Mild Solutions of Stochastic Navier-Stokes Equations-III

Navier-Stokes Equations and Stokes Operator

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Outline

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1. Navier-Stokes Equations
   - Helmholtz-Leray Decomposition
   - Variational Formulation
   - Function Spaces
   - Weak Formulation of NSE
2. The Stokes Operator
   - The Stokes Operator in the No-Slip Case
   - Fractional Powers of A
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Helmholtz-Leray decomposition

The Helmholtz decomposition rewrites a vector field u in $\mathbb{R}^d$ ($d = 2, 3$) into the sum of a gradient and a curl vector. For vector fields on a bounded set, by taking into account the boundary conditions of the problem, we have its generalization called the Helmholtz-Leray decomposition.

For a given vector field w, we seek an orthogonal decomposition of the form

$$w = \nabla q + v, \quad \text{with} \quad \operatorname{div} v = 0. \tag{1}$$

Locally, v is a curl vector, $v = \operatorname{curl}\zeta$.

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Locally, v is a curl vector, $v = \text{curl}\zeta$.

Boundary conditions

Since $\operatorname{div} v = 0$, it implies

$$\triangle q = \operatorname{div} w. \tag{2}$$

Using equation (2) and boundary conditions, we can derive q from w. From (1), we obtain v.

In no-slip case, w vanishes at the boundary and we require only that

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Neumann problem

This implies that $\nabla q \cdot n = w \cdot n$, that is

$$\frac{\partial q}{\partial n} = w \cdot n \quad \text{on} \quad \partial\Omega. \tag{3}$$

Thus q is solution of the Neumann problem (2) and (3). It is uniquely defined up to an additive constant and v is equally well-defined.

Since $\text{div} v = 0$, it follows that v is the curl of a single-valued function ζ defined locally. If the boundary $\partial\Omega$ is connected-that is, if Ω has no holes (i.e., it is a simply connected set), then the conditions

$$\text{div} v = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad v \cdot n = 0 \quad \text{on} \partial\Omega$$

imply that ζ is a single-valued function in the whole domain Ω, with $v = \text{curl}\zeta$, as in the usual Helmholtz decomposition.

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Uniqueness

Different to the usual Helmoltz decomposition, the Helmholtz-Leray decomposition of w is unique (up to an additive constant for q). In fact

If $q = q_1 - q_2$ and $v = v_1 - v_2$, where (q_1, v_1) and (q_2, v_2) correspond to two such decompositions, then

$$\nabla q + v = 0$$

hence

$$\triangle q = 0.$$

In no-slip case, by (3), q is constant.

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Leray projector

Since the map $w \mapsto v$ is well defined, we denote it by

$$P_L : w \mapsto v(w)$$

This map is a projector; that is, if w is already divergence-free then $P_L w = w$. We will call it the Leray projector (for the corresponding boundary conditions).

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Projection using P_L

Recall the Navier-Stokes equations for a viscous, compressible, homogeneous flow:

$$\frac{\partial u}{\partial t} - \nu \triangle u + (u \cdot \nabla) u + \nabla p = f,$$
$$\nabla \cdot u = 0$$

Apply P_L to both sides of the above equation and use the divergence free condition, we have

$$P_L u = u, \quad P_L \frac{\partial u}{\partial t} = \frac{\partial u}{\partial t}, \quad \text{and} \quad P_L \nabla p = 0.$$

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Variational formulation of NSE

Therefore, we find that

$$\frac{du}{dt} + \nu Au + B(u) = P_L f, \tag{4}$$

where

$$Au = -P_L \triangle u, \quad B(u) = B(u, u) = P_L((u \cdot \nabla)u).$$

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Stokes operator

The operator A is the Stokes operator. In the space-periodic case,

$$Au = -P_L \triangle u = -\triangle u.$$

However, in the no-slip case,

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Simpler form

In general, we assume for simplicity that f is divergence free, so that $P_L f = f$; this can always be done, with the term $(I - P_L)f$ being added to the pressure.

Then we write

$$u' = F(t, u),$$

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Two important spaces

There are two fundamental spaces, denoted as H and V, for each choice of boundary conditions. They are natural spaces that take into account the boundary conditions, the incompressibility condition, and the physical quantities $e(u)$ and $E(u)$ (resp., the kinetic energy and the enstrophy).

The space H is the space of incompressible vector fields with finite kinetic energy and with the appropriate boundary conditions required by each initial and boundary value problem, and V is the space of incompressible vector fields with finite enstrophy and also with appropriate boundary conditions.

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Definitions

we consider a bounded domain Ω in $\mathbb{R}^d$ with $d = 2$ or 3, and the starting point is the space $\mathrm{L}^2(\Omega)$ of square integrable vector fields from Ω into $\mathbb{R}^d$. As previously remarked, this is the space of finite kinetic energy vector fields.

This space is endowed with the inner product

$$(u, v) = \int_\Omega u(x) \cdot v(x) dx$$

and the associated norm

$$|u| = (u, u)^{1/2} = \left\{ \int_\Omega |u(x)|^2 dx \right\}^{1/2}$$

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Sobolev space H^1

Another important space, which is associated with the notion of enstrophy, is the Sobolev space $H^1(\Omega)$. It consists of the space of vector fields on Ω that are square integrable (finite kinetic energy) and whose gradient is square integrable (finite enstrophy).

The associated inner product and norm are

$$((u, v))_1 = \frac{1}{L^2} \int_\Omega u(x) \cdot v(x) dx + \int_\Omega \sum_{i=1}^{d} \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i} dx$$

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with L as typical length (for example, the diameter of Ω).

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Norm for enstrophy

It is useful to distinguish the term related to the enstrophy in the inner product and in the norm in $H^1(\Omega)$.

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V and H

A mathematically rigorous and physically intuitive definition of the spaces V and H is as follows:

V is made up of all the limit points (in the distributional sense) of all the possible sequences of smooth vector fields u_m which are divergence-free, which satisfy the boundary conditions of the problem, and whose enstrophy remains bounded, that is, $E(u_m) < \infty$.

The space H is defined in a similar way, replacing the boundedness of the enstrophy by the boundedness of the kinetic energy, $e(u_m) < \infty$.

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No-slip boundary conditions

In the no-slip case, the domain Ω is assumed to be bounded and to have a smooth boundary. More precisely, we assume that:

Ω is open, bounded, and connected, with a C^2 boundary $\partial\Omega$ and such that Ω is on only one side of $\partial\Omega$.

By a C^2 boundary we mean that the boundary can be represented locally as the graph of a C^2 function (i.e., a twice differentiable function).

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H_{nsp} and V_{nsp}

It can be proved that

$$H_{nsp} = \{u \in \mathrm{L}^2(\Omega); \nabla \cdot u = 0, u \cdot n|_{\partial\Omega} = 0\}.$$

Similarly

$$V_{nsp} = \{u \in H^1(\Omega); \nabla \cdot u = 0, u|_{\partial\Omega} = 0\}$$

where H_{nsp} is endowed with norm $|\cdot|$ and inner product $(\cdot, \cdot)$ from $\mathrm{L}^2(\Omega)$. V_{nsp} is endowed with the norm $||\cdot||$ and inner product $((\cdot, \cdot))$.

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Equivalent definitions

In the mathematical literature, we usually define V_{nsp} and H_{nsp} as the closure in $H^1(\Omega)$ and in $\mathrm{L}^2(\Omega)$ (respectively) of the space

$$\mathcal{V}_{nsp} = \{u \in C_c^\infty(\Omega); \nabla \cdot u = 0\},$$

where $C_c^\infty(\Omega)$ denotes the space of infinitely differentiable vector fields with compact support in Ω.

The space $\mathcal{V}_{nsp}$ resembles the space of test functions in the theory of distributions by Schwartz. In fact, Leray introduced it before the theory of distributions and the Sobolev spaces had even been developed. It is easily checked that this definition coincides with the previous one.

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Test function space

We will derive a weak form of the original Navier-Stokes equation in this section. First, a test function v is assumed to be divergence-free and to satisfy the same boundary conditions as u.

Consider at each instant of time the vector field

$$x \in \Omega \mapsto u(x, t).$$

For simplicity, we will denote this vector field at time t by $u(\cdot, t)$ or simply by $u(t)$.

Then, let $v = v(x)$ be a test function belonging to V. Take the inner product of the NSE with $v(x)$ and integrate over Ω.

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Derivation of weak formulation

Using integration by parts, we have

$$\int_\Omega \frac{\partial u}{\partial t}(x,t)\cdot v(x)dx = \frac{d}{dt}\int_\Omega u(x,t)\cdot v(x)dx$$

and

$$-\int_\Omega \triangle u(x,t)\cdot v(x)dx = -\sum_{i,j=1}^{d}\int_{\partial\Omega}\frac{\partial u_i}{\partial n}(x,t)\cdot v_i(x)dS(x) + \sum_{i,j=1}^{d}\int_\Omega \frac{\partial u_i}{\partial x_j}(x,t)\frac{\partial v_i}{\partial x_j}(x)dx = \sum_{i,j=1}^{d}\int_\Omega \frac{\partial u_i}{\partial x_j}(x,t)\frac{\partial v_i}{\partial x_j}(x)dx$$

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Derivation of weak formulation

By boundary and divergence-free conditions on the test function v, we have

$$\int_\Omega \nabla p(x,t)\cdot v(x)dx = \int_{\partial\Omega} p(x,t)v(x)\cdot n(x)dS(x) - \int_\Omega p(x,t)\mathrm{div}\, v(x)dx$$

Hence

$$\frac{d}{dt}\int_\Omega u(x,t)\cdot v(x)dx + \nu \sum_{i,j=1}^{d}\int_\Omega \frac{\partial u_i}{\partial x_j}(x,t)\frac{\partial v_i}{\partial x_j}(x)dx$$

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Derivation of weak formulation

By boundary and divergence-free conditions on the test function v, we have

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Weak formulation of NSE

To simplify the notation, we define

$$b(\varphi, \psi, \theta) = \sum_{i,j=1}^{d} \int_\Omega \varphi_i(x) \frac{\partial \psi_j(x)}{\partial x_i} \theta_j(x) dx.$$

Using previous defined norms, the weak equation can be expressed as follows:

The function $t \mapsto u(t)$ takes its values in V and satisfies

$$\frac{d}{dt}(u(t), v) + \nu((u(t), v)) + b(u(t), u(t), v) = (f(t), v)$$

for every test function $v \in V$. With initial conditions

$$u(0) = u_0$$

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Definition

The Stokes operator was formally defined by $Au = -P_L \triangle u$, where P_L is the Helmholtz-Leray projector and $\triangle$ is the Laplacian. For a rigorous definition we must also define the domain $D(A)$ of A, that is, the space of functions in H for which Au makes sense.

One can show that

$$Au = -P_L \triangle u \quad \text{for} \quad u \in D(A) = V \cap H^2(\Omega),$$

where $d = 2$ or 3, Ω is bounded in $\mathbb{R}^d$ and, in the no-slip case, is smooth.

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We have

$$Au = -P_L \triangle u \quad \text{for} \quad u \in D(A) = V_{nsp} \cap H^2(\Omega) \tag{5}$$

and A is one-to-one from $D(A)$ onto H.

Note that, since a vector field u in $D(A)$ belongs to $H^2(\Omega)$, the Laplacian $\triangle u$ makes sense and is square integrable, so that the Helmholtz-Leray projector P_L can be applied to it to yield a vector field Au in H.

It can be shown that, in fact, the Stokes operator maps $D(A)$ onto H. Hence, the inverse A^{-1} is well-defined and takes H onto $D(A)$. Because Ω is bounded and smooth, we have by Rellich's theorem that $H^2(\Omega)$ is compactly embedded into $\mathrm{L}^2(\Omega)$. It follows also that $D(A)$ is compactly embedded into H.

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Properties of A

Integration by parts shows that the Stokes operator is symmetric

$$(Au, v) = (u, Av) \quad \text{for all} \quad u, v \in D(A).$$

It turns out that A^{-1} is also self-adjoint. There exists an orthonormal basis $\{w_m\}_{m\in\mathbb{N}}$ in H and a sequence of real eigenvalues $\{\sigma_m\}_{m\in\mathbb{N}}$ accumulating at zero, so that

$$A^{-1}w_m = \sigma_m w_m, \quad m = 1, 2, \cdots,$$

Setting $\lambda_m = 1/\sigma_m$, we see that

$$Aw_m = \lambda_m w_m, \quad m = 1, 2, \cdots .$$

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Properties of A

Since A is a positive definite operator (Integration by parts),

$$(Au, u) = ||u||^2 > 0 \quad \text{for all} \quad u \in D(A), u \neq 0$$

each λ_m is positive.

Moreover,

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots, \lambda_m \to +\infty \quad as \quad m \to +\infty.$$

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Fourier expansion

Because $\{w_m\}_{m\in\mathbb{N}}$ is an orthonormal basis in H, we can expand each vector field u in terms of its projection onto each eigenspace:

$$u = \sum_{m=1}^{\infty} (u, w_m) w_m \quad \text{for} \quad u \in H$$

Denote $\hat{u}_m = (u, w_m)$, by Parseval identity, $|u|^2 = \sum_{m=1}^{\infty} |\hat{u}_m|^2$. Similarly,

$$||u||^2 = \sum_{m=1}^{\infty} \lambda_m |\hat{u}_m|^2, \quad |Au|^2 = \sum_{m=1}^{\infty} \lambda_m^2 |\hat{u}_m|^2.$$

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Fractional powers of A

Define spaces V_{2s}, for all $s \geq 0$, by setting

$$u \in V_{2s} \Leftrightarrow \sum_{m=1}^{\infty} \lambda_m^{2s} |\hat{u}_m|^2 < \infty$$

The powers A^s $(s \geq 0)$ of A are defined by

$$A^s u = \sum_{m=1}^{\infty} \lambda_m^s \hat{u}_m w_m.$$

The domain of A^s in H is $D(A^s) = V_{2s}$. Moreover, A^r maps $D(A^s) = V_{2s}$ into $D(A^{s-r}) = V_{2(s-r)}$.

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Norm and inner product for $D(A^s)$

The domain $D(A^s) = V_{2s}$ is endowed with the inner product

$$(u, v)_{D(A^s)} = (A^s u, A^s v) = \sum_{m=1}^{\infty} \lambda_m^{2s} \hat{u}_m \hat{v}_m,$$

where $\hat{v}_m = (v, w_m)$, and with the norm

$$|u|_{D(A^s)}^2 = |A^s u|^2 = \sum_{m=1}^{\infty} \lambda_m^{2s} |\hat{u}_m|^2.$$

For $s = 1/2$, we have $D(A^{1/2}) = V$. In fact,

$$|u|_{D(A^{1/2})}^2 = \sum_{m=1}^{\infty} \lambda_m |\hat{u}_m|^2 = \sum_{m=1}^{\infty} (u, \lambda_m w_m)(u, w_m)$$

$$= \sum_{m=1}^{\infty} (u, Aw_m)(u, w_m) = \sum_{m=1}^{\infty} (Au, w_m)(u, w_m) = (Au, u) = ||u||^2.$$

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Negative powers of A

For $s > 0$, we define $D(A^{-s})$ to be the completed space of H for the norm

$$|u|_{D(A^{-s})} = \sum_{m=1}^{\infty} \lambda_m^{-2s} |(u, w_m)|^2.$$

We also define

$$A^{-s}u = \sum_{m=1}^{\infty} \lambda_m^{-s}(u, w_m) w_m \quad \text{for} \quad u \in D(A^{-s}).$$

Moreover,

$$D(A^{-s}) \sim D(A^s)', \quad D(A^s) \subset H \subset D(A^{-s}).$$

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Interpolation inequalities

$$|A^s u| \leq |A^{s_1} u|^\theta |A^{s_2} u|^{1-\theta},$$

for any real $s_1 \leq s \leq s_2$, where θ is given by

$$s = s_1\theta + s_2(1-\theta)$$

Assume that the domain Ω is of class C^{m+2}, where $m \in \mathbb{N}$. Then, if f belongs to $H \cap H^m(\Omega)$, the solution u of the Stokes problem $Au = f$ belongs to $H \cap H^{m+2}(\Omega)$. Moreover, the corresponding pressure p belongs to $H^{m+1}(\Omega)$.

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Linear functional in V'

For each vector field u in V, the map

$$v \mapsto ((u, v))$$

defines a linear functional in V.

It is continuous, since the Cauchy-Schwarz inequality implies that

$$((u, v)) \leq ||u|| \cdot ||v|| \leq C_u ||v||$$

for some constant C_u and for all $v \in V$.

This linear functional belongs to V' and can be represented by an element l in V'. Each u determines uniquely an element $l(u)$in V' in this way.

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The map $u \mapsto l(u)$ is linear, so we denote it by Au. By definition,

$$A : V \to V', \qquad (Au, v) = ((u, v)) \quad \text{for all} \quad u, v \in V.$$

For smooth u, using integration by parts, for all $v \in V$,

$$(Au, v) = ((u, v)) = -(\triangle u, v) = (-P_L \triangle u, v),$$

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Properties

Map A is an isomorphism between V and its dual V'. By previous calculation, $A = -P_L\triangle$, so that $-A$ is in general not the Laplace operator.

By the Riesz representation theorem, A is one-to-one from V onto V' with

$$||Au||_{V'} = ||u|| \quad \text{for all} \quad u \in V$$

The domain $D(A)$ of A in H in this context can be defined as

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Remark

In the no-slip case, if the domain Ω is not regular enough, then one would not recover the characterization (5) for the domain of A, and the two definitions of the Stokes operator are not known to be identical.