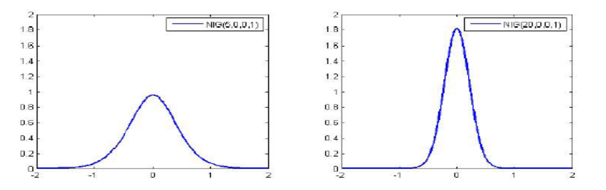
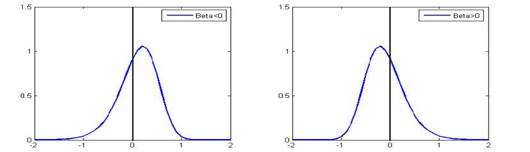


Abbildung: Example for different alphas

The Normal Inverse Gaussian distribution:

Interpretation and Visualization

- β is the Skwness parameter
  Indeed, a $\beta < 0$ implies a density skew to the left and $\beta > 0$ implies a density skew to the right. The skewness of the density increases as β increases. In the case where β is equal to 0, the density is symmetric around μ



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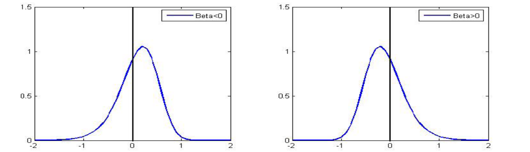

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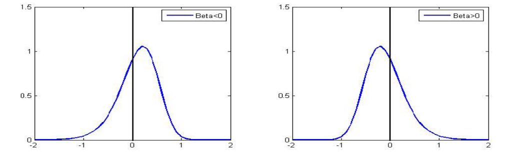

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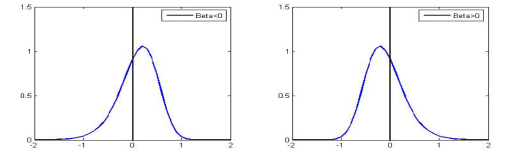

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Motivation
Normal Inverse Gaussian distribution
Calibration
The NIG Levy Process
Simulation

The NIG densities has the following properties:

Properties of the NIG Distribution

- The NIG distribution tends to the normal distribution $N\left(\mu, \sigma^2\right)$ when $\beta = 0,\ \alpha \to \infty\ \ \textit{and}\ \ \frac{\delta}{\alpha} = \sigma^2$.
- The NIG distribution has the scaling property:

$$X \sim NIG\left(\alpha, \beta, \delta_1, \mu_1\right) \Leftrightarrow cX \sim NIG\left(\alpha, \beta, \delta_1, \mu_1\right)$$

- The NIG distribution is closed under convolution:

$$NIG\left(\alpha, \beta, \delta_1, \mu_1\right) * NIG\left(\alpha, \beta, \delta_2, \mu_2\right) = NIG\left(\alpha, \beta, \delta_1 + \delta_2, \mu\right.$$

NIG Parameters Estimation: Method of Moment

The method of moments for the estimation of the NIG distribution parameters consists of solving a nonlinear system of equations for α, β, δ *and* μ.

$$E[X] = \mu + \frac{\kappa\delta}{(1-\kappa)^{1/2}}$$

$$V[X] = \frac{\delta^2}{\alpha(1-\kappa^2)^{3/2}}$$

$$S[X] = \frac{3\kappa}{\alpha^{1/2}(1-\kappa^2)^{1/4}}$$

$$K[X] = 3\frac{a\kappa^2+1}{\alpha(1-\kappa^2)^{3/2}},$$

where $\kappa = \frac{\beta}{\alpha}$ and $E[X],\ V[X],\ S[X],\ K[X]$ are The expected value, variance skewness and kurtosis of a random variable $X \sim NIG(\alpha, \beta, \delta, \mu)$ respectively.

NIG Parameters Estimation: Maximum Likelihood

Maximum Likelihood Estimation determines the parameter values that make the empirical data mmore likelytto have a NIG distribution than any other parameter values from a probabilistic viewpoint. in the praxis it consist to maximize the log likelihood function $\mathcal{L}_{NIG}\left(\alpha,\beta,\delta,\mu\right)$ with is given by:

$$\mathcal{L}_{NIG}\left(\alpha,\beta,\delta,\mu\right) = log\left(\frac{\left(\alpha^2-\beta^2\right)^{-1/4}}{\sqrt{2\pi\alpha^{-1}}\delta^{-1/2}K_{-1/2}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\right) - \frac{1}{2}\sum_{i=1}^{n} log\left(\delta^2+(x_i-\mu)^2\right)\sum_{i=1}^{n}\left[K_1\left(\alpha\sqrt{\delta^2+(x_i-\mu)^2}\right)+\beta\left(x_i-\mu\right)\right]$$

PS: This method assumes that the observations $x_i, \ldots x_n$ are independent

Software: useful tools

- matlab: function „fmincon“.
- R : packages „ghyp“, function „fit.NIGuv“

Levy Processes

Given a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, a Levy process $L = \{L_t, t \geq 0\}$ can be defined as an infinitely divisible continuous time stochastic process, $L_t : \Omega \longmapsto \mathbb{R}$ with stationary and independent increments.

Levy processes are more realistic than Gaussian driven processes, their structure allows the representation of:

- Jumps
- Skewness
- Excess kurtosis

NIG Process as a Levy Process

The characteristic function of the NIG is given by:

$$\theta_{NIG}(u;\alpha,\beta,\delta) = \exp\left(-\delta\left(\sqrt{\alpha^2-(\beta+iu)^2}-\sqrt{\alpha^2-\beta^2}\right)+i\mu u\right)$$

One can see that this is an infinitely divisible characteristic function ($\delta_n = \frac{\delta}{n}$).
Hence according to the previous definition the NIG Levy process

$$X^{NIG} = \left\{X_t^{NIG}, t \geq 0\right\} \tag{8}$$

with $X_0^{NIG} = 0$ and stationary and independent NIG distributed increments.

Definition: Levy Processes

A cadlag stochastic process $\{L_t, t \geq 0\}$ (Process whit paths which are right-continuous and have limits from the left) on $(\Omega, \mathcal{F}, \mathcal{P})$, with values in $\mathbb{R}$ such that $L_0 = 0$ is called a Lvy process if it possesses the following properties:

- Independent increments: for every increasing sequence of times $t_0 \ldots t_n$, the random variables $L_{t_0}, L_{t_1} - L_{t_1}, \ldots, L_{t_n} - L_{t_{n-1}}$are independent
- Stationary increments: the law of $L_{t+h} - L_t$ does not depend on t
- Stochastic continuity: $\forall \epsilon > 0. \lim_{h \to 0} \mathcal{P}(|L_{t+h} - L_t| \geq \epsilon)$

The distribution of the increments of a Levy process is describes by its characteristic exponent $\psi(u) = log(\phi(u))$, which satisfies the LevyKhintchine representation:

Definition: Levy Khintchine representation

$$\psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty} \left(exp(iux) - 1 - iux1_{\{|x|<1\}}\right)\nu(dx)$$

where $\gamma \in \mathbb{R}, \sigma^2 \geq 0$ and ν is a measure on $\mathbb{R}\setminus\{0\}$ with

$$\int_{\mathbb{R}\setminus\{0\}} \min\{x^2, 1\}\nu(dx) < \infty.$$

From the Levy Khintchine representation, a Levy process can be seen as having three parts:

1. a linear deterministic part (**drift**) controlled by the drift coefficient γ
2. a Brownian part (**diffusion**) controlled by the diffusion coefficient σ^2
3. a **pure jump** part controlled by The Levy measure $\nu(dx)$, which dictates how the jumps occur.

So, Levy processes can be represented by their Levy characteristics (or triplet) $\left[\gamma, \sigma^2, \nu(dx)\right]$

Levy Process: Levy-Triple for NIG Process

An NIG process has no Brownian component its diffusion coefficient σ^2 is also a pure jump process and its Levy triplet $\left[\gamma, \sigma^2, \nu\left(dx\right)\right]$ is given by:

$$\gamma = \frac{2\alpha\gamma}{\pi}\int_0^1 sinh\left(\beta x\right) K_1\left(\alpha x\right) dx$$

$$\nu\left(dx\right) = \frac{\delta\alpha}{\pi}\frac{exp\left(\beta x\right) K_1\left(\alpha\left|x\right|\right)}{\left|x\right|} dx$$

$$\sigma^2 = 0$$

Simulation of the Normal Inverse Gaussian Process

To simulate an NIG process we can consider it as a time-changed Brownian motion. namely we can see it has a Brownian motion subordinated by an Inverse Gaussian process. Recall that an NIG process $\{X_t, t \geq 0\}$ with parameters $\alpha > 0,\ -\alpha < \beta < \alpha$ *and* $\delta > 0$ can be obtained by time-changing a standard Brownian motion $\{W_t,\ t \geq 0\}$ with drift by an IG process $\{I_t,\ t \geq 0\}$ with parameters $a = 1$ and $b = \delta\sqrt{\alpha^2 - \beta^2}$. The stochastic process

$$X_t = \beta\delta^2 I_t + \delta W_{I_t} \tag{9}$$

is thus an NIG process with parameters $\alpha, \beta,$ *and* δ

Simulation of NIG as a subordinated Brownian Motion

Algorithm: simulation of Normal Inverse Gaussian Process

- *Set* $a = 1, \ b = \sqrt{\alpha^2 - \beta^2}$
- Simulate n IG variables IG_t with parameters $(ah, \ b)$
- Simulate n $i.i.d.$ $N(0,1)$ random variables
- Set $X_1 = 0$
- Simulate $X_t = \beta\delta^2 IG_t + \delta W_{IG_t}$

where W is the Brownian Motion and h the discretization step

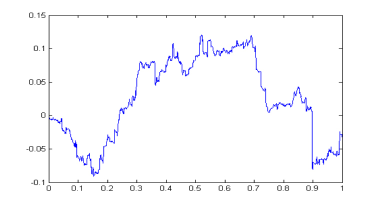

Abbildung: One paths of an NIG(40,-8,1) process